

# The bounded Borel class and complex representations of 3-manifold groups

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**Abstract** If  $\Gamma < \mathrm{PSL}(2, \mathbb{C})$  is a lattice, we define an invariant of a representation  $\Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  using the Borel class  $\beta(n) \in H_c^3(\mathrm{PSL}(n, \mathbb{C}), \mathbb{R})$ . We show that this invariant satisfies a Milnor–Wood type inequality and its maximal value is attained precisely by the representations conjugate to the restriction to  $\Gamma$  of the irreducible complex  $n$ -dimensional representation of  $\mathrm{PSL}(2, \mathbb{C})$  or its complex conjugate.

Major ingredients of independent interest are the extension to degenerate configuration of flags of a cocycle defined by Goncharov and its study, as well as the identification of  $H_b^3(\mathrm{SL}(n, \mathbb{C}), \mathbb{R})$  as a normed space.

## 1 Introduction

Let  $M = \Gamma \backslash \mathbb{H}^3$  be a finite volume quotient of the real hyperbolic 3-space  $\mathbb{H}^3$ , where  $\Gamma < \mathrm{PSL}(2, \mathbb{C})$  is a torsion free lattice. There is a considerable body of work concerning the representation variety  $\mathrm{Hom}(\Gamma, \mathrm{PSL}(n, \mathbb{C}))$ , the problem of finding explicit parametrizations of (large parts of) it, and expressing various invariants of such representations like the “volume”, the Dehn invariant and the Chern–Simons invariant in those parameters, [?, ?, ?]. In fact this representation variety is particularly rich when  $M$  is not compact, say with  $h \geq 1$  cusps, since for instance in this case the character variety of  $\Gamma$  into  $\mathrm{PSL}(n, \mathbb{C})$  locally has dimension  $(n - 1)h$  near

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$\pi_n|_{\Gamma}$ , where  $\pi_n : \mathrm{PSL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(n, \mathbb{C})$  is the irreducible complex representation, [?].

In this paper, we will study the volume of a representation  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  that we will rename as the *Borel invariant of  $\rho$* . Indeed, the continuous cohomology of  $\mathrm{PSL}(n, \mathbb{C})$  in degree 3 is generated by a specific class called the *Borel class  $\beta(n)$* . Our normalization of  $\beta(n)$  is that under the Van Est isomorphism it corresponds to the  $\mathrm{PSU}(n)$ -invariant 3-form on the space  $\mathfrak{p}_n$  of Hermitian matrices of trace zero given by

$$\omega_n(X, Y, Z) = 2 \operatorname{tr}(i[X, Y]Z).$$

In particular for  $\mathrm{PSL}(2, \mathbb{C})$  it corresponds to the volume form of the hyperbolic metric of constant curvature  $-1$ . We will not use it in this paper, but  $2i\beta(n)$  coincides with the Borel element  $Bo_2$  in [?, Definition 9.24]. When  $M$  is compact, the definition of the Borel invariant of  $\rho$  is straightforward as it is the evaluation on the fundamental class  $[M]$  of the pullback by  $\rho$  of the Borel class. If  $M$  has cusps, the definition of this invariant presents interesting difficulties which we overcome by the use of bounded cohomology. More precisely,  $\beta(n)$  can be represented by a bounded cocycle, which gives rise to a bounded continuous class

$$\beta_b(n) \in H_{\mathrm{cb}}^3(\mathrm{PSL}(n, \mathbb{C}), \mathbb{R}).$$

The Borel invariant of  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  is then defined as

$$\mathcal{B}(\rho) = \langle \rho^*(\beta_b(n)), [N, \partial N] \rangle,$$

where  $N$  is a compact core of  $M$ . We refer the reader to Section ?? for a precise interpretation of this formula. This definition does not use any triangulation, it is independent of the choice of compact core and can be made for any compact oriented 3-manifold whose boundary has amenable fundamental group.

The bounded cocycle entering the definition of  $\beta_b(n)$  is constructed by means of an invariant

$$B_n : \mathcal{F}(\mathbb{C}^n)^4 \longrightarrow \mathbb{R}$$

of 4-tuples of complete flags, which on generic 4-tuples has been defined and studied by A.B. Goncharov, [?]. It generalizes the volume function in the case  $\mathcal{F}(\mathbb{C}^2) = \mathbb{P}(\mathbb{C}^2) = \partial\mathbb{H}^3$  (see Section ?? for a detailed discussion). This invariant can also be used to give an efficient formula for  $\mathcal{B}(\rho)$ . To this end assume that  $M$  has toric cusps. Let  $\varphi : \mathcal{C} \rightarrow \mathcal{F}(\mathbb{C}^n)$  be a decoration, that is any  $\Gamma$ -equivariant map from the set of cusps  $\mathcal{C} \subset \partial\mathbb{H}^3$  of  $\Gamma$  into  $\mathcal{F}(\mathbb{C}^n)$ , and let  $P_1, \dots, P_r$  be a family of oriented ideal tetrahedra with vertices in  $\mathcal{C}$  whose projections to  $M$  form an ideal triangulation of  $M$ . If  $(P_i^0, P_i^1, P_i^2, P_i^3)$  are the vertices of  $P_i$ , then

$$\mathcal{B}(\rho) = \sum_{i=1}^r B_n(\varphi(P_i^0), \varphi(P_i^1), \varphi(P_i^2), \varphi(P_i^3)) \quad (1)$$

(see Section ?? for a proof). Notice that our formula for the volume does not involve any barycentric subdivision of the ideal triangulation, nor any conditions on

the decoration. Upon passing to a barycentric subdivision, or restricting to generic decorations, one recovers from the right hand side of (??) the formulas in [?, ?, ?].

Our first result is that on the character variety  $\Gamma$  into  $\mathrm{PSL}(n, \mathbb{C})$ , the invariant  $\mathcal{B}$  attains a unique maximum at  $[\pi_n|_\Gamma]$ .

**Theorem 1.** *Let  $\Gamma = \pi_1(M)$  be the fundamental group of a complete finite volume real hyperbolic 3-manifold and let  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  be any representation. Then*

$$|\mathcal{B}(\rho)| \leq \frac{n(n^2 - 1)}{6} \mathrm{Vol}(M),$$

with equality if and only if  $\rho$  is conjugate to  $\pi_n|_\Gamma$  or to its complex conjugate  $\bar{\pi}_n|_\Gamma$ .

The case of the character variety of  $\Gamma$  into  $\mathrm{PSL}(3, \mathbb{C})$  is instructive: in [?] the authors study the derivative of  $\mathcal{B}$  on a Zariski open subset and show that it is entirely expressed in terms of the eigenvalues of the holonomy at the cusps. In particular boundary unipotent representations are critical points of  $\mathcal{B}(\rho)$ . The example of the complement of the figure eight knot [?] suggests that in general there are many boundary unipotent representations and therefore many critical points for  $\mathcal{B}$ .

A large part of this paper is devoted to the study of the invariant  $B_n : \mathcal{F}(\mathbb{C}^n)^4 \rightarrow \mathbb{R}$  on 4-tuples of flags (see Theorem ?? below), to the bounded class it defines and the consequences, in combination with stability results by N. Monod in [?], for the bounded cohomology of  $\mathrm{PSL}(n, \mathbb{C})$ . Our main result about the bounded cohomology of these groups in degree 3 is:

**Theorem 2.** *The class  $\beta_b(n)$  is a generator of  $H_{\mathrm{cb}}^3(\mathrm{PSL}(n, \mathbb{C}), \mathbb{R})$  and its Gromov norm is*

$$\|\beta_b(n)\| = \frac{n(n^2 - 1)}{6} v_3,$$

where  $v_3$  is the volume of a maximal ideal tetrahedron in  $\mathbb{H}^3$ . In addition  $\beta_b(n)$  restricts to  $\beta_b(n-1)$  under the left corner injection  $\mathrm{SL}(n-1, \mathbb{C}) \hookrightarrow \mathrm{SL}(n, \mathbb{C})$  and to  $(n(n^2 - 1)/6) \cdot \beta_b(2)$  under the irreducible representation  $\pi_n : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(n, \mathbb{C})$ .

The case  $n = 2$  follows from work of Bloch [?]. Theorem ?? gives additional evidence for the conjecture that for connected simple Lie groups with finite center, the comparison map between bounded continuous and continuous cohomology is an isomorphism. So far this conjecture has been established only in degree 2 [?], in degree 3 for the isometry group of real hyperbolic  $n$ -space [?], and in degree 3 and 4 for  $\mathrm{SL}(2, \mathbb{R})$  ([?] and [?] respectively).

## 2 Outline of the Paper and the definition of the Borel invariant

### *The cocycle representing $\beta_b(n)$*

We start in Sections ?? and ?? by setting up a homological machinery involving chains on configuration spaces; this is largely borrowed from Goncharov, [?]. The aim is to define an invariant

$$B_n : \mathcal{F}(\mathbb{C}^n)^4 \longrightarrow \mathbb{R}$$

on the space of 4-tuples of complete flags in  $\mathbb{C}^n$  and to show that it is a strict cocycle, that is a cocycle that is everywhere defined and that satisfies everywhere the cocycle identity. The definition of the cocycle in general is rather technical, so we will illustrate here only the case  $n = 3$  in the dual setting, that is interchanging lines and planes through the origin. Observe that interestingly, Falbel and Wang have just proven [?] that this dual cocycle  $\check{B}_n$  is not equal to Goncharov's cocycle  $B_n$  but only cohomologous to it. Furthermore the authors give an explicit coboundary for the difference of the two cocycles [?, Proposition 3.10].

A complete flag in  $\mathbb{C}^2$  is a choice of a line in  $\mathbb{C}^2$  or, equivalently, of a point  $P \in \mathbb{P}(\mathbb{C}^2)$ . Using the identification  $\mathbb{P}^2(\mathbb{C}) = \partial\mathbb{H}^3$ , the invariant  $B_2 = \check{B}_2$  associates to four points in  $\mathbb{P}(\mathbb{C}^2)$  the signed volume of the ideal tetrahedron that they define.

After projectivization, a complete flag  $F$  in  $\mathbb{P}(\mathbb{C}^3)$  is given by a projective line  $L \subset \mathbb{P}(\mathbb{C}^3)$  and a point  $P \in L$ . We denote it by  $F = \{P \in L \subset \mathbb{P}(\mathbb{C}^3)\} \in \mathcal{F}(\mathbb{P}(\mathbb{C}^3))$ . Given a complete flag  $F \in \mathcal{F}(\mathbb{P}(\mathbb{C}^3))$  and a projective line  $L' \subset \mathbb{P}(\mathbb{C}^3)$ , we define the intersection  $F \cap L'$  to be the point in  $\mathbb{P}(\mathbb{C}^3)$  given by

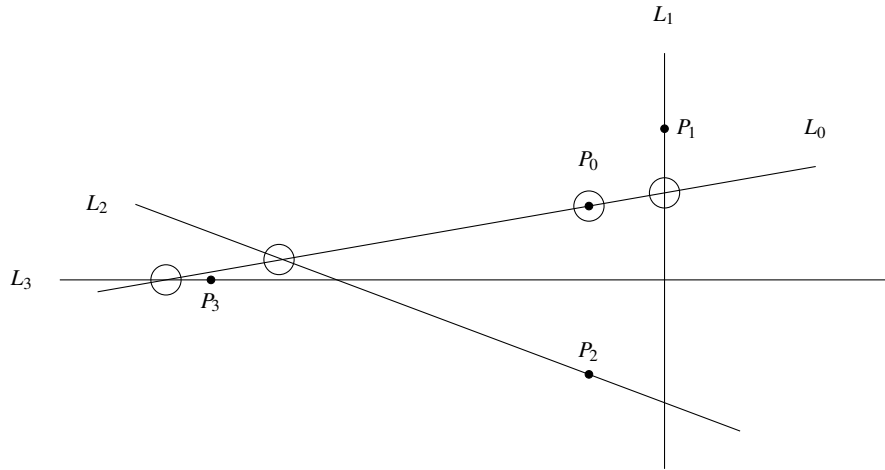
$$F \cap L' = \begin{cases} L \cap L' & \text{if } L \neq L', \\ P & \text{if } L = L'. \end{cases}$$

Now we define a cocycle  $\check{B}_3 : \mathcal{F}(\mathbb{P}(\mathbb{C}^3))^4 \rightarrow \mathbb{R}$  by sending four flags  $F_0, \dots, F_3$ , where  $F_i = \{P_i \in L_i \subset \mathbb{P}(\mathbb{C}^3)\}$ , to

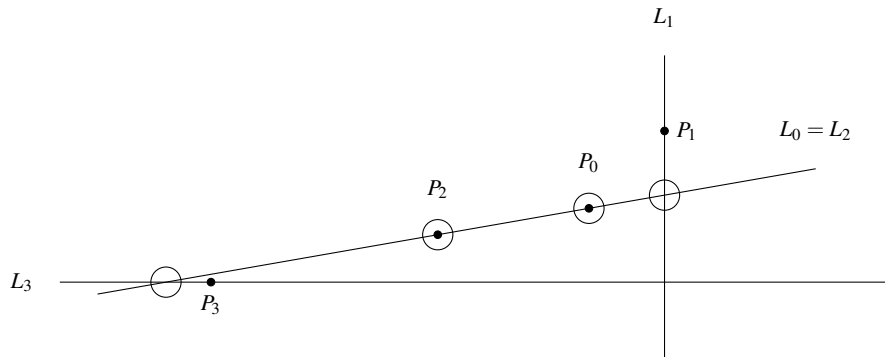
$$\check{B}_3(F_0, \dots, F_3) = \begin{cases} \text{Vol}_{L_i}(F_0 \cap L_i, \dots, F_3 \cap L_i) & \text{if } \exists i \neq j \text{ with } L_i = L_j, \\ \text{Vol}_L(F_0 \cap L, \dots, F_3 \cap L) & \text{if } \cap_{j=0}^3 L_j \text{ is a point,} \\ \sum_{i=0}^3 \text{Vol}_{L_i}(F_0 \cap L_i, \dots, F_3 \cap L_i) & \text{otherwise,} \end{cases}$$

where in the second case,  $L$  is any projective line not containing the point  $\cap_{j=0}^3 L_j$  and  $\text{Vol}_L = B_2 = \check{B}_2$  (respectively  $\text{Vol}_{L_i} = B_2 = \check{B}_2$ ) after the identification of  $L_i$  (respectively of  $L$ ) with  $\mathbb{P}(\mathbb{C}^2)$ . To check that  $\beta_3$  is well defined we need some observations.

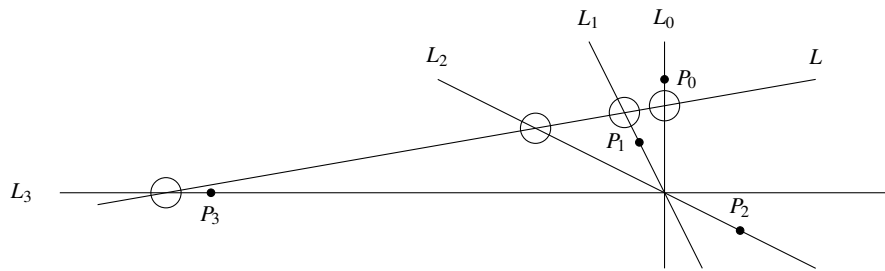
Let  $P \in \mathbb{P}(\mathbb{C}^3)$  be a point and  $L \subset \mathbb{P}(\mathbb{C}^3)$  be a projective line not containing  $P$ . We define a projection  $p : \mathbb{P}(\mathbb{C}^3) \setminus \{P\} \rightarrow L$  by sending a point  $x \in \mathbb{P}(\mathbb{C}^3) \setminus \{P\}$  to the intersection of the line through  $P$  and  $x$  and the line  $L$ . Note that  $p$  is induced by the projection  $\mathbb{C}^3 = \bar{P} \oplus \bar{L} \rightarrow \bar{L}$ , where  $\bar{P}$  and  $\bar{L}$  are the vector spaces corresponding



**Fig. 1** The generic case.



**Fig. 2** The case in which two lines, in this case  $L_0$  and  $L_2$ , coincide (but  $P_0 \neq P_2$  otherwise the two flags  $(P_0, L_0)$  and  $(P_2, L_2)$  would be equal, in which case  $\text{Vol} = 0$ ).



**Fig. 3** The case in which  $\cap_{j=0}^3 L_j$  is a point.

to  $P$  and  $L$  respectively. The following lemma is immediate, using the fact that  $p$  induces an isomorphism between  $L'$  and  $L$ .

**Lemma 3** *Let  $L \subset \mathbb{P}(\mathbb{C}^3)$  be a projective line and  $P \in \mathbb{P}(\mathbb{C}^3) \setminus L$  a point. If  $p$  is the projection  $p : \mathbb{P}(\mathbb{C}^3) \setminus \{P\} \rightarrow L$  defined above, then*

$$\text{Vol}_{L'}(x_0, \dots, x_3) = \text{Vol}_L(p(x_0), \dots, p(x_3)),$$

for any projective line  $L' \subset \mathbb{P}(\mathbb{C}^3)$  not containing  $P$  and any  $x_0, \dots, x_3 \in L'$ .

Now we can verify that in the second case, the definition is independent of the choice of  $L$ . Indeed, let  $L'$  be another projective line not containing the point  $P = L_0 \cap L_1 \cap L_2 \cap L_3$ . Then the projection  $p : \mathbb{P}(\mathbb{C}^3) \setminus \{P\} \rightarrow L$  sends  $F_i \cap L' = L_i \cap L'$  to  $F_i \cap L = L_i \cap L$  (because  $L_i$  is the line containing  $P$  and  $F_i \cap L'$ ) and the conclusion follows from Lemma ???. Second, observe that the projective line  $L_i = L_j$  of the first case might not be uniquely determined. This happens precisely if there are two pairs of lines among the four flags which are equal. Since  $\text{Vol}$  is alternating, we can without loss of generality suppose that  $L_0 = L_1 \neq L_2 = L_3$ . But in this case we have  $F_0 \cap L_2 = F_1 \cap L_2 = L_0 \cap L_2$  and  $F_2 \cap L_0 = F_3 \cap L_0 = L_2 \cap L_0$ , so that for any choice of  $i \in \{0, 1, 2, 3\}$ , two of the four points on which the alternating cocycle  $\text{Vol}_{L_i}$  will be evaluated are equal, so the evaluation is 0. Finally, it is possible that the first and second case happen simultaneously, in which case one easily checks that in both cases one obtains 0.

We refer the reader to (??) in Section ?? for the definition of  $B_n$  for all  $n \geq 2$  and to [?], where  $B_3 - \beta_3$  is written as an explicit coboundary.

**Theorem 4** *1. The function  $B_n$  is a  $\text{GL}(n, \mathbb{C})$ -invariant alternating strict Borel cocycle.*

*2. Its absolute value satisfies the inequality*

$$|B_n(F_0, \dots, F_3)| \leq \frac{1}{6} n(n^2 - 1) v_3 \quad (2)$$

with equality if and only if  $F_0, \dots, F_3$  are, up to the action of  $\text{GL}(n, \mathbb{C})$ , images under the Veronese embedding of vertices of a regular ideal simplex in  $\mathbb{P}(\mathbb{C}^2) = \partial \mathbb{H}^3$ .

Before outlining the proof of this theorem, we remark that, by evaluation on  $(g_0 F, g_1 F, g_2 F, g_3 F)$ , where  $F$  is a fixed flag in  $\mathcal{F}(\mathbb{C}^n)$ , the cocycle  $B_n$  defines a continuous bounded cohomology class of  $\text{PSL}(n, \mathbb{C})$ , which we denote by  $\beta_b(n) \in H_{\text{cb}}^3(\text{PSL}(n, \mathbb{C}), \mathbb{R})$  and call the *bounded Borel class*. The fact that this class is bounded already follows from Goncharov's almost everywhere defined cocycle, for which the bound in (??) holds almost everywhere. However, the stability properties in Theorem ?? under the left corner injection as well as the exact determination of the norm require the use of the strict cocycle. The proof of Theorem ?? is presented in Section ??.

The strategy of the proof of Theorem ?? (1) is similar to the one of the Key Lemma of Goncharov [?, Key Lemma 2.1]. The main modification consists in the

fact that we work with spaces of configurations of vectors allowed to be nongeneric. To show that the function  $B_n$  is a strict Borel cocycle we will show that it can be realized as the pullback via a map of complexes of a cocycle on an appropriate space of “decorated” vector spaces. More precisely, we first introduce the space  $\sigma_k$  of isomorphism classes of objects  $[V; x_0, \dots, x_k]$  consisting of a vector space  $V$  and a  $(k+1)$ -tuple of vectors spanning it and proceed to construct a complex  $(\mathbb{Z}[\sigma_k], D_k)$ . Then we define  $\text{Vol} : \sigma_3 \rightarrow \mathbb{R}$  using the hyperbolic volume as in Section ?? (and hinted at above). We proceed to show that  $D_4^* \text{Vol} = 0$  in Theorem ?. If  $\mathcal{F}_{\text{aff}}(\mathbb{C}^n)$  is the space of *affine flags* (see Section ??),  $\mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1})^{\text{GL}(n, \mathbb{C})}$  is the space of  $\text{GL}(n, \mathbb{C})$ -invariant alternating functions on  $\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}$  and

$$\mathbb{R}_{\text{alt}}(\sigma_k) = \{f : \sigma_k \rightarrow \mathbb{R} \mid f \text{ is alternating w.r.t. the } S_{k+1}\text{-action}\},$$

we finally construct a map of complexes

$$T_k^* : \mathbb{R}_{\text{alt}}(\sigma_k) \rightarrow \mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1})^{\text{GL}(n, \mathbb{C})}$$

which allows us to view  $B_n$  as the pullback  $B_n = T_3^*(\text{Vol})$  of the cocycle  $\text{Vol}$  on  $\sigma_3$  to the space of flags and conclude the proof of (1) of Theorem ?. In Section ?? we show the upper bound of  $B_n$  by induction in Theorem ?. In Section ?? we analyze the equality case in (2) of Theorem ?. The proof relies in particular on the noteworthy relationship

$$B_n(\varphi_n(\xi_0), \dots, \varphi_n(\xi_3)) = \frac{1}{6} n(n^2 - 1) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3),$$

for all  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$  (see Proposition ??), where  $\varphi_n : \mathbb{P}(\mathbb{C}^2) \rightarrow \mathcal{F}(\mathbb{C}^n)$  is the Veronese embedding. In addition, configurations of maximal 4-tuples of flags have the same property than configurations of 4-points in  $\partial\mathbb{H}^3$  of maximal volume: namely, if  $B_n(F_0, \dots, F_3)$  is maximal, then  $F_3$  is completely determined by  $F_0, F_1, F_2$  and the sign of  $B_n(F_0, \dots, F_3)$ .

### ***The Borel invariant***

Let  $\Gamma < \text{PSL}(2, \mathbb{C})$  be a lattice and  $\rho : \Gamma \rightarrow \text{PSL}(n, \mathbb{C})$  a representation. We consider first the case in which  $\Gamma < \text{PSL}(2, \mathbb{C})$  is torsion-free, so that the quotient  $M = \Gamma \backslash \mathbb{H}^3$  is a hyperbolic three-manifold and its cohomology is canonically isomorphic to the cohomology of  $\Gamma$ .

If  $M$  is compact, then the top dimensional cohomology groups  $H^3(\Gamma, \mathbb{R}) \cong H^3(M, \mathbb{R})$  are canonically isomorphic to  $\mathbb{R}$ , where the isomorphism is given by evaluation on the fundamental class  $[M]$ <sup>1</sup>. We define

<sup>1</sup> Since we will only consider cohomology with real coefficients, from § ?? on for ease of notation the coefficients will be not explicitly stated.

$$\mathcal{B}(\rho) = \langle \rho^*(\beta(n)), [M] \rangle,$$

where  $\rho^* : H_c^3(\mathrm{PSL}(n, \mathbb{C}), \mathbb{R}) \rightarrow H^3(\Gamma, \mathbb{R})$  denotes the pull-back via  $\rho$ .

If  $M$  is not compact, the above definition fails since  $H^3(\Gamma, \mathbb{R}) \cong H^3(M, \mathbb{R}) = 0$ . To circumvent this problem we use bounded cohomology, following the approach initiated in [?] and used also in [?]. Analogously to what happens in the ordinary group cohomology, a representation  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  induces a pullback in bounded cohomology  $\rho^* : H_{\mathrm{cb}}^3(\mathrm{PSL}(n, \mathbb{C}), \mathbb{R}) \rightarrow H_b^3(\Gamma, \mathbb{R})$  and the latter group is canonically isometrically isomorphic to the bounded singular cohomology  $H_b^3(M, \mathbb{R})$  of the manifold  $M$ . (The latter fact is true for any CW complex [?, ?], but in our case it is a simple consequence of the fact that  $M$  is aspherical.) Let  $N \subset M$  be a compact core of  $M$ , that is the complement in  $M$  of a disjoint union of finitely many horospherical neighborhoods  $E_i$ ,  $i = 1, \dots, k$ , of cusps. These have amenable fundamental groups and thus the map  $(N, \partial N) \rightarrow (M, \emptyset)$  induces an isometric isomorphism in cohomology,  $H_b^3(M, \mathbb{R}) \cong H_b^3(N, \partial N, \mathbb{R})$ , [?], by means of which we can consider  $\rho^*(\beta_b(n))$  as a bounded relative class in  $H_b^3(N, \partial N, \mathbb{R})$ . Finally, the image of  $\rho^*(\beta_b(n))$  via the comparison map  $c : H_b^3(N, \partial N, \mathbb{R}) \rightarrow H^n(N, \partial N, \mathbb{R})$  is an ordinary relative class whose evaluation on the relative fundamental class  $[N, \partial N]$  gives the definition of the Borel invariant of the representation  $\rho$ ,

$$\mathcal{B}(\rho) := \langle (c \circ \rho^*)(\beta_b(n)), [N, \partial N] \rangle, \quad (3)$$

which is independent of the choice of the compact core  $N$ . If  $M$  is compact, we recover the invariant previously defined. We complete the definition in the case in which  $\Gamma$  has torsion by setting

$$\mathcal{B}(\rho) := \frac{\mathcal{B}(\rho|_\Lambda)}{[\Gamma : \Lambda]},$$

where  $\Lambda < \Gamma$  is a torsion free subgroup of finite index.

In order to give a geometric interpretation of this definition when  $\Gamma$  is torsion free, we need to understand the composition of the maps

$$H_{\mathrm{cb}}^3(\mathrm{PSL}(n, \mathbb{C})) \longrightarrow H_b^3(\Gamma) \cong H_b^3(M) \cong H_b^3(N, \partial N)$$

at the cocycle level. The difficulty here lies in the isomorphism  $H_b^3(\Gamma) \cong H_b^3(N, \partial N)$  and we recall from [?, Section 3] that it admits the following explicit description: we identify the universal cover  $\tilde{N}$  of  $N$  with  $\mathbb{H}^3$  minus a  $\Gamma$ -invariant collection of open horoballs centered at the cusps  $\mathcal{C}$ . Let  $p : \tilde{N} \rightarrow \mathcal{C}$  be the  $\Gamma$ -equivariant map sending each horosphere to the corresponding cusp, and for the interior of  $\tilde{N}$ , fix a fundamental domain for the  $\Gamma$ -action, map this fundamental domain to a chosen cusp and extend  $\Gamma$ -equivariantly. The bounded cohomology of  $\Gamma$  can be represented by  $\Gamma$ -invariant bounded cocycles on the set of cusps of  $\Gamma$  in  $\partial\mathbb{H}^3$ , and given such a cocycle  $c : \mathcal{C}^4 \rightarrow \mathbb{R}$ , we obtain a relative cocycle on  $(N, \partial N)$  which we canonically describe as a  $\Gamma$ -equivariant cocycle on  $(\tilde{N}, \partial\tilde{N})$  as follows:



$$\{\sigma : \Delta^3 \rightarrow \tilde{N}\} \mapsto c(p(\sigma(e_0)), \dots, p(\sigma(e_3))),$$

where  $e_0, \dots, e_3$  denote the vertices of the standard simplex  $\Delta^3$ .

Given a representation  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  with  $\rho$ -equivariant decoration  $\varphi : \mathcal{C} \rightarrow \mathcal{F}(\mathbb{C}^n)$ , it follows from [?], using the crucial fact that the cocycle  $B_n$  is a Borel cocycle defined everywhere, that the class  $\rho^*(\beta_b(n)) \in H^3(\Gamma)$  is represented by the cocycle  $\varphi^*(B_n)$ . Thus, given any relative triangulation of  $(N, \partial N)$ , the Borel invariant of the representation is computable as

$$\mathcal{B}(\rho) = \sum_{i=1}^r B_n(\varphi(P_i^1), \varphi(P_i^2), \varphi(P_i^3), \varphi(P_i^4)), \quad (4)$$

where the  $(P_i^0, \dots, P_i^3)$  are the simplices of the triangulation of  $N$  lifted to  $\tilde{N}$ . This works as well replacing the triangulation by any cycle representing the fundamental class  $[N, \partial N]$ .

From an ideal triangulation of  $M$  as in [?], where degenerate tetrahedra – meaning tetrahedra contained in planes – are allowed, we obtain a relative cycle representing the fundamental class  $[N, \partial N]$  by triangulating the intersection of the ideal triangulation with  $\tilde{N}$ . The formula (??) now follows.

Finally, a simple cohomological argument using the naturality of the transfer maps  $H_b^3(\Gamma) \rightarrow H_{\mathrm{cb}}^3(\mathrm{PSL}(2, \mathbb{C}))$  allows us to reinterpret our Borel invariant in terms of a multiplicative constant in Proposition ?? of Section ?. It is this interpretation of the Borel invariant which we will use for the proof of our main Theorem ?? in Section ?.

### 3 The cocycle representing $\beta_b(n)$

In this section we define a cocycle representing  $\beta_b(n)$ . As already mentioned in the introduction, the construction is a modification of a cocycle defined by Goncharov [?] for generic configurations of flags. Notice however that the extension to all configuration of flags requires substantial modifications.

#### 3.1 Configuration spaces

For  $k, m \geq 0$ , let  $\sigma_k(m)$  be the quotient of the set of all spanning  $(k+1)$ -tuples  $(x_0, \dots, x_k)$  of vectors in  $\mathbb{C}^m$  by the diagonal action of  $\mathrm{GL}(m, \mathbb{C})$ . Observe that for  $k+1 < m$ , the set  $\sigma_k(m)$  is empty.

Given an  $m$ -dimensional complex vector space  $V$  and a  $(k+1)$ -tuple  $(x_0, \dots, x_k)$  of vectors spanning  $V$ , we obtain by choosing an isomorphism  $V \rightarrow \mathbb{C}^m$  a well defined element of  $\sigma_k(m)$  denoted  $[V; (x_0, \dots, x_k)]$ .

On

$$\sigma_k := \sqcup_{m \geq 0} \sigma_k(m) = \sigma_k(0) \sqcup \cdots \sqcup \sigma_k(k+1),$$

there are two kinds of face maps

$$\varepsilon_i^{(k)}, \eta_i^{(k)} : \sigma_k \longrightarrow \sigma_{k-1},$$

for  $0 \leq i \leq k$ , given as

$$\varepsilon_i^{(k)}([\mathbb{C}^m; (x_0, \dots, x_k)]) = [x_0, \dots, \widehat{x}_i, \dots, x_k; (x_0, \dots, \widehat{x}_i, \dots, x_k)],$$

$$\eta_i^{(k)}([\mathbb{C}^m; (x_0, \dots, x_k)]) = [\mathbb{C}^m / \langle x_i \rangle; (x_0, \dots, \widehat{x}_i, \dots, x_k)],$$

where in the last definition,  $x_j$  is understood as the image of  $x_j$  in  $\mathbb{C}^m / \langle x_i \rangle$  and  $\langle y_0, \dots, y_\ell \rangle$  denotes the linear subspace spanned by the set  $\{y_0, \dots, y_\ell\}$ . Observe that on  $\sigma_k(m)$  both face maps take values in  $\sigma_{k-1}(m) \sqcup \sigma_{k-1}(m-1)$ .

One can easily verify the following relations between these face maps; for all  $0 \leq j < i \leq k$ :

$$\varepsilon_j^{(k-1)} \varepsilon_i^{(k)} = \varepsilon_{i-1}^{(k-1)} \varepsilon_j^{(k)}, \quad (\text{R1})$$

$$\eta_j^{(k-1)} \eta_i^{(k)} = \eta_{i-1}^{(k-1)} \eta_j^{(k)}, \quad (\text{R2})$$

$$\eta_j^{(k-1)} \varepsilon_i^{(k)} = \varepsilon_{i-1}^{(k-1)} \eta_j^{(k)}. \quad (\text{R3})$$

Let us denote, for  $k \geq 0$ , by  $\mathbb{Z}[\sigma_k]$  the free abelian group on  $\sigma_k$  and set  $\mathbb{Z}[\sigma_k] = 0$  for  $k \leq -1$ . We extend the face maps to morphisms  $\mathbb{Z}[\sigma_k] \rightarrow \mathbb{Z}[\sigma_{k-1}]$ . For  $k \geq 1$ , we define  $\partial_k, d_k$  and  $D_k : \mathbb{Z}[\sigma_k] \rightarrow \mathbb{Z}[\sigma_{k-1}]$  by

$$\partial_k(\tau) := \sum_{i=0}^k (-1)^i \varepsilon_i^{(k)}(\tau),$$

$$d_k(\tau) := \sum_{i=0}^k (-1)^i \eta_i^{(k)}(\tau),$$

for  $\tau \in \sigma_k$ , and extend this definition to  $\partial_k = d_k = 0$  for  $k \leq 0$ . Finally, we set  $D_k := \partial_k - d_k$ , for any  $k \in \mathbb{Z}$ .

From the relations (R1-R3), we immediately deduce:

**Lemma 5** *Let  $k \in \mathbb{Z}$ . Then*

- $\partial_{k-1} \partial_k = 0$ ,
- $d_{k-1} d_k = 0$ ,
- $\partial_{k-1} d_k + d_{k-1} \partial_k = 0$ ,
- $D_{k-1} D_k = 0$ .

We have thus established that  $(\mathbb{Z}[\sigma_k], D_k)$  is a chain complex.

Observe that the symmetric group  $S_{k+1}$  acts on  $\sigma_k(m)$  and hence on  $\sigma_k$ . We let

$$\mathbb{R}_{\text{alt}}(\sigma_k) = \{f : \sigma_k \rightarrow \mathbb{R} \mid f \text{ is alternating w.r.t the } S_{k+1}\text{-action}\},$$

and let  $D_k^* : \mathbb{R}(\sigma_{k-1}) \rightarrow \mathbb{R}(\sigma_k)$  denote the dual of  $D_k \otimes_{\mathbb{R}} \mathbb{1} : \mathbb{R}[\sigma_k] \rightarrow \mathbb{R}[\sigma_{k-1}]$ . Then we obtain from Lemma ??:

**Lemma 6**  $(\mathbb{R}_{\text{alt}}(\sigma_k), D_k^*)$  is a cochain complex.

### 3.2 The volume cocycle

We identify  $\mathbb{P}(\mathbb{C}^2)$  with  $\partial\mathbb{H}^3$ , so that we have a map  $\mathbb{C}^2 \setminus \{0\} \rightarrow \partial\mathbb{H}^3$  induced by the canonical projection. We consider the map  $\text{Vol}_{\mathbb{H}^3} : (\partial\mathbb{H}^3)^4 = \mathbb{P}(\mathbb{C}^2)^4 \rightarrow \mathbb{R}$  that to a quadruple of points  $(x_0, x_1, x_2, x_3) \in (\partial\mathbb{H}^3)^4$  associates the signed volume of the corresponding hyperbolic simplex. This extends to a function

$$\text{Vol} : \sigma_3 \longrightarrow \mathbb{R}$$

by setting  $\text{Vol}|_{\sigma_3(m)} = 0$  for all  $m \neq 2$  and

$$\text{Vol}([\mathbb{C}^2; v_0, \dots, v_3]) := \begin{cases} \text{Vol}_{\mathbb{H}^3}(v_0, \dots, v_3) & \text{if } v_i \neq 0 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 7** The function  $\text{Vol} \in \mathbb{R}_{\text{alt}}(\sigma_3)$  satisfies  $D_4^* \text{Vol} = 0$ .

*Proof.* First observe that  $D_4^* \text{Vol}(\tau) = \text{Vol}(D_4 \tau) = 0$  if  $\tau \in \sigma_4(0) \sqcup \sigma_4(1) \sqcup \sigma_4(4) \sqcup \sigma_4(5)$ . Thus we have two cases to consider, namely  $\tau \in \sigma_4(2)$  and  $\tau \in \sigma_4(3)$ .

**Lemma 8** Let  $\tau \in \sigma_4(2)$ . Then  $\text{Vol}(D_4 \tau) = 0$ .

*Proof.* Let  $\tau = [\mathbb{C}^2; v_0, \dots, v_4]$ , then

$$\begin{aligned} D_4^* \text{Vol}(\tau) &= \sum_{i=0}^4 (-1)^i \text{Vol}([\langle v_0, \dots, \widehat{v}_i, \dots, v_4 \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4]) \\ &\quad - \sum_{i=0}^4 (-1)^i \text{Vol}([\mathbb{C}^2 / \langle v_i \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4]). \end{aligned}$$

Suppose that  $v_j \neq 0$  for every  $j$ . Observe that whether  $v_0, \dots, \widehat{v}_i, \dots, v_4$  generate  $\mathbb{C}^2$  or not, we have

$$\text{Vol}([\langle v_0, \dots, \widehat{v}_i, \dots, v_4 \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4]) = \text{Vol}_{\mathbb{H}^3}(v_0, \dots, \widehat{v}_i, \dots, v_4).$$

Indeed, if the  $v_i$ 's do generate  $\mathbb{C}^2$ , this is the definition of  $\text{Vol}$ , and if not, then on the one hand  $\text{Vol}([\langle v_0, \dots, \widehat{v}_i, \dots, v_4 \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4]) = 0$  and on the other hand  $\text{Vol}_{\mathbb{H}^3}(v_0, \dots, \widehat{v}_i, \dots, v_4) = 0$  since the projection of  $v_j$ 's in  $\partial\mathbb{H}^3 = \mathbb{P}(\mathbb{C}^2)$  coincide. As all the  $v_j \neq 0$  and  $\text{Vol}_{\mathbb{H}^3}$  is a cocycle on  $\partial\mathbb{H}^3 = \mathbb{P}(\mathbb{C}^2)$ , the first sum vanishes. So does each term of the second sum since  $\mathbb{C}^2 / \langle v_i \rangle$  is 1-dimensional.

If  $v_j = 0$  for some  $j$ , then for every  $i \neq j$ , the  $i$ -th term of the first and second sums vanish since the null vector appears in the argument each time. It follows that  $D_4^* \text{Vol}(\tau)$  is equal to

$$(-1)^j \left\{ \underbrace{\text{Vol}(\langle v_0, \dots, \widehat{v}_j, \dots, v_4 \rangle; v_0, \dots, \widehat{v}_j, \dots, v_4)}_{\mathbb{C}^2} - \underbrace{\text{Vol}(\mathbb{C}^2 / \langle v_j \rangle; v_0, \dots, \widehat{v}_j, \dots, v_4)}_{\mathbb{C}^2} \right\} = 0,$$

which finishes the proof of the lemma.  $\square$

In the case of a generic configuration of flags the following lemma is what Goncharov called the ‘‘five line identity’’.

**Lemma 9** *Let  $\tau \in \sigma_4(3)$ . Then  $\text{Vol}(D_4 \tau) = 0$ .*

*Proof.* Let  $\tau = [\mathbb{C}^3; v_0, \dots, v_4]$ , then

$$D_4^* \text{Vol}(\tau) = \sum_{i=0}^4 (-1)^i \text{Vol}(\langle v_0, \dots, \widehat{v}_i, \dots, v_4 \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4) - \sum_{i=0}^4 (-1)^i \text{Vol}(\mathbb{C}^3 / \langle v_i \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4).$$

We distinguish several cases:

**$v_j = 0$  for some  $j$ :** For every  $i \neq j$ , the  $i$ -th term of the first and second sums vanish. The  $j$ -th terms are also both zero since both spaces  $\langle v_0, \dots, \widehat{v}_j, \dots, v_4 \rangle$  and  $\mathbb{C}^3 / \langle v_j \rangle$  are 3- and not 2-dimensional.

From now on we suppose that  $v_j \neq 0$  for every  $j$ .

**$\langle v_i \rangle = \langle v_j \rangle$  for some pair  $i \neq j$ :** By alternation we can suppose that  $i = 0, j = 1$ . In this case,  $\langle v_1, v_2, v_3, v_4 \rangle = \langle v_0, v_2, v_3, v_4 \rangle = \mathbb{C}^3$  and the two first terms of the first sum vanish. Since  $\langle v_0 \rangle = \langle v_1 \rangle$  the three last terms of the first sum vanish also. In the second sum, the two first terms vanish since the null vector appears in the argument, while the last three terms vanish since either  $v_0$  and  $v_1$  are zero in the corresponding quotients or they span the same line.

We suppose from now on that all lines generated by the  $v_i$ 's are distinct.

**$\text{Dim}(\langle v_0, \dots, \widehat{v}_j, \dots, v_4 \rangle) = 2$  for some  $j$ :** By alternation we can suppose that  $j = 4$ . Since no two vectors lie on the same line, it follows that  $\langle v_0, \dots, \widehat{v}_i, \dots, v_3 \rangle \cong \mathbb{C}^2$  for every  $i \in \{0, \dots, 3\}$ ; since the 5-tuple generates  $\mathbb{C}^3$ , we further get  $\langle v_0, \dots, \widehat{v}_i, \dots, v_3, v_4 \rangle = \mathbb{C}^3$ . The two sums for  $D_4^* \text{Vol}(\tau)$  thus reduce to

$$\text{Vol}(\langle v_0, \dots, v_3 \rangle; v_0, \dots, v_3) - \sum_{i=0}^4 (-1)^i \text{Vol}(\mathbb{C}^3 / \langle v_i \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4).$$

The composition of injection and projection

$$\langle v_0, \dots, v_3 \rangle \hookrightarrow \mathbb{C}^3 \longrightarrow \mathbb{C}^3 / \langle v_4 \rangle$$

gives an isomorphism identifying

$$[\langle v_0, \dots, v_3 \rangle; v_0, \dots, v_3] = [\mathbb{C}^3 / \langle v_4 \rangle; v_0, \dots, v_3].$$

Thus, the remaining term for the first sum cancels with the last term of the second sum. We are left with

$$- \sum_{i=0}^3 (-1)^i \text{Vol}([\mathbb{C}^3 / \langle v_i \rangle; v_0, \dots, \widehat{v}_i, \dots, v_3, v_4]).$$

But since  $\langle v_0, \dots, \widehat{v}_i, \dots, v_3 \rangle = \langle v_0, v_1, v_2, v_3 \rangle$  the projections of  $v_j$ , for any  $i \neq j$  between 0 and 3 generate the same complex line in the quotient  $\mathbb{C}^3 / \langle v_i \rangle$  so that Vol vanishes on such configurations, finishing the proof in this case.

**For every  $j$ ,  $\langle v_0, \dots, \widehat{v}_j, \dots, v_4 \rangle = \mathbb{C}^3$  and  $(v_0, \dots, v_4)$  is not generic:** Recall that a  $q$ -tuple of vectors  $(w_1, \dots, w_q)$  in  $\mathbb{C}^m$  is generic if and only if  $\text{Dim} \langle w_{i_1}, \dots, w_{i_k} \rangle = k$  whenever  $k \leq m$  and the  $1 \leq i_1, \dots, i_k \leq q$  are distinct. As we have assumed that none of the  $v_j$  vanish and no 4-subtuple generate  $\mathbb{C}^2$ , the only way this 5-tuple can be non-generic is if 3 among the vectors generate a 2-dimensional subspace. We can without loss of generality suppose that  $\text{Dim}(\langle v_2, v_3, v_4 \rangle) = 2$ . Since  $\langle v_0, \dots, \widehat{v}_j, \dots, v_4 \rangle = \mathbb{C}^3$  for every  $j$ , the first sum vanishes. As in the previous case, the images of  $v_3$  and  $v_4$  generate the same line in  $\mathbb{C}^3 / \langle v_2 \rangle$ ; the analogous statement holds for  $\mathbb{C}^3 / \langle v_3 \rangle$  and  $\mathbb{C}^3 / \langle v_4 \rangle$ , so that the  $i$ -th term of the second sum vanishes for  $i = 2, 3, 4$ . Finally, we have

$$\mathbb{C}^3 = \langle v_2, v_3, v_4 \rangle \oplus \langle v_1 \rangle = \langle v_2, v_3, v_4 \rangle \oplus \langle v_0 \rangle.$$

Since there exists  $g \in \text{GL}(3, \mathbb{C})$  fixing the plane  $\langle v_2, v_3, v_4 \rangle$  and sending  $v_1$  to  $v_0$  it follows that

$$\text{Vol}([\mathbb{C}^3 / \langle v_0 \rangle; v_1, v_2, v_3, v_4]) = \text{Vol}([\mathbb{C}^3 / \langle v_1 \rangle; v_0, v_2, v_3, v_4]),$$

finishing the proof of this case.

$(v_0, \dots, v_4)$  **is generic:** As in the previous case, the first sum cancels since the spaces in the arguments are all 3-dimensional. Using the surjective linear maps  $f_i$  of the next Lemma ??, we have

$$[\mathbb{C}^3 / \langle v_i \rangle; v_0, \dots, \widehat{v}_i, \dots, v_4] = [\mathbb{C}^2; f_i(v_0), \dots, f_i(\widehat{v}_i), \dots, f_i(v_4)].$$

Then Lemma ?? below implies that  $w_k := \langle f_i(v_k) \rangle$  for  $i \neq k$  is independent of  $i$ , so that the second sum now reduces to  $\sum_{i=0}^4 (-1)^i \text{Vol}_{\mathbb{H}^3}(w_0, \dots, \widehat{w}_i, \dots, w_4)$ , for  $w_j \in f_i(v_j)$  for  $i \neq j$ , which is equal to 0 since  $\text{Vol}_{\mathbb{H}^3}$  is a cocycle.

□

**Lemma 10** *Let  $v_0, \dots, v_4$  be a generic 5-tuple in  $\mathbb{C}^3$  and let  $L_i = \langle v_i \rangle$ . Then there are 5 lines  $\ell_0, \dots, \ell_4$  in  $\mathbb{C}^2$  and surjective linear maps  $f_i : \mathbb{C}^3 \rightarrow \mathbb{C}^2$  with  $f_i(L_i) = 0$  and  $f_i(L_j) = \ell_j$  for  $j \neq i$ .*

*Proof.* We use  $\{v_2, v_3, v_4\}$  as a basis of  $\mathbb{C}^3$  and express everything in those coordinates. Then  $v_0 = (a_2, a_3, a_4)$  and  $v_1 = (b_2, b_3, b_4)$  with all coordinates nonzero. Let

$$g = \begin{pmatrix} a_2/b_2 & 0 & 0 \\ 0 & a_3/b_3 & 0 \\ 0 & 0 & a_4/b_4 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{C}),$$

so that  $g(v_1) = v_0$  and  $g(v_i) \in \mathbb{C}^* v_i$  for  $i \in \{2, 3, 4\}$ . Choose a surjection  $f_0 : \mathbb{C}^3 \rightarrow \mathbb{C}^2$  with kernel  $L_0$  and define

$$\ell_0 = f_0 \circ g(L_0), \quad \ell_1 = f_0(L_1), \quad \ell_2 = f_0(L_2), \quad \ell_3 = f_0(L_3), \quad \ell_4 = f_0(L_4).$$

Then  $f_0$  automatically satisfies the conclusion of the lemma, and so does  $f_1 = f_0 \circ g$ . We proceed to construct  $f_i$  for  $i = 2, 3, 4$ . By symmetry, it is enough to construct  $f_2$ . Set  $f_2 = \mu_0 f_0 + \mu_1 f_1$ , with  $\mu_0, \mu_1 \neq 0$  chosen such that  $f_2(v_2) = 0$ . Note that such  $\mu_0$  and  $\mu_1$  exist since  $f_0(v_2)$  and  $f_1(v_2)$  belong to the same line  $\ell_2 \setminus \{0\}$  and do not vanish. Since for  $i = 3, 4$   $f_0(v_i)$  and  $f_1(v_i)$  both belong to  $\ell_i$ , then  $f_2(v_i)$  belongs to  $\ell_i$  as well. To show that  $f(L_i) = \ell_i$  it is thus enough to show that  $f_2(v_i)$ , say  $f_2(v_3) \neq 0$ . Suppose the contrary, then together with  $f_2(v_2) = 0$  we get

$$\begin{aligned} \mu_0 a_2 + \mu_1 b_2 &= 0, \\ \mu_0 a_3 + \mu_1 b_3 &= 0, \end{aligned}$$

and hence  $b_2/a_2 = b_3/a_3$ . But this implies  $b_2 v_0 - a_2 v_1 = (b_2 a_4 - a_2 b_4) v_4$  which is a contradiction with the assumption that  $v_0, \dots, v_4$  is generic. Also,  $f_2(v_0) = \mu_0 f_0(v_0) + \mu_1 f_1(v_0) = \mu_1 f(v_0) \in \ell_1 \setminus \{0\}$  so that indeed  $f(L_0) = \ell_0$ , and similarly  $f(L_1) = \ell_1$ , finishing the proof of the lemma.  $\square$

Putting together Lemma ?? and ?? finishes the proof of Theorem ??.  $\square$

## 4 Affine flags

A complete flag  $F$  in  $\mathbb{C}^n$  is a sequence of  $n+1$  vector subspaces  $F^0 \subset F^1 \subset \dots \subset F^n$  with  $\dim(F^j) = j$ . By an abuse of terminology, in the sequel by a ‘‘flag’’ we will mean a complete flag. An affine flag  $(F, v)$  is a pair consisting of a flag  $F$  and an  $n$ -tuple of vectors  $v = v^1, v^2, \dots, v^n$  such that

$$F^j = \mathbb{C}v^j + F^{j-1}, \quad j \geq 1.$$

The group  $\mathrm{GL}(n, \mathbb{C})$  acts naturally on the space  $\mathcal{F}(\mathbb{C}^n)$  of all flags and the space  $\mathcal{F}_{\mathrm{aff}}(\mathbb{C}^n)$  of affine flags.

We consider  $(\mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}], \partial_k)$ , where  $\mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}]$  is the free abelian group on  $\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}$  and  $\partial_k$  is the boundary map associated to the usual face maps  $\varepsilon_i^{(k)} : \mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1} \rightarrow \mathcal{F}_{\text{aff}}(\mathbb{C}^n)^k$  consisting in dropping the  $i$ -th component for  $k \geq 0$ . We extend the definition to  $\partial_0 : \mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)] \rightarrow 0$ . Our aim is to construct a morphism of complexes, or almost so,

$$T_k : (\mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}], \partial_k) \longrightarrow (\mathbb{Z}[\sigma_k], D_k).$$

To this end, for every multiindex  $\mathbb{J} \in [0, n-1]^{k+1}$ , we define the map

$$t_{\mathbb{J}} : \mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1} \rightarrow \sigma_k$$

by

$$t_{\mathbb{J}}((F_0, v_0), \dots, (F_k, v_k)) := \left[ \frac{\langle F_0^{j_0+1}, \dots, F_k^{j_k+1} \rangle}{\langle F_0^{j_0}, \dots, F_k^{j_k} \rangle}; (v_0^{j_0+1}, \dots, v_k^{j_k+1}) \right]$$

and finally  $T_k : \mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}] \rightarrow \mathbb{Z}[\sigma_k]$  by

$$T_k((F_0, v_0), \dots, (F_k, v_k)) = \sum_{\mathbb{J} \in [0, n-1]^{k+1}} t_{\mathbb{J}}((F_0, v_0), \dots, (F_k, v_k)).$$

**Lemma 11** *Let  $k \geq 1$  and let us consider the following diagram:*

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}] & \xrightarrow{T_k} & \mathbb{Z}[\sigma_k] \\ \partial_k \downarrow & & \downarrow D_k \\ \mathbb{Z}[\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^k] & \xrightarrow{T_{k-1}} & \mathbb{Z}[\sigma_{k-1}]. \end{array}$$

We have:

1. If  $k$  is odd,  $T_{k-1}\partial_k - D_kT_k = 0$ .
2. If  $k$  is even,  $T_{k-1}\partial_k - D_kT_k$  evaluated on an affine flag equals

$$n^k [0; \underbrace{(0, \dots, 0)}_k] \in \mathbb{Z}[\sigma_{k-1}(0)].$$

*Proof.* One verifies the following relations for every  $0 \leq i \leq k$  and  $\mathbb{J} \in [0, n-1]^{k+1}$ :

- (a) If  $j_i \leq n-2$ , then  $\eta_i^{(k)} t_{\mathbb{J}} = \varepsilon_i^{(k)} t_{\mathbb{J}+\delta_i}$ , where  $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$  with the 1 in the  $i$ -th position.
- (b) If  $j_i = n-1$ , then  $\eta_i^{(k)} t_{\mathbb{J}}((F_0, v_0), \dots, (F_k, v_k)) = [0; (0, \dots, 0)]$ .
- (c) If  $j_i = 0$ , then  $\varepsilon_i^{(k)} t_{\mathbb{J}} = t_{\mathbb{J}(i)} \varepsilon_i^{(k)}$ , where  $\mathbb{J}(i) \in [0, n-1]^k$  is obtained from  $\mathbb{J}$  by dropping  $j_i$ .

We evaluate

$$D_k T_k((F_0, v_0), \dots, (F_k, v_k)) = \sum_{i=0}^k (-1)^i \left( \sum_{\mathbb{J}} \varepsilon_i^{(k)} t_{\mathbb{J}}((F_0, v_0), \dots, (F_k, v_k)) \right. \\ \left. - \sum_{\mathbb{J}} \eta_i^{(k)} t_{\mathbb{J}}((F_0, v_0), \dots, (F_k, v_k)) \right).$$

Splitting the first inner sum into a sum over  $\mathbb{J} \in [0, n-1]^{k+1}$  with  $j_i = 0$  and  $\mathbb{J}$  with  $j_i \geq 1$ , we obtain using (c) from the first contribution the value  $T_{k-1} \varepsilon_i^{(k)}((F_j, v_j))$  while the second contribution together with the second inner sum adds up to  $-n^k [0; (0, \dots, 0)]$  using (a) and (b).  $\square$

Now we dualize the objects so far considered, as in Section ???. For the natural  $S_{k+1}$ -action on  $\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}$ , the spaces  $\mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1})$  of alternating cochains together with the dual  $\partial_k^*$  of  $\partial_k \otimes_{\mathbb{R}} \mathbb{1}$  form a complex. Denoting  $T_k^*$  the dual of  $T_k \otimes_{\mathbb{R}} \mathbb{1}$  we obtain immediately from Lemma ???:

**Proposition 12** *The map  $T_k^* : \mathbb{R}_{\text{alt}}(\sigma_k) \rightarrow \mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1})$  is a morphism of complexes, taking values in the subcomplex  $\mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1})^{\text{GL}(n, \mathbb{C})}$  of  $\text{GL}(n, \mathbb{C})$ -invariants.*

In particular, defining now

$$B_n((F_0, v_0), \dots, (F_3, v_3)) := T_3^* \text{Vol}((F_0, v_0), \dots, (F_3, v_3)) \\ = \sum_{\mathbb{J} \in [0, n-1]^4} \text{Vol} \left( \left[ \frac{\langle F_0^{j_0+1}, \dots, F_3^{j_3+1} \rangle}{\langle F_0^{j_0}, \dots, F_3^{j_3} \rangle}; (v_0^{j_0+1}, \dots, v_3^{j_3+1}) \right] \right), \quad (5)$$

where  $\text{Vol} \in \mathbb{R}_{\text{alt}}(\sigma_3)$  is the function on  $\sigma_3$  defined in Section ??? we obtain

**Corollary 13** *The function  $B_n$  is a  $\text{GL}(n, \mathbb{C})$ -invariant alternating cocycle defined on all 4-tuples of affine flags in  $\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^4$ , which descends to a well defined  $\text{GL}(n, \mathbb{C})$ - hence  $\text{PGL}(n, \mathbb{C})$ -invariant function on the space  $\mathcal{F}(\mathbb{C}^n)^4$  of 4-tuples of flags.*

*Proof.* That  $B_n$  is alternating follows from the same property of  $\text{Vol}$ . The fact that it is a cocycle follows from Proposition ??? and Theorem ???. Finally, it descends to  $\mathcal{F}(\mathbb{C}^n)^4$  since  $\text{Vol}([\mathbb{C}^2; v_0, \dots, v_4])$  only depends on the lines generated by  $v_0, \dots, v_4$ .  $\square$

## 5 Boundedness of $B_n$

The aim of this section is to establish the following

**Theorem 14** *Let  $F_0, \dots, F_3 \in \mathcal{F}(\mathbb{C}^n)$  be arbitrary flags. Then*



$$B_n(F_0, \dots, F_3) \leq \frac{1}{6}n(n^2 - 1) \cdot v_3.$$

where  $v_3$  denotes the volume of the maximal ideal tetrahedron in  $\mathbb{H}^3$ .

We start by introducing some useful notation and giving an argument for Theorem ?? in the case in which the configuration of flags is generic.

Given four flags  $F_0, \dots, F_3 \in \mathcal{F}(\mathbb{C}^n)$  we denote by  $\mathbb{F} = (F_0, \dots, F_3)$  the corresponding quadruple of flags. For any multi-index  $\mathbb{J} = (j_0, \dots, j_3)$  with  $0 \leq j_i \leq n-1$  we let  $Q(\mathbb{F}, \mathbb{J})$  be the quotient

$$Q(\mathbb{F}, \mathbb{J}) := \frac{\langle F_0^{j_0+1}, \dots, F_3^{j_3+1} \rangle}{\langle F_0^{j_0}, \dots, F_3^{j_3} \rangle}$$

and denote by  $f(\mathbb{F}, \mathbb{J}) \subset Q(\mathbb{F}, \mathbb{J})$  the 4-tuple of 0 or 1-dimensional subspaces obtained by projecting  $(F_0^{j_0+1}, \dots, F_3^{j_3+1})$  to  $Q(\mathbb{F}, \mathbb{J})$ .

Furthermore, for any nonnegative integers  $k, n$  with  $k \geq 1$  we set

$$C_k(n) = \# \left\{ (a_0, \dots, a_{k-1}) \mid a_i \in \mathbb{N}, \sum_{i=0}^{k-1} a_i = n \right\}.$$

Note that  $C_k(0) = 1$ ,  $C_k(1) = k$ ,  $C_1(n) = 1$  and we have the recursive relation

$$C_k(n) = C_{k-1}(n) + C_k(n-1), \quad (6)$$

for  $k \geq 2$ ,  $n \geq 1$ . Indeed the set underlying  $C_k(n)$  is the disjoint union of the  $k$ -tuples with  $a_{k-1} = 0$  giving the term  $C_{k-1}(n)$  and the  $k$ -tuples with  $a_{k-1} \geq 1$  which is in bijection with the set underlying  $C_k(n-1)$  via  $a_{k-1} \mapsto a_{k-1} - 1$ . Using the relation (??) it is straightforward to conclude that

$$\begin{aligned} C_4(n) &= \binom{n+1}{1} C_3(0) + \binom{n+1}{2} C_2(1) + \binom{n+1}{3} C_1(2) \\ &= \frac{1}{6}(n+1)(n+2)(n+3). \end{aligned}$$

Observe that  $C_4(n-2)$  is exactly the number of  $\mathbb{J} = (j_0, \dots, j_3)$  with  $0 \leq j_i \leq n-2$  for which the dimension

$$\dim(Q(\mathbb{F}, \mathbb{J})) = 2,$$

for quadruple of flags  $\mathbb{F} = (F_0, \dots, F_3) \in (\mathcal{F}(\mathbb{C}^n))^4$  in general position. Let us outline the short argument assuming  $n \geq 3$ . We distinguish three cases:

- If  $j_0 + \dots + j_3 \leq n-3$ , then the dimension of the vector space  $\langle F_0^{j_0+1}, \dots, F_3^{j_3+1} \rangle$  is equal to the minimum between  $n$  and  $j_0 + \dots + j_3 + 4$  so that the quotient has dimension

$$\min\{n, j_0 + \dots + j_3 + 4\} - (j_0 + \dots + j_3) \geq \min\{3, 4\} \geq 3.$$

- If  $j_0 + \dots + j_3 = n - 2$  then  $\langle F_0^{j_0+1}, \dots, F_3^{j_3+1} \rangle$  has dimension  $n$  so that the quotient has dimension 2.
- If  $j_0 + \dots + j_3 \geq n - 1$ , then the quotient has dimension 0 or 1.

We conclude that  $C_4(n-2)$  is the number of nonzero summand in  $B_n$  for generic flags. This proves Theorem ?? for generic 4-tuples of flags. The non-generic case is more involved and we start with the following simple observation.

**Lemma 15** *Let  $\mathbb{F} = (F_0, \dots, F_3) \in \mathcal{F}(\mathbb{C}^n)^4$  be arbitrary 4-tuple of flags. Then for every  $0 \leq j_0, j_1, j_2 \leq n-2$  there exists at most one  $0 \leq j_3 \leq n-2$  such that*

$$\text{Vol}(Q(\mathbb{F}, (j_0, \dots, j_3)); f(\mathbb{F}, (j_0, \dots, j_3))) \neq 0.$$

*Proof.* If there is no  $j_3$  with  $0 \leq j_3 \leq n-2$  such that  $\dim(Q(\mathbb{F}, \mathbb{J})) = 2$  and  $F_3^{j_3+1} \neq 0$  in  $Q(\mathbb{F}, \mathbb{J})$ , we are done. Otherwise take  $j_3$  minimal satisfying these two conditions. This implies that  $F_0^{j_0+1}, F_1^{j_1+1}, F_2^{j_2+1}$  all lie on the same line in  $\mathbb{C}^n / \langle F_3^{j_3+1} \rangle$  and hence in  $\mathbb{C}^n / \langle F_3^j \rangle$  for any  $j > j_3$  and also in  $\langle F_0^{j_0+1}, \dots, F_3^{j_3+1} \rangle / \langle F_0^j, \dots, F_3^j \rangle$ . Thus the volume vanishes for  $j > j_3$ . It clearly vanishes for  $j < j_3$  by definition of  $j_3$ .  $\square$

Notice that the lemma implies immediately that

$$\left| \sum_{\mathbb{J} \in \left\{ \begin{array}{l} j_0 = j_1 = j_2 = 0, \\ 0 \leq j_3 \leq n-2 \end{array} \right\}} \text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})) \right| \leq v_3 \quad (7)$$

and

$$\left| \sum_{\mathbb{J} \in \left\{ \begin{array}{l} j_0 = j_1 = 0 \\ 0 \leq j_2, j_3 \leq n-2 \end{array} \right\}} \text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})) \right| \leq C_2(n-2) \cdot v_3; \quad (8)$$

in fact there are  $C_2(n-2) = n-1$  choices for  $j_2$  giving by Lemma ?? each at most one nonzero summand. We will further show:

**Lemma 16** *Let  $\mathbb{F} = (F_0, \dots, F_3) \in (\mathcal{F}(\mathbb{C}^n))^4$  be an arbitrary quadruple of flags. Then*

$$\left| \sum_{\mathbb{J} \in \left\{ \begin{array}{l} j_0 = 0 \\ 0 \leq j_1, j_2, j_3 \leq n-2 \end{array} \right\}} \text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})) \right| \leq C_3(n-2) \cdot v_3.$$

*Proof (of Theorem ?? and Lemma ??).* We prove the theorem and the lemma simultaneously by induction on  $n$ .

For  $n = 2$  there is only one summand  $(j_0, \dots, j_3) = (0, \dots, 0)$  in both the theorem and the lemma, so the inequalities are immediate. Suppose that the theorem and the lemma are proven for  $n - 1$ . By definition, we have

$$B_n(F_0, \dots, F_4) = \sum_{\mathbb{J} \in \{(j_0, \dots, j_3) \mid 0 \leq j_i \leq n-2\}} \text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})).$$

Indeed if  $j_i = n - 1$  then the quotient is 1 or 0-dimensional and the volume vanishes. We split the above sum into three, summing over

$$\begin{aligned} J_1 &= \{(j_0, \dots, j_3) \mid j_0 = j_1 = 0, 0 \leq j_2, j_3 \leq n-2\}, \\ J_2 &= \{(j_0, \dots, j_3) \mid j_0 = 0, 0 < j_1 \leq n-2, 0 \leq j_2, j_3 \leq n-2\}, \\ J_3 &= \{(j_0, \dots, j_3) \mid 0 < j_0 \leq n-2, 0 \leq j_1, j_2, j_3 \leq n-2\}. \end{aligned}$$

We first analyze the sum over  $J_3$ . Denote by  $\bar{V}$  the image of a subspace  $V \subset \mathbb{C}^n$  under the projection onto  $\mathbb{C}^n / \langle F_0^1 \rangle$ . If  $F \in \mathcal{F}(\mathbb{C}^n)$  is a complete flag, then we denote by  $\bar{F} \in \mathcal{F}(\mathbb{C}^n / \langle F_0^1 \rangle)$  the complete flag we obtain as the projection of  $F$ . More precisely, the  $n + 1$  subspaces of  $F$  project onto  $n$  distinct subspaces in the quotient, giving the  $n$  distinct subspaces of  $\bar{F}$ :

$$\underbrace{\bar{F}^0}_{=\bar{F}^0} \subset \dots \subset \underbrace{\bar{F}^{i-1}}_{=\bar{F}^{i-1}} \subset \underbrace{\bar{F}^i = \bar{F}^{i+1}}_{=\bar{F}^i} \subset \underbrace{\bar{F}^{i+2}}_{=\bar{F}^{i+1}} \subset \dots \subset \underbrace{\bar{F}^n}_{=\bar{F}^{n-1}} = \mathbb{C}^n / \langle F_0^1 \rangle,$$

for some  $0 \leq i \leq n - 1$ . Note in particular that  $\bar{F}^j$  is equal to either  $\bar{F}^j$  or  $\bar{F}^{j+1}$  (or both in the unique case of  $j = i$ ). The projection of  $F_0$  is

$$\bar{F}_0^0 = \bar{F}_0^1 \subset \bar{F}_0^2 \subset \dots \subset \bar{F}_0^n = \mathbb{C}^n / \langle F_0^1 \rangle,$$

so in this case, the  $j$ -th space of  $\bar{F}_0$  is always the projection of the  $(j + 1)$ -th space of  $F_0$ .

Note that since in the sum over  $J_3$ , the index  $j_0$  is greater or equal to 1, we have for each summand  $1 \leq j_0 \leq n - 2, 0 \leq j_1, j_2, j_3 \leq n - 2$  that

$$\text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})) = \text{Vol} \left( \frac{\langle \bar{F}_0^{j_0+1}, \dots, \bar{F}_3^{j_3+1} \rangle}{\langle \bar{F}_0^{j_0}, \dots, \bar{F}_3^{j_3} \rangle}; \bar{F}_0^{j_0+1}, \dots, \bar{F}_3^{j_3+1} \right).$$

Furthermore, if  $j_k$ , for  $1 \leq k \leq 3$  is such that  $\bar{F}_k^{j_k} = \bar{F}_k^{j_k+1}$  then the space  $\bar{F}_k^{j_k+1}$  is 0 in the quotient and thus the volume vanishes. Instead of summing on the dimensions of the spaces of the flags  $F_0, \dots, F_3$ , we can thus sum over the dimensions of the spaces of the quotient flags  $\bar{F}_0, \dots, \bar{F}_3$  and the sum over  $J_3$  becomes

$$\sum_{0 \leq i_0, \dots, i_3 \leq n-3} \text{Vol} \left( \frac{\langle \overline{F}_0^{i_0+1}, \dots, \overline{F}_3^{i_3+1} \rangle}{\langle \overline{F}_0^{i_0}, \dots, \overline{F}_3^{i_3} \rangle}; \overline{F}_0^{i_0+1}, \dots, \overline{F}_3^{i_3+1} \right) \leq C_4(n-3) \cdot v_3, \quad (9)$$

where the last inequality follows by the induction hypothesis in the theorem.

Similarly, for the sum over  $J_2$  we quotient by  $F_1^1$  to obtain

$$\begin{aligned} & \sum_{\mathbb{J} \in \left\{ \begin{array}{l} j_0 = 0, 1 \leq j_1 \leq n-2 \\ 0 \leq j_1, j_2, j_3 \leq n-2 \end{array} \right\}} \text{Vol}(Q(\mathbb{F}, \mathbb{J}); f(\mathbb{F}, \mathbb{J})) \\ &= \sum_{\substack{i_0 = 0 \\ 0 \leq i_1, i_2, i_3 \leq n-3}} \text{Vol} \left( \frac{\langle \overline{F}_0^{i_0+1}, \dots, \overline{F}_3^{i_3+1} \rangle}{\langle \overline{F}_0^{i_0}, \dots, \overline{F}_3^{i_3} \rangle}; \overline{F}_0^{i_0+1}, \dots, \overline{F}_3^{i_3+1} \right) \\ &\leq C_3(n-3) \cdot v_3, \end{aligned}$$

by the induction hypothesis in the lemma. Since the sum over  $J_1$  is by (??) bounded by  $C_2(n-2) \cdot v_3$ , it follows that the sum over  $J_1$  and  $J_2$  is indeed bounded by

$$(C_2(n-2) + C_3(n-3)) \cdot v_3 = C_3(n-2) \cdot v_3,$$

which proves the lemma. It follows that  $B_n(F_0, \dots, F_3)$  is bounded by

$$(C_3(n-2) + C_4(n-3)) \cdot v_3 = C_4(n-2) \cdot v_3,$$

which proves the theorem.  $\square$

In the next section we will characterize the equality case, for which it will be useful to know, as a preliminary case, that equality can happen only if the flags are in general position, i.e. flags for which

$$\dim(\langle F_0^{j_0}, \dots, F_3^{j_3} \rangle) = j_0 + \dots + j_3$$

whenever  $j_0 + \dots + j_3 \leq n$ . The proof of the following lemma uses some of the arguments in the proof of Theorem ??.

**Lemma 17** *If equality holds in Theorem ?? then the flags  $F_0, \dots, F_3$  are in general position.*

*Proof.* We prove the lemma by induction on  $n$ . For  $n = 2$  the four flags  $F_0, \dots, F_3$  (which are given by their lines  $F_i^1$ ) are in general position if and only if  $\dim(\langle F_i^1, F_j^1 \rangle) = 2$  for every  $i \neq j$ , i.e. if and only if the lines are distinct. But if two lines are equal, then  $B_2(F_0, \dots, F_3) = 0$ . Suppose the lemma proven for  $n - 1$ . As before, denote by  $\overline{F}$  the projection of a complete flag  $F \in \mathcal{F}(\mathbb{C}^n)$  to a complete flag in  $\mathcal{F}(\mathbb{C}^n / \langle F_0^1 \rangle)$ . Recall that in particular,  $\overline{F}^j = \overline{F}^j$  or  $\overline{F}^{j-1}$ . Suppose that  $j$  is minimal

such that  $\overline{F_1^j} = \overline{F_1^{j-1}}$  or equivalently such that  $F_0^1 \subset F_1^j$ . By the proof of Theorem ??,  $B_n(F_0, \dots, F_3)$  is maximal if and only if each of the sums over  $J_1, J_2$  and  $J_3$  is maximal. In particular, by symmetry, the sum over  $j_0 = j_2 = 0$  is also maximal and hence

$$\begin{aligned} & \sum_{\substack{j_0 = j_2 = 0 \\ 0 \leq j_1, j_3 \leq m-2}} \text{Vol} \left( \frac{\langle F_0^1, F_1^{j_1+1}, F_2^1, F_3^{j_3+1} \rangle}{\langle F_1^{j_1}, F_3^{j_3} \rangle}; F_0^1, F_1^{j_1+1}, F_2^1, F_3^{j_3+1} \right) \\ &= C_2(n-2) \cdot v_3. \end{aligned}$$

But for  $j_1 \geq j$ , the space  $F_0^1$  is 0 in the quotient  $\frac{\langle F_0^1, F_1^{j_1+1}, F_2^1, F_3^{j_3+1} \rangle}{\langle F_1^{j_1}, F_3^{j_3} \rangle}$ , while for  $j_1 = j-1$ , the spaces  $F_0^1$  and  $F_1^j$  are equal in the quotient. In both cases, the volume vanishes. By Lemma ?? it follows that the above sum is smaller or equal to  $(j-1) \cdot v_3$ , hence we must have  $j-1 \geq C_2(n-2) = n-1$ .

We have thus established that  $F_0^1 \subset F_1^n \setminus F_1^{n-1}$  and by symmetry the same holds for the flags  $F_2$  and  $F_3$ . In particular,  $\overline{F_k^j} = \overline{F_k^j}$  for  $k = 1, 2, 3$  and  $0 \leq j \leq n-1$ , while  $\overline{F_0^j} = \overline{F_0^{j-1}}$  for  $j \geq 1$ .

Let  $0 \leq j_0, j_1, j_2, j_3 \leq n$  be such that  $j_0 + \dots + j_3 \leq n$ . Since the case  $j_0 = \dots = j_3 = 0$  is trivial we can by symmetry suppose that  $j_0 \geq 1$ . Again, it follows from the proof of the theorem that  $B_n(F_0, \dots, F_3)$  is maximal if and only if the sum over  $J_3$  is maximal. This sum is rewritten in (??) as

$$B_{n-1}(\overline{F_0}, \dots, \overline{F_3}) = C_4(n-3) \cdot v_3.$$

Thus by induction, the flags  $\overline{F_0}, \dots, \overline{F_3}$  are in general position. It remains to compute

$$\begin{aligned} \dim(\langle F_0^{j_0}, \dots, F_3^{j_3} \rangle) &= \dim(\langle \overline{F_0^{j_0}}, \dots, \overline{F_3^{j_3}} \rangle) + 1 \\ &= \dim(\langle \overline{F_0^{j_0-1}}, \overline{F_1^{j_1}}, \dots, \overline{F_3^{j_3}} \rangle) + 1 \\ &= (j_0 - 1) + j_1 + j_2 + j_3 + 1 = j_0 + j_1 + j_2 + j_3, \end{aligned}$$

which finishes the proof of the lemma.  $\square$

## 6 Maximality properties of the cocycle

The main result of this section is Theorem ??, in which we characterize the configuration of 4-tuples of flags on which  $B_n$  is maximal. This configuration is related to the Veronese embedding to which we now turn. The irreducible representation  $\pi_n : \text{PSL}(2, \mathbb{C}) \rightarrow \text{PSL}(n, \mathbb{C})$  induces a  $\pi_n$ -equivariant boundary map

$$\varphi_n : \mathbb{P}(\mathbb{C}^2) \longrightarrow \mathcal{F}(\mathbb{C}^n),$$

also known as the Veronese embedding. It is defined as follows:  $\varphi_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$  is the complete flag with  $(n-1)$ -dimensional space with basis

$$\left\{ \begin{pmatrix} x \\ y \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ y \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \\ y \end{pmatrix} \right\},$$

where  $\begin{bmatrix} x \\ y \end{bmatrix}$  are homogeneous coordinates on  $\mathbb{P}(\mathbb{C}^2)$ . The lower dimensional spaces are then obtained inductively with basis  $v'_i = xv_i + yv_{i+1}$ , for  $i = 1, \dots, k-1$ , where  $\{v_1, \dots, v_k\}$  is the basis of the  $k$ -dimensional space. More precisely, the basis of the  $(n-2)$ -dimensional space is

$$\left\{ \begin{pmatrix} x^2 \\ 2xy \\ y^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^2 \\ 2xy \\ y^2 \end{pmatrix} \right\}.$$

The  $(n-i)$ -dimensional space has as basis the vectors

$$\left( \underbrace{0, \dots, 0}_k, x^i, \binom{i}{1} x^{i-1} y^1, \dots, \binom{i}{j} x^{i-j} y^j, \dots, \binom{i}{i-1} x^1 y^{i-1}, y^i, \underbrace{0, \dots, 0}_{n-i-k-1} \right)^T, \quad (10)$$

for  $k = 0, \dots, n-i-1$ . We give another useful description of this complete flag. For  $i = 1, \dots, n-1$ , set

$$z_i^n = \left( x^i, \binom{i}{1} x^{i-1} y, \binom{i}{2} x^{i-2} y^2, \dots, \binom{i}{i} y^i, \underbrace{0, \dots, 0}_{n-i-1} \right)^T.$$

Note that  $z_i^n$  is the first vector of the basis in (??) of the  $(n-i)$ -th space of  $\varphi_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$ , that is the one corresponding to  $k = 0$ . Furthermore, since  $z_i^n$  does not belong to the space generated by  $z_{n-1}^n, z_{n-2}^n, \dots, z_{i+1}^n$ , the  $(n-i)$ -th space admits the alternative basis

$$\{z_{n-1}^n, z_{n-2}^n, \dots, z_{i+1}^n, z_i^n\}.$$

With this at hand, it is easy to prove the following:

**Lemma 18** *Let  $D$  be the  $(n-1) \times (n-1)$  diagonal matrix with diagonal entries  $1, 2, \dots, n-1$ . Let  $p$  be the projection  $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1} \cong \langle e_2, \dots, e_n \rangle$  with kernel  $\langle e_1 \rangle$ . Then*

$$D \cdot p \left( \varphi_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \right) = \varphi_{n-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right),$$

for any  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{P}(\mathbb{C}^2)$ .

Note that the projection  $p$  induces a map from the set of complete flags in  $\mathbb{C}^n$  to the set of complete flags in  $\mathbb{C}^{n-1}$ . (See the proof of Theorem ?? and Lemma ?? for a detailed description of the induced map.)

*Proof.* For  $y = 0$ , the complete flag  $\varphi_n \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$  is given as

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_{n-1} \rangle.$$

Its projection by  $p$  is the complete flag

$$\langle e_2 \rangle \subset \langle e_2, e_3 \rangle \subset \dots \subset \langle e_2, \dots, e_{n-1} \rangle.$$

Multiplication by  $D$  leaves this complete flag invariant, and this is indeed the image of  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  under  $\varphi_{n-1}$ .

Suppose now that  $y \neq 0$ . We may assume that  $y = 1$ . Note that the projection by  $p$  of the  $(n-i)$ -dimensional space of  $\varphi_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$  is  $(n-i)$ -dimensional, so  $p(z_{n-1}^n), \dots, p(z_i^n)$  is a basis of it. We show that  $D \cdot p(z_i^n) = i \cdot z_{i-1}^{n-1}$ , from which the lemma follows immediately. Indeed, projection by  $p$  erases the first entry of  $z_i^n$ . For the remaining entries, we have that the  $j$ -th entry of  $z_i^n$ , for  $2 \leq j \leq n$  is

$$\binom{i}{j} x^{i-j}.$$

Multiplication by  $D$  will multiply this entry by  $j$ , giving

$$i \cdot \binom{i-1}{j-1} x^{i-j},$$

which is precisely  $i$  times the  $(j-1)$ -th entry of  $z_{i-1}^{n-1}$ .

**Theorem 19** *Let  $F_0, \dots, F_3 \in \mathcal{F}(\mathbb{C}^n)$ . Then*

$$B_n(F_0, \dots, F_3) = \frac{1}{6} n(n^2 - 1) \cdot v_3$$

*if and only if there exists  $g \in \mathrm{GL}(n, \mathbb{C})$  and a positively oriented regular simplex with vertices  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$  such that*

$$F_i = g(\varphi_n(\xi_i)),$$

for  $i = 0, \dots, 3$ .

**Corollary 20** *Let  $F_0, \dots, F_3 \in \mathcal{F}(\mathbb{C}^n)$  be a maximal 4-tuple, in the sense that*

$$|B_n(F_0, \dots, F_3)| = \frac{1}{6}n(n^2 - 1) \cdot v_3.$$

*If for  $F \in \mathcal{F}(\mathbb{C}^n)$ , there is equality*

$$B_n(F_0, \dots, F_2, F_3) = B_n(F_0, \dots, F_2, F),$$

*then  $F = F_3$ .*

For the rest of this section, we will use the notation introduced after Lemma ?? at the beginning of Section ?. The first direction of Theorem ?? will follow from the following more general computation:

**Proposition 21** *Let  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$  and set  $F_i = \varphi_n(\xi_i)$ . Then*

$$B_n(F_0, \dots, F_3) = \frac{1}{6}n(n^2 - 1) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3).$$

To prove Proposition ?? by induction, we first prove:

**Lemma 22** *Let  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$  and set  $F_i = \varphi_n(\xi_i)$ . Then*

$$\begin{aligned} & \sum_{\substack{j_0 = j_1 = 0 \\ 0 \leq j_2, j_3 \leq n-2}} \text{Vol} \left( \frac{\langle F_0^1, F_1^1, F_2^{j_2+1}, F_3^{j_3+1} \rangle}{\langle F_2^{j_2}, F_3^{j_3} \rangle}; F_0^1, F_1^1, F_2^{j_2+1}, F_3^{j_3+1} \right) \\ & = C_2(n-2) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3). \end{aligned}$$

*Proof.* Let  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$ . If  $\xi_i = \xi_j$  for  $i \neq j$  then both sides of the equality vanish. By transitivity of  $\text{SL}_2\mathbb{C}$  on distinct triples of points, it is enough to prove the lemma for the four points

$$\xi_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \xi_1 = \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where  $z \in \mathbb{C}$ . Then the line of the flag  $\varphi_n(\xi_0)$  is generated by the vector

$$\left( 1, \binom{n-1}{1}, \binom{n-1}{2}, \dots, \binom{n-1}{n-1}, 1 \right)^T$$

and the the line of the flag  $\varphi_n(\xi_1)$  is generated by the vector

$$\left( z^{n-1}, \binom{n-1}{1} z^{n-2}, \binom{n-1}{2} z^{n-3}, \dots, \binom{n-1}{n-1} z, 1 \right)^T.$$



The flag  $\varphi_n(\xi_2)$  is

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_{n-1} \rangle$$

and the flag  $\varphi_n(\xi_3)$  is

$$\langle e_n \rangle \subset \langle e_n, e_{n-1} \rangle \subset \langle e_n, e_{n-1}, e_{n-2} \rangle \subset \dots \subset \langle e_n, e_{n-1}, \dots, e_2 \rangle.$$

The quotient  $\langle F_0^1, F_1^1, F_2^{j_2+1}, F_3^{j_3+1} \rangle / \langle F_2^{j_2}, F_3^{j_3} \rangle$  can only be 2-dimensional if  $j_2 + j_3 = n - 2$ . Fix  $0 \leq j_2 \leq n - 2$ , and notice that there are exactly  $C_2(n - 2) = n - 1$  such  $j_2$ 's. Let  $j_3 = n - 2 - j_2$ . The space generated by  $\varphi_n(\xi_2)^{j_2}$  and  $\varphi_n(\xi_3)^{n-2-j_2}$  is the space

$$\langle \varphi_n(\xi_2)^{j_2}, \varphi_n(\xi_3)^{n-2-j_2} \rangle = \langle e_1, \dots, e_{j_2}, e_{j_2+2}, \dots, e_n \rangle.$$

We choose as isomorphism between

$$\mathbb{C}^n / \langle \varphi_n(\xi_2)^{j_2}, \varphi_n(\xi_3)^{n-2-j_2} \rangle \cong \mathbb{C}^2 \cong \langle e_{j_2+1}, e_{j_2+2} \rangle$$

the map which is induced by the orthogonal projection from  $\mathbb{C}^n$  onto  $\langle e_{j_2+1}, e_{j_2+2} \rangle$ . Then the four points defined by  $F_0^1, F_1^1, F_2^{j_2+1}, F_3^{j_3+1}$  in the projectivisation of the quotient are

$$\begin{aligned} & \left[ \binom{n-1}{j_2}, \binom{n-1}{j_2+1} \right], \\ & \left[ \binom{n-1}{j_2} z^{n-(j_2+1)}, \binom{n-1}{j_2+1} z^{n-(j_2+2)} \right], \\ & [1, 0] \text{ and } [0, 1]. \end{aligned}$$

Acting with the diagonal 2 by 2 matrix with entries  $\binom{n-1}{j_2}^{-1}$ ,  $\binom{n-1}{j_2+1}^{-1}$ , and rescaling the second vector by  $z^{-n+(j_2+2)}$ , the four points become

$$[1, 1], \quad [z, 1], \quad [1, 0], \quad [0, 1],$$

which are exactly the original vertices  $\xi_0, \dots, \xi_3$ . It follows that

$$\text{Vol} \left( \frac{\langle F_0^1, F_1^1, F_2^{j_2+1}, F_3^{n-1-j_2} \rangle}{\langle F_2^{j_2}, F_3^{n-2-j_2} \rangle}; F_0^1, F_1^1, F_2^{j_2+1}, F_3^{n-1-j_2} \right) = \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3),$$

which proves the lemma.  $\square$

*Proof (of Proposition ??).* We prove the proposition by induction on  $n$ , establishing first the cases of  $n = 2$  and  $n = 3$ . For  $n = 2$ , there is nothing to prove. For  $n = 3$ , let  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$ . The volume  $B_3(\varphi_3(\xi_0), \dots, \varphi_3(\xi_3))$  is written as a sum over  $0 \leq j_0, \dots, j_3 \leq 1$ . For  $(j_0, \dots, j_3) = (0, \dots, 0)$  the quotient is 3-dimensional so the summand is 0. We thus have at most four nonzero summands given by letting one of the  $j_k$ 's be equal to 1. The set  $\{(0, 0, 1, 0), (0, 0, 0, 1)\}$  is exactly the set summed over in Lemma ?? for  $n = 3$ , so the value of the volume on these

two multi-indices is equal to  $2 \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3)$ . By symmetry, the same holds for  $\{(1, 0, 0, 0), (0, 1, 0, 0)\}$ , so that the value of  $B_3(\varphi_n(\xi_0), \dots, \varphi_n(\xi_3))$  is indeed  $4 \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3) = C_4(1) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3)$ .

Suppose that  $n \geq 4$  and let  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$ . As usual, the volume  $B_n(\varphi_n(\xi_0), \dots, \varphi_n(\xi_3))$  is written as a sum over  $0 \leq j_0, \dots, j_3 \leq n-2$ . We rewrite this sum as a sum over the three sets

$$\begin{aligned} & \{1 \leq j_0 \leq n-2, 0 \leq j_1, j_2, j_3 \leq n-2\}, \\ & \{1 \leq j_1 \leq n-2, 0 \leq j_0, j_2, j_3 \leq n-2\}, \\ & \{j_0 = j_1 = 0, 0 \leq j_2, j_3 \leq n-2\} \end{aligned}$$

minus the sum over

$$\{1 \leq j_0, j_1 \leq n-2, 0 \leq j_2, j_3 \leq n-2\}.$$

It follows from Lemma ?? that the third term is equal to  $C_2(n-2) \cdot v_3$ . Taking the quotient by  $F_0^1$ , the first term becomes

$$B_{n-1}(\overline{\varphi_n(\xi_0)}, \dots, \overline{\varphi_n(\xi_3)}).$$

But by Lemma ??, and with  $D$  therein,

$$\overline{\varphi_n(\xi_i)} = D^{-1} \varphi_{n-1}(\xi_i),$$

for  $0 \leq i \leq 3$ . In particular, the first term of the sum can be rewritten as

$$B_{n-1}(\varphi_{n-1}(\xi_0), \dots, \varphi_{n-1}(\xi_3)),$$

which is equal to  $C_4(n-3) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3)$  by induction. By symmetry, the same holds for the second term of the sum. For the fourth (and last) term, we take first the quotient by  $F_0^1$  and then by  $F_1^1$ , apply twice Lemma ?? to conclude that it is equal to

$$B_{n-2}(\varphi_{n-2}(\xi_0), \dots, \varphi_{n-2}(\xi_3)),$$

which by induction is equal to  $C_4(n-4) \cdot \text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3)$ .

In conclusion,  $B_n(\varphi_n(\xi_0), \dots, \varphi_n(\xi_3))$  is equal to  $\text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3)$  times

$$\begin{aligned} & 2 \cdot C_4(n-3) + \underbrace{C_2(n-2)}_{=C_3(n-2)-C_3(n-3)} - C_4(n-4) \\ & = C_4(n-3) + C_3(n-2) - C_3(n-3) + \underbrace{C_4(n-3) - C_4(n-4)}_{=C_3(n-3)} \\ & \qquad \qquad \qquad \underbrace{\hspace{10em}}_{=0} \\ & = C_4(n-2), \end{aligned}$$

which finishes the proof of the proposition.  $\square$

**Lemma 23** *The group  $\mathrm{GL}(n, \mathbb{C})$  acts transitively on triples  $(F_0, F_1, L)$ , where  $F_0, F_1 \in \mathcal{F}(\mathbb{C}^n)$  and  $L$  is a line in  $\mathbb{C}^n$  such that  $(F_0, F_1, L)$  is in general position, that is such that for every  $0 \leq j_0, j_1 \leq n$ ,*

$$\begin{aligned} \dim \langle F_0^{j_0}, F_1^{j_1} \rangle &= \min\{j_0 + j_1, n\}, \\ \dim \langle F_0^{j_0}, F_1^{j_1}, L \rangle &= \min\{j_0 + j_1 + 1, n\}. \end{aligned}$$

*Proof.* It is well known that  $\mathrm{GL}(n, \mathbb{C})$  acts transitively on the set of pairs of transverse flags. As a result we may assume that

$$\begin{aligned} F_0 &= \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_{n-1} \rangle, \\ F_1 &= \langle e_n \rangle \subset \langle e_n, e_{n-1} \rangle \subset \dots \subset \langle e_n, e_{n-1}, \dots, e_2 \rangle. \end{aligned}$$

Let  $L = \langle v \rangle$ ; if  $v_j = 0$  for some  $1 \leq j \leq n$ , then  $\dim \langle F_0^{j-1}, F_1^{n-j}, L \rangle = n - 1$ , contradicting the genericity assumption. Thus all the co-ordinates of  $v$  are non-zero; the diagonal matrix  $\mathrm{diag}(1/v_1, \dots, 1/v_n)$  then stabilises  $F_0, F_1$  and sends  $v$  to  $e_1 + \dots + e_n$ .  $\square$

**Lemma 24** *For any generic (in the sense of Lemma ??) triple  $(F_0, F_1, F_2^1)$ , where  $F_0, F_1 \in \mathcal{F}(\mathbb{C}^n)$  and  $F_2^1$  is a line, there exists a unique line  $F_3^1$  such that*

$$\sum_{\substack{0 \leq j_0, j_1 \leq n-2 \\ j_0 + j_1 = n-2}} \mathrm{Vol} \left( \frac{\mathbb{C}^n}{\langle F_0^{j_0}, F_1^{j_1} \rangle}; F_0^{j_0+1}, F_1^{j_1+1}, F_2^1, F_3^1 \right) = C_2(n-2) \cdot v_3.$$

*Proof.* By Lemma ?? we may assume that

$$\begin{aligned} F_0 &= \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_{n-1} \rangle, \\ F_1 &= \langle e_n \rangle \subset \langle e_n, e_{n-1} \rangle \subset \dots \subset \langle e_n, e_{n-1}, \dots, e_2 \rangle \end{aligned}$$

and

$$L = \langle e_1 + \dots + e_n \rangle.$$

For  $0 \leq j_0 \leq n-2$  let  $j_1 = n-2-j_0$ . The space generated by  $F_0^{j_0}$  and  $F_1^{n-2-j_0}$  is the space

$$\langle F_0^{j_0}, F_1^{n-(j_0+1)} \rangle = \langle e_1, \dots, e_{j_0}, e_{j_0+2}, \dots, e_n \rangle.$$

The orthogonal projection of  $\mathbb{C}^n$  onto  $\langle e_{j_0+1}, e_{j_0+2} \rangle$  induces an isomorphism

$$\mathbb{C}^n / \langle F_0^{j_0}, F_1^{j_1} \rangle \cong \langle e_{j_0+1}, e_{j_0+2} \rangle.$$

Let  $v = (v_1, \dots, v_n)^T$  be a generator of  $F_3^1$ . The points  $F_0^{j_0+1}, F_1^{j_1+1}, F_2^1, F_3^1$  are mapped, in the projectivization of  $\langle e_{j_0+1}, e_{j_0+2} \rangle$ , to

$$[1, 0], \quad [0, 1], \quad [1, 1], \quad [v_{j_0+1}, v_{j_0+2}].$$

For this 4-tuple to be the vertices of a positively oriented regular simplex, we need  $v_{j_0+1}/v_{j_0+2} = \omega = e^{i\pi/3}$ , for every  $0 \leq j_0 \leq n-2$ . Thus,  $F_3^1$  is generated by

$$(\omega^{n-1}, \omega^{n-2}, \dots, \omega, 1)^T,$$

which proves the lemma.  $\square$

*Proof (of Theorem ??).* The first direction of the theorem follows from the more general Proposition ?. For the other direction, fix a positively oriented simplex with vertices  $\xi_0, \dots, \xi_3 \in \mathbb{P}(\mathbb{C}^2)$ . Let  $F_0, \dots, F_3$  be flags such that

$$B_n(F_0, \dots, F_3) = C_4(n-2) \cdot v_3.$$

By Lemma ?? this implies that the flags are in general position. By the transitivity of  $\mathrm{GL}(n, \mathbb{C})$  on pairs of flags and 1-dimensional space all in generic positions established in Lemma ??, we can assume that  $F_0 = \varphi_n(\xi_0)$ ,  $F_1 = \varphi_n(\xi_1)$  and the 1-dimensional space of  $F_2$  is  $F_2^1 = \varphi_n(\xi_2)^1$ . Maximality and genericity imply that

$$\mathrm{Vol} \left( \frac{\langle F_0^{j_0+1}, F_1^{j_1+1}, F_2^{j_2+1}, F_3^{j_3+1} \rangle}{\langle F_0^{j_0}, F_1^{j_1}, F_2^{j_2}, F_3^{j_3} \rangle}; F_0^{j_0+1}, F_1^{j_1+1}, F_2^{j_2+1}, F_3^{j_3+1} \right) = v_3$$

for any  $j_0 + \dots + j_3 = n-2$ . Thus it follows by Lemma ?? that  $F_3^1$  is uniquely determined and since  $\varphi_n(\xi_3)^1$  by the other direction of the proof also satisfies the condition of the lemma, it follows that  $F_3^1 = \varphi_n(\xi_3)^1$ .

Inductively suppose that  $F_3^j = \varphi_n(\xi_3)^j$ . We will show that  $F_3^{j+1} = \varphi_n(\xi_3)^{j+1}$ . Indeed, look at the quotient  $\mathbb{C}^n/F_3^j$ . By the genericity of  $\varphi_n(\xi_0), \dots, \varphi_n(\xi_3)$ , the projections  $\overline{F_0}, \overline{F_1}$  of the flags  $F_0$  and  $F_1$  are still in general position and moreover, the line  $F_2^1$  projects to a line  $\overline{F_2^1}$  in general position with respect to  $\overline{F_0}, \overline{F_1}$ . Note that the complete flag  $F_3$  projects to a complete flag  $\overline{F_3}$  with  $\overline{F_3}^{j_3} = \overline{F_3}^{j_3+j+1}$ . Maximality implies that the volume is equal to  $v_3$  for the 4-tuple  $(j_0, \dots, j_3) = (k, n-k-j-2, 0, j)$  for any  $0 \leq k \leq n-j-2$ , which we can rewrite as

$$\mathrm{Vol} \left( \frac{\langle \overline{F_0}^{j_0+1}, \overline{F_1}^{j_1+1}, \overline{F_2}^1, \overline{F_3}^1 \rangle}{\langle \overline{F_0}^{j_0}, \overline{F_1}^{j_1} \rangle}; \overline{F_0}^{j_0+1}, \overline{F_1}^{j_1+1}, \overline{F_2}^1, \overline{F_3}^1 \right) = v_3.$$

Thus by Lemma ??,  $\overline{F_3}^1 = \overline{F_3}^{j+1}$  is completely determined by this maximality condition. In particular,  $F_3^{j+1}$  is completely determined by the fact that

$$\mathrm{Vol} \left( \frac{\langle F_0^{j_0+1}, F_1^{j_1+1}, F_2^1, F_3^{j+1} \rangle}{\langle F_0^{j_0}, F_1^{j_1}, F_2^1 \rangle}; F_0^{j_0+1}, F_1^{j_1+1}, F_2^1, F_3^{j+1} \right) = v_3,$$

for any  $0 \leq j_0 \leq n-j-2$  and  $j_1 = n-j_0-j-2$ . Since  $\varphi_n(\xi_3)^{j+1}$  also satisfies this maximality condition by the other direction of the theorem, it follows that  $F_3^{j+1} = \varphi_n(\xi_3)^{j+1}$ .

We have thus shown that  $F_3 = \varphi_n(\xi_3)$ . By symmetry, we can apply the same argument to show that  $F_2 = \varphi_n(\xi_2)$ , which finishes the proof of the theorem.  $\square$

## 7 Proof of Theorem ??

Recall that the space  $\sigma_k(n)$  is the quotient of  $\{(x_0, \dots, x_k) \in (\mathbb{C}^n)^{k+1} \mid \langle x_0, \dots, x_k \rangle = \mathbb{C}^n\}$  by the diagonal  $\mathrm{GL}(n, \mathbb{C})$ -action and is thus in a natural way a complex manifold of dimension  $(k+1-n) \cdot n$ . The symmetric group  $S_{k+1}$  acts on  $\sigma_k(n)$  and we let  $\mathcal{B}_{\mathrm{alt}}^\infty(\sigma_k)$  denote the Banach space of bounded alternating Borel functions on  $\sigma_k$ . Together with  $D_k^*$ , the dual of  $D_k \otimes_{\mathbb{R}} \mathbb{1} : \mathbb{R}[\sigma_k] \rightarrow \mathbb{R}[\sigma_{k-1}]$ , we obtain a complex of Banach spaces  $(\mathcal{B}_{\mathrm{alt}}^\infty(\sigma_*)^*, D_*)$ .

Using Proposition ??, we deduce that the restriction of  $T_k^*$  to the subcomplexes of bounded Borel functions gives a morphism of complexes

$$T_k^* : \mathcal{B}_{\mathrm{alt}}^\infty(\sigma_k) \rightarrow \mathcal{B}_{\mathrm{alt}}^\infty(\mathcal{F}_{\mathrm{aff}}(\mathbb{C}^n)^{k+1})^{\mathrm{GL}(n, \mathbb{C})}.$$

Recall now that  $(\mathcal{B}_{\mathrm{alt}}^\infty(\mathcal{F}_{\mathrm{aff}}(\mathbb{C}^n)^{**+1}), \partial_*)$  is a strong resolution of  $\mathbb{R}$  by  $\mathrm{GL}(n, \mathbb{C})$ -Banach modules (see [?]) and thus we have a canonical map  $c_*$  from the cohomology of the complex of  $\mathrm{GL}(n, \mathbb{C})$ -invariants to the bounded continuous cohomology  $H_{\mathrm{cb}}^*(\mathrm{GL}(n, \mathbb{C}))$  of  $\mathrm{GL}(n, \mathbb{C})$ . (As announced in § ??, we are dropping the explicit dependence on the coefficients.) As a result, we obtain by composing  $c_k$  with the map induced in cohomology by  $T_k^*$  a map

$$S^k(n) : H^k(\mathcal{B}_{\mathrm{alt}}^\infty(\sigma_*)) \longrightarrow H_{\mathrm{cb}}^k(\mathrm{GL}(n, \mathbb{C}), \mathbb{R}).$$

**Proposition 25** For  $k \geq 2$  the diagram

$$\begin{array}{ccc} & & H_{\mathrm{cb}}^k(\mathrm{GL}(n+1, \mathbb{C})) \\ & \nearrow^{S^k(n+1)} & \downarrow \\ H^k(\mathcal{B}_{\mathrm{alt}}^\infty(\sigma_*)) & & \\ & \searrow_{S^k(n)} & \\ & & H_{\mathrm{cb}}^k(\mathrm{GL}(n, \mathbb{C})), \end{array}$$

where the vertical arrow is induced by the left corner injection, commutes.

*Proof.* If  $n \geq 2$ , let  $i_n : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$  denote the embedding

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix}.$$

We define  $i_n : \mathcal{F}_{\text{aff}}(\mathbb{C}^n) \rightarrow \mathcal{F}_{\text{aff}}(\mathbb{C}^{n+1})$  by  $i_n((F, v)) := (F', v')$ , where  $F'^j := i_n(F^j)$ ,  $v'^j = i_n(v^j)$  and  $v'^{n+1} = e_{n+1}$ , for  $0 \leq j \leq n$ .

Let now  $\mathbb{J} \in [0, n]^{k+1}$  and  $I = \{i : 0 \leq i \leq k \text{ such that } j_i = n\}$ .

One verifies that if  $I = \emptyset$ , then  $\mathbb{J} \in [0, n-1]^{k+1}$  and

$$t_{\mathbb{J}}(i_n((F_j, v_j))) = t_{\mathbb{J}}((F_j, v_j)), \quad (11)$$

while if  $I \neq \emptyset$ , then

$$t_{\mathbb{J}}(i_n((F_j, v_j))) = [\mathbb{C}; \delta_i^I], \quad (12)$$

where  $\delta_i^I = 1$  if  $i \in I$  and 0 otherwise.

We deduce from (??) and (??) that  $i_n$  induces for  $k \geq 2$  a commutative diagram of complexes

$$\begin{array}{ccc} & \mathcal{B}_{\text{alt}}^{\infty}(\mathcal{F}_{\text{aff}}(\mathbb{C}^{n+1})^{k+1}) & \\ & \nearrow T_k^* & \downarrow i_n^* \\ \mathcal{B}_{\text{alt}}^{\infty}(\sigma_k) & & \mathcal{B}_{\text{alt}}^{\infty}(\mathcal{F}_{\text{aff}}(\mathbb{C}^n)^{k+1}). \\ & \searrow T_k^* & \end{array}$$

Indeed, for  $k \geq 2$  alternating functions vanish on  $[\mathbb{C}; (\delta_i^I)]$ . On the other hand  $i_n^*$  implements the restriction map in bounded cohomology associated to the left corner injection.  $\square$

*Proof (of Theorem ??).* We have  $\beta_b(n) = S^3(n)([\text{Vol}])$ , where  $\text{Vol} \in \mathcal{B}_{\text{alt}}^{\infty}(\sigma_3)$  was defined in Section ?? . The compatibility under the left corner injection then follows from the above proposition. Now  $H_{\text{cb}}^3(\text{GL}(2, \mathbb{C}))$  is one dimensional, generated by  $\beta_b(2)$ . Thus we deduce that  $\beta_b(n) \neq 0$  and  $\dim H_{\text{cb}}^3(\text{GL}(n, \mathbb{C})) \geq 1$ . We will conclude by using the stability results from Monod [?]. For  $n \geq 2$ , the diagram of short exact sequence

$$\begin{array}{ccccccc} (1) & \longrightarrow & \mathbb{C}^{\times} \text{Id} & \hookrightarrow & \text{GL}(n, \mathbb{C}) & \twoheadrightarrow & \text{PGL}(n, \mathbb{C}) \longrightarrow (1) \\ & & \uparrow & & \uparrow & & \downarrow \cong \\ (1) & \longrightarrow & \mu_n \text{Id} & \hookrightarrow & \text{SL}(n, \mathbb{C}) & \twoheadrightarrow & \text{PSL}(n, \mathbb{C}) \longrightarrow (1) \end{array}$$

induces a diagram of isometric isomorphisms in bounded cohomology

$$\begin{array}{ccc} H_{\text{cb}}^*(\text{GL}(n, \mathbb{C})) & \xleftarrow{\cong} & H_{\text{cb}}^*(\text{PGL}(n, \mathbb{C})) \\ \cong \downarrow & & \parallel \\ H_{\text{cb}}^*(\text{SL}(n, \mathbb{C})) & \xleftarrow{\cong} & H_{\text{cb}}^*(\text{PSL}(n, \mathbb{C})). \end{array} \quad (13)$$

Hence [?, Theorem 1.1 and Proposition 3.4] can be rephrased by saying that for  $0 \leq q \leq n$ , the standard embedding  $\mathrm{GL}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}(n+1, \mathbb{C})$  induces an isomorphism

$$H_{\mathrm{cb}}^q(\mathrm{GL}(n+1, \mathbb{C})) \xrightarrow{\cong} H_{\mathrm{cb}}^q(\mathrm{GL}(n, \mathbb{C}))$$

and an injection

$$H_{\mathrm{cb}}^q(\mathrm{GL}(q, \mathbb{C})) \hookrightarrow H_{\mathrm{cb}}^q(\mathrm{GL}(q-1, \mathbb{C})).$$

Applying this to  $q = 3$  we obtain that  $\dim H_{\mathrm{cb}}^3(\mathrm{GL}(n, \mathbb{C})) = 1$ , which proves the first part of Theorem ???. As for the second part, it follows from Section ?? that  $\|\beta_{\mathbf{b}}(n)\|_{\infty} \leq (1/6)n(n^2 - 1)v_3$ . For the other inequality, let  $\varphi_n : \mathbb{P}(\mathbb{C}^2) \rightarrow \mathcal{F}(\mathbb{C}^n)$  be the Veronese embedding. Then

$$B_n(\varphi_n(\xi_0), \dots, \varphi_n(\xi_3)) = \frac{n(n^2 - 1)}{6} B_2(\xi_0, \dots, \xi_3)$$

by Proposition ?? and as a result,  $T_n^*(\beta_{\mathbf{b}}(n)) = \frac{n(n^2 - 1)}{6} \beta_{\mathbf{b}}(2)$ . Since  $\|\beta_{\mathbf{b}}(2)\|_{\infty} = v_3$ , we deduce

$$\frac{n(n^2 - 1)}{6} v_3 = \|\pi_n^*(\beta_{\mathbf{b}}(n))\|_{\infty} \leq \|\beta_{\mathbf{b}}(n)\|_{\infty},$$

which, using (??), concludes the proof of Theorem ???.  $\square$

## 8 Proof of Theorem ???

### 8.1 The Borel invariant as a multiplicative constant

The aim of this subsection is to identify the Borel invariant  $\mathcal{B}(\rho)$  as a multiplicative factor in the composition of certain bounded cohomology maps (Proposition ??) and to establish the simple direction of Theorem ??? (Lemma ??). The proof is identical to the corresponding statement in [?, Proposition 3.3] and is based on the existence of a natural transfer map

$$H_{\mathbf{b}}^*(\Gamma) \xrightarrow{\mathrm{trans}_{\Gamma}} H_{\mathrm{cb}}^*(\mathrm{PSL}(2, \mathbb{C}))$$

defined by integrating a  $\Gamma$ -invariant cocycle over  $\Gamma \backslash G$  to make it  $G$ -invariant.

We refer the reader to [?, Section 3.2] for the complete definitions and proofs.

**Proposition 26** *Let  $\Gamma$  be a lattice in  $\mathrm{PSL}(2, \mathbb{C})$  and  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  be a representation. The composition*

$$H_{\mathrm{cb}}^3(\mathrm{PSL}(n, \mathbb{C})) \xrightarrow{\rho^*} H_{\mathbf{b}}^3(\Gamma) \xrightarrow{\mathrm{trans}_{\Gamma}} H_{\mathrm{cb}}^3(\mathrm{PSL}(2, \mathbb{C})) \cong \mathbb{R}$$

maps  $\beta_b(n)$  to  $\frac{\mathcal{B}(\rho)}{\text{Vol}(\Gamma \backslash \text{PSL}(2, \mathbb{C}))} \beta_b(2)$  and

$$\left| \frac{\mathcal{B}(\rho)}{\text{Vol}(\Gamma \backslash \text{PSL}(2, \mathbb{C}))} \right| \leq \frac{1}{6} n(n^2 - 1).$$

**Lemma 27** *Let  $i : \Gamma \hookrightarrow \text{PSL}(2, \mathbb{C})$  be a lattice embedding. Then*

$$(\pi_n \circ i)^* \beta_b(n) = \frac{1}{6} (n-1)n(n+1) \text{Vol}(i(\Gamma) \backslash \text{PSL}(2, \mathbb{C})) \beta_b(2).$$

*Proof.* Setting  $\rho = \pi_n \circ i$  in Proposition ??, we see that the pullback  $\rho^* : H_{\text{cb}}^3(\text{PSL}(n, \mathbb{C})) \rightarrow H_b^3(\Gamma)$  factors through  $H_{\text{cb}}^3(\text{PSL}(2, \mathbb{C}))$ . The composition of maps of the proposition thus becomes

$$H_{\text{cb}}^3(\text{PSL}(n, \mathbb{C})) \xrightarrow{\pi_n^*} H_{\text{cb}}^3(\text{PSL}(2, \mathbb{C})) \xrightarrow{i^*} H_b^3(\Gamma) \xrightarrow{\text{trans}_\Gamma} H_{\text{cb}}^3(\text{PSL}(2, \mathbb{C})) \cong \mathbb{R}.$$

The conclusion is immediate from the fact that

$$\pi_n^*(\beta_b(n)) = \frac{1}{6} (n-1)n(n+1) \cdot \beta_b(2)$$

(Theorem ??) and that  $\text{trans}_\Gamma \circ i^* = \text{Id}$ .  $\square$

## 8.2 Proof of Theorem ??

An essential aspect of bounded cohomology is that it can be implemented by alternating  $L^\infty$  invariant cocycles on an appropriate boundary, [?]. Then if a bounded cohomology class can be represented by a strict cocycle, its pullback via a representation, for example  $\rho : \Gamma \rightarrow \text{PSL}(n, \mathbb{C})$  in our case, can be implemented by boundary maps, [?]. Recall that by Furstenberg, given any representation of  $\Gamma$  into  $\text{PSL}(n, \mathbb{C})$ , there is always an equivariant measurable map

$$\varphi : \mathbb{P}(\mathbb{C}^2) \longrightarrow M^1(\mathcal{F}(\mathbb{C}^n)),$$

where  $M^1(\mathcal{F}(\mathbb{C}^n))$  denotes the space of probability measures on the flag space. More precisely, for every  $\gamma \in \Gamma$  and almost every  $\xi \in \mathbb{P}(\mathbb{C}^2)$ , we have

$$\varphi(i(\gamma) \cdot \xi) = \rho(\gamma) \cdot \varphi(\xi). \quad (14)$$

The bounded cohomology groups  $H_{\text{cb}}^3(\text{PSL}(n, \mathbb{C}))$  and  $H_b^3(\Gamma)$  can both be computed from the corresponding  $L^\infty$  equivariant cochains on  $\mathcal{F}(\mathbb{C}^n)$  and  $\partial\mathbb{H}^3 = \mathbb{P}(\mathbb{C}^2)$  respectively. The image of  $\beta_b(n)$  by  $\rho^* : H_{\text{cb}}^3(\text{PSL}(n, \mathbb{C})) \rightarrow H_b^3(\Gamma)$  is represented at the cochain level by the pullback by  $\varphi$ , or more precisely, by the following cocycle:



$$\begin{aligned} (\partial\mathbb{H}^3)^4 &\longrightarrow \mathbb{R} \\ (\xi_0, \dots, \xi_3) &\longmapsto \varphi(\xi_0) \otimes \dots \otimes \varphi(\xi_3)[B_n], \end{aligned}$$

where the last expression means that the cocycle  $B_n$  is integrated with respect to the product of the four measures  $\varphi(\xi_0), \dots, \varphi(\xi_3)$ , [?]. It should however be noted that the pullback in bounded cohomology cannot in general be implemented by boundary maps, unless the class to pull back can be represented by a strict invariant Borel cocycle [?].

The further composition with the transfer map amounts to integrating the preceding cocycle over a fundamental domain for  $\Gamma \backslash \mathrm{PSL}(2, \mathbb{C})$ . In conclusion, since  $\mathrm{trans}_\Gamma \circ \rho^*(B_n)$  is by Proposition ?? equal to  $\frac{\mathcal{B}(\rho)}{\mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))} \cdot \beta_b(2)$  and at the cohomology level there are no coboundaries in degree 3 [?], the map  $\mathrm{trans}_\Gamma \circ \rho^*$  sends the cocycle  $B_n$  to  $\frac{\mathcal{B}(\rho)}{\mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))} \mathrm{Vol}_{\mathbb{H}^3}$ . Thus, for almost every  $\xi_0, \dots, \xi_3 \in \partial\mathbb{H}^3$ , we have

$$\int_{\Gamma \backslash \mathrm{PSL}(2, \mathbb{C})} \varphi(g\xi_0) \otimes \dots \otimes \varphi(g\xi_3)[B_n] d\mu(g) = \frac{\mathcal{B}(\rho)}{\mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))} \mathrm{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3). \quad (15)$$

It is however shown in [?, Proposition 4.2 for  $n = 3$ ] that this almost everywhere equality is in fact a true equality. As a consequence, we show that in the maximal case, the map  $\varphi$  takes essentially values in the set of Dirac masses:

**Corollary 28** *Let  $i : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be a lattice embedding,  $\rho : \Gamma \rightarrow \mathrm{PSL}(n, \mathbb{C})$  a representation and  $\varphi : \partial\mathbb{H}^3 \rightarrow M^1(\mathcal{F}(\mathbb{C}^n))$  a  $\rho$ -equivariant measurable map. Suppose that  $|\mathcal{B}(\rho)| = \frac{1}{6}n(n^2 - 1) \cdot \mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))$ . Then for almost every  $\xi \in \mathbb{P}(\mathbb{C}^2)$  the image  $\varphi(\xi)$  is a Dirac mass.*

*Proof.* Upon conjugating  $\rho$  by the anti-holomorphic map  $I$  induced by  $z \mapsto \bar{z}$ , we can without loss of generality assume that  $\mathcal{B}(\rho) = \frac{1}{6}n(n^2 - 1) \cdot \mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))$ . Assume  $\mathrm{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3) = v_3$ . Then (??) holds. We deduce then from the fact that  $|B_n|$  is bounded by  $\frac{1}{6}n(n^2 - 1)v_3$  (see Theorem ??) that

$$\varphi(g\xi_0) \otimes \dots \otimes \varphi(g\xi_3)[B_n] = \frac{1}{6}n(n^2 - 1)v_3$$

for almost every  $g \in \mathrm{SL}(2, \mathbb{C})$ . As a consequence, for almost every  $(F_0, \dots, F_3) \in \mathcal{F}(\mathbb{C}^n)^4$  with respect to the product measure  $\varphi(g\xi_0) \otimes \dots \otimes \varphi(g\xi_3)$ , we have equality

$$B_n(F_0, \dots, F_3) = \frac{1}{6}n(n^2 - 1)v_3.$$

Fix a triple  $(F_0, F_1, F_2)$  such that the previous equality holds for  $\varphi(g\xi_3)$ - almost every  $F_3$ . However, by Corollary ??, this  $F_3$  is unique which implies that the support of  $\varphi(g\xi_3)$  is reduced to one point. Since this holds for almost every  $g \in \mathrm{SL}(2, \mathbb{C})$ , the corollary is proven.  $\square$

If equality  $|\mathcal{B}(\rho)| = \frac{1}{6}n(n^2 - 1) \mathrm{Vol}(\Gamma \backslash \mathrm{PSL}(2, \mathbb{C}))$  holds, then upon conjugating  $\rho$  by the anti-holomorphic map  $I$  which has the effect of changing the sign of  $\mathcal{B}(\rho)$

and composing  $\varphi$  with the induced boundary map  $I$ , we can suppose that  $\mathcal{B}(\rho) = \frac{1}{6}n(n^2 - 1)\text{Vol}(\Gamma \backslash \text{PSL}(2, \mathbb{C}))$ . It then follows from the above that  $\varphi$  maps almost every maximal 4-tuples in  $P^1(\mathbb{C})$  to maximal 4-tuples in  $\mathcal{F}(\mathbb{C}^n)$ .

**Theorem 29.** *Let  $\varphi : \mathbb{P}(\mathbb{C}^2) \rightarrow \mathcal{F}(\mathbb{C}^n)$  be a measurable map sending almost every maximal 4-tuple in  $\mathbb{P}(\mathbb{C}^2)$  to a maximal 4-tuple in  $\mathcal{F}(\mathbb{C}^n)$ . Then there exists  $g \in \text{PSL}(n, \mathbb{C})$  such that*

$$\varphi = g \cdot \varphi_n$$

almost everywhere.

The theorem is a straightforward generalization of the corresponding statement with  $\mathcal{F}(\mathbb{C}^n)$  replaced by  $\partial\mathbb{H}^3$  and  $\text{PSL}(n, \mathbb{C})$  replaced by  $\text{PSL}(2, \mathbb{C})$  which was proven by Thurston for the generalization of Gromov's proof of Mostow rigidity for 3-dimensional hyperbolic manifolds. Our proof is a reformulation of Dunfield's detailed version [?, pp. 654-656] of Thurston's proof [?, two last paragraphs of Section 6.4] in the language of ergodic theory.

Let  $T$  denote the set of 4-tuples in  $\partial\mathbb{H}^3$  whose convex hull is a regular simplex. Denote by  $\Lambda_{\underline{\xi}} < \text{Isom}(\mathbb{H}^3)$  the reflection group generated by the reflections in the faces of the simplex  $\underline{\xi}$ . For  $\varphi : \partial\mathbb{H}^3 \rightarrow \mathcal{F}(\mathbb{C}^n)$ , we let  $T^\varphi$  be the subset of  $T$  of regular simplices being mapped to maximal 4-tuples (up to sign). More precisely, we set

$$T^\varphi := \left\{ (\xi_0, \dots, \xi_3) \in T \mid B_n(\varphi(\xi_0), \dots, \varphi(\xi_3)) = \frac{1}{6}n(n^2 - 1)\text{Vol}_{\mathbb{H}^3}(\xi_0, \dots, \xi_3) \right\}.$$

Exactly the same proof as in [?, Lemma 4.6] shows the following:

**Lemma 30** *Let  $\underline{\xi} = (\xi_0, \dots, \xi_3) \in T$ . Suppose that  $\varphi : \partial\mathbb{H}^3 \rightarrow \mathcal{F}(\mathbb{C}^n)$  is a map such that for every  $\gamma \in \Lambda_{\underline{\xi}}$ , the translate  $(\gamma\xi_0, \dots, \gamma\xi_3)$  belongs to  $T^\varphi$ . Then there exists a unique  $g \in \text{PSL}(n, \mathbb{C})$  such that  $g\varphi_n(\xi) = \varphi(\xi)$  for every  $\xi \in \bigcup_{i=0}^3 \Lambda_{\underline{\xi}}\xi_i$ .*

In the proof of Mostow Rigidity in dimension greater than or equal to 4, Lemma ?? was sufficient to prove the corresponding Theorem ?. In dimension 3 however, an additional difficulty is due to the fact that the group  $\Lambda_{\underline{\xi}}$  is discrete in  $\text{Isom}(\mathbb{H}^3)$  and in particular does not act ergodically on  $\text{Isom}(\mathbb{H}^3)$ . For this reason, we introduce the bigger group  $\Gamma_{\underline{\xi}}$  which will act ergodically on  $\text{Isom}(\mathbb{H}^3)$  (Proposition ??) and for which we can prove the corresponding statement of Lemma ?? (Proposition ??). We set

$$\Gamma_{\underline{\xi}} := \langle \Lambda_{\underline{\xi}}, \gamma_{\underline{\xi}} \rangle,$$

where  $\gamma_{\underline{\xi}}$  is defined as follows: If  $\underline{\xi} = (+\infty, 0, \xi_2, \xi_3)$  the isometry  $\gamma_{\underline{\xi}}$  induces the map  $\gamma := z \mapsto 2z$  on  $\partial\mathbb{H}^3 = \mathbb{C} \cup \{+\infty\}$ . If  $\underline{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3)$  is any regular simplex, let  $g \in \text{PSL}(2, \mathbb{C})$  be an isometry such that  $g\xi_0 = +\infty$  and  $g\xi_1 = 0$ . Set then  $\gamma_{\underline{\xi}} = g^{-1}\gamma_2g$ .

**Proposition 31** *Let  $\underline{\xi} = (\xi_0, \dots, \xi_3) \in T$ . Suppose that  $\varphi : \partial\mathbb{H}^3 \rightarrow \mathcal{F}(\mathbb{C}^n)$  is a map such that for every  $\gamma \in \Gamma_{\underline{\xi}}$ , the translate  $(\gamma\xi_0, \dots, \gamma\xi_3)$  belongs to  $T^\varphi$ . Then there exists a unique  $g \in \text{PSL}(n, \mathbb{C})$  such that  $g\varphi_n(\xi) = \varphi(\xi)$  for every  $\xi \in \bigcup_{i=0}^3 \Gamma_{\underline{\xi}}\xi_i$ .*

*Proof.* For every  $\underline{\xi} \in \partial\mathbb{H}^3$ , let  $S_{\underline{\xi}}$  denote the natural set of generators of  $\Gamma_{\underline{\xi}}$  consisting of the reflections with respect to the faces of  $\underline{\xi}$  and  $\gamma_{\underline{\xi}}^{\pm 1}$ . Exactly as for reflection groups, one shows that every  $\gamma \in \Gamma_{\underline{\xi}}$  can be written as a product  $\gamma = r_k \dots r_2 \cdot r_1$ , where  $r_i \in S_{r_{i-1} \dots r_1 \underline{\xi}}$ . Indeed by definition  $\gamma = s_k \dots s_2 \cdot s_1$  for  $s_i \in S_{\underline{\xi}}$  and we can take

$$r_i := (s_1 s_2 \dots s_{i-1}) s_i (s_1 s_2 \dots s_{i-1})^{-1} \in S_{r_{i-1} \dots r_1 \underline{\xi}},$$

where

$$S_{r_{i-1} \dots r_1 \underline{\xi}} = (s_1 s_2 \dots s_{i-1}) S_{\underline{\xi}} (s_1 s_2 \dots s_{i-1})^{-1}.$$

Let now  $\underline{\xi}$  be as in the assumption of the proposition. By Theorem ??, for every  $\gamma \in \Gamma_{\underline{\xi}}$ , there exists a unique  $g_\gamma \in \text{PSL}(n, \mathbb{C})$  such that  $g_\gamma \varphi_n(\gamma\xi) = \varphi(\gamma\xi_i)$ , for  $i = 0, \dots, 3$ . We need to show that  $g_\gamma$  is independent of  $\gamma$ . Let  $\gamma = r_k \dots r_2 \cdot r_1$  be as above. We prove the independence of  $\gamma$  by showing  $g_{r_k \dots r_2 \cdot r_1} = g_{r_{k-1} \dots r_2 \cdot r_1}$ , where for  $k = 1$ , the latter element of  $\text{PSL}(n, \mathbb{C})$  is  $g_{\text{id}}$ . If  $r_k$  is a reflection in one of the faces of the simplex  $r_{k-1} \dots r_2 \cdot r_1 \underline{\xi}$  the claim follows by Lemma ?. Up to conjugation, we can suppose that the simplex  $r_{k-1} \dots r_2 \cdot r_1 \underline{\xi}$  has the form  $\underline{\eta} = (+\infty, 0, \eta_2, \eta_3)$ . In the case where  $r_k = \gamma_{\underline{\eta}}^{\pm 1}$ , the simplex  $r_k \underline{\eta}$  has the form  $(+\infty, 0, 2^{\pm 1} \eta_2, 2^{\pm 1} \eta_3)$  and in particular  $\gamma_{\underline{\eta}} = \gamma_{\gamma_{\underline{\eta}} \underline{\eta}}$ . It is thus enough to treat the case  $r_k = \gamma_{\underline{\eta}}$ . In this case, the vertices of  $\gamma_{\underline{\eta}}$  are vertices of the tessellation of  $\underline{\eta}$  by  $\Lambda_{\underline{\eta}}$ , which is a subgroup of  $\Gamma_{\underline{\xi}}$ , so the claim follows by Lemma ?.  $\square$

**Lemma 32** *For every  $\underline{\xi} \in T$ , the group  $\Gamma_{\underline{\xi}}$  acts ergodically on  $\text{Isom}(\mathbb{H}^3)$ .*

*Proof.* We could show by hand that  $\Gamma_{\underline{\xi}} \cap \text{PSL}(2, \mathbb{C})$  is dense in  $\text{PSL}(2, \mathbb{C})$ , which is equivalent to the ergodicity statement. We give however instead a geometric argument.

We claim that  $\Lambda_{\underline{\xi}}$  is of infinite index in  $\Gamma_{\underline{\xi}}$ . Let  $\Lambda'_{\underline{\xi}} < \Lambda_{\underline{\xi}}$  be a torsion free subgroup of finite index. Then  $M := \Lambda'_{\underline{\xi}} \backslash \mathbb{H}^3$  is a finite volume manifold and  $\xi_0, \xi_1$  are cusps of  $M$ . Therefore the geodesic in  $\mathbb{H}^3$  joining  $\xi_0$  and  $\xi_1$  maps properly to a biinfinite geodesic in  $M$ . If  $\Lambda_{\underline{\xi}}$  were of finite index in  $\Gamma_{\underline{\xi}}$ , then some finite power of the hyperbolic element  $\gamma_{\underline{\xi}}$  with fixed points  $\xi_0$  and  $\xi_1$  would be in  $\Lambda'_{\underline{\xi}}$  and the geodesic joining  $\xi_0$  to  $\xi_1$  would map to a periodic geodesic in  $M$ , which is a contradiction.

We conclude now by observing that Zariski density of  $\Lambda_{\underline{\xi}}$  implies density of  $\Gamma_{\underline{\xi}}$  in the usual topology; in particular  $\Gamma_{\underline{\xi}}$  acts ergodically on  $\text{Isom}(\mathbb{H}^3)$   $\square$

The proof of Theorem ?? follows now completely analogously as in [?, Proposition 4.7], taking into account Lemma ??

We have thus established that  $\varphi$  is essentially equal to  $g \cdot \varphi_n$ . It remains to see that  $g$  realizes the conjugation between  $\rho$  and  $\pi_n \circ i$ . Indeed, replacing  $\varphi$  by  $g \cdot \varphi_n$  in (??) we have

$$(g \cdot \varphi_n)(i(\gamma) \cdot \xi) = \rho(\gamma) \cdot (g \cdot \varphi_n)(\xi),$$

for every  $\xi \in \partial\mathbb{H}^3$  and  $\gamma \in \Gamma$ . The  $\pi_n$ -equivariance of  $\varphi_n$  allows us to rewrite this equation as

$$g \cdot \pi_n(i(\gamma)\varphi_n(\xi)) = \rho(\gamma) \cdot (g \cdot \varphi_n)(\xi),$$

Thus,  $g\pi_n(i(\gamma))$  and  $\rho(\gamma) \cdot g$  which both belong to  $\mathrm{PSL}(n, \mathbb{C})$  act identically on the image of  $\varphi_n$ , from which we conclude that they are equal. This concludes the proof of Theorem ??.

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