11

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ON RELATIVE PROPERTY (T)

TO BOB ZIMMER ON HIS 60TH BIRTHDAY

Abstract

We present families of pairs ($H \ltimes A, A$) with relative property (T), where H is a locally compact group acting continuously by automorphisms on a locally compact abelian group A. The paper is completely self-contained.

1. Introduction

Property (T) for groups and for pairs (also referred to as relative property (T)) was first introduced by D. Kazhdan in his thesis [14] to show that lattices in semisimple Lie groups are finitely generated. His proof that $\mathrm{SL}(n, \mathbb{R})$ has property (T) for $n \geq 3$ follows from the fact that the pair $(\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has the relative property (T).

Recall that a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of a locally compact group G almost has invariant vectors if for every compact subset $K \subset G$ and for every $\epsilon > 0$ there exists $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ such that

$$\sup_{g \in K} \|\pi(g)\xi - \xi\| < \epsilon .$$

Equivalently, one says that the representation π weakly contains the trivial representation $\mathbb{1}_G$. Then a group G has property (T) if any representation that almost has invariant vectors actually has invariant vectors. A typical example of a group with property (T) is a connected semisimple real Lie group all of whose factors have real rank at least 2, for example $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$, or any lattice therein.

DEFINITION 1.1. Let $L < G$ be a subgroup of a locally compact group G. The pair (G, L) has *relative property (T)* if whenever $(\pi, \mathcal{H})$ is a continuous unitary representation of G, which almost has invariant vectors; there are nonzero L-invariant vectors.

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Property (T) has several diverse applications in representation theory, ergodic theory, operator algebras, lattices in algebraic groups over local fields, geometric group theory, and the theory of networks. Aside the implicit use of relative property (T) in [14], among its first applications there is the construction of an infinite family of expanders [15], or the solution of Ruziewicz problem for $\mathbb{R}^n$, [16], both due to Margulis. Further applications include Gaboriau and Popa's construction [11] of a noncountable family of free ergodic measure-preserving non-orbit equivalent actions of the free group $\mathbb{F}_r$ in $r \geq 2$ generators on a standard probability space (see also [21] or [10] for generalizations) or Popa's construction [20, 22, 23] of a factor of type II_1 with trivial fundamental group. Finally, although the fact that a pair (G, L) has relative property (T) is a weaker condition than requiring that the group G itself has property (T), in some situations relative property (T) will suffice; for example, Navas [19] has proven that for the nonexistence of "interesting" C^2-actions on the circle, the relative property (T) of a pair (Γ, Γ_0), where Γ_0 is normal in Γ, will suffice.

Just to give a flavor of how relative property (T) differs from property (T), although we will not use these facts here, observe that if (G, L) has relative property (T) this does not imply that either G or L have property (T), and moreover, $(\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T) but neither $\mathrm{SL}(2, \mathbb{Z})$ nor $\mathbb{Z}^2$ have property (T). Furthermore, the fact that (G, L) has relative property (T) does not imply that either G or L is compactly generated, as the example of $(\mathrm{SL}(3, \mathbb{Z}) \times \Lambda, \mathbb{F}_\infty \times \{e\})$, where Λ is any group that is not finitely generated shows.[1]

Just as for property (T), several different characterizations of relative property (T) are available, for example, in terms of strongly ergodic actions, in terms of Von Neumann algebras, of positive definite functions, of isometric actions of Hilbert spaces, and so on; moreover, it is possible to define relative property (T) for pairs (G, X) where X is any subset [6]. We refer the reader to [13] and [6] and to the references therein.

It goes without saying that if either G or L is a group with property (T), then the pair (G, L) has the relative property (T); nontrivial examples of pairs with relative property (T) have been constructed in [5], in [8] (see Theorem 1.3 and the comments thereafter) or in [25] (see the last paragraph of this introduction). The scope of this note is to present families of pairs (G, L) with relative property (T) whose general form is the semidirect product $G = H \ltimes A$ of a

1. Note, however, that if (G, L) has relative property (T), there exists a compactly generated subgroup H containing L such that (H, L) has relative property (T); see [6].

locally compact group H acting continuously by automorphisms on a locally compact abelian group A and $L = A$.

1.1.

Let Γ be a discrete group, S a finite set of prime numbers, $\mathbb{Z}[S]$ the ring obtained inverting the primes in S and $\rho : \Gamma \to \mathrm{GL}_N(\mathbb{Z}[S])$ a homomorphism. We will denote by $\rho_p : \Gamma \to \mathrm{GL}_N(\mathbb{Q}_p)$ the representation obtained by injecting $\mathbb{Z}[S]$ into $\mathbb{Q}_p$, where $\mathbb{Q}_\infty := \mathbb{R}$.

THEOREM 1.2. *Assume that the image $\rho(\Gamma)$ of Γ is Zariski dense in SL_N. Then the following are equivalent:*

i) *the pair $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ has relative property (T);*
ii) *the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[S]^N$ is finitely generated;*
iii) *for every $p \in S \cup \{\infty\}$ the image $\rho_p(\Gamma) < \mathrm{SL}_N(\mathbb{Q}_p)$ is not bounded; and*
iv) *there is no $\rho_p^*(\Gamma)$-invariant probability measure on $\mathbb{P}((\mathbb{Q}_p^N)^*)$, for $p \in S \cup \{\infty\}$.*

We remark that (ii) is in contrast with the fact that, unless $S = \emptyset$, the ring $\mathbb{Z}[S]$ is not finitely generated as a $\mathbb{Z}$-module; moreover, it is remarkable that although the relative property (T) is analytic in nature, the equivalent property in (ii) above is purely algebraic. As it will be clear from the proof (see Proposition 4.5) Zariski density is not needed for the equivalence of the assertions (i) and (iv).

The motivation to establish results of the type of Theorem 1.2 comes from [8], where the following is shown:

THEOREM 1.3. (FERNÓS, [8]) *Let Γ be a finitely generated group. Then the following are equivalent:*

i) *there exists a representation $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{R})$ such that the real points $\overline{\rho(\Gamma)}^Z(\mathbb{R})$ of the Zariski closure of the image of ρ is not an amenable group; and*
ii) *there exists a finite set of primes S, an integer $N \in \mathbb{N}$, and a representation $\Gamma \to \mathrm{SL}_N(\mathbb{Z}[S])$ such that $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ has relative property (T).*

EXAMPLE 1.4.

1) According to the Tits alternative, if k is any local field of zero characteristic and $\Gamma < \mathrm{GL}(n, k)$ is a finitely generated subgroup, then either Γ is

virtually solvable or it contains a free subgroup $\mathbb{F}_2$ in two generators that is Zariski dense in Γ. Since $\Gamma = \mathrm{SL}_N(\mathbb{Z})$ is not virtually solvable, then there is a Zariski-dense $\mathbb{F}_2 < \mathrm{SL}_N(\mathbb{Z})$ and thus $(\mathbb{F}_2 \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has relative property (T).

2) A topological version of the Tits alternative shown by Breuillard and Gelander [4] asserts that if k_i are local fields of zero characteristic and $\Gamma < \prod \mathrm{GL}(n, k_i)$ is a finitely generated subgroup, then either Γ contains an open solvable subgroup, or a finitely generated free subgroup that is dense in Γ.

Using this result we show the following:

COROLLARY 1.5. *For every natural number $N \in \mathbb{N}$ and every nonempty finite set of primes S, there is a finitely generated free group $\Gamma < \mathrm{SL}_N(\mathbb{Z}[S])$ such that the pair $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ has relative property (T).*

1.2.

Now let $\rho : \mathrm{G} \to \mathrm{GL}(V)$ be a rational representation defined over $\mathbb{Q}$, where G is a connected, semisimple $\mathbb{Q}$-group. Let $\rho_{\mathbb{R}} : \mathrm{G}_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ be the representation on the level of real points. We say that $\rho_{\mathbb{R}}$ is *totally unbounded* if for any $\mathrm{G}_{\mathbb{R}}$-invariant subspace $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ of positive dimension, the group $\rho(\mathrm{G}_{\mathbb{R}})|_{W_{\mathbb{R}}} < \mathrm{GL}(W_{\mathbb{R}})$ is unbounded.

THEOREM 1.6. *Let $\Lambda \subset V_{\mathbb{Q}}$ be a $\mathbb{Z}$-module of maximal rank invariant under $\mathrm{G}_{\mathbb{Z}}$. Let $\Gamma < \mathrm{G}_{\mathbb{Z}}$ and assume that Γ is Zariski dense in G. The following are equivalent:*

i) *the pair $(\Gamma \ltimes \Lambda, \Lambda)$ has relative property (T); and*
ii) *the representation $\rho_{\mathbb{R}} : \mathrm{G}_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is totally unbounded.*

A remark is in order regarding the hypothesis of Zariski density of Γ in G. There is a result of Borel [3, theorem 1] to the extent that if G contains no connected normal $\mathbb{Q}$-subgroups $N \neq \{e\}$ such that $N_{\mathbb{R}}$ is compact, then $\mathrm{G}_{\mathbb{Z}}$ is Zariski dense in G; this avoids the situation in which if $G = G_1 \times G_2$ with G_i $\mathbb{Q}$-groups, then $G_{1,\mathbb{Z}} \times G_{2,\mathbb{Z}} = G_{\mathbb{Z}} \subset G_{\mathbb{R}} = G_{1,\mathbb{R}} \times G_{2,\mathbb{R}}$ with $G_{2,\mathbb{R}}$ compact, which would prevent the Zariski density of $G_{\mathbb{Z}}$. So Theorem 1.6 applies to any finite-index subgroup Γ of $\mathrm{G}_{\mathbb{Z}}$, provided $G_{\mathbb{R}}$ has no compact factors.

If $\rho_{\mathbb{R}}$ is irreducible with unbounded image, it is trivially totally unbounded and hence the pair $(\Gamma \ltimes \Lambda, \Lambda)$ has relative property (T). The construction of such a representation for a group defined over $\mathbb{Q}$, which is either adjoint or simply connected will be given in Lemma 4.7. As an application we have the following:

COROLLARY 1.7. *If G is a connected real algebraic semisimple Lie group without compact factors and $\Gamma < G$ is an arithmetic lattice, then there is a linear representation $\Gamma' \to \mathrm{SL}_N(\mathbb{Z})$ of a finite-index subgroup $\Gamma' < \Gamma$ such that $(\Gamma' \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has relative property (T).*

Theorem 1.6 is motivated by [25, theorems 1 and 4], where the same assertion as in Corollary 1.7 is proven, under the additional hypothesis that the group G is absolutely simple. Corollary 1.7 can also be deduced from [8, theorems 3 and 7.1] by observing that such an arithmetic lattice Γ satisfies property (F_∞) in [8]. Just like in [25] the value of the integer N is explicitly given (see the proof of Lemma 4.7). Moreover, if $\Gamma' < \Gamma$ is of finite-index $k := [\Gamma, \Gamma']$ and $(\Gamma' \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has relative property (T), then $(\Gamma \ltimes \mathbb{Z}^{kN}, \mathbb{Z}^{kN})$ has relative property (T), [8, step B in the proof of theorem 7.1].

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2. Generalities

We collect here, with proofs, few classical facts about relative property (T). We start by recording here the following observation:

LEMMA 2.1. *Let H_i, B_i be locally compact groups such that H_i acts on B_i via a continuous action $H_i \to \mathrm{Aut}(B_i)$, $i = 1, 2$, and let $q : H_1 \ltimes B_1 \to H_2 \ltimes B_2$ be a continuous homomorphism such that $q(B_1) = B_2$. If $(H_1 \ltimes B_1, B_1)$ has relative property (T), then also $(H_2 \ltimes B_2, B_2)$ has relative property (T).*

LEMMA 2.2. *Let $H, A_1, A_2, \dots, A_n$ be locally compact groups such that H acts on A_j via a continuous action $H \to \mathrm{Aut}(A_j)$. The following are equivalent:*

i) *the pair $(H \ltimes (A_1 \times \cdots \times A_n), A_1 \times \cdots \times A_n)$ has relative property (T); and*
ii) *the pair $(H \ltimes A_j, A_j)$ has relative property (T) for $j = 1, \dots, n$.*

Proof. The implication (i)⇒(ii) follows from Lemma 2.1 by considering the homomorphisms $H \ltimes (A_1 \times \cdots \times A_n) \to H \ltimes A_j$, while (ii)⇒(i) is in [8, lemma 5.2], where it is used that relative property (T) is a property closed under certain extensions. □

The following corollary is essential in the proof of Proposition 4.5.

COROLLARY 2.3. *Let H and M be locally compact groups, $\Lambda < M$ a cocompact lattice and let $\rho : H \to \mathrm{Aut}(M)$ be a continuous action by automorphisms preserving Λ. Then the pair $(H \ltimes \Lambda, \Lambda)$ has relative property (T) if and only if the same holds for the pair $(H \ltimes M, M)$.*

Although the above result is all we need, we are going to prove the following more general result from which the corollary can be obtained at once by setting $L := H \ltimes \Lambda$ and $A := M \trianglelefteq H \ltimes M =: G$.

PROPOSITION 2.4. *Let G be a locally compact group, $A \trianglelefteq G$ a normal subgroup, and $L < G$ a closed subgroup.*

1) *If (G, A) has relative property (T) and $L\backslash G$ has a finite G-invariant measure, then the pair $(L, L \cap A)$ also has relative property (T).*
2) *If $A/L \cap A$ is compact with finite A-invariant measure and $(L, L \cap A)$ has relative property (T), then (G, A) has relative property (T).*

In order to prove this, we will use a strengthening of relative property (T). We recall first that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a locally compact group G, we say that a sequence $\{\xi_n\} \subset \mathcal{H}$ of vectors is *asymptotically $\pi(G)$-invariant* if $\|\xi_n\| = 1$ for all n and for every compact subset $K \subset G$ we have that

2.1 $$\lim_{n\to\infty} \sup_{k\in K} \|\pi(k)\xi_n - \xi_n\| = 0\,.$$

If a group is σ-compact, a representation almost has invariant vectors if and only if it has a sequence of asymptotically invariant vectors. Denote by $\mathcal{H}^G$ the subspace of $\mathcal{H}$ consisting of $\pi(G)$-invariant vectors. Then it is well known that if G is a group with property (T) and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G with a sequence $\{\xi_n\}$ of asymptotically $\pi(G)$-invariant vectors, then

$$\lim_{n\to\infty} d(\xi_n, \mathcal{H}^G) = 0,$$

Where d denotes the distance to a subspace.

Similarly one has the following:

LEMMA 2.5. *Let $A \trianglelefteq G$ be a normal subgroup of a locally compact group G such that the pair (G, A) has relative property (T). If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous* *unitary representation and* $\{\xi_n\} \subset \mathcal{H}$ *is a sequence of asymptotically* $\pi(G)$*-invariant vectors, then*

$$\lim_{n\to\infty} d(\xi_n, \mathcal{H}^A) = 0\,.$$

Proof. Since A is normal in G, the Hilbert subspace $\mathcal{H}^A$ is $\pi(G)$-invariant, so that

$$\mathcal{H} = \mathcal{H}^A \oplus (\mathcal{H}^A)^\perp$$

is an orthogonal decomposition into $\pi(G)$-invariant subspaces. Let $\{\xi_n\} \subset \mathcal{H}$ be a sequence of asymptotically invariant vectors and let $\xi_n = \zeta_n + \zeta'_n$ be the corresponding orthogonal decomposition, so that showing that $\lim_{n\to\infty} d(\xi_n, \mathcal{H}^A) = 0$ is equivalent to showing that $\lim_{n\to\infty} \|\zeta'_n\| = 0$.

Let us assume by contradiction that $\limsup_{n\to\infty} \|\zeta'_n\| \neq 0$, that is, for some $\epsilon > 0$, let $\{\zeta'_{n_k}\} \in (\mathcal{H}^A)^\perp$ be a subsequence such that

2.2 $$\text{for all } k \in \mathbb{N} \text{ we have } \|\zeta'_{n_k}\| \geq \epsilon\,.$$

Since $(\mathcal{H}^A)^\perp$ is $\pi(G)$-invariant and the orthogonal projection is norm decreasing, we have that for all $g \in G$

$$\|\pi(g)\zeta'_{n_k} - \zeta'_{n_k}\| \leq \|\pi(g)\xi_{n_k} - \xi_{n_k}\|$$

from which, using Equation **2.2**, we obtain that

$$\left\| \pi(g)\left(\frac{\zeta'_{n_k}}{\|\zeta'_{n_k}\|}\right) - \frac{\zeta'_{n_k}}{\|\zeta'_{n_k}\|} \right\| \leq \frac{1}{\epsilon}\|\pi(g)\xi_{n_k} - \xi_{n_k}\|\,.$$

If now $K \subset G$ is any compact set, since the sequence $\{\xi_n\} \subset \mathcal{H}$ is asymptotically $\pi(G)$-invariant, we have

$$\lim_{k\to\infty} \sup_{k\in K} \left\| \pi(g)\left(\frac{\zeta'_{n_k}}{\|\zeta'_{n_k}\|}\right) - \frac{\zeta'_{n_k}}{\|\zeta'_{n_k}\|} \right\| \leq \lim_{k\to\infty} \sup_{k\in K} \frac{1}{\epsilon}\|\pi(g)\xi_{n_k} - \xi_{n_k}\| = 0\,,$$

from which we deduce that the sequence $\{\zeta'_{n_k}/\|\zeta'_{n_k}\|\}$ is also asymptotically $\pi(G)$-invariant. Hence $\pi(G)|_{(\mathcal{H}^A)^\perp}$ has non-zero $\pi(A)$-invariant vectors, which is a contradiction. □

Proof of Proposition 2.4.

1) Let $(\pi, \mathcal{H})$ be a representation of L that almost has invariant vectors. Since weak containment is preserved under induction, the representation $\omega := \mathrm{Ind}_L^G(\pi)$ of G induced from π weakly contains the representation $\mathrm{Ind}_L^G(\mathbb{1}_L)$ induced to G from the identity representation of L; the latter in turn has invariant vectors since $L\backslash G$ has finite G-invariant measure and hence, by transitivity of weak containment, ω almost has invariant vectors. Thus, since (G, A) has relative property (T), the Hilbert space

$$\begin{aligned}\mathcal{L}_\omega := \{f : G \to \mathcal{H} : f \text{ is measurable}, f(\ell g) = \pi(\ell)(f(g)), \\ \text{for almost every } g \in G \text{ and for all } \ell \in L \\ \text{and } \int_{L\backslash G} \|f(g)\|^2 d\mu(g) < \infty\}\end{aligned}$$

of the representation ω has nonzero A-invariant vectors with respect to the action $(\omega(g)f)(x) = f(xg)$ for all $g \in G$ and almost all $x \in G$. Let $f \in \mathcal{L}_\omega$ be such nonzero A-invariant vector. Then for all $a \in A \cap L$ and almost all $x \in G$ we have

$$\pi(a)(f(x)) = f(ax) = f(x(x^{-1}ax)) = f(x),$$

which shows that the space of $(L \cap A)$-invariant vectors in $\mathcal{H}$ is not trivial and hence shows (1).

2) Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a representation of G and $\{\xi_n\} \in \mathcal{H}$ be a sequence of asymptotically $\pi(G)$-invariant vectors.[2] Then $\{\xi_n\}$ is also asymptotically $\pi(L)$-invariant and since by hypothesis $(L, L \cap A)$ has relative property (T) and $L \cap A$ is normal in L, by Lemma 2.5 there exists a sequence $\{\zeta_n\} \in \mathcal{H}^{L\cap A}$ such that $\lim_{n\to\infty} \|\xi_n - \zeta_n\| = 0$. After rescaling if necessary, we may assume that $\|\zeta_n\| = 1$. For all $n \in \mathbb{N}$ the vectors

$$\eta_n := \int_{A/A\cap L} \pi(a)\zeta_n \, d\lambda(a) \in \mathcal{H},$$

where λ is the A-invariant probability measure on $A/A \cap L$, are obviously $\pi(A)$-invariant, and we only need to show that there exists an $n \in \mathbb{N}$ such that $\eta_n \neq 0$.

2. For the sake of simplicity we assume in the sequel that all locally compact groups are σ-compact. If not, the same arguments apply by replacing sequences with generalized sequences.

To see this, let $F \subset A \subset G$ be a compact fundamental domain for the action of $A \cap L$ on A. Then for all $a \in A$ there exists $f_a \in F$ and $\ell_a \in A \cap L$ such that $a = f_a \ell_a$, so that, by $\pi(L \cap A)$-invariance of ζ_n,

$$\|\eta_n - \zeta_n\| = \left\| \int_{A/A\cap L} \pi(a)\zeta_n \, d\lambda(a) - \zeta_n \right\| \leq \int_{A/A\cap L} \|\pi(a)\zeta_n - \zeta_n\| \, d\lambda(a)$$
$$\leq \sup_{f_a \in F} \|\pi(f_a)\zeta_n - \zeta_n\| .$$

Now we observe that $\{\zeta_n\}$ is asymptotically $\pi(G)$-invariant: indeed for all $g \in G$,

$$\|\pi(g)\zeta_n - \zeta_n\| \leq 2\|\zeta_n - \xi_n\| + \|\pi(g)\xi_n - \xi_n\| .$$

It follows that $\lim_{n\to\infty} \|\eta_n - \zeta_n\| = 0$ and thus in particular $\eta_n \neq 0$ for some $n \in \mathbb{N}$. □

LEMMA 2.6. *Let $\Gamma \ltimes A$ be a semidirect product of discrete groups, with A abelian, and assume that $(\Gamma \ltimes A, A)$ has relative property (T). Then the $\mathbb{Z}[\Gamma]$-module A is finitely generated.*

Proof. Let $A_n \uparrow A$ be an increasing sequence of Γ-invariant subgroups finitely generated over $\mathbb{Z}[\Gamma]$ and $\cup A_n = A$. Consider the $\Gamma \ltimes A$-regular action on

$$\bigoplus_{n\geq 1} \ell^2(\Gamma \ltimes A/\Gamma \ltimes A_n) ;$$

the sequence $\delta_{e(\Gamma \ltimes A_n)}$ is asymptotically invariant, hence there is an A-invariant vector in some $\ell^2(\Gamma \ltimes A/\Gamma \ltimes A_n)$; this implies that there is a finite A-orbit in $(\Gamma \ltimes A/\Gamma \ltimes A_n) \cong A/A_n$ and hence $|A/A_n| < +\infty$. □

3. Algebraic Actions and Measure Theory

We start by recalling the classical fact that if k is a local field and H_k consists of the k-points of an algebraic group defined over k, then any k-algebraic action of H_k on the k-points of a k-variety has orbits, which are locally closed in the Hausdorff topology [2]. Recall, moreover, that an action is almost effective if the intersection of the stabilizers is a finite group.

At the heart of what we are doing is the following theorem of Gromov: the proof we recall here is more elementary than the original one and appears in [1, théorème 6.5].

THEOREM 3.1. (GROMOV) *Let k be a local field, H a k-algebraic group, W a k-algebraic variety and $H \times W \to W$ a k-algebraic action. Let $\mu \in \mathcal{M}^1(W_k)$ be a probability measure on W_k such that the Zariski closure of its support is W, and assume that the action is almost effective. Then*

$$\mathrm{Stab}_{H_k}(\mu) := \{h \in H_k : h_*\mu = \mu\}$$

is compact.

Proof. Observe first of all that since $\mathrm{supp}(\mu) \subset W_k$, then $\overline{\mathrm{supp}(\mu)}^Z \subset \overline{W_k}^Z$, which, together with the hypothesis on the support of μ implies that W_k is Zariski dense in W. This and the fact that the action is almost effective imply that

3.1 $$\bigcap_{x \in W_k} \mathrm{Stab}_{H_k}(x) = \bigcap_{x \in W} \mathrm{Stab}_{H_k}(x) \text{ is finite}.$$

The Noetherian property for the k-algebraic group $\mathrm{Stab}_{H_k}(x)$ implies that there exist $x_1, \dots, x_n \in W_k$ such that

3.2 $$\bigcap_{j=1}^{n} \mathrm{Stab}_{H_k}(x_j) = \bigcap_{x \in W_k} \mathrm{Stab}_{H_k}(x).$$

If we now let H act diagonally on W^n and define

$$\mathcal{O} := \{p \in W^n : \mathrm{Stab}_{H_k}(p) \text{ is finite}\},$$

we deduce immediately from Equations **3.1** and **3.2** that $\mathcal{O}$ is not empty. Likewise it is easy to see that $\mathcal{O}$ is Zariski open and hence, if $\nu := \mu^{\otimes n}$, $\nu(\mathcal{O}_k) > 0$. Since, moreover, $\mathcal{O}_k$ is H_k-invariant, then we conclude that $\nu|_{\mathcal{O}_k}$ is a finite measure left-invariant by $\mathrm{Stab}_{H_k}(\mu)$.

Since $\mathcal{O}_k$ is open in W_k^N and the H_k-orbits in W_k are locally closed, then also the $\mathrm{Stab}_{H_k}(\mu)$-orbits in $\mathcal{O}_k$ are locally closed in the Hausdorff topology. By decomposing $\nu|_{\mathcal{O}_k}$ if necessary into ergodic components and applying general considerations (see [26] or [12, 7]), we deduce that the finite Stab_{H_k}

(μ)-invariant measure on $\mathcal{O}_k$ is supported on an orbit $\mathrm{Stab}_{H_k}(\mu)\cdot p_0$, with $p_0 \in \mathcal{O}_k$. But

$$\mathrm{Stab}_{H_k}(\mu)\cdot p_0 \cong \mathrm{Stab}_{H_k}(\mu)/\,\mathrm{Stab}_{H_k}(p_0)\cap \mathrm{Stab}_{H_k}(\mu)$$

and, by hypothesis, $\mathrm{Stab}_{H_k}(p_0)$ is finite. It follows that $\mathrm{Stab}_{H_k}(\mu)$ supports a finite measure that is invariant by translations and hence is compact. □

We can draw from this theorem some useful consequences on the structure of the stabilizer of a probability measure in projective space. The point is to be able to deal with measures whose support is not necessarily Zariski dense and with actions that are not necessarily effective. To apply the previous theorem and for further reference, let us set up some notation. If k is a local field, let E be a finite-dimensional vector space defined over k. If $\mu \in \mathcal{M}^1(\mathbb{P}(E_k))$, let us set

3.3 $$W := \overline{\operatorname{supp}\mu}^{Z} \subseteq \mathbb{P}(E)$$

to be the projective subvariety defined as the Zariski closure of the support of μ and define the algebraic subgroups

3.4 $$\begin{aligned} N(W) &:= \{h \in \mathrm{PGL}(E) : h(W) = W\} \\ I(W) &:= \{h \in \mathrm{PGL}(E) : h|_W = Id_W\}, \end{aligned}$$

where $I(W)$ is normal in $N(W)$ and both are defined over k. The above notation will be typically applied when E is either a vector space as in the next corollary or the dual of a vector space (the only difference of course is lying in the action).

COROLLARY 3.2. *Let V_k be a finite-dimensional k-vector space seen as the set of k-points of the corresponding vector space V over an algebraic closure of k. The stabilizer $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)$ of a probability measure $\mu \in \mathcal{M}^1(\mathbb{P}(V_k))$ on the projective space $\mathbb{P}(V_k)$ has a cocompact normal k-subgroup. More precisely, if $W \subset \mathbb{P}(V)$ and $I(W)$ are defined as in Equations (3.3) and (3.4), respectively, then $I(W)$ is a k-subgroup of $\mathrm{PGL}(V)$ such that $I(W)_k$ is normal in $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)$ and the quotient $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)/I(W)_k$ is compact.*

Proof. By construction the group $H := N(W)/I(W)$ is defined over k and so is the algebraic action $H \times W \to W$, which is in addition effective by construction. Since $N(W)_k/I(W)_k$ is the $N(W)_k$-orbit of the identity in $(N(W)/I(W))_k = H_k$, it is locally closed in the Hausdorff topology and, being a topological group, is also closed; but $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)/I(W)_k$ being closed in $N(W)_k/I(W)_k$ it is also closed in H_k. Since the quotient $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)/I(W)_k$ is closed and contained in the compact $\mathrm{Stab}_{H_k}(\mu)$, it is itself compact. □

REMARK 3.3. *Furstenberg's lemma [9] asserts that either the stabilizer of a probability measure μ on $\mathbb{P}(V_k)$ is compact or the support of the measure is contained in the union of two proper subspaces. Using this lemma Zimmer has shown [26, theorem 3.2.4] (see also [18]) that under the same hypotheses as in Corollary 3.2* $\mathrm{Stab}_{\mathrm{PGL}(V_k)}(\mu)$ *has a normal subgroup of finite index that contains a cocompact normal subgroup consisting of the k-points of a k-algebraic group.*

4. Applications to Group Pairs

Let V_k be as in §3 a k-vector space identified with the k-points of a vector space V over an algebraic closure of k, and let V_k^* be its dual. If $\mu \in \mathcal{M}^1(\mathbb{P}(V_k^*))$, define

4.1 $$G_\mu := \{h \in \mathrm{GL}(V_k) : [h^*] \in \mathrm{Stab}_{\mathrm{PGL}(V_k^*)}(\mu)\}$$

where h^* denotes the transpose of h. Then we have

PROPOSITION 4.1. *Let $\mu \in \mathcal{M}^1(\mathbb{P}(V_k^*))$. Then the pair $(G_\mu \ltimes V_k, V_k)$ does not have relative property (T).*

Proof. Let us consider $W \subset \mathbb{P}(V^*)$ and $I(W)$ as defined in Equations (3.3) and (3.4) with $E = V^*$ and let us define

$$\begin{aligned} I_\mu &:= \{h \in G_\mu : [h^*] \in I(W)_k\} \\ &= \{h \in \mathrm{GL}(V_k) : [h^*] \in I(W)_k\}. \end{aligned}$$

Let $[\lambda] \in \operatorname{supp}\mu$, where $\lambda : V_k \to k$ is a nonzero linear form. Then, by definition of I_μ, there exists a continuous homomorphism $\chi_\lambda : I_\mu \to k^*$ such that for every $h \in I_\mu$

4.2 $$h^*\lambda = \chi_\lambda(h)\lambda.$$

But this is equivalent to saying that the map

$$\begin{aligned} q_\lambda : I_\mu \ltimes V_k \to &\quad k^* \ltimes k \\ (h, \nu) \mapsto &\ (\chi_\lambda(h), \lambda(\nu)) \end{aligned}$$

is a homomorphism. Since k is not compact and $k^* \ltimes k$ is amenable, then $(k^* \ltimes k, k)$ does not have relative property (T) (see, for instance, [8, lemma 8.3]), and since q_λ is continuous with $q_\lambda(V_k) = k$, then also $(I_\mu \ltimes V_k, V_k)$ does not have relative property (T) (Lemma 2.1). But I_μ is a normal subgroup of G_μ and G_μ / I_μ is compact by Corollary 3.2, so that Proposition 2.4(1) implies that also $(G_\mu \ltimes V_k, V_k)$ does not have relative property (T). □

Our source of examples is the following:

THEOREM 4.2. *Let k be a local field, V_k a finite-dimensional k-vector space endowed with its structure of (additive) locally compact group, H a locally compact group, and $\rho : H \to \mathrm{GL}(V_k)$ a continuous representation. Then the following are equivalent:*

i) *the pair $(H \ltimes V_k, V_k)$ has relative property (T); and*
ii) *there is no H-invariant probability measure on $\mathbb{P}(V_k^*)$.*

The proof of the implication (ii)⇒(i) was proven in [5, proposition 7 and examples] and is recalled below for completeness. The case in which $k = \mathbb{R}$ and H is a semisimple connected Lie group was established in [24, proposition 2.3]. The implication (i)⇒(ii) in the general case appears in [6, proposition 3.1.9], but it follows here immediately from the above proposition. In fact, assume that via the contragredient action ρ^* of ρ, the group H fixes a measure $\mu \in \mathcal{M}^1(\mathbb{P}(V_k^*))$. With G_μ as in Equation **4.1**, consider the continuous homomorphism

$$H \ltimes V_k \xrightarrow{\rho \otimes Id} G_\mu \ltimes V_k\ .$$

Then Proposition 4.1 and the contrapositive of Lemma 2.1 imply that $(H \ltimes V_k, V_k)$ does not have relative property (T).

We recall now the proof of the implication (ii)⇒(i). Let $\rho : H \to \mathrm{GL}(V_k)$ be a continuous representation and let $G := H \ltimes V_k$. By fixing a nontrivial character $\chi \in \mathrm{Hom_c}(k, \mathbb{T})$ we get the usual isomorphism as locally compact groups of the Pontryagin dual $\widehat{V}_k$ of V_k with the dual V_k^*, via

$$\begin{aligned} V_k^* &\longrightarrow \widehat{V}_k \\ \lambda &\longmapsto \chi \circ \lambda . \end{aligned}$$

Given a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, we have the spectral measure

$$P : \mathcal{B}(V_k^*) \to \mathcal{P}(\mathcal{H})$$

of $\pi|_{V_k}$, that is, a map associating to every Borel set $B \subset V_k^*$ an orthogonal projection satisfying certain additional properties. Essential is the fact that for every $\xi \in \mathcal{H}$,

$$\mu_\xi(B) := \langle P(B)\xi, \xi\rangle$$

defines a bounded positive Radon measure $\mu_\xi \in \mathcal{M}^+(V_k^*)$ on V_k^* and

$$\langle \pi(v)\xi, \xi\rangle = \hat{\mu}_\xi(v) = \int_{V_k^*} \chi(\lambda(v))\, d\mu_\xi(\lambda) .$$

In other words, μ_ξ is determined uniquely by the diagonal coefficients on V_k associated to ξ and it is easy to see that $P(B)$ is uniquely determined by the map $\xi \mapsto \mu_\xi(B)$; furthermore, we have for all $h \in H$ and $B \in \mathcal{B}(V_k^*)$

4.3 $$P(\rho(h)^* B) = \pi(h)^{-1}\, P(B)\, \pi(h) .$$

Now let $\xi \in \mathcal{H}$ and $\mu_\xi \in \mathcal{M}^+(V_k^*)$ and define

$$m_\xi := p_*(\mu_\xi|_{V_k^* \setminus \{0\}})$$

to be the push- forward of the measure $\mu_\xi|_{V_k^*\setminus\{0\}}$ under the projection $p : V_k^* \setminus \{0\} \to \mathbb{P}(V_k^*)$. If $B \in \mathcal{B}(\mathbb{P}(V_k^*))$ is a Borel subset in $\mathbb{P}(V_k^*)$ and we set $B' = p^{-1}(B)$, using Equation **4.3** we have

$$\begin{aligned} m_\xi(\rho^*(h)B) - m_\xi(B) &= \langle P(\rho^*(h)B')\xi, \xi\rangle - \langle P(B')\xi, \xi\rangle \\ &= \langle P(B')\pi(h)\xi, \pi(h)\xi\rangle - \langle P(B')\xi, \xi\rangle \\ &= \operatorname{Re}\langle P(B')(\pi(h)\xi + \xi), \pi(h)\xi - \xi\rangle , \end{aligned}$$

so that

$$|m_\xi(\rho^*(h)B) - m_\xi(B))| \le 2\|\xi\|\, \|\pi(h)\xi - \xi\| .$$

Now for any subset $K \subset H$, introduce the following quantity

$$\alpha(K, \rho) := \inf_{m \in \mathcal{M}^1(\mathbb{P}(V_k^*))} \sup_{B \in \mathcal{B}(\mathbb{P}(V_k^*))} \sup_{h \in K} |m(\rho^*(h)B) - m(B)|$$

which is somehow the extent to which "measures are moved by $\rho^*(h)$."

Since the total mass of the positive measure m_ξ is

$$\mu_\xi(\mathbb{P}(V_k^*)) = \mu_\xi(V_k^*) - \mu_\xi(\{0\}),$$

and since there are no $\rho(V_k)$-invariant vectors if and only if $\mu_\xi(\{0\}) = 0$ for every $\|\xi\| = 1$, we obtain the following:

PROPOSITION 4.3. *Let $\rho : G \to \mathcal{U}(\mathcal{H})$ be a continuous unitary representation of $H \ltimes V_k$ with no nonzero V_k-invariant vectors. Then for every compact subset $K \subset H$ and for every $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, we have that*

$$\max_{h\in K} \|\rho(h)\xi - \xi\| \geq \frac{1}{2}\alpha(K,\rho).$$

The proof of Theorem 4.2(ii)⇒(i) will be complete if we show that H has no invariant measure on $\mathbb{P}(V_k^*)$ via ρ if and only if there exists a compact set $K \subset H$ with $\alpha(K,\rho) > 0$; but this is just an exercise.

4.1. Proof of the Results in §1.1

We adopt the same notation as in §1.1, namely, Γ is a discrete group, S a finite set of primes, $\mathbb{Z}[S]$ the ring obtained inverting the primes in S, $\rho : \Gamma \to \mathrm{GL}_N(\mathbb{Z}[S])$ a representation, and $\rho_p : \Gamma \to \mathrm{GL}_N(\mathbb{Q}_p)$ is the representation obtained by composing ρ with the embedding $\mathbb{Z}[S] \hookrightarrow \mathbb{Q}_p$, where $\mathbb{Q}_\infty := \mathbb{R}$. Since the diagonal embedding

$$\mathbb{Z}[S]^N \hookrightarrow \mathbb{R}^N \times \prod_{p\in S} \mathbb{Q}_p^N$$

realizes $\mathbb{Z}[S]^N$ as a cocompact lattice, Corollary 2.3 can be translated into the following:

COROLLARY 4.4. *The pair $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ has relative property (T) if and only if the pair $(\Gamma \ltimes (\mathbb{R}^N \times \prod_{p\in S} \mathbb{Q}_p^N), \mathbb{R}^N \times \prod_{p\in S} \mathbb{Q}_p^N)$ has relative property (T).*

From this we deduce the equivalence of (i) and (iv) in Theorem 1.2, which we record here separately as it does not require the hypothesis of Zariski density of $\rho(\Gamma)$ in SL_N.

PROPOSITION 4.5. *With the above hypotheses, the following are equivalent:*

i) $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ *has relative property (T); and*

ii) *for every* $p \in S \cup \{\infty\}$, *there is no* $\rho_p^*(\Gamma)$*-invariant probability measure on* $\mathbb{P}((\mathbb{Q}_p^N)^*)$.

Proof. Using Corollary 4.4 and Lemma 2.2 we deduce that the pair $(\Gamma \ltimes \mathbb{Z}[S]^N, \mathbb{Z}[S]^N)$ has relative property (T) if and only if $(\Gamma \ltimes \mathbb{Q}_p^N, \mathbb{Q}_p^N)$ has relative property (T) for every $p \in S \cup \{\infty\}$. Theorem 4.2 then implies the equivalence with (ii). □

Proof of Theorem 1.2. The chain of implications that we shall prove is as follows: (iii)⇒(iv)⇒(i)⇒(ii)⇒(iii), where, however, the equivalence (iv)⇔(i) is the content of Proposition 4.5.

(iii)⇒(iv). We shall show the contrapositive statement; namely, we shall assume that for some $p \in S \cup \{\infty\}$ there exists a $\rho_p^*(\Gamma)$-invariant probability measure μ and we shall prove that then the image $\rho_p(\Gamma) \subset \mathrm{SL}_N(\mathbb{Q}_p)$ is bounded.

Let $W \subset \mathbb{P}((\overline{\mathbb{Q}_p}^N)^*)$ and $I(W)$ be defined as in Equations **3.3** and **3.4** with $E = (\overline{\mathbb{Q}_p}^N)^*$ and $\overline{\mathbb{Q}_p}$ an algebraic closure of $\mathbb{Q}_p$, and let us consider $I'(W) = I(W) \cap \mathrm{PSL}_N$. Since $I'(W)$ is normalized by $\rho(\Gamma)$ and $\rho(\Gamma)$ is Zariski dense in SL_N, then $I'(W)$ is a normal subgroup of PSL_N and hence either trivial or the full group. Obviously it cannot be the full group PSL_N since this cannot fix pointwise any nonempty subset in projective space. So since $I'(W)$ is trivial, then by Corollary 3.2 $\mathrm{Stab}_{\mathrm{PSL}(\mathbb{Q}_p^N)}(\mu)$ is compact, so that $\rho_p(\Gamma)$ is bounded.

(i)⇒(ii). This follows from Lemma 2.6.

(ii)⇒(iii). If $p = \infty$, then it is easy to see that $\rho_\infty(\Gamma)$ cannot be bounded because otherwise it would be conjugate into a maximal compact subgroup that is a real algebraic group, thus contradicting the Zariski density of $\rho(\Gamma)$ in SL_N.

Let us assume that $\mathbb{Z}[S]^N$ is finitely generated as a module over $\mathbb{Z}[\Gamma]$, namely there exist $a_j \in \mathbb{Z}[S]^N$, with $1 \le j \le r$, such that $\mathbb{Z}[S]^N = \rho(\mathbb{Z}(\Gamma))a_1 + \cdots + \rho(\mathbb{Z}(\Gamma))a_r$. For $p \in S$, $n_i \in \mathbb{Z}$, and $\gamma_i \in \Gamma$, using the ultrametric inequality, we have

$$\left\| \sum_i^{<\infty} \sum_{j=1}^r n_i \rho_p(\gamma_i) a_j \right\| \le \max_{j,i} \|n_i \rho_p(\gamma_i) a_j\|$$
$$= \max_{j,i} \|\rho_p(\gamma_i) a_j\| \le \max_j \sup_{\gamma\in\Gamma} \|\rho_p(\gamma) a_j\|$$

so that if $\rho_p(\Gamma)$ were to be bounded for any of the p's in S, then $\mathbb{Z}[S]^N$ would also be bounded in $\mathbb{Q}_p^N$, which is not the case. □

In the course of the proof of the implication (iii)$\Rightarrow$(i) in Theorem 1.2 we have proven the following fact, which we record as it might be of independent interest and which could be deduced also from Furstenberg's Lemma [9].

LEMMA 4.6. *Let $\Lambda < \mathrm{PSL}_N(\mathbb{Q}_p)$ be an unbounded closed subgroup that is Zariski dense in PSL_N. Then there exists no Λ-invariant probability measure on $\mathbb{P}(\mathbb{Q}_p^N)$.*

Proof of Corollary 1.5. Since $\mathrm{SL}_N(\mathbb{Z}[S])$ is an irreducible lattice in the product $\mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S} \mathrm{SL}_N(\mathbb{Q}_\ell)$, if $p \in S$ is a fixed prime, the projection

$$\alpha_p : \mathrm{SL}_N(\mathbb{Z}[S]) \to \mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S, \ell \neq p} \mathrm{SL}_N(\mathbb{Q}_\ell)$$

has dense image. We assume for the moment that we have proven that $\alpha_p(\mathrm{SL}_N(\mathbb{Z}[s]))$ cannot contain an open solvable subgroup, so that by the topological Tits alternative [4, theorem 1.6] there exists a free dense subgroup $\Lambda < \alpha_p(\mathrm{SL}_N(\mathbb{Z}[S]))$ of finite rank, which is also dense in $\mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S, \ell \neq p} \mathrm{SL}_N(\mathbb{Q}_\ell)$. This implies in particular that for all $\ell \in S$, $\ell \neq p$, the projection $\rho_\ell : \alpha_p^{-1}(\Lambda) \to \mathrm{SL}_N(\mathbb{Q}_\ell)$ is unbounded. If we show that also the projection $\rho_p : \Gamma \to \mathrm{SL}_N(\mathbb{Q}_p)$ is unbounded where $\Gamma := \alpha_p^{-1}(\Lambda) < \mathrm{SL}_N(\mathbb{Z}[S])$, then Theorem 1.2 will imply that the pair $(\Gamma \ltimes \mathbb{Z}[s]^N, \mathbb{Z}[s]^N)$ has relative property (T). In fact, if the projection $\rho_p : \Gamma \to \mathrm{SL}_N(\mathbb{Q}_p)$ were bounded with compact closure K, then Γ would be a discrete subgroup of $\mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S, \ell \neq p} \mathrm{SL}_N(\mathbb{Q}_\ell) \times K$, contradicting the fact that its projection in $\mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S, \ell \neq p} \mathrm{SL}_N(\mathbb{Q}_\ell)$ is dense.

To complete the proof we need to verify that $\alpha_p(\mathrm{SL}_N(\mathbb{Z}[S]))$ does not have open solvable subgroups. Let us start with the general observation that if $L < H < G$ are topological groups such that L is open in H and H is dense in G, then the closure $\overline{L}$ of L is an open subgroup of G. By applying this to $H = \alpha_p(\mathrm{SL}_N(\mathbb{Z}[S])) < \mathrm{SL}_N(\mathbb{R}) \times \prod_{\ell \in S, \ell \neq p} \mathrm{SL}_N(\mathbb{Q}_p)$, we have that if such open solvable subgroup $L < \alpha_p(\mathrm{SL}_N(\mathbb{Z}[S]))$ were to exist, then $\rho_\infty(\overline{L})$ would be an open solvable subgroup of $\mathrm{SL}_N(\mathbb{R})$, hence closed, and hence $\rho_\infty(\overline{L}) = \mathrm{SL}_N(\mathbb{R})$, since $\mathrm{SL}_N(\mathbb{R})$ is connected. This is not possible and the proof is completed. □

4.2. *Proof of the results in §1.2*

We now move on to the proof of the results in §1.2.

Proof of Theorem 1.6. In view of Theorem 4.2 it will be enough to prove the equivalence of the following statements:

1) There exists a $\rho^*(\Gamma)$-invariant probability measure on $\mathbb{P}(V_{\mathbb{R}}^*)$; and
2) The representation $\rho_{\mathbb{R}} : G_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is not totally unbounded.

Remark first of all that one should avoid the temptation of trying to deduce Theorem 1.6 as an application of Theorem 1.2 (with $p = \infty$), as in order to do so one should require in addition to the hypotheses of Theorem 1.6 also the Zariski density of the image $\rho(\Gamma)$ in SL_N.
(2)⇒(1). By hypothesis let $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ be a $\rho_{\mathbb{R}}(G)$-invariant subspace such that $\rho_{\mathbb{R}}(G)|_{W_{\mathbb{R}}}$ is bounded. Then on $\mathbb{P}(W_{\mathbb{R}}) \subset \mathbb{P}(V_{\mathbb{R}})$ there exists a $\rho(\Gamma)$-invariant probability measure.
(1)⇒(2). Let $\mu \in \mathcal{M}^1(\mathbb{P}(V_{\mathbb{R}}^*)$ be a $\rho^*(\Gamma)$-invariant probability measure and let $W \subset \mathbb{P}(V^*)$, $N(W)$, and $I(W)$ be as usual as defined in Equations **3.3** and **3.4** with $E = V^*$. If ρ^* is the contragredient representation, by hypothesis for all $g \in G$,

$$[\rho^*(g)] \in N(W)\,.$$

If we define $N := \{g \in G : [\rho^*(g)] \in I(W)\}$, then we have an injective homomorphism

$$G/N \hookrightarrow N(W)/I(W)\,,$$

and, by passing to the real points, we have an induced homomorphism

$$h : G_{\mathbb{R}}/N_{\mathbb{R}} \longrightarrow N(W)_{\mathbb{R}}/I(W)_{\mathbb{R}}$$

that is at most finite-to-one. If $p : G_{\mathbb{R}} \to G_{\mathbb{R}}/N_{\mathbb{R}}$ denotes the usual projection, then $h \circ p(\Gamma)$ is relatively compact since it is contained in $\mathrm{Stab}_{\mathrm{PGL}(V_{\mathbb{R}}^*)}(\mu)/I(W)_{\mathbb{R}}$, which is compact by Corollary 3.2. Since h is at most finite-to-one, we infer that $p(\Gamma)$ is bounded. Since $(N_{\mathbb{R}})^\circ$ is of finite index in $N_{\mathbb{R}}$, if $p_1 : G_{\mathbb{R}} \to G_{\mathbb{R}}/(N_{\mathbb{R}})^\circ$, then $p_1(\Gamma)$ is also bounded in $G_{\mathbb{R}}/(N_{\mathbb{R}})^\circ$.

The rest of the proof will consist just in identifying the quotient $G_{\mathbb{R}}/(N_{\mathbb{R}})^\circ$ to deduce that $\rho_{\mathbb{R}}$ is not totally unbounded. To this purpose, observe that the connected component N° of N fixes pointwise every vector in the linear span E of $\pi^{-1}(W)$, where $\pi : V^* \to \mathbb{P}(V^*)$. In fact, for every $[\lambda] \in W$, $\lambda : V \to \mathbb{C}$, we have $\chi_\lambda : N \to \mathbb{C}^*$ given by

$$\rho^*(n)\lambda = \chi_\lambda(n)\lambda$$

for every $n \in N$. Thus, since N° is connected and a product of almost $\mathbb{R}$-simple factors of G, we get that $\chi_\lambda(N^\circ) = 1$. Since

$$\begin{aligned} N =&\{g \in G : \rho^*(g)|_W = Id_W\} \\ \supset&\{g \in G : \rho^*(g)|_E = Id_E\} \supset N^\circ \end{aligned}$$

then $G_\mathbb{R}/(N_\mathbb{R})^\circ$ surjects onto $G_\mathbb{R}/\ker\{g \mapsto \rho^*(g)|_{E_\mathbb{R}}\}$. The fact that

$$G_\mathbb{R}/\ker\{g \mapsto \rho^*(g)|_{E_\mathbb{R}}\} \cong \mathrm{im}\{g \mapsto \rho^*(g)|_{E_\mathbb{R}}\} \subset \mathrm{GL}(E_\mathbb{R}),$$

together with the fact that $p_1(\Gamma)$ was bounded in $G_\mathbb{R}/(N_\mathbb{R})^\circ$, implies that $\rho^*(\Gamma)|_{E_\mathbb{R}}$ is bounded in $\mathrm{GL}(E_\mathbb{R})$; again by Zariski density, $\rho^*_\mathbb{R}(G_\mathbb{R})|_{E_\mathbb{R}}$ is bounded, thus showing that $\rho_\mathbb{R}$ is not totally unbounded. □

LEMMA 4.7. *Let H be a connected $\mathbb{Q}$-semisimple group that is either simply connected or adjoint and with no factors of $\mathbb{Q}$-rank $= 0$. Then there exists an irreducible representation $\rho : H \to \mathrm{GL}(V)$ such that $\rho_\mathbb{R} : H_\mathbb{R} \to \mathrm{GL}(V_\mathbb{R})$ is irreducible and unbounded.*

Proof. It will be enough to prove the assertion under the hypothesis that H is $\mathbb{Q}$-simple. In fact, since $H = H_1 \times \cdots \times H_n$, if a representation $\rho_j : H_j \to \mathrm{GL}(V_j)$ with the desired properties exists for each factor H_i, then the Kronecker product $\rho := \bigotimes_{j=1}^n \rho_j : H \to \mathrm{GL}\left(\bigotimes_{j=1}^n V_j\right)$ will have the desired properties for the group H.

So, recall (see for instance [17, pp. 47-48]) that if H is a connected almost $\mathbb{Q}$-simple group, there is a number field k and an absolutely simple k-group $\mathbb{L}$ such that $H = \mathrm{Res}_{k/\mathbb{Q}}\mathbb{L}$. Thus, over $\mathbb{C}$, $H = \prod_{\sigma:k\to\mathbb{C}} \mathbb{L}^\sigma$, where the product is over all Archimedean places of k. If $\mathfrak{l}$ denotes the Lie algebra of $\mathbb{L}$, let $\mathrm{Ad}_1 := \mathrm{Ad} : \mathbb{L} \to \mathrm{Aut}(\mathfrak{l})$ be the adjoint representation of $\mathbb{L}$ and $\mathrm{Ad}_i : \mathbb{L}^{\sigma_i} \to \mathrm{Aut}(\mathfrak{l}^i)$. Then by Weil's criterion $\rho := \bigotimes_{i=1}^n \mathrm{Ad}_i : \prod \mathbb{L}^{\sigma_i} \to \mathrm{Aut}(\bigotimes_{i=1}^n \mathfrak{l}^i)$ can be defined over $\mathbb{Q}$ and, setting $V := \bigotimes \mathfrak{l}^i$, we get an irreducible representation $\rho : H \to \mathrm{GL}(V)$; then $\rho_\mathbb{R} : H_\mathbb{R} \to \mathrm{GL}(V_\mathbb{R})$ is also irreducible and with unbounded image. □

Proof of Corollary 1.7. By hypothesis, there exists a connected semisimple simply connected algebraic group H defined over $\mathbb{Q}$ and a surjective homomorphism $h : H_\mathbb{R} \twoheadrightarrow G$ such that $h(H_\mathbb{Z})$ is commensurable with Γ. By passing to a subgroup of finite index $\Gamma_0 \leq H_\mathbb{Z}$, we may assume that $p|_{\Gamma_0}$ is injective and has image $\Gamma' < \Gamma < G$ of finite index in Γ. If $\rho : H \to \mathrm{GL}(V)$ is the representation in Lemma 4.7 and $\dim V = N$, Theorem 1.6 implies that $(H_k \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has relative property (T); hence $(\Gamma' \ltimes \mathbb{Z}^N, \mathbb{Z}^n)$ has relative property (T) where the action of Γ' on $\mathbb{Z}^N$ is via $\rho \circ h^{-1}$. $\square$

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