STABILITY PROPERTIES OF MULTIPLICATIVE REPRESENTATIONS OF THE FREE GROUP

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Abstract. In 2004 the second and third author introduced a large family of representations of a free group Γ weakly contained in the regular representation.

In this paper we enlarge a little bit this class for Γ so that the new class $\mathbf{Mult}(\Gamma)$, of the *multiplicative representations*, is stable under taking finite direct sums, under restriction to and induction from a finite index subgroup.

As an application, using the properties of multiplicative representations we define a new class of tempered unitary represenations for a class of groups that includes for example all lattices of unimodular subgroups of automorphisms of a locally finite regular tree.

The main tool is the detailed study of the properties of the action of a free group on its Cayley graph with respect to a change of generators, as well as the relative properties of the action of a subgroup of finite index after the choice of a nice fundamental domain.

1. Introduction

In 2004 the second and third author introduced in [KS04] a large family of representations of a free group Γ, constructed from *vector valued multiplicative* functions. These representations are all *tempered*, that

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is they are weakly contained in the regular representation or, equivalently, they are representations of the reduced C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$. Such an algebra is traceable, simple (see [Pow75]) and Type II, so that its unitary dual cannot be parametrized by any standard measure space. Even if one will never be able to "list" all possible tempered unitary representations of Γ, the construction in [KS04] covers all specific irreducible tempered representations of Γ presented in the literature that are obtained by embedding Γ in the automorphism group of its Cayley graph. Moreover, multiplicative representations are defined very explicitly and hence amenable to be an excellent source of tests samples. Furthermore, they can be extended in a very natural way to *boundary representations*, that is to representations of the cross product C^*-algebra $\Gamma \ltimes \mathcal{C}(\partial\Gamma)$ where $\mathcal{C}(\partial\Gamma)$ is the C^*-algebra of the continuous functions on the boundary $\partial\Gamma$ of Γ. Because of all of these nice properties it is hence natural to try to extent the scope of the definition to a class of groups as large as possible.

As defined in [KS04], this class of representations had however the inconvenient property of being not very stable with respect to natural operations such as inducing or restricting to a subgroup. Moreover, as is, the definition is dependent on the choice of the generating set. In this paper we propose a small, yet important modification of the definition, that will insure that the class of representations so defined has nice stability properties that we then proceed to prove. We finally apply these results to extend the family of representations in [KS04] to a class of groups that include, among others, virtually free groups and appropriate subgroups of the automorphisms group of a regular tree.

More precisely, we recall that the definition in [KS04] of these representations requires a set of data, called *matrix system with inner product*, consisting of a (complex) vector space and a positive definite sesquilinear form for each generator, as well as linear maps between any two pairs of vector spaces, all subject to some compatibility condition (recalled in § 2). In [KS04] the authors required the matrix system to be *irreducible*, which resulted in representations that are irreducible as representations of the cross product $\Gamma \ltimes \mathcal{C}(\partial\Gamma)$. However, to ensure enough stability for basic operations such as unitary induction and restriction to a finite index subgroup, we need to free ourselves from the irreducibility of the matrix system.

The starting point in this paper is the following result, according to which irreducibility of the matrix system is not essential: representations arising from non-irreducible matrix systems are anyway finitely reducible in the following sense:

Theorem 1. *Every representation $(\pi, \mathcal{H})$ constructed from a matrix system with inner products (V_a, H_{ba}, B_a) decomposes into the orthogonal direct sum with respect to $\mathcal{B} = (B_a)$ of a finite number of representations constructed from irreducible matrix systems.*

We call such a representation *multiplicative* and, if Γ is a free group, we denote by $\mathbf{Mult}(\Gamma)$ the class of representations that are unitarily equivalent to a multiplicative representation (see § 3 for the precise definition). That we are allowed to drop the dependence of the set of free generators follows from the following important result:

Theorem 2. *Let A and A' be two symmetric sets of free generators of a free group Γ, and let us denote by $\mathbb{F}_A$ and $\mathbb{F}_{A'}$ the group Γ as generated respectively by A and A'. Then for every $\pi \in \mathbf{Mult}(\mathbb{F}_{A'})$ there exists a matrix system with inner product indexed on A, such that $\pi \in \mathbf{Mult}(\mathbb{F}_A)$.*

In particular the class $\mathbf{Mult}(\Gamma)$ is $\mathrm{Aut}(\Gamma)$-invariant.

In [KS04] it is shown that the representations π_s of the principal spherical series of Figà-Talamanca and Picardello [FTP82], associated to a generating set A, can be realized as multiplicative representations with respect to scalar matrices acting on one dimensional spaces. The reader should notice that spherical series arising from different generating sets A and A' are inequivalent unless A is obtainable by A' by an automorphism of the Cayley graph associated to A' (see [PS96]). Nonetheless the spherical series associated to A' can be realized as a *multiplicative representation* with respect to A: in this case the new matrices will fail to be scalars, as one can see in Example 6.10.

As announced, we can also prove stability properties with respect to the induction and the restriction to a subgroup:

Theorem 3. *Assume that Γ is a finitely generated non-abelian free group and let $\Gamma' < \Gamma$ be a subgroup of finite index.*

(1) *If $\pi \in \mathbf{Mult}(\Gamma)$, then the restriction of π to Γ' belongs to $\mathbf{Mult}(\Gamma')$.*
(2) *If $\pi \in \mathbf{Mult}(\Gamma')$, then the induction of π to Γ belongs to $\mathbf{Mult}(\Gamma)$.*

Let us turn now to more general groups. Let Λ be any locally compact group satisfying the following two conditions:

($*$) it admits an embedding of a free group Γ as a lattice, and
($**$) any two such free lattices are commensurable up to conjugation.

As an application of the above results, in this paper we define a new class of representations, called $\mathbf{Mult}(\Lambda)$, for groups Λ satisfying the conditions $(*)$ and $(**)$ above, by inducing to Λ a multiplicative representation from a (in fact, any) free subgroup Γ embedded as a lattice. Note that an immediate consequence of Theorem 3 is the fact that the class $\mathbf{Mult}(\Gamma)$ is well-defined if Γ is a free group, that is the class of representations defined using matrix systems coincides with the class obtained by inducing a multiplicative representation from a finite index free subgroup. We prove the following:

Theorem 4. *Let Λ be any locally compact group admitting a free group Γ embedded as a lattice and such that any two such free lattices are commensurable. Then:*

(1) *the class $\mathbf{Mult}(\Lambda)$ is not empty and consists of representations that are weakly contained in the regular representation of Λ.*
(2) *the class $\mathbf{Mult}(\Lambda)$ does not depend on Γ*

Examples of groups for which the theorem holds include for example all virtually free groups. A less obvious example is obtained by taking a unimodular subgroup $\Lambda < \mathrm{Aut}(\mathcal{T})$ acting cocompactly on $\mathcal{T}$, where $\mathcal{T}$ is a regular tree of locally finite valency. Then there exists cocompact lattices in Λ, and such cocompact lattices are always virtually free, [BK90]. Furthermore, since any two finite graphs that admit a common covering admit also a common finite covering, [Lei82], such group Λ satisfies also condition $(**)$.

As we mentioned before, the representations of the class $\mathbf{Mult}(\Gamma)$ are also representations of the cross product C^*-algebra $\Gamma \ltimes \mathcal{C}(\partial\Gamma)$ and hence they admit a *boundary realization*, that is, a relization on a Hilbert space which is the direct integral over $\partial\Gamma$ with respect to some quasi-invariant measure.

It is proved in [IKS13] that *every* tempered representation of a torsion-free not almost cyclic Gromov hyperbolic group Λ extends to a representation of the crossed product $\Lambda \ltimes \mathcal{C}(\partial\Lambda)$ and hence has a boundary realization.

However, while the existence of such a boundary realization for a representation of a Gromov hyperbolic group follows from general C^*-algebra inclusion and extension properties using Hahn–Banach theorem, and is hence highly non-constructive, for representations in the class $\mathbf{Mult}(\Gamma)$ the boundary realization is more accessible and sometimes very concrete. Its uniqueness is also studied in details in the scalar case in [KS01], but remains in general an open question.

We summarise in the next section the results needed from [KS04], while we define the classes **Mult** and outline the structure of the paper in § 3.

2. Prolegomenon

Fix a symmetric set A of free generators for $\mathbb{F}_A$, $A = A^{-1}$. Throughout, when we use a, b, c, d, a_j, for $j \in \mathbf{N}$, for elements of $\mathbb{F}_A$, it is intended that they are elements of A. There is a unique *reduced word* for every $x \in \mathbb{F}_A$:

$$x = a_1 a_2 \dots a_n \qquad \text{where for all } j,\ a_j \in A \text{ and } a_j a_{j+1} \neq e.$$

The *Cayley graph* of $\mathbb{F}_A$ has as vertices $\mathcal{V}$ the elements of $\mathbb{F}_A$ and as undirected edges the couples $\{x, xa\}$ for $x \in \mathbb{F}_A$, $a \in A$. This is a tree $\mathcal{T}$ with $\#A$ edges attached to each vertex and the action of $\mathbb{F}_A$ on itself by left translation preserves the tree structure.

A sequence $(x_0, x_1, \dots, x_n)$ of vertices in the tree is a *geodesic segment* if for all j, x_{j+1} is adjacent to x_j and $x_{j+2} \neq x_j$. We denote such geodesic segment joining x_0 with x_n with

$$[x_0, x_1, \dots, x_n] \qquad \text{or} \qquad [x_0, x_n]\,,$$

whenever the intermediate vertices are not important. If the vertex $z \in \mathcal{V}$ is on the geodesic from x_0 to x_n, we write $z \in [x_0, x_n]$. We define the distance between two vertices of the tree as the number of edges in the geodesic segment joining them. This gives $d(e, x) = |x|$, $d(x, y) = |x^{-1}y|$.

Definition 2.1. A *matrix system* or simply *system* (V_a, H_{ba}) is a choice of

- a complex finite dimensional vector space V_a for each $a \in A$, and
- a linear map $H_{ba} : V_a \to V_b$ for each pair $a, b \in A$, where $H_{ba} = 0$ whenever $ab = e$.

Definition 2.2. A tuple of linear subspaces $W_a \subseteq V_a$ is called an *invariant subsystem* of (V_a, H_{ba}) if

$$H_{ba} W_a \subseteq W_b \qquad \text{for all } a,\ b.$$

For any given invariant subsystem (W_a, H_{ba}) of (V_a, H_{ba}) the *quotient system* $(\widetilde{V}_a, \widetilde{H}_{ba})$ is defined on $\widetilde{V}_a = V_a / W_a$ in the obvious way:

$$\widetilde{H}_{ba} \widetilde{v}_a := \widetilde{H_{ba} v_a} \qquad \text{where } v_a \text{ is any representative for } \widetilde{v}_a.$$

The system (V_a, H_{ba}) is called *irreducible* if it is nonzero and if it admits no invariant subsystems except for itself and the zero subsystem.

Definition 2.3. A map from the system (V_a, H_{ba}) to the system (V'_a, H'_{ba}) is a tuple (J_a) where $J_a : V_a \to V'_a$ is a linear map and

$$H'_{ab} J_b = J_a H_{ab}\ .$$

The tuple (J_a) is called an *equivalence* if each J_a is a bijection. Two systems are called *equivalent* if there is an equivalence between them.

Remark 2.4. A map (J_a) between irreducible systems (V_a, H_{ba}) and (V_a', H_{ba}') is either 0 or an equivalence. This is because the kernels (respectively, the images) of the maps J_a constitute an invariant subsystem.

For $x \in \mathcal{V}$ we set once and for all

$$
\begin{aligned}
E(x) &:= \{y \in \mathcal{V} : \text{ the reduced word for } y \text{ ends with } x\} \\
&= \{y \in \mathcal{V} : \ x^{-1} \in [e, y^{-1}]\} \\
C(x) &:= \{y \in \mathcal{V} : \text{ the reduced word for } y \text{ starts with } x\} \\
&= \{y \in \mathcal{V} : \ x \in [e, y]\} \,.
\end{aligned}
\tag{2.1}
$$

Definition 2.5. A *(vector-valued) multiplicative function* is a function

$$
f : \mathbb{F}_A \ \to \ \coprod_{a \in A} V_a
$$

for which there exists $N = N(f) \geq 0$ such that for every $x \in \mathcal{V}$, with $|x| \geq N$

$$
\begin{aligned}
&f(x) \in V_a && \text{if } x \in E(a) \\
&f(xb) = H_{ba} f(x) && \text{if } x \in E(a) \text{ and } |xb| = |x| + 1 \,.
\end{aligned}
\tag{2.2}
$$

We denote by $\mathcal{H}_0^\infty(V_a, H_{ba})$ (or $\mathcal{H}_0^\infty$ is there is no risk of confusion) the space of multiplicative functions with respect to the system (V_a, H_{ba}).

Note that if f satisfies (2.2) for some $N = N_0$, it also satisfies (2.2) for all $N \geq N_0$. We define two multiplicative functions f and g to be equivalent, $f \sim g$, if $f(x) = g(x)$ for all but finitely many elements of $\mathcal{V}$ and $\mathcal{H}^\infty$ is defined as the quotient of the space of multiplicative functions with respect to this equivalence relation $\mathcal{H}^\infty := \mathcal{H}_0^\infty / \sim$. The vector space structure on $\mathcal{H}^\infty$ is given by pointwise multiplication by scalars and pointwise addition, where we choose an arbitrary value for $(f_1 + f_2)(x)$ for those finitely many x for which $f_1(x)$ and $f_2(x)$ do not belong to the same space V_a.

In the following we will need a particular type of multiplicative function which we now define.

Definition 2.6. Let x be a reduced word in $E(a)$ and $v_a \in V_a$. A *shadow* $\mu[x, v_a]$ is (the equivalence class of) a multiplicative function supported on the cone $C(x)$, such that

$$
N\big(\mu[x, v_a]\big) = |x| \ \text{ and } \mu[x, v_a](x) := v_a \,.
$$

It is clear that every multiplicative function can be written as the sum of a finite number of shadows.

For each $a \in A$ choose a positive definite sesquilinear form B_a on $V_a \times V_a$ and set

$$\langle f_1, f_2 \rangle := \sum_{|x|=N} \sum_{\substack{a \\ |xa|=|x|+1}} B_a\big(f_1(xa), f_2(xa)\big) \tag{2.3}$$

where N is large enough so that both f_i satisfy (2.2). It is easy to verify that for the definition to be independent of N the B_a must satisfy the condition $B_a(v_a, v_a) = \sum_b B_b(H_{ba}v_a, H_{ba}v_a)$, for all $a \in A$ and $v_a \in V_a$.

Definition 2.7. The triple (V_a, H_{ba}, B_a) is a *system with inner products* if (V_a, H_{ba}) is a matrix system, B_a is a positive definite sesquilinear form on V_a for each $a \in A$ and for $v_a \in V_a$

$$B_a(v_a, v_a) = \sum_{b \in A} B_b(H_{ba}v_a, H_{ba}v_a)\,. \tag{2.4}$$

We refer to (2.4) as to a *compatibility condition.*

Assuming that such a family exists, define a unitary representation π of $\mathbb{F}_A$ on $\mathcal{H}^\infty$ by the rule

$$(\pi(x)f)(y) = f(x^{-1}y)\,. \tag{2.5}$$

The existence of a family of sesquilinear forms satisfying the compatibility condition was shown in [KS04] as follows.

Definition 2.8. For each $a \in A$, let S_a be the real vector space of symmetric sesquilinear forms on $V_a \times V_a$. Let $\mathcal{S} = \bigoplus_{a \in A} S_a$. We say that a tuple $\mathcal{B} = (B_a) \in \mathcal{S}$ is *positive definite* (resp. *positive semi-definite*) if each of its components is positive definite (resp. positive semi-definite), in which case we write $\mathcal{B} > 0$ (resp. $\mathcal{B} \geq 0$).

Let $\mathcal{K} \subseteq \mathcal{S}$ denote the solid cone consisting of positive semi-definite tuples. Define a linear map $\mathcal{L} : \mathcal{S} \to \mathcal{S}$ by the rule

$$(\mathcal{L}\mathcal{B})_a(v_a, v_a) = \sum_b B_b(H_{ba}v_a, H_{ba}v_a)\,, \tag{2.6}$$

where $\mathcal{B} = (B_a)$, and observe that $\mathcal{L}(\mathcal{K}) \subseteq \mathcal{K}$.

The existence of a tuple $(B_a)_{a \in A}$ compatible with (V_a, H_{ba}) depends on the eigenvalues of $\mathcal{L}$, as stated in the following:

Lemma 2.9 ([KS04, § 4])**.** *The spectral radius ρ of $\mathcal{L}$ is an eigenvalue and there exists a tuple of positive semi-definite sesquilinear forms (B_a)*

on V_a such that

$$\sum_b B_b(H_{ba}v_a, H_{ba}v_a) = \rho B_a(v_a, v_a)\,. \tag{2.7}$$

If the matrix system is irreducible then each B_a is strictly positive definite and, up to positive multiplicative scalars, there exists a unique tuple satisfying (2.7)*.*

We refer to ρ as the *Perron–Frobenius eigenvalue* of the system (V_a, H_{ba}).

It follows that, up to a normalization of the matrices H_{ba}, every matrix system becomes a system with inner products. Complete now $\mathcal{H}^\infty$ to $\mathcal{H} = \mathcal{H}(V_a, H_{ab}, B_a)$ with respect to the norm defined in (2.3) (where, again, we shall drop the dependence from (V_a, H_{ab}, B_a) unless necessary) and extend the representation π defined in (2.5) to a unitary representation on $\mathcal{H}$.

Two equivalent systems (V_a, H_{ba}, B_a) and (V'_a, H'_{ba}, B'_a) give rise to equivalent representations π and π' on $\mathcal{H} = \mathcal{H}(V_a, H_{ab}, B_a)$ and $\mathcal{H} = \mathcal{H}(V'_a, H'_{ab}, B'_a)$. In fact, if the tuple (J_a) gives the equivalence of the two systems in Definition 2.3, the operator defined by

$$U\big(\mu[x, v_a]\big) := \mu[x, J_a v_a]$$

for $v_a \in V_a$ extends to a unitary equivalence between $(\pi, \mathcal{H}(V_a, H_{ab}, B_a))$ and $(\pi', \mathcal{H}(V'_a, H'_{ab}, B'_a))$. Notice that the converse is not true, namely non-equivalent systems can give rise to equivalent representations: the simplest example is given by any spherical representation of the principal series of Figà-Talamanca and Picardello corresponding to a non-real parameter $q^{-\frac{1}{2}+is}$ [KS04, Example 6.3].

The irreducibility condition in the last statement in Lemma 2.9 is only sufficient. In fact, even if the matrix system is reducible, we can always assume that the B_a are strictly positive definite by passing to an appropriate quotient, as the following shows:

Lemma 2.10. *Let (V_a, H_{ba}, B_a) be a matrix system with inner product and let π a multiplicative representation on $\mathcal{H}(V_a, H_{ba}, B_a)$. Then there exist a matrix system with inner product $(\widetilde{V}_a, \widetilde{H}_{ba}, \widetilde{B}_a)$ and a representation $\widetilde{\pi}$ on $\widetilde{\mathcal{H}}(\widetilde{V}_a, \widetilde{H}_{ab}, \widetilde{B}_a)$ equivalent to π such that $\widetilde{\mathcal{B}} = (\widetilde{B}_a) > 0$.*

Proof. If (B_a) is not strictly positive definite, then for some $a \in A$,

$$W_a := \{w_a \in V_a \setminus \{0\} :\ B_a(w_a, w_a) = 0\} \neq \emptyset\,.$$

Since for $w_a \in W_a$

$$0 = B_a(w_a, w_a) = \sum_b B_b(H_{ba}w_a, H_{ba}w_a)$$

and all the terms $B_b(H_{ba}w_a, H_{ba}w_a)$ on the right are non-negative, each of these must be zero. Thus, $H_{ba}w_a \in W_b$ and we conclude that (W_a) is a nontrivial invariant subsystem.

Let $(\widetilde{V}_a, \widetilde{H}_{ba})$ be the quotient system. The tuple $(\widetilde{B}_a)$ given by

$$\widetilde{B}_a(\widetilde{v}_a, \widetilde{v}_a) = B_a(v_a, v_a) \qquad \text{for some } v_a \in \widetilde{v}_a$$

is well-defined and strictly positive on $(\widetilde{V}_a)$. In the representation space $\mathcal{H}^\infty(V_a, H_{ba})$ define the invariant subspace

$$\mathcal{H}^\infty_W = \{f \in \mathcal{H}^\infty(V_a, H_{ba}) : \ f(xa) \in W_a \text{ for all } a \in A \text{ and for all } x \in \mathbb{F}_A \text{ with } |x| \geq N(f) \text{ and } |xa| = |x| + 1\} .$$

and consider the quotient representation π_W on $\mathcal{H}^\infty(V_a, H_{ba})/\mathcal{H}^\infty_W$. Then the representation space $\mathcal{H}^\infty(V_a, H_{ba})/\mathcal{H}^\infty_W$ may be identified with the space $\mathcal{H}^\infty(\widetilde{V}_a, \widetilde{H}_{ba})$ of vector-valued multiplicative functions taking values in $\bigoplus_{a\in A} \widetilde{V}_a$ and, after the appropriate completion, π is equivalent to π_W. $\square$

3. The Class **Mult**

Definition 3.1. Given a free group $\mathbb{F}_A$ on a symmetric set of generators A, we say that a representation (ρ, H) belongs to the class $\mathbf{Mult}(\mathbb{F}_A)$ if there exists a system with inner products (V_a, H_{ba}, B_a), a dense subspace $M \subseteq H$ and a unitary operator $U : H \to \mathcal{H} = \mathcal{H}(V_a, H_{ba}, B_a)$ such that

- U is an isomorphism between M and the space $\mathcal{H}^\infty(V_a, H_{ba}, B_a)$ of vector-valued multiplicative functions.
- $U(\rho(x)m) = \pi(x)(Um)$ for every $m \in M$ and $x \in \mathbb{F}_A$.

We call such a representation *multiplicative.*

We emphasise again that the representations defined in [KS04], were built up from irreducible matrix systems. A representation that arises from an irreducible matrix system with inner product is always irreducible as a representation of the crossed product the C^*-algebra $\mathcal{C}(\partial\Gamma)$, while, as a representation of Γ is either irreducible or, in some special cases, sum of two irreducible ones. We analyze here instead representations arising from non-irreducible matrix systems showing that they are still well behaved in the sense that they decompose into an orthogonal direct sum of a finite number of representations obtained from irreducible matrix systems (see Theorem 2 for the statement and § 5 for the proof).

We can now proceed to analyse the properties of the family of the multiplicative representations of a free group. The first thing to observe is that the class $\mathbf{Mult}(\mathbb{F}_A)$ is independent of the generating system A, and hence is invariant under group automorphisms (see Theorem 2 for the statement and § 6 for the proof). This allows us to refine at once the definition of the class of multiplicative representations.

Definition 3.2. Given a non abelian finitely generated free group Γ, we say that a representation π belongs to the class $\mathbf{Mult}(\Gamma)$ if there exists a symmetric set of generators A such that $\pi \in \mathbf{Mult}(\mathbb{F}_A)$.

Observe that the property of being invariant under a change of generators is enjoyed by the class $\mathbf{Mult}(\Gamma)$, but not by single representations, as will be shown in the Example 6.10 at the end of § 6.

Finally the stability properties of the class $\mathbf{Mult}(\Gamma)$ with respect to the restriction and the induction to a subgroup (stated in Theorem 3 and proven in § 7.1 and § 7.2) allows us to proceed to define the class of multiplicative representations for a group satisfying $(*)$ and $(**)$ in the introduction, as follows: recall that if G is a locally compact topological group and $H < G$ is a discrete subgroup, we say that H is a lattice in G if the quotient $H\backslash G$ admits a finite G-invariant measure.

Definition 3.3. Let Λ be a locally compact group admitting a free group Γ as a lattice. We say that a representation π of Λ belongs to the class $\mathbf{Mult}_\Gamma(\Lambda)$ if there exists and a representation ρ in the class $\mathbf{Mult}(\Gamma)$ such that π is contained in $\mathrm{Ind}_\Gamma^\Lambda(\rho)$,

$$\mathbf{Mult}_\Gamma(\Lambda) := \big\{\pi :\ \exists\, \rho \in \mathbf{Mult}(\Gamma) \text{ such that } \pi \leq \mathrm{Ind}_\Gamma^\Lambda(\rho)\big\}$$

In general the class of representations $\mathbf{Mult}_\Gamma(\Lambda)$ depends on Γ. The next theorem shows however that the class $\mathbf{Mult}_\Gamma(\Lambda)$ does not depend on the subgroup Γ within a commensurability class, up to conjugation. Recall that two subgroups $H_1 < G$ and $H_2 < G$ are commensurable if their intersection is of finite index in both and they are commensurable up to conjugation if there exists $g \in G$ such that $H_1 \cap g^{-1}H_2 g$ has finite index both in H_1 and in $g^{-1}H_2 g$.

Theorem 3.4. *Let Γ be a free group embedded as a lattice in a locally compact group Λ. Assume that any other free group embedded as a lattice in Λ is commensurable to Γ up to conjugacy. Then* $\mathbf{Mult}_\Gamma(\Lambda)$ *is independent of* Γ.

Proof. Let Γ_i, $i = 1, 2$, be free subgroups embedded as lattices in Λ. We shall prove that $\mathbf{Mult}_{\Gamma_1}(\Lambda) = \mathbf{Mult}_{\Gamma_2}(\Lambda)$. Choose $\pi \in \mathbf{Mult}_{\Gamma_1}(\Lambda)$.

By definition there exists $\rho_1 \in \mathbf{Mult}(\Gamma_1)$ such that

$$\pi \leq \mathrm{Ind}_{\Gamma_1}^{\Lambda}(\rho_1)\,, \tag{3.1}$$

and we look for $\rho_2 \in \mathbf{Mult}(\Gamma_2)$ such that $\pi \leq \mathrm{Ind}_{\Gamma_2}^{\Lambda}(\rho_2)$ so that $\pi \in \mathbf{Mult}(\Gamma_2)$. By symmetry we will have proven the equality of the two classes.

To this purpose, choose $g \in \Lambda$ so that $\Gamma_0 := \Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index both in Γ_1 and $\Gamma_2^g := g\Gamma_2 g^{-1}$ and let

$$\rho_0 = \rho_1|_{\Gamma_0} \text{ and } \pi_2 := \mathrm{Ind}_{\Gamma_0}^{\Gamma_2^g}(\rho_0)\,.$$

By Theorem 3 $\rho_0 \in \mathbf{Mult}(\Gamma_0)$ and $\pi_2 \in \mathbf{Mult}(\Gamma_2^g)$.

By the general properties of induction (see for example [Mac76]), we have that

$$\pi \leq \mathrm{Ind}_{\Gamma_1}^{\Lambda}(\rho_1) \leq \mathrm{Ind}_{\Gamma_1}^{\Lambda}\mathrm{Ind}_{\Gamma_0}^{\Gamma_1}(\rho_0) = \mathrm{Ind}_{\Gamma_2^g}^{\Lambda}\mathrm{Ind}_{\Gamma_0}^{\Gamma_2^g}(\rho_0) = \mathrm{Ind}_{\Gamma_2^g}^{\Lambda}(\pi_2)\,,$$

Fix now a free symmetric generating set A for Γ_2 and define a representation ρ_2 of Γ_2 by letting $\rho_2(\gamma) := \pi_2(g\gamma g^{-1})$. Then ρ_2 can be realized as a multiplicative representation with respect to the generating set $g^{-1}Ag$ and the theorem will be proved as soon as we show that

$$\Pi := \mathrm{Ind}_{\Gamma_2^g}^{\Lambda}(\pi_2) \quad \text{and } \tilde{\Pi} := \mathrm{Ind}_{\Gamma_2}^{\Lambda}(\rho_2)$$

are equivalent. To this purpose, let $\mathcal{H}_2$ denote the representation space of π_2. Recall that Π is acting on

$$\mathrm{Ind}_{\Gamma_2^g}^{\Lambda}(\mathcal{H}_2) := \left\{ f : \Lambda \to \mathcal{H}_2 :\ \pi_2(\gamma)f(v) = f(v\gamma^{-1}),\ \text{for all } \gamma \in \Gamma_2^g, v \in \Lambda \right\},$$

as the left regular representation, while $\tilde{\Pi}$ is also acting as the left regular representation on the space

$$\mathrm{Ind}_{\Gamma_2}^{\Lambda}(\mathcal{H}_2) := \left\{ f : \Lambda \to \mathcal{H}_2 :\ \rho_2(\gamma)f(v) = f(v\gamma^{-1}),\ \text{for all } \gamma \in \Gamma_2, v \in \Lambda \right\},$$

For every $f \in \mathrm{Ind}_{\Gamma_2^g}^{\Lambda}(\mathcal{H}_2)$ define $\tilde{f}$ by letting

$$\tilde{f}(v) := f(gvg^{-1})\,.$$

It is easy to see that $\tilde{f} \in \mathrm{Ind}_{\Gamma_2}^{\Lambda}(\mathcal{H}_2)$ and that

$$\widetilde{\Pi(h)f}(v) = (\tilde{\Pi}^g(h)\tilde{f})(v)$$

where $\widetilde{\Pi^g}$ is the representation defined as $\tilde{\Pi}^g(h) = \tilde{\Pi}(g^{-1}hg)$. Since $\tilde{\Pi}^g$ and $\tilde{\Pi}$ are equivalent as representations of Λ, the proof is concluded. $\square$

We remarked already that Theorem 3(1) and Theorem 3(2) guarantee that if Λ is itself a free group the definition given here gives the same class $\mathbf{Mult}(\Gamma)$ as defined in 3.2.

Remark 3.5. Our Theorem 3.4 applies to any virtually free group and hence to any *non-uniform lattice* $\Lambda < PSL(2,\mathbf{R})$. Moreover, when Λ is virtually free one can say more about the class $\mathbf{Mult}(\Lambda)$: the reader may consult [IKS13].

Remark 3.6. To get another example of groups that fulfill the hypothesis of our Theorem 3.4 let $\mathcal{T}$ be a *uniform tree* $\mathcal{T}$, that is a locally finite tree such that the group of automorphisms $\mathrm{Aut}(\mathcal{T})$ is unimodular and $\mathrm{Aut}(\mathcal{T})\backslash\mathcal{T}$ is finite. Let $H < \mathrm{Aut}(\mathcal{T})$ be any unimodular subgroup acting cocompactly on $\mathcal{T}$. Then there exists uniform lattices Γ in H, and such lattices are always virtually free, [BK90, Theorem 4.7]. Moreover, every two cocompact lattices are commensurable *up to conjugation in* $\mathrm{Aut}(\mathcal{T})$, [BK90, Corollary 4.8]. In particular our theorem applies to $\mathrm{Aut}(\mathcal{T})$ as well asto any uniform lattice in $PGL(2,\mathbf{Q}_p)$ ($\mathbf{Q}_p$ denotes the field of p-adic numbers).

Question 3.7. Our Theorem however does not apply to $\Lambda = PGL(2,\mathbf{Q}_p)$ itself, since two uniform lattices Γ_1 and Γ_2 need not be commensurable in $PGL(2,\mathbf{Q}_p)$: indeed there are infinitely many conjugacy classes of uniform lattices ([Lub91], the reader should consult also [BL01]) in $PGL(2,\mathbf{Q}_p)$. It is known that it is possible to embed a free group Γ_1 as a lattice in $PGL(2,\mathbf{Q}_p)$ in such a way that the representations of the *spherical* principal series of $PGL(2,\mathbf{Q}_p)$ restrict to the spherical principal series of Γ_1 with respect to a given set of generators (see [FTP84]). Essentially Frobenius reciprocity (see [CS91, Proposition 1.1]) says the spherical principal series of $PGL(2,\mathbf{Q}_p)$ belongs to the class $\mathbf{Mult}_{\Gamma_1}PGL(2,\mathbf{Q}_p)$: are there other irreducible representations of $PGL(2,\mathbf{Q}_p)$ in this class? Is $\mathbf{Mult}_{\Gamma_1}(PGL(2,\mathbf{Q}_p))$ still independent of Γ_1? What happens of the supercuspidal representations of $PGL(2,\mathbf{Q}_p)$?

4. Preliminary Results

4.1. **The Compatibility Condition and the Norm of a Multiplicative Function.** Let f be a function multiplicative for $|x| \geq N$. Fix any vertex x such that $d(e,x) \geq N$ and denote by $t(x)$ the last letter in the reduced word for x. Then the compatibility condition can be rewritten as

$$B_{t(x)}\big(f(x), f(x)\big) = \sum_{\substack{y \\ |y|=|x|+1}} B_{t(y)}\big(f(y), f(y)\big)\,, \tag{4.1}$$

so that, from (2.3),

$$\|f\|_{\mathcal{H}}^2 = \sum_{|x|=N} \|f(x)\|^2 \,,$$

where

$$\|f(x)\|^2 := B_{t(x)}(f(x), f(x))\,.$$

The hypothesis of compatibility (2.4) has further consequences in the computation of the norm of a function, that we illustrate now. We start with some definitions and notation.

Definition 4.1. Let $\mathcal{T}$ be a tree of degree $q+1$ and $\mathcal{X}$ a finite subtree. We say that $\mathcal{X}$ is *non-elementary* if it contains at least two vertices. If x is a vertex of $\mathcal{X}$, its *degree relative* to $\mathcal{X}$ is the number of neighborhoods of x that lie in $\mathcal{X}$. A finite subtree $\mathcal{X}$ is called *complete* if all its vertices have relative degree equal either to 1 or to $q+1$. The vertices having degree 1 are called *terminal* while the others are called *interior*.

The set of terminal vertices is denoted by $T(\mathcal{X})$. If $\mathcal{X}$ is a complete nonelementary subtree not containing e as an interior vertex, we denote by $\bar{x}_e$ the unique vertex of $\mathcal{X}$ which minimizes the distance from e and x_e the unique vertex of $\mathcal{X}$ connected to $\bar{x}_e$ (which exists since $\bar{x}_e \in T(\mathcal{X})$). We call $\mathcal{X}$ a *complete (nonelementary) subtree based at* x_e. We set moreover $T_e(\mathcal{X}) := T(\mathcal{X}) \setminus \{\bar{x}_e\}$ and denote by $B(x,N) = \{y \in \mathcal{T} :\ d(x,y) \le N\}$ the (closed) ball of radius N centered at $x \in \mathcal{T}$.

Lemma 4.2. *Let $\mathcal{X}$ be any complete nonelementary subtree not containing e as an interior vertex. With the above notation, assume that f is a function multiplicative outside the ball $B\big(e, |x_e|\big)$. Then*

$$\|f(x_e)\|^2 = \sum_{t\in T_e(\mathcal{X})} \|f(t)\|^2 \,. \tag{4.2}$$

Proof. Let

$$n = \sup_{x\in\mathcal{X}} d(x_e, x)\,.$$

The statement can be easily proved by induction on n. When $n = 1$ the subtree $\mathcal{X}$ must be exactly $B\big(x_e, 1\big)$ and (4.2) reduces to (4.1). Assume now that (4.2) is true for n and pick any y_1 such that

$$d(x_e, y_1) = n + 1 = \sup_{x\in\mathcal{X}} d(x_e, x)\,.$$

Denote by $[x_e, \dots, \bar{y}_1, y_1]$ the geodesic joining x_e to y_1. By construction y_1 is a terminal vertex while $\bar{y}_1$ is an interior vertex. Let $\mathcal{X}_1$ be the subtree obtained from $\mathcal{X}$ by removing all the q neighbors of $\bar{y}_1$ at distance $n+1$ from x_e. Now $\bar{y}_1$ is a terminal vertex of $\mathcal{X}_1$. If the supremum over all the vertices of the new complete subtree $\mathcal{X}_1$ of the distances $d(x_e,x)$ is n use induction, otherwise, if

$$n+1=\sup_{x\in\mathcal{X}_1} d(x_e,x)\ ;$$

pick any y_2 such that $n+1=d(x_e,y_2)$ and proceed as before. In a finite number of steps we shall end with a finite complete subtree $\mathcal{X}_k$ satisfying

$$n=\sup_{x\in\mathcal{X}_k} d(x_e,x)$$

for which (4.2) holds. Since by inductive hypothesis $\mathcal{X}$ can be obtained from $\mathcal{X}_k$ by adding all the q neighbors of each point $\bar{y}_i$ which are at distance $n+1$ from x_e, $i=1,\dots,k$, again (4.2) follows from (4.1). $\square$

We saw that the norm of a multiplicative function can be computed as the sum of the values of $\|f(x)\|^2$, where x ranges over all terminal vertices in $B(e,N)$ for N large enough; building on the previous lemma, the next result asserts that branching off in some direction along a complete subtree and considering again all terminal vertices does not change the norm.

Lemma 4.3. *Let $\mathcal{X}$ be any finite complete subtree containing $B(e,N)$ and let f be multiplicative for $|x|\geq N$. Then*

$$\|f\|^2_{\mathcal{H}}=\sum_{x\in T(\mathcal{X})}\|f(x)\|^2\,.$$

Proof. Let $L\geq N$ be the radius of the largest ball $B(e,L)$ completely contained in $\mathcal{X}$, so that $\|f\|^2_{\mathcal{H}}=\sum_{|x|=L}\|f(x)\|^2$.

If $B(e,L)\neq\mathcal{X}$, the set of points

$$I:=\big\{x\in\mathcal{X}:\ d(e,x)=L \text{ and } x\notin T(\mathcal{X})\big\}$$

is not empty. Apply now Lemma 4.2 to the complete subtree $\mathcal{X}_x$ of $\mathcal{X}$ based at x for all $x\in I$. $\square$

4.2. **The Perron–Frobenius Eigenvalue.** Before we conclude this section we prove the following two lemmas, which shed some light on the possible values of the Perron–Frobenius eigenvalue of a given matrix system. Both lemmas, together with Lemma 2.10, will be necessary in the proof of Theorem 1.

Lemma 4.4. *Let (V_a,H_{ba},B_a) be a matrix system with inner product, (W_a,H_{ba}) an invariant subsystem. Let π be the multiplicative representation on $\mathcal{H}(V_a,H_{ba},B_a)$ and let π_W be the restriction of π to a multiplicative representation on $\mathcal{H}(W_a,H_{ba},B_a)$. Assume that the quotient system $(\widetilde{V}_a,\widetilde{H}_{ba})$ is irreducible. If the Perron–Frobenius eigenvalue ρ of* *the quotient system* $(\widetilde{V}_a, \widetilde{H}_{ba})$ *is less than 1 then the representations* π *and* π_W *are equivalent.*

Proof. By Lemma 2.10 we may assume that the B_a are strictly positive definite. For each a let

$$W_a^{\perp} := \{v_a \in V_a : \ B_a(w_a, v_a) = 0 \text{ for all } w_a \in W_a\}$$

be the orthogonal complement (with respect to B_a) of W_a in V_a. Let $\varphi_a : V_a \to \widetilde{V}_a$, respectively $P_a : V_a \to W_a^{\perp}$, denote the projection of V_a onto $\widetilde{V}_a$ and the orthogonal projection of V_a onto $W_a^{\perp}$. Set $H_{ba}^{\perp} := P_b H_{ba} P_a$. The following diagram

$$\begin{tikzcd} V_a \arrow[r, "\varphi_a"] & \widetilde{V}_a \\ V_a \arrow[u, "="] \arrow[r, "P_a"'] & W_a^{\perp} \arrow[u, "\varphi_a|_{W_a^{\perp}}"'] \end{tikzcd}$$

is commutative, so that the system $(W_a^{\perp}, H_{ba}^{\perp})$ may be viewed as an invariant subsystem of the quotient system $(\widetilde{V}_a, \widetilde{H}_{ba})$. Since the dimensions are the same, the two systems must be equivalent.

Denote by ρ the Perron-Frobenius eigenvalue of the system $(\widetilde{V}_a, \widetilde{H}_a)$. By Lemma 2.9 there exists an essentially unique tuple $\widetilde{B}_a$ of sesquilinear forms on $\widetilde{V}_a$ such that

$$\sum_{b \in A} \widetilde{B}_b(\widetilde{H}_{ba}\widetilde{v}_a, \widetilde{H}_{ba}\widetilde{v}_a) = \rho \widetilde{B}_a(\widetilde{v}_a, \widetilde{v}_a)\,, \tag{4.3}$$

which can be chosen to be positive definite since the system $(\widetilde{V}_a, \widetilde{B}_a)$ is irreducible. By identifying the finite dimensional subspaces $W_a^{\perp}$ and $\widetilde{V}_a$, the norms induced on $W_a^{\perp}$ by B_a and on $\widetilde{V}_a$ by $\widetilde{B}_a$ are equivalent and there exists a constant K so that

$$B_a\big(P_a(v_a), P_a(v_a)\big) \le K \widetilde{B}_a\big(\varphi(v_a), \varphi(v_a)\big)$$

for all $a \in A$.

Define, as in Lemma 2.10,

$$\mathcal{H}_W^{\infty} = \{f \in \mathcal{H}^{\infty}(V_a, H_{ba}) : \ f(xa) \in W_a \text{ for all } a \in A \text{ and for all } x \in \mathbb{F}_A \text{ with } |x| \ge N(f) \text{ and } |xa| = |x| + 1\}\,.$$

Under the assumption that $\rho < 1$, we shall prove that $\mathcal{H}_W^{\infty}$ is dense in $\mathcal{H}^{\infty}(V_a, H_{ba})$ with respect to the norm induced by the B_a, from which the assertion will follow. Choose f in $\mathcal{H}^{\infty}(V_a, H_{ba})$ and $\epsilon > 0$. Let $N = N(f)$ be such that f is multiplicative for $n \ge N$ and let us fix

$x \in \mathbb{F}_A$ and $a \in A$ such that $|x| \geq N$ and $|xa| = |x| + 1$. Write $f(xa) = w_a + w_a^\perp$, where $w_a \in W_a$ and $w_a^\perp \in W_a^\perp$, and observe that

$$\begin{aligned} P_b\big(f(xab)\big) &= P_b\big(H_{ba} f(xa)\big) = P_b\big(H_{ba}(w_a + w_a^\perp)\big) \\ &= P_b H_{ba} w_a^\perp = H_{ba}^\perp w_a^\perp \,. \end{aligned} \tag{4.4}$$

Define now

$$g_0 := \sum_{b:\ ab \neq e} \mu[xab, f(xab) - P_b(f(xab))]$$

and compute

$$\begin{aligned} \|f - g_0\|^2_{\mathcal{H}} &= \sum_{\substack{b \\ |xab| = |x|+2}} B_b\big(f(xab) - g_0(xab), f(xab) - g_0(xab)\big) \\ &= \sum_{\substack{b \\ |xab| = |x|+2}} B_b(H_{ba}^\perp w_a^\perp, H_{ba}^\perp w_a^\perp) \\ &\leq K \sum_{\substack{b \\ |xab| = |x|+2}} \widetilde{B}_b(H_{ba}^\perp w_a^\perp, H_{ba}^\perp w_a^\perp) \\ &= K \rho \widetilde{B}_a(w_a^\perp, w_a^\perp) \,. \end{aligned}$$

Let n be large enough so that

$$K \rho^n \widetilde{B}_a(w_a^\perp, w_a^\perp) < \epsilon \,.$$

Let $z := a_1 \dots a_n$ a reduced word of length n so that $y = xazb$ has length $|y| = |x| + 2 + n$. Define $H_y^\perp = H_{ba_n}^\perp \dots H_{a_1 a}^\perp$ and use induction and (4.4) to see that

$$P_b(f(y)) = H_y^\perp w_a^\perp \,.$$

A repeated application of (4.3) yields

$$\sum_{b \in A} \sum_{\substack{y \in C(xa) \cap E(b) \\ |y| = |x|+2+n}} \widetilde{B}_b(H_y^\perp w_a^\perp, H_y^\perp w_a^\perp) \ = \ \rho^{n+1} \widetilde{B}_a(w_a^\perp, w_a^\perp) \,.$$

If we set, as before,

$$g_n := \sum_{b \in A} \sum_{\substack{y \in C(xa) \cap E(b) \\ |y| = |x|+2+n}} \mu[y, f(y) - P_b(f(y))] \,,$$

then

$$\begin{aligned}\|f-g_n\|_{\mathcal{H}}^2 &= \sum_{b\in A}\sum_{\substack{y\in C(xa)\cap E(b)\\ |y|=|x|+2+n}} B_b\big(P_b(f(y)),P_b(f(y))\big)\\ &\le K\sum_{b\in A}\sum_{\substack{y\in C(xa)\cap E(b)\\ |y|=|x|+2+n}} \widetilde{B}_b(H_y^\perp w_a^\perp, H_y^\perp w_a^\perp)\\ &= K\rho^{n+1}\widetilde{B}_a(w_a^\perp, w_a^\perp)\,,\end{aligned}$$

and hence

$$\|f-g_n\|_{\mathcal{H}}^2 \le K\rho^{n+1}\widetilde{B}_a(w_a^\perp,w_a^\perp) < \epsilon\,.$$

Since g_n belongs to $\mathcal{H}_W$ this concludes the proof. □

Lemma 4.5. *Let (V_a, H_{ba}, B_a) be a matrix system with inner products and (W_a, H_{ba}) a maximal nontrivial invariant subsystem with quotient $(\widetilde{V}_a, \widetilde{H}_{ba})$. Then there exists a tuple of strictly positive definite forms on $\widetilde{V}_a$ with Perron–Frobenius eigenvalue $\rho = 1$.*

Proof. We may assume that $\mathcal{B} := (B_a) > 0$. The maximality of (W_a, H_{ba}) implies that the quotient system $(\widetilde{V}_a, \widetilde{H}_{ba})$ is irreducible, hence by Lemma 2.9 there exists a tuple of strictly positive definite forms $(\widetilde{B}_a)$ satisfying

$$\sum_b \widetilde{B}_b(\widetilde{H}_{ba}\widetilde{v}_a, \widetilde{H}_{ba}\widetilde{v}_a) = \rho\widetilde{B}_a(\widetilde{v}_a,\widetilde{v}_a)$$

for some positive ρ.

If the Perron–Frobenius eigenvalue ρ relative to $(\widetilde{V}_a, \widetilde{H}_{ba})$ were strictly smaller than one, by Lemma 4.4 the representations π on $\mathcal{H}(V_a, H_{ba}, B_a)$ and π_W on $\mathcal{H}(W_a, H_{ba}, B_a)$ would be equivalent and we could restrict ourselves to the new system (W_a, H_{ba}, B_a) of strictly smaller dimension.

We may assume therefore that $\rho \ge 1$.

Assume, by way of contradiction, that $\rho > 1$. Lift the $\widetilde{B}_a$ to a positive semi-definite form on V_a by setting it equal to zero on W_a. Rewrite our conditions in terms of the operator $\mathcal{L}$ defined in (2.6):

$$\mathcal{L}\mathcal{B} = \mathcal{B} \quad \text{and} \quad \mathcal{L}\widetilde{\mathcal{B}} = \rho\widetilde{\mathcal{B}}$$

where $\mathcal{B} = (B_a)_{a\in A}$ and $\widetilde{\mathcal{B}} = (\widetilde{B}_a)_{a\in A}$. Since all the B_a are strictly positive definite, there exists a positive number k such that $kB_a - \widetilde{B}_a$ is strictly positive definite on V_a for each $a \in A$. Hence for every integer n

$$\mathcal{L}^n(k\mathcal{B} - \widetilde{\mathcal{B}}) = k\mathcal{L}^n(\mathcal{B}) - \mathcal{L}^n(\widetilde{\mathcal{B}}) = k\mathcal{B} - \rho^n\widetilde{\mathcal{B}} \ge 0$$

Choose now $v_a \in V_a$ so that $\widetilde{B}_a(v_a, v_a) \neq 0$ and n large enough to get a contradiction. □

5. Stability Under Orthogonal Decomposition

Proof of Theorem 1. Let (V_a, H_{ba}, B_a) be a matrix system with inner products and assume that $\mathcal{B} = (B_a) > 0$ (see Lemma 2.10).

Let (W_a, H_{ba}) be a maximal nontrivial invariant subsystem with irreducible quotient $(\widetilde{V}_a, \widetilde{H}_{ba})$ and let $(\widetilde{B}_a)$ be a tuple of strictly positive definite forms with Perron–Frobenius eigenvalue $\rho = 1$, whose existence follows from Lemma 4.5. Pull back the forms $(\widetilde{B}_a)$ to obtain a tuple of positive semi-definite forms on V_a which have W_a as the kernel and which we still denote by $\widetilde{B}_a$. Define

$$\lambda_0 = \sup\{\lambda > 0 : \ B_a - \lambda \widetilde{B}_a \geq 0 \text{ for all } a \in A\}$$

Since $(\widetilde{B}_a)$ are strictly positive on $W_a^{\perp}$, λ_0 is finite. Moreover, for such λ_0, $B_a - \lambda_0 \widetilde{B}_a$ is not strictly positive for some a and hence, for these a's

$$W_a^0 := \{v_a \in V_a : \ (B_a - \lambda_0 \widetilde{B}_a)(v_a, v_a) = 0\} \neq \{0\}\,.$$

Set

$$(\mathcal{B}^0)_a := B_a - \lambda_0 \widetilde{B}_a$$

and observe that

$$\mathcal{B}^0 = \mathcal{B} - \lambda_0 \widetilde{\mathcal{B}} \geq 0$$
$$\mathcal{L}(\mathcal{B} - \lambda_0 \widetilde{\mathcal{B}}) = \mathcal{L}\mathcal{B} - \lambda_0 \mathcal{L}\widetilde{\mathcal{B}} = \mathcal{B} - \lambda_0 \widetilde{\mathcal{B}}\,.$$

Arguing as in Lemma 2.10 one can see that also the (W_a^0), and hence the $(W_a + W_a^0)$, constitute an invariant subsystem. We claim that $V_a = W_a \oplus W_a^0$. In fact, since $\widetilde{B}_a|_{W_a} \equiv 0$, then $W_a \cap W_a^0 = 0$ for all a. Moreover, if $\varphi_a : V_a \to \widetilde{V}_a$ denotes the projection, the system $\varphi_a(W_a \oplus W_a^0)$ would be invariant and hence, by irreducibility of $(\widetilde{V}_a)$, the image $\varphi_a(W_a \oplus W_a^0)$ has to be all of $\widetilde{V}_a$, that is to say $V_a = W_a \oplus W_a^0$ for all a. Moreover

$$B_a = B_a^0 + \lambda_0 \widetilde{B}_a$$

is the sum of two orthogonal forms. The representation $(\pi, \mathcal{H})$ constructed from the system (V_a, H_{ba}, B_a) decomposes as the sum of the two sub-representations corresponding to the systems (W_a, H_{ba}, B_a^0) and $(W_a^0, H_{ba}, \widetilde{B}_a)$ where the latter is an irreducible system. To complete the proof repeat the above argument for the system (W_a, H_{ba}, B_a^0): since all the V_a are finite dimensional, this reduction process will stop with an irreducible subsystem. □

6. Stability Under Change of Generators

We begin with some notation. Write a_i, b_i, c_i, and $\alpha_j, \beta_j, \gamma_j$, for generic elements of A or A', respectively. Denote by $\mathcal{T}$ and $\mathcal{T}'$ the tree relative to the generating set A and A', and by $|x|$, $|x|'$ the tree distance of x from e in $\mathcal{T}$ and $\mathcal{T}'$. Every element has a unique expression as a reduced word in both alphabets and we shall write $z = a_1 \dots a_n$ or $z = \alpha_1 \dots \alpha_k$. If $\ell(A, A')$ denotes the maximum length of the elements of A with respect to the elements of A', then

$$|z|' \leq \ell(A, A')|z| \, .$$

The two trees $\mathcal{T}$ and $\mathcal{T}'$ have the same set of vertices $\mathcal{V}$, but, for a given vertex $y \in \mathcal{V}$, the geodesic $[e, y]$ in $\mathcal{T}$ might be quite different form the geodesic $[e, y]'$ in $\mathcal{T}'$. We recall from (2.1) that

$$C(z) = \{y \in \mathcal{V} : \ z \in [e, y]\}$$

and we define analogously

$$C'(z) = \{y \in \mathcal{V} : \ z \in [e, y]'\} \, ,$$

Hence, if $z = \alpha_1 \dots \alpha_k \in \mathbb{F}_{A'}$ and $z = a_1 \dots a_n \in \mathbb{F}_A$, $C'(z)$ consists of all reduced words in the alphabet A' of the form $y = \alpha_1 \dots \alpha_k s$ with $|y|' = k + |s|'$ while $C(z)$ consists of all reduced words in the alphabet A of the form $w = a_1 \dots a_n t$ with $|w| = n + |t|$.

We remark that, for $xy \neq e$, in general we have that

$$C(xy) \subseteq xC(y) \, ,$$

as $xC(y)$ might contain the identity and hence need not be a cone. The following lemma gives conditions under which there is, in fact, equality.

Lemma 6.1. *Let $x, y \in \mathcal{V}$.*

(i) *$xC(y) = C(xy)$ if and only if y does not belong to the geodesic fom e to x^{-1} in $\mathcal{T}$.*

(ii) *Let $a \in A$ be such that $|xa| = |x| + 1$ and assume that $C'(y) \subseteq C(a)$. Then $xC'(y) = C'(xy)$.*

Proof. The identity is not in $xC(y)$ if and only if x does not cancel y, that is, if and only if $y \notin [e, x^{-1}]$.

To prove the second assertion, observe that, since $|xa| = |x| + 1$, the element x^{-1} does not belong to $C(a)$ and, *a fortiori* to $C'(y)$ by hypothesis. Hence y does not belong the geodesic $[e, x^{-1}]'$ in $\mathcal{T}'$, which, by (i) is equivalent to saying that $xC'(y) = C'(xy)$. □

The following easy lemma will be useful in the definition of the matrices and the proof of their compatibility.

Lemma 6.2. *Let $a \in A$ and $z \in \mathcal{V}$ such that $C'(z) \subseteq C(a)$. Then for every $b \in A$, $ab \neq e$, the last letter of z and of bz in the alphabet A' coincide.*

Proof. If not, multiplication by b on the left would delete z, that is the reduced expression in the alphabet A' of the generator $b \in A$ would be $b = \alpha_1 \dots \alpha_t z^{-1}$. Taking the inverses one would have $b^{-1} = z{\alpha_t}^{-1} \dots {\alpha_1}^{-1}$, thus contradicting the hypothesis that $C'(z) \subseteq C(a)$. $\square$

We have seen in the last two lemmas the first consequences of the inclusion of cones with respect to the two different sets of generators. Analogous inclusions follow from the fact that, given two generating systems A and A', for every $k \geq 0$ there exists an integer $N = N(k)$ such that the first $N(k)$ letters of a word z in the alphabet A' determine the first k letters of z in the alphabet A. In other words, for any given $z \in \mathcal{V}$ there exists $N(|z|)$ and y with $|y|' \leq N(|z|)$ so that

$$C'(y) \subseteq C(z)\,. \tag{6.1}$$

The set of $y \in \mathcal{V}$ with this property is not necessarily unique. To refine the study of the consequences of this cone inclusion, we need to consider, among the y that satisfy (6.1), those that are the "shortest" with this property, in the appropriate sense. To make this precise, we use the following notation:

$\bar{y}$ is the last vertex before y in the geodesic $[e, \dots, \bar{y}, y]' \subset \mathcal{T}'$

$\widetilde{y}_z$ is the first vertex in the geodesic $[e, y]'$ such that $C'(\widetilde{y}_z) \subseteq C(z)\,.$

(For ease of notation, we will remove the subscript z whenever this does not cause any confusion.) For any $z \in \mathcal{V}$ we then define

$$\begin{aligned} Y(z) &= \{y \in \mathcal{V} : \ C'(y) \subseteq C(z) \text{ and } C'(\bar{y}) \not\subseteq C(z)\} \\ &= \{y \in \mathcal{V} : \ C'(y) \subseteq C(z) \text{ and } y = \widetilde{y}_z\} \end{aligned}$$

Then we have the following analogue of Lemma 6.1:

Corollary 6.3. *For every $a, b \in A$, $ab \neq e$, we have*

$$aY(b) = Y(ab)\,.$$

Proof. Let $y \in Y(b)$. By Lemma 6.2, $\overline{ay} = a\bar{y}$. Since $C'\ (\bar{y}) \not\subseteq C(b)$ and $C(b) \supseteq C'(y)$ there exists a reduced word $\bar{y}t$ in the alphabet A' so that $\bar{y}t \in C(d)$ for some $d \in A$ with $d \neq b$. Hence the element $a\bar{y}t$ will not be contained in $C(ab)$. $\square$

For any given π' in $\mathbf{Mult}(\mathbb{F}_{A'})$ we shall now construct π in $\mathbf{Mult}(\mathbb{F}_A)$ so that π' is either a subrepresentation or a quotient of π. Namely, if we are given a matrix system with inner products $(V'_\alpha, H'_{\beta\alpha}, B'_\alpha)$, we need to define a new system (V_a, H_{ba}, B_a) in such a way that the original system appears as a quotient or as a subsystem of the new one.

Definition 6.4. Let $z = \alpha_1 \dots \alpha_{k-1}\alpha_k \in \mathbb{F}_{A'}$ and define

$$V'_z = V'_{\alpha_k} \quad B'_z = B'_{\alpha_k}\,.$$

We set

$$V_a = \bigoplus_{z\in Y(a)} V'_z \qquad B_a = \bigoplus_{z\in Y(a)} B'_z$$

We need now to define the new matrices $H_{ba} : V_a \to V_b$, for $b \neq a^{-1}$. To this extent, take $z \in Y(b)$. Since $b \neq a^{-1}$, then $az \in C(a)$ and hence, by definition, $\widetilde{(az)}_a \in Y(a)$. Then we have two cases: either $az = \widetilde{(az)}_a$ and hence $az \in Y(a)$; or $az = \widetilde{(az)}_a x$ with $x \neq e$. In this case, if the reduced expression for x in the alphabet A' is $x = \alpha_1 \dots \alpha_n$ and $\alpha \neq \alpha_1^{-1}$ is the last letter (in A') of $\widetilde{(az)}_a$, define

$$H'_{az,\widetilde{az}} := H'_{\alpha_n\alpha_{n-1}} \dots H'_{\alpha_1\alpha}$$

where we wrote $az, \widetilde{az}$ for $az, \widetilde{(az)}_a$ for ease of notation. The new matrices $H_{ba} : V_a \to V_b$ can hence be defined to be block matrices indexed by pairs (z, w), with $z \in Y(b)$ and $w \in Y(a)$, as follows:

$$(H_{ba})_{z,w} := \begin{cases} \mathrm{Id} & \text{if } w = az = \widetilde{(az)}_a \\ H'_{az,\widetilde{az}} & \text{if } w = \widetilde{(az)}_a \neq az \end{cases} \tag{6.2}$$

and $(H_{ba})_{z,w} = 0$ for all other $w \in Y(a)$ with $w \neq \widetilde{(az)}_a$.

In the course of the definition we have shown that

$$\bigcup_{\substack{z\in Y(b)\\ b\neq a^{-1}}} \widetilde{(az)}_a \subseteq Y(a)\,,$$

but to show that the matrices so defined give a compatible matrix system we need to show that the above inclusion is in fact an equality, namely:

Proposition 6.5. *We have that*

$$Y(a) = \bigcup_{\substack{z\in Y(b)\\ b\neq a^{-1}}} \widetilde{(az)}_a$$

Proof. Take any $w \in Y(a)$ so that $C'(w) \subseteq C(a)$. Hence either there exists $b \neq a^{-1}$ such that $C'(w) \subseteq C(ab)$, in which case $w \in Y(ab)$, or $C'(w) \nsubseteq C(ab)$ for all $b \neq a^{-1}$. In this case, according to the discussion after Lemma 6.2, there exists $b \neq a^{-1}$ and $t_b \in \mathcal{V}$ with the following properties:

(1) $|wt_b|' = |w|' + |t_b|'$;
(2) $C'(wt_b) \subseteq C(ab)$;
(3) t_b is minimal with the above properties, that is $C'(w\bar{t}_b) \nsubseteq C(ab)$.

In the last case one has, by definition, $wt_b \in Y(ab)$. By Corollary 6.3 $Y(ab) = aY(b)$, so that either $w = az$ or $wt_b = az$ for some $z \in Y(b)$. Since $w \in Y(a)$, it is obvious that $w = \widetilde{(az)}_a$ when $w = az$. To finish we must show that $w = \widetilde{(wt_b)}_a$ when $wt_b = az$. By definition $\widetilde{(az)}_a$ is the first vertex in the geodesic $[e, wt_b]' = [e, az]'$ such that $C'(\widetilde{(az)}_a) \subset C(a)$. But by hypothesis $w \in Y(a)$, that is $C'(w) \subset C(a)$ and $C'(\overline{w}) \nsubseteq C(a)$. Thus $\widetilde{(az)}_a = w$. $\square$

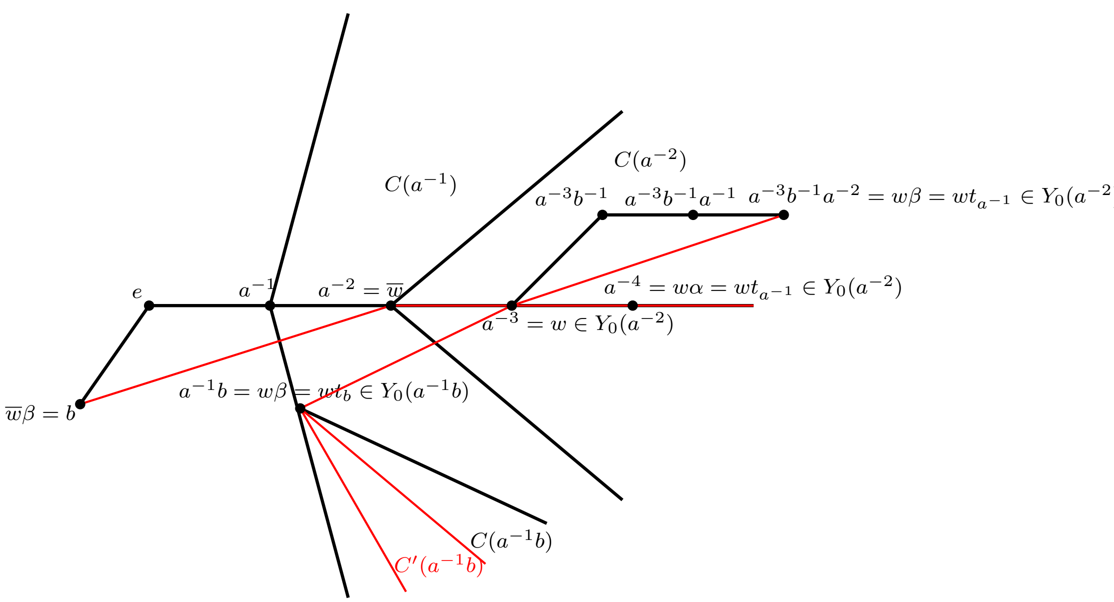

FIGURE 1: The trees $\mathcal{T}$ (in black) and $\mathcal{T}'$ (in red) associated respectively to $\mathbb{F}_A$ and $\mathbb{F}_{A'}$, where $A = \{a, b, a^{-1}, b^{-1}\}$ and A' is obtained with the change of generators $a \mapsto \alpha$ and $b \mapsto \beta = a^2 b$.

In the course of the proof of the above proposition we have distinguished two types of elements of $Y(a)$, and we can consequently conclude the following:

Corollary 6.6. *We have*

$$Y(a) = Y_0(a) \sqcup Y_1(a)\,,$$

where

$$\begin{aligned} Y_1(a) :&= \bigcup_{b\neq a^{-1}} (Y(a)\cap Y(ab)) \\ &= \Big\{w \in Y(a):\ \textit{there exists } b \neq a^{-1} \textit{ and } z \in Y(b),\ \textit{such that} \\ &\qquad\qquad\qquad\qquad\qquad\qquad w = az = \widetilde{(az)}_a\Big\} \end{aligned}$$

and

$$\begin{aligned} Y_0(a) :&= \Big\{w \in Y(a):\ \textit{for all } b \neq a^{-1},\ C'(w) \nsubseteq C(ab)\Big\} \\ &= \Big\{w \in Y(a):\ \textit{for some } b \neq a^{-1} \textit{ there exists } z \in Y(b),\ \textit{such} \\ &\qquad\qquad \textit{that } w = \widetilde{(az)}_a \ \textit{ and } az = w\,x,\ \textit{with } x \neq e\Big\}\,. \end{aligned}$$

To prove the compatibility condition we will make use of Lemma 4.2, so that we need to construct an appropriate finite complete subtree in $\mathcal{T}'$. Notice that for all $w \in \mathcal{V}\setminus\{e\}$, the set $\overline{w}\cup C'(w)$ is a complete subtree, but infinite. To "prune" it so that it will be finite and still complete, consider an element $w \in Y_0(a)$ and the following decomposition

$$\begin{aligned} C'(w) =&\Big\{y \in C'(w):\ \ C'(y) \nsubseteq C(ab) \text{ for all } b \neq a^{-1}\Big\} \\ &\cup \Big\{y \in C'(w):\ \ C'(y) \subseteq C(ab) \text{ for some } b \neq a^{-1}\Big\} \\ =& I'_w \cup \bigcup_{b\neq a^{-1}} \Big\{y \in C'(w):\ C'(y) \subseteq C(ab)\Big\}\,, \end{aligned}$$

where we have set

$$I'_w := \Big\{y \in C'(w):\ \ C'(y) \nsubseteq C(ab) \text{ for all } b \neq a^{-1}\Big\}\,.$$

Since the set I'_w is finite and $w \in I'_w$, we need to prune the other set.

Proposition 6.7. *Let $w \in Y_0(a)$ and define*

$$\begin{aligned} T'_w :&= \bigcup_{b\neq a^{-1}} \Big\{y \in C'(w):\ C'(y) \subseteq C(ab),\, C'(\overline{y}) \nsubseteq C(ab)\Big\} \\ &= \bigcup_{b\neq a^{-1}} \big(C'(w)\cap Y(ab)\big)\,. \end{aligned}$$

The set

$$\mathcal{X}'_w := \{\overline{w}\} \cup I'_w \cup T'_w$$

is a finite complete subtree in $\mathcal{T}'$ whose terminal vertices are $\overline{w}$ and T'_w.

Before proceeding to the proof, we remark that this kind of construction will be performed also in other parts of the paper, whenever we need to associate to a closed ball in $\mathcal{T}$ a finite complete subtree in $\mathcal{T}'$ (see for example Lemmas 7.12, 7.13 and 7.14 in § 7.2).

Proof. By definition if $y \in I'_w \setminus \{w\}$, then $\overline{y} \in I'_w$ and if $y \in T'_w$, then $\overline{y} \in I'_w$. This shows in particular that $T'_w \subset T(\mathcal{X}'_w)$. To see that the set of terminal vertices consists of $\{\overline{w}\} \cup T'_w$, observe that if $y \in I'_w$ and $y\alpha \in \mathcal{T}'$ is such that $|y\alpha|' = |y|' + 1$, then by construction either $y\alpha \in I'_w$ or $y\alpha \in T'_w$. $\square$

We are now finally ready to prove the compatibility condition.

Proposition 6.8. *The system (V_a, B_a, H_{ba}) is a compatible matrix system in the sense of* (2.4).

Proof. We need to show that if $v_a \in V_a$, then

$$B_a(v_a, v_a) = \sum_{b \neq a^{-1}} B_b(H_{ba}v_a, H_{ba}v_a)\,. \tag{6.3}$$

Use Proposition (6.5) and Corollary (6.6) to write

$$Y(a) = \bigcup_{\substack{z \in Y(b) \\ b \neq a^{-1}}} \widetilde{(az)}_a = Y_0(a) \cup Y_1(a)\,.$$

By definition of B_a we can write the left hand side of (6.3) as

$$B_a(v_a, v_a) = \sum_{w \in Y_0(a)} B'_w(v'_w, v'_w) + \sum_{w \in Y_1(a)} B'_w(v'_w, v'_w)$$

and, likewise the right hand side as

$$\begin{aligned}
&\sum_{b \neq a^{-1}} B_b(H_{ba}v_a, H_{ba}v_a) = \sum_{b \neq a^{-1}} \sum_{z \in Y(b)} \sum_{w = \widetilde{az}} B'_z(H'_{az,\widetilde{az}}\, v'_w, H'_{az,\widetilde{az}}\, v'_w) = \\
&\sum_{b \neq a^{-1}} \sum_{z \in Y(b)} \sum_{\substack{w = \widetilde{az} \neq az \\ w \in Y_0(a)}} B'_z(H'_{az,\widetilde{az}}\, v'_w, H'_{az,\widetilde{az}}\, v'_w) + \\
&\sum_{b \neq a^{-1}} \sum_{z \in Y(b)} \sum_{\substack{w = \widetilde{az} = az \\ w \in Y_1(a)}} B'_z(v'_w, v'_w)\,,
\end{aligned}$$

where we used the definition of the H_{ba} (6.2).

Write $Y_1(a) = \coprod_{b:\, b \neq a^{-1}} (Y(a) \cap Y(ab))$, a disjoint union. Since, for every $b \neq a^{-1}$, the set $Y(a) \cap Y(ab)$ consists of those elements w of the form $w = az = \widetilde{az}$ for some $z \in Y(b)$, using Lemma 6.2 we get

$$\sum_{w \in Y_1(a)} B'_w(v'_w, v'_w) = \sum_{b \neq a^{-1}} \sum_{z \in Y(b)} \sum_{w = az \in Y_1(a)} B'_{az}(v'_w, v'_w)\,,$$

so that showing (6.3) reduces to showing that

$$\sum_{w \in Y_0(a)} B'_w(v'_w, v'_w) = \sum_{b \neq a^{-1}} \sum_{z \in Y(b)} \sum_{w = \widetilde{az} \in Y_0(a)} B'_z(H'_{az,\widetilde{az}}\, v'_w, H'_{az,\widetilde{az}}\, v'_w)\,.$$

To this purpose, observe that, for any element $w \in Y_0(a)$ there exists a geodesic $[w, wt_b]'$ which starts at the vertex w and ends up in the cone $C(ab)$ for some $b \neq a^{-1}$ (see Proposition 6.5 and Figure 1). This geodesic is "minimal" in the sense that $C'(w\bar{t_b})$ would fail to be in the cone $C(ab)$. The endpoints wt_b of these geodesics, for all possible b, are exactly the terminal points T'_w of the tree $\mathcal{X}'_w$. Hence, for each $w \in Y_0(a)$, by Lemma 4.2 applied to the shadow $\mu[w, v'_w]$ at the point w and the tree $\mathcal{X}'_w$, one has

$$B'_w(v'_w, v'_w) = \sum_{b \neq a^{-1}} B'_{wt_b}(v'_{wt_b}, v'_{wt_b})\ .$$

We need now to compare the two quantities $B'_{wt_b}(v'_{wt_b}, v'_{wt_b})$ and $B'_z(H_{az,\widetilde{az}} v'_w, H_{az,\widetilde{az}} v'_w)$.

By Proposition 6.5 we have seen that such terminal vertices can be written as $wt_b = az$ for some $z \in Y(b)$ and that $\widetilde{az}_a = w$. By definition of H_{ba} one has

$$B'_{wt_b}(v'_{wt_b}, v'_{wt_b}) = B'_z(H_{az,\widetilde{az}} v'_w, H_{az,\widetilde{az}} v'_w)$$

where we have used again Lemma 6.2. Summing over $w \in Y_0(a)$ (or, what is the same, over $az \in Y_0(a)$), we obtain the desired assertion. □

Let now π be the left regular action of $\mathbb{F}_A$ on $\mathcal{H}^\infty(V_a, H_{ba})$ and let $\mathcal{H}(V_a, H_{ba}, B_a)$ be the completion of $\mathcal{H}^\infty(V_a, H_{ba})$ with respect to the norm induced by the (B_a).

We define now the intertwining operator

$$U : \mathcal{H}^\infty(V'_\alpha, H'_{\beta\alpha}, B'_\alpha) \to \mathcal{H}^\infty(V_a, H_{ba}, B_a)\,.$$

For every $f \in \mathcal{H}^\infty(V'_\alpha, H'_{\beta\alpha})$ and a reduced word xa in the alphabet A we set

$$(Uf)(xa) := \sum_{y \in Y(xa)} f(y)\,.$$

To see that U intertwines π' to π fix any $y \in \mathcal{V}$ and assume that $|y| \leq |x| + 1$. For any such x and y one has

$$\begin{aligned}\pi(y)Uf(xa) = Uf(y^{-1}xa) &= \sum_{z\in Y(y^{-1}xa)} f(z) = \sum_{z\in y^{-1}Y(xa)} f(z) \\ &= \sum_{u\in Y(xa)} f(y^{-1}u) = U\big(\pi'(y)f\big)(xa)\end{aligned}$$

since $Y(y^{-1}xa) = y^{-1}Y(xa)$ if $|y| \leq |x|+1$. It follows that $U\pi'(y)f(xa)$ and $\pi(y)Uf(xa)$ differ only for a finite set of values of x, and hence are equal in $\mathcal{H}^\infty(V_a, H_{ba})$.

We conclude with the following

Theorem 6.9. *U is unitary.*

Proof. Assume that $f \in \mathcal{H}^\infty(V'_\alpha, H'_{\beta\alpha})$ is multiplicative for $|y|' \geq N$. We may also assume that f is zero if $|y'| \leq N-1$. By the discussion after Lemma 6.2 there exists an integer k such that $|y| \leq k$ whenever $|y|' \leq N$. Define

$$S_k^0 = \{z \in \mathcal{V} : \ C'(z) \nsubseteq C(x) \text{ for all } x \text{ with } |x| = k\}$$

and

$$\mathcal{S}'(k) = \{e\} \cup S_k^0 \cup \bigcup_{\substack{x\in\mathcal{V}\\ |x|=k}} Y(x)\,.$$

Arguing as in the proof of Proposition 6.7 one can show that $\mathcal{S}'(k)$ is a finite complete subtree in $\mathcal{T}'$ whose terminal vertices are the elements of $Y(x)$ for all x with $|x| = k$. Since every y belongs to $C'(y)$, we see that $\mathcal{S}'(k)$ contains the ball of radius N about the origin in $\mathcal{T}'$. Use now Lemma 4.3 to conclude the proof. $\square$

We conclude this section with an example illustrating the effect of a nontrivial change of generators on a given multiplicative representation.

Example 6.10. Let $\Gamma = \mathbb{F}_A$, where $A = \{a, b, a^{-1}, b^{-1}\}$. Consider the change of generators given by $\alpha = a$ and $\beta = ab$ and let π_s be the spherical series representation of Figà–Talamanca and Picardello [FTP82] constructed from the set of generators $A' = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$. Denote by a', b' the generic elements of A'. In [KS04] it is shown that π_s can be realized as a multiplicative representation with respect to the following matrix system:

$$\begin{aligned} V_{a'} &= \mathbf{C} && \forall a' \in A' \\ H_{b'a'} &= 3^{-\frac{1}{2}+is} =: \lambda && \forall a', b' \in A' \\ B_{a'}(v,v) &= \frac{|v|^2}{4} && . \end{aligned}$$

Moreover, in [PS96] it is also shown that it is impossible to realize π_s as any spherical representation arising from the generators a and b. We show here that it is however possible to realize π_s as a multiplicative representation with respect to the other generators a and b. In fact one can verify that

$$\begin{aligned} Y(a) &= \{\alpha, \beta\} \\ Y(b) &= \{\alpha^{-1}\beta\} \\ Y(a^{-1}) &= \{\alpha^{-2}, \alpha^{-1}\beta^{-1}\} \\ Y(b^{-1}) &= \{\beta^{-1}\} \end{aligned}$$

According to Definition 6.4 the spaces V_a and $V_{a^{-1}}$ are two dimensional while $V_b = V_{b^{-1}} = \mathbf{C}$. The matrices appearing in (6.2) are:

$$\begin{aligned} H_{aa} = H_{a^{-1}a^{-1}} &= \begin{pmatrix} \lambda & 0 \\ \lambda & 0 \end{pmatrix} \\ H_{ba^{-1}} = H_{b^{-1}a} &= \begin{pmatrix} \lambda & 0 \end{pmatrix} \\ H_{ba} = H_{b^{-1}a^{-1}} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \\ H_{bb} = H_{b^{-1}b^{-1}} &= \lambda^2 \\ H_{ab} = H_{a^{-1}b^{-1}} &= \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \\ H_{ab^{-1}} = H_{a^{-1}b} &= \begin{pmatrix} \lambda^2 \\ \lambda^2 \end{pmatrix} \end{aligned}$$

.

Let W_a (respectively $W_{a^{-1}}$) denote the subspace of V_a (respectively $V_{a^{-1}}$) generated by the vector $(1,1)$. The reader can verify that the subspaces W_a, $W_{a^{-1}}$, $W_b = V_b = \mathbf{C}$ and $W_{b^{-1}} = V_{b^{-1}} = \mathbf{C}$ constitute an invariant subsystem and that the quotient system has Perron–Frobenius eigenvalue zero. According to Lemma 4.4 the representation π_s is equivalent to the multiplicative representation constructed from the subsystem W.

7. Stability Under Restriction and Unitary Induction

In this section the set A of generators for Γ is fixed once and for all. As before, we write $\bar{x}$ for the (reduced) word obtained from x by deleting the last letter of the reduced expression for x. Set also $\bar{a} = e$ if a belongs to A.

Definition 7.1. A Schreier system S in Γ is a nonempty subset of Γ satisfying the following conditions:

(1) $e \in S$;
(2) if $x \in S$, then $\bar{x} \in S$.

Assume that Γ' is a subgroup of finite index in Γ. Essential in the following will be a choice of an appropriate fundamental domain D for the action of Γ' on the Cayley graph of Γ with respect to a symmetric set of generators A. It is well known (see for example [Mas77, Chapter VI]) that one can choose in Γ a set S' of representatives for the left cosets $\Gamma'\gamma$ which is also a Schreier set. Identifying S' with an appropriate set of vertices D of $\mathcal{T}$, it turns out that D has the following properties:

- D is a finite subtree containing e.
- D is a fundamental domain with respect to the left action on the vertices of $\mathcal{T}$ in the sense that the set of vertices of $\mathcal{T}$ is the disjoint union of the subtrees $x'D$ with $x' \in \Gamma'$.

We shall refer to every such D as to a *fundamental subtree.*

Corresponding to that choice of D one has also a natural choice of generators for Γ', namely one can prove that Γ' is generated by the set

$$A' := \left\{ a'_j \in \Gamma : \ d(D, a'_j D) = 1 \right\} . \tag{7.1}$$

We shall assume in this Section that D is a fixed fundamental subtree and that A' is the corresponding generating set defined as in 7.1. We write $a', b', \dots$ to denote a generic element of A'.

The following lemma summarizes the properties of the translates of D which will be used in several occasions to build finite complete subtrees.

Lemma 7.2. *Let $\gamma' a' \neq e$ be a reduced word in Γ'.*

(1) *There exists $x \in \Gamma$ such that $\gamma' a' D \subset C(x)$ but $\gamma' D \not\subset C(x)$. Moreover $\gamma' a' b' D \subset C(x)$ for all b' such that $a'b' \neq e$.*
(2) *The geodesic in $\mathcal{T}$ connecting $\gamma' a' D$ and e crosses $\gamma' D$.*

Proof. Let $a' \in A'$ be a generator of Γ' and D a fundamental subtree. Let $x(a') \in a'D$ be the vertex of $a'D$ closest to D. Since the distance between D and $a'D$ is one, there exists a unique edge $(x, x(a'))$ such that $x \in D$ and $x(a') \in a'D$. We claim that $a'D \subset C(x(a'))$. Assume the contrary: namely assume that there exists $v \in a'D$ whose reduced word does not start with $x(a')$. Since $a'D$ is a subtree it must contain the geodesic $[v, x(a')]$ connecting v to $x(a')$, but this is impossible since $x \in [v, x(a')]$. Let $b' \in A'$ be such that $a'b' \neq e$. Denote by (w, w') ($w \in a'D$, $w' \in a'b'D$) the unique edge connecting $a'b'D$ to $a'D$. If $a'b'D \not\subset C(x(a'))$ it must be $w = x(a')$ and $w' = x$, which is impossible. By induction one has $a'\gamma'D \subset C(x(a'))$ for every γ' so that $a'\gamma' = 1 + |\gamma'|$.

Let now $\gamma'a'$ be a reduced word in Γ' and let $x(\gamma'a')$ denote the vertex of $\gamma'a'D$ closest to D. Translating the picture by γ'^{-1} one can see that $\gamma'^{-1}x(\gamma'a') = x(a')$, that is

$$x(\gamma'a') = \gamma'x(a')\,. \tag{7.2}$$

Since we have

$$\gamma'a'D \subset \gamma'C(x(a'))$$

(1) will be proved as soon as we show that $\gamma'C(x(a')) = C(\gamma'x(a'))$. Let d'^{-1} denote the last letter of γ', so that $d'^{-1} \neq a'$. Since the two subtrees $d'D$ and $a'D$ are both at distance one from D they cannot be contained in the same cone: so that neither $x(a')$ is the first part of $x(d')$ nor the converse. In particular $x(a')$ does not belong to the geodesic, in $\mathcal{T}$, $[e, \gamma'^{-1}]$ so that, by Lemma 6.1, $\gamma'C(x(a')) = C(\gamma'x(a'))$.

To complete the proof observe that, since $a'b'D \subset C(x(a'))$ and $e \in D$, the geodesic connecting D and $a'b'D$ must cross $x(a')$.

$\square$

7.1. **Stability Under Restriction.** The goal of this section is to prove Theorem 3(1).

Choose D and A' as in Definition 7.1. Although D is a finite subtree, it is not complete. The strategy of the proof consists of completing D to a complete subtree D', then translating D' by a generator of Γ', so that most of it (in fact, all of it with the exception of the unique edge closer to the identity) is contained in a cone at distance one from D. A wise definition of $(V_{a'}, H_{b'a'})$ and $B_{a'}$, together with the help of a shadow supported on the cone, will provide the construction of a matrix system with inner product for the subgroup Γ'.

Let $x(a')$ denote the vertex of $a'D$ closest to D, as in the proof of Lemma 7.2. Let D' be the subtree obtained by adding to D the vertices $x(a')$ (and the relative edges) corresponding to all $a' \in A'$. Write $x(a')$ in the generators of Γ and denote by $q(a')$ the last letter of its reduced expressions, that, with the notation used in (4.1), we have that $q(a') = t(x(a'))$.

Lemma 7.3. *Let D, D', $x(a')$ as above.*

(1) *The subtree D' is complete and its terminal vertices consist of exactly all the $x(a')_{a'\in A'}$.*
(2) *For every $a', b' \in A'$, the vertex of $a'b'D$ closest to $a'D$ is $a'x(b')$.*
(3) *Assume that $a'b' \neq e$. Then the geodesic joining e and $a'x(b')$ crosses $x(a')$ and the last letter of $a'x(b')$ is $q(b')$.*

Proof. (1) Let $v \in D$ and assume that v' is a neighbor of v. If $v' \notin D$ there exists $x' \in \Gamma'$ and $u \in D$ such that $v' = x'u$. Hence the distance between D and $x'D$ is one: this implies that $x' = a'$ for some $a' \in A'$ and $v' = x(a')$. This proves that every vertex of D is an interior vertex of D'. Choose now any $x(a')$ and consider its $q+1$ neighbors: one of them belongs to D and the others, being at distance two from D, cannot be in D'. This proves that D' is complete with terminal vertices $x(a')_{a'\in A'}$.
(2) follows immediately from (7.2). In particular the vertex of $a'b'D$ closest to $a'D$ is $a'x(b') = x(a'b')$.
(3) By Lemma 7.2, the geodesic joining e and $x(a'b')$, crosses $x(a')$. In terms of the generators of Γ this means that $x(a')$ is the first piece of the reduced word for $a'x(b')$ and, in particular, passing from $x(a')$ to $a'x(b')$, the last letter of $x(a')$ is not canceled. To prove the second assertion, observe that e does not belong to $x(b')^{-1}(a')^{-1}D$. In fact, if it did, one would have $e = x(b')^{-1}(a')^{-1}\xi_0$ for some $\xi_0 \in D$: but since we also have $x(b') = b'\xi_1$ this would imply that $\xi_0 = \xi_1$ and $b' = (a')^{-1}$. Hence the subtree $x(b')^{-1}(a')^{-1}D$ is contained in the cone $C(c)$ for some $c \in A$. Since

$$d(x(b')^{-1}D, x(b')^{-1}(a')^{-1}D) = d(D, (a')^{-1}D) = 1\,,$$

the subtree $x(b')^{-1}D$ is at distance one from $x(b')^{-1}(a')^{-1}D$. This is possible only in two ways: either $x(b')^{-1}D$ is contained in $C(c)$ or $x(b')^{-1}D$ contains the identity e. The second possibility is ruled out because $x(b') \notin D$. This implies that the last letter of $x(b')$ is the same as the last letter of $a'x(b')$. □

We collect here the results as they will be needed later.

Corollary 7.4. *With the above notation the subtree $a'D'$ is a non-elementary tree based at $x(a')$ whose terminal vertices are $T(a'D') = \{a'x(b') : b' \in A'\}$. The terminal vertex closest to e is $a'x(a'^{-1})$, so that*

$$T_e(a'D') = \{a'x(b') : b' \in A',\, a'b' \neq e\}$$

and

$$a'x(b') = x(a')a_1a_2\dots a_kq(b') = a'x(a'^{-1})\dots q(a')a_1a_2\dots a_kq(b') \tag{7.3}$$

is the reduced expression of $a'x(b')$ in the alphabet A.

We are now ready to define the matrix system $(V_{a'}, H_{b'a'})$.

Definition 7.5. With (7.3) in mind, we set

$$V_{a'} := V_{q(a')}\,, \quad \text{and}$$
$$H_{b'a'} := \begin{cases} H_{q(b')a_k} \dots H_{a_2a_1} H_{a_1q(a')} & \text{if } b'a' \neq e \\ 0 & \text{if } b'a' = e\,. \end{cases}$$

Lemma 7.6. *The tuple $(B_{a'})_{a'\in A'}$ defined by*

$$B_{a'} := B_{q(a')}$$

is compatible with the matrix system $(V_{a'}, H_{b'a'})$.

Proof. We need to prove that, for every $v_{a'} \in V_{a'}$

$$B_{a'}(v_{a'}, v_{a'}) = \sum_{b' :\, a'b' \neq e} B_{b'}(H_{b'a'}v_{a'}, H_{b'a'}v_{a'})\,. \tag{7.4}$$

Let $\mu[x(a'), v_{a'}]$ be the shadow as in Definition 2.6. Since by definition

$$B_{a'}(v_{a'}, v_{a'}) = \left\|\mu[x(a'), v_{a'}](x(a'))\right\|^2,$$

showing (7.4) is equivalent to showing that

$$\left\|\mu[x(a'), v_{a'}](x(a'))\right\|^2 = \sum_{b' :\, a'b' \neq e} \left\|H_{b'a'}\mu[x(a'), v_{a'}](x(a'))\right\|^2.$$

Moreover, since $\mu[x(a'), v_{a'}]$ is multiplicative, according to the definition of $H_{b'a'}$ we have

$$\mu[x(a'), v_{a'}](a'x(b')) = H_{b'a'}\mu[x(a'), v_{a'}](x(a'))\,. \tag{7.5}$$

By Lemma 4.2, Corollary 7.4 and (7.5) it follows that

$$\begin{aligned} \left\|\mu[x(a'), v_{a'}](x(a'))\right\|^2 &= \sum_{t\in T_e(a'D')} \left\|\mu[x(a'), v_{a'}](t)\right\|^2 \\ &= \sum_{b' :\, b'a' \neq e} \left\|\mu[x(a'), v_{a'}](a'x(b'))\right\|^2 \\ &= \sum_{b' :\, a'b' \neq e} \left\|H_{b'a'}\mu[x(a'), v_{a'}](x(a'))\right\|^2, \end{aligned}$$

which completes the proof. □

We need to define now the intertwining operator between the restriction $\pi|_{\Gamma'}$ to Γ' of the representation π on $\mathcal{H}(V_a, H_{ba}, B_a)$ and the representation ρ of Γ' on $\mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})$ defined by

$$\rho(x')f(y') := f(x'^{-1}y')\,,$$

for $x', y' \in \Gamma'$ and $f \in \mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})$.

Definition 7.7. Let $f \in \mathcal{H}^\infty(V_a, H_{ba}, B_a)$. If $x' = y'a' \in \Gamma'$ with $a' \in A'$ and $|x'|_{\Gamma'} = |y'|_{\Gamma'} + 1$ (in the word metric with respect to the generators A'), define

$$(Uf)(x') := f\big(y'x(a')\big)\,.$$

Proof of Theorem 3(1). It is easy to check that the operator U maps the restriction to Γ' of multiplicative functions in $\mathcal{H}^\infty(V_a, H_{ba}, B_a)$ to multiplicative functions in $\mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'})$. In fact, if $x' = y'a' \in \Gamma$ with $a' \in \Gamma'$ and $|x'|_{\Gamma'} = |y'|_{\Gamma'} + 1$, then, By Lemma 7.3 (3)

$$(Uf)(x') = f\big(y'x(a')\big) \in V_{t(y'x(a'))} = V_{q(a')}\,.$$

Moreover, if $y'a'b' \in \Gamma'$ with $a', b' \in A'$ and $|y'a'b'|_{\Gamma'} = |y'|_{\Gamma'} + 2$, then

$$(Uf)(y'a'b') = f\big(y'a'x(b')\big) = H_{b'a'}\big(f(y'a')\big)\,.$$

Furthermore, it is straightforward to check that

$$U\big(\pi|_{\Gamma'}(x')f\big) = \rho(x')(Uf)\,,$$

thus completing the proof. $\square$

7.2. **Stability Under Unitary Induction.** The goal of this section is to prove Theorem 3(2).

Let $\Gamma' \leq \Gamma$ be a subgroup of finite index and let D be a fundamental subtree for the action of Γ' on $\mathcal{T}$. By Theorem 2 we may assume that A' is the generating set of Γ' corresponding to D as in (7.1).

Suppose that we are given a matrix system with inner products $(V_{a'}, H_{b'a'}, B_{a'})$ relative to Γ' and hence a representation π' of the class $\mathbf{Mult}(\Gamma')$ acting on $\mathcal{H}_s := \mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})$. Because of Theorem 1 we may always assume that the system is irreducible. Let $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\pi')$ denote the induced representation acting on $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s)$. We recall that

$$\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s) := \big\{f : \Gamma \to \mathcal{H}_s :\ \pi'(h)f(g) = f(gh^{-1}),\ \text{for all } h \in \Gamma', g \in \Gamma\big\}\,,$$

on which Γ acts by

$$\big(\mathrm{Ind}_{\Gamma'}^{\Gamma}(\pi')(g_0)f\big)(g) := f({g_0}^{-1}g)\,,$$

for all $g_0, g \in \Gamma$. In particular $f(g)$ is uniquely determined by its values on a set of representatives for the right cosets of Γ' in Γ, which can be taken to be the set $D^{-1} = \{u^{-1} :\ u \in D\}$.

Denote by $\mathcal{H}_s^\infty := \mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'})$ the dense subspace $\mathcal{H}_s$ consisting of multiplicative functions and define, with a slight abuse of notation, the dense subset

$$\begin{aligned}\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s^\infty) := \big\{f : \Gamma \to \mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'}) :\ &\pi'(h)f(g) = f(gh^{-1}),\\ &\text{for all } h \in \Gamma', g \in \Gamma\big\}\end{aligned}$$

which, by definition of $\mathcal{H}_s^\infty$, can be identified with

$$\operatorname{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s^\infty) \cong \Big\{\varphi : D^{-1}\cdot\Gamma' \to \coprod_{a'\in A'} V_{a'} : \pi'(h)\varphi(g) = \varphi(gh^{-1}),$$
$$\text{for all } h\in\Gamma', g\in\Gamma \text{ and } \varphi \text{ is multiplicative as a function of } \Gamma'\Big\}$$

via the map $f \mapsto \Phi(f)$, where $\Phi(f)(x) := f(u^{-1})(h)$, for $x = u^{-1}h$, with $h \in \Gamma'$ and $u \in D$. The invariance property of functions in $\operatorname{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s^\infty)$ imply that $\Phi(f)$ is well defined.

We want to show that there exists a matrix system with inner product (V_a, H_{ba}, B_a) on Γ so that $\operatorname{Ind}_{\Gamma'}^{\Gamma}(\pi')$ is equivalent to a multiplicative representation π on $\mathcal{H}(V_a, H_{ba}, B_a)$. The vector spaces V_a will be direct sums of possibly multiple copies of the vector spaces $V_{a'}$, according to some appropriately chosen "coordinates" on subsets of the cones $C(a)$. To this purpose, let us define for any generator a of Γ, the set

$$P(a) = (D^{-1}\cdot A')\cap C(a)\,.$$

The following lemma is technical, but only specifies the multiplicative property of the chosen coordinates.

Lemma 7.8. *Let us fix $a \in A$ and $v \in D$.*

1. *Assume that $va^{-1} \in D$ and let $c' \in A'$ be any generator. Then $av^{-1}c' \in P(a)$ if and only if $v^{-1}c' \in P(b)$ for some $b \in A$ with $ab \neq e$.*
2. *Assume that $va^{-1} \notin D$. Then*
   - (a) *there exists $c' \in A'$ and $u \in D$ such that $av^{-1} = u^{-1}c' \in P(a)$;*
   - (b) *furthermore for every $d' \in A'$ such that $c'd' \neq e$, there exists a unique $b \in A$ with $ab \neq e$ such that $v^{-1}d' \in P(b)$.*

Proof. (1) Let $b \in A$ be such that $v^{-1}c' \in P(b)$. Then in particular $v^{-1}c'$ starts with b and hence $av^{-1}c' \in C(a)$ if $ab \neq e$. Since by hypothesis $va^{-1} \in D$, it follows that $av^{-1}c' \in P(a)$.

Conversely, let $b \in A$ be such that $v^{-1}c' \in C(b)$. Since $av^{-1}c' \in P(a)$, it follows that $ab \neq e$. Moreover, since $v \in D$, we have that $v^{-1}c' \in P(b)$.

(2a) Since $v \in D$ but $va^{-1} \notin D$ and D is a Schreier system, then $|va^{-1}| = |v| + 1$, that is $d(va^{-1}, D) = 1$. By (7.1), there exist $u \in D$ and $(c')^{-1} \in A'$ such that $va^{-1} = (c')^{-1}u$, from which it follows that $av^{-1} = u^{-1}c' \in P(a)$.

(2b) Choose $d' \in A'$. By (7.1), D and $d'D$ are disjoint subtrees at distance one from each other. We claim that if $d' \neq (c')^{-1}$, neither of their translates $av^{-1}D$ and $av^{-1}d'D$ contains the identity e. In fact, if e were to belong to $av^{-1}D$, we would have that $va^{-1} \in D$, which is excluded by hypothesis. If on the other hand e were to belong to $av^{-1}d'D$, then we would have that for some $u_0 \in D$, $av^{-1} = u_0^{-1}(d')^{-1}$. But by (2a) we know that $av^{-1} = u^{-1}c'$, so that, by uniqueness of the decomposition, one would conclude that $c' = (d')^{-1}$, which is also excluded by hypothesis.

Hence both subtrees are contained in some cone $C(b)$, where $b \in A$ and, since they are at distance one from each other, this cone must be the same for both. But since $v \in D$, then $a \in av^{-1}D$, so that $av^{-1}D$, and hence $av^{-1}d'D$, are contained in $C(a)$.

Since $e \in D$, this means in particular that $av^{-1}d' \in C(a)$, so that $v^{-1}d' \in C(b)$ for some b such that $ab \neq e$. Hence $v^{-1}d' \in P(b)$. $\square$

We are now ready to define the matrix system (V_a, H_{ba}).

Definition 7.9. For every $u \in D$ and a in A let $V_{u,a}$ be the direct sum of the spaces $V_{c'}$ for all c' such that $u^{-1}c'$ belongs to $P(a)$, namely

$$V_{u,a} := \bigoplus \left\{ V_{c'} : \quad c' \in A' \text{ and } u^{-1}c' \in P(a) \right\},$$

and set

$$V_a := \bigoplus_{u \in D} V_{u,a} = \bigoplus \left\{ V_{c'} : \; u \in D,\, c' \in A' \text{ and } u^{-1}c' \in P(a) \right\}. \tag{7.6}$$

In other words, we can think of the V_a's as consisting of blocks, corresponding to elements $u \in D$ each of them containing a copy of $V_{c'}$ whenever $u^{-1}c' \in P(a)$. With this definition of the V_a's, we can now define a map

$$U : \mathrm{Ind}_{\Gamma'}^{\Gamma} \left(\mathcal{H}_s^\infty(V_{a'}, H_{b'a'}) \right) \to \left\{ \Gamma \to \bigoplus_{a \in A} V_a \right\}$$

with the idea in mind that the target will have to be the space of multiplicative functions on some matrix system with inner product (V_a, H_{ba}, B_a). Fix $a \in A$ and let $u^{-1}c' \in P(a)$. Then for all $x \in \Gamma$ such that $|xa| = |x| + 1$ and for $f \in \mathrm{Ind}_{\Gamma'}^{\Gamma} \left(\mathcal{H}_s^\infty(V_{a'}, H_{b'a'}) \right)$, we define $Uf(xa)$ to be the vector whose (u, c')-component is given by

$$Uf(xa)_{u,c'} := \Phi(f)(xu^{-1}c')$$

or, equivalently,

$$Uf(xa) = \bigoplus_{(u,c')} f(xu^{-1})(c') \tag{7.7}$$

It is not difficult to convince oneself on how to construct the linear maps H_{ba} so that the functions Uf will be multiplicative: we give here an explanation, and one can find the formula in (7.8).

Since the functions Uf will have to be multiplicative, if $|xab| = |x|+2$ they will have to satisfy

$$f(xav^{-1}d') = (Uf)(xab)_{v,d'} = \big(H_{ba}(Uf)(xa)\big)_{v,d'}$$

whenever $v^{-1}d' \in P(b)$ for some $H_{ba} : V_a \to V_b$ to be specified. Thinking of the "block decomposition" alluded to above, the linear maps H_{ba} will also be block matrices that will perform three kinds of operations on a vector $w_a \in V_a$ with coordinates $w_a = (w_{u,c'})_{u^{-1}c' \in P(a)}$.

- If there exists $d' \in A'$ such that for some $v \in D$, $av^{-1}d' \in P(a)$ and $v^{-1}d' \in P(b)$, (see Lemma 7.8(1)), then H_{ba} will just move the (va^{-1}, d')-component of w_a to the (v, d')-component of $H_{ba}w_a$. According to Lemma 7.8(1) this happens precisely when $va^{-1} \in D$.
- If $va^{-1} \notin D$, write, as in Lemma 7.8(2a), $av^{-1} = u^{-1}c'$ and take any d' with $c'd' \neq e$. If b is such that $v^{-1}d' \in P(b)$ (cf. Lemma 7.8(2b)) then $H_{ba}|_{V_{u,c'}} : V_{u,c'} \to V_{v,d'}$ will be nothing but $H_{d'c'}$.
- In all other cases H_{ba} will be set equal to zero .

More precisely we define

(7.8)
$$(H_{ba}w_a)_{v,d'} := \begin{cases} (w_a)_{va^{-1},d'} & \text{if } va^{-1} \in D \\ H_{d'c'}(w_a)_{u,c'} & \text{if } va^{-1} \notin D \text{ and } av^{-1} = u^{-1}c' \\ 0 & \text{otherwise}\,. \end{cases}$$

That this makes sense follows directly from Lemma 7.8 as we explained above.

The definition of a tuple of positive definite forms is now obvious, namely the (u, c')-component of B_a is given by the following

$$(B_a)_{u,c'} := B_{c'} \qquad \text{where } u^{-1}c' \in P(a) \tag{7.9}$$

Proposition 7.10. *The tuple $(B_a)_{a\in A}$ is compatible with the system H_{ba} defined in* (7.8).

Proof. We must check that, for every $w_a \in V_a$ one has

$$B_a(w_a, w_a) = \sum_{b:\, ab\neq e} B_b(H_{ba}w_a, H_{ba}w_a)\,.$$

Remembering that, by definition of V_a and B_a

$$B_a(w_a, w_a) = \sum_{u\in D} \sum_{u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big)\,, \tag{7.10}$$

we must prove that

$$\begin{aligned} &\sum_{u\in D} \sum_{u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big) = \\ &\sum_{b:\ ab\neq e} \sum_{v\in D} \sum_{v^{-1}d'\in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big)\,. \end{aligned} \tag{7.11}$$

Fix a in A and define

$$D_a = \{u \in D:\ u = va^{-1} \text{ for some } v \in D\}\,,$$

so that

$$D_a \cdot a = \{v \in D:\ v = ua \text{ for some } u \in D_a\}$$

is in bijective correspondence with D_a. Viceversa, to every $v \in D\backslash D_a{\cdot}a$, corresponds a unique $u \in D \setminus D_a$ and $c' \in A'$ such that $av^{-1} = u^{-1}c'$.

Split the sums on each side of (7.11) to obtain

$$\begin{aligned} &\sum_{u\in D_a} \sum_{u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big) \\ &+ \sum_{u\in D\backslash D_a} \sum_{u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big) \\ &= \sum_{v\in D_a\cdot a} \sum_{b:\ ab\neq e} \sum_{v^{-1}d'\in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big) \\ &+ \sum_{v\in D\backslash D_a\cdot a} \sum_{b:\ ab\neq e} \sum_{v^{-1}d'\in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big)\,. \end{aligned} \tag{7.12}$$

We will show the equality

$$\begin{aligned} &\sum_{u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big) \\ &= \sum_{b:\ ab\neq e} \sum_{v^{-1}d'\in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big) \end{aligned} \tag{7.13}$$

in the two cases

(1) $u \in D_a$ and $v = ua \in D_a \cdot a$,
(2) $u \notin D_a$ and $v \neq ua$ but $v \in D$.

Then (7.12) will follow by summing (7.13) once over D_a and once over $D \setminus D_a$ and adding the resulting equations.

(1) Let $u \in D_a$, $v = ua$ and $u^{-1}c' \in P(a)$ for some $c' \in A'$. According to Lemma 7.8(1) one has $v^{-1}c' \in P(b)$ for some $b \neq a^{-1}$. The definition of the matrices H_{ba} implies that

$$\begin{aligned}&B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big) =\\ &B_{c'}\big((w_a)_{va^{-1},c'}, (w_a)_{va^{-1},c'}\big) =\\ &B_{c'}\big((H_{ba}w_a)_{v,c'}, (H_{ba}w_a)_{v,c'}\big)\,,\end{aligned}$$

so that

$$\begin{aligned}&\sum_{c'\colon u^{-1}c'\in P(a)} B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big)\\ &= \sum_{c'\colon av^{-1}c'\in P(a)} B_{c'}\big((w_a)_{va^{-1},c'}, (w_a)_{va^{-1},c'}\big)\\ &= \sum_{b\colon ab\neq e}\ \sum_{c'\colon v^{-1}c'\in P(b)} B_{c'}\big((H_{ba}w_a)_{v,c'}, (H_{ba}w_a)_{v,c'}\big)\end{aligned}$$

which proves (7.13) in the case $u \in D_a$ and $v = ua \in D_a \cdot a$.

(2) Fix now any v in $D \setminus D_a \cdot a$ and write $av^{-1} = u^{-1}c'$ (Lemma 7.8(2a)). Choose any d' with $c'd' \neq e$ and let $b \in A$ with $ab \neq e$ be the unique b such that $v^{-1}d' \in P(b)$ (Lemma 7.8(2b)) By definition of H_{ba}

$$(H_{ba}w_a)_{v,d'} = H_{d'c'}(w_a)_{u,c'}\,.$$

To every b corresponds a subset A'_b of A' consisting of all d' such that $v^{-1}d'$ belongs to $P(b)$ and we observed before that $\bigcup_b A'_b = A' \setminus (c')^{-1}$. Hence

$$\begin{aligned}&\sum_{b\colon ab\neq e}\ \sum_{v^{-1}d'\in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big) =\\ &\sum_{b\colon ab\neq e}\ \sum_{d'\in A'_b} B_{d'}(H_{d'c'}(w_a)_{u,c'}, H_{d'c'}(w_a)_{u,c'}) =\\ &\sum_{d'\in A'\setminus(c')^{-1}} B_{d'}(H_{d'c'}(w_a)_{u,c'}, H_{d'c'}(w_a)_{u,c'}) = B_{c'}((w_a)_{u,c'}, (w_a)_{u,c'})\,,\end{aligned}$$

where the last equality is nothing but the compatibility of the $(B_{a'})$. In particular to every v in $D \setminus D_a \cdot a$ corresponds a unique u in $D \setminus D_a$

and a unique $c' \in A'$ such that $u^{-1}c' \in P(a)$ and

$$\sum_{b:\, ab \neq e} \sum_{v^{-1}d' \in P(b)} B_{d'}\big((H_{ba}w_a)_{v,d'}, (H_{ba}w_a)_{v,d'}\big) \\ = B_{c'}\big((w_a)_{u,c'}, (w_a)_{u,c'}\big)\,,$$

which proves (7.13) in the case $v \neq ua$. □

The upshot of the above discussion is that we have shown that the map U takes values in the space of multiplicative functions. We still need to show that U is an unitary operator and hence it extends to a unitary equivalence between $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}(V_{a'}, H_{b'a'}, B_{a'}))$ and $\mathcal{H}(V_a, H_{ba}, B_a)$. The following theorem will complete the proof.

Theorem 7.11. *Let V_a, H_{ba} and B_a be as in* (7.6), (7.8) *and* (7.9) *and let*

$$U : \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'})) \to \mathcal{H}^\infty(V_a, H_{ba}, B_a)$$

be as in (7.7). *Then U is an unitary operator and hence it extends to a unitary equivalence*

$$U : \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})) \to \mathcal{H}(V_a, H_{ba}, B_a)\,.$$

Proof. Let us simply write as before $\mathcal{H}_s^\infty$ for $\mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'})$ and $\mathcal{H}^\infty$ for $\mathcal{H}^\infty(V_a, H_{ba}, B_a)$.

For every $f \in \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s^\infty)$ we have by definition of the induced norm that

$$\|f\|^2_{\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathcal{H}_s^\infty)} = \sum_{u \in D} \|f(u^{-1})\|^2_{\mathcal{H}_s^\infty}\,,$$

and, since the above sum is orthogonal, we may assume that f is supported on $z \cdot \Gamma'$ for some $z \in D^{-1}$.

For such an f it will be hence enough to show that

$$\|Uf\|^2_{\mathcal{H}^\infty} = \|f(z)\|^2_{\mathcal{H}_s^\infty}\,.$$

Using the definition of the norm in (2.3) as well as the definitons of U in (7.7) and of B_a in (7.10) we obtain that for N large enough

$$\begin{aligned}\|Uf\|^2_{\mathcal{H}^\infty} &= \sum_{a \in A} \sum_{\substack{|x|=N \\ |xa|=|x|+1}} B_a\big(Uf(xa), Uf(xa)\big) \\ &= \sum_{a \in A} \sum_{\substack{|x|=N \\ |xa|=|x|+1}} \sum_{u^{-1}c' \in P(a)} B_{c'}\big(f(xu^{-1})(c'), f(xu^{-1})(c')\big)\,.\end{aligned}$$

Since $f(z) \in \mathcal{H}_s^\infty$, there exists $M > 0$ such that $f(z)$ is multiplicative outside the ball $B'(e, M)$ in $\mathcal{T}'$ of radius M. To complete the proof it will be hence enough to show the following

Lemma 7.12. *There exists a finite complete subtree $\mathcal{S}' \subset \mathcal{T}'$ containing $B'(e, M)$ whose terminal elements are*

$$T(\mathcal{S}') = \{\gamma' = z^{-1}xy \in \Gamma' : \, |x| = N, \, |xa| = N + 1, \, y \in P(a)\} .$$

Observe that since, according to the above lemma, $\gamma' \in T(\mathcal{S}')$ has the form $\gamma' = z^{-1}xu^{-1}c'$ with $u \in D$ and $c' \in A'$, the invariance property of f translates into the equality

$$f(z)(\gamma') = f(xu^{-1})(c') .$$

From this in fact, using Lemma 4.3 and denoting $\overline{\gamma'}$ to be as before the reduced word obtained by deleting the last letter (in Γ') of γ', we deduce that

$$\begin{aligned}
\|f(z)\|^2_{\mathcal{H}_s^\infty} &= \sum_{\substack{\gamma' \in T(\mathcal{S}') \\ \gamma' = \overline{\gamma'}c'}} B_{c'}\big(f(z)(\gamma'), f(z)(\gamma')\big) \\
&= \sum_{a \in A} \sum_{\substack{|x| = N \\ |xa| = |x|+1}} \sum_{u^{-1}c' \in P(a)} B_{c'}\big(f(xu^{-1})(c'), f(xu^{-1})(c')\big) ,
\end{aligned}$$

thus concluding the proof. □

We need now to show Lemma 7.12. We start recording the following obvious fact, which follows immediately from the observation that left translates of D are subtrees (hence convex) and that cones are disjoint and convex.

Lemma 7.13. *Let $\Gamma' \leq \Gamma$ be a subgroup of a free group with associated trees $\mathcal{T}'$, $\mathcal{T}$. Let D be a fundamental subtree in $\mathcal{T}$ for the action of Γ'. Then for any $w \in \Gamma$ and $N > |w|$ we can write*

$$\mathcal{T} = w\, B(e, N) \sqcup \bigsqcup_{\substack{|x| = N \\ |xa| = N+1}} w\, C(xa)$$

and

$$\begin{aligned}
\mathcal{T}' = &\big\{\gamma' \in \Gamma' : \, \gamma' D \,\cap\, w\, B(e, N) \neq \emptyset\big\} \sqcup \\
&\sqcup \bigsqcup_{\substack{|x| = N \\ |xa| = N+1}} \big\{\gamma' \in \Gamma' : \, \gamma' D \subseteq w\, C(xa)\big\} .
\end{aligned}$$

Clearly there are finitely many $\gamma' \in \Gamma'$ such that $\gamma' D \cap w\, B(e,N) \neq \emptyset$, but infinitely many $\gamma' \in \Gamma'$ such that $\gamma' D \subseteq w\, C(xa)$ for some fixed x and a. The right finiteness condition is imposed in the following lemma which implies Lemma 7.12.

Lemma 7.14. *Fix any $z \in \Gamma$ and choose $N > |z|$ large enough so that $\gamma' D \cap z^{-1}B(e,N) \neq \emptyset$ for all $|\gamma'| \leq M$. Define*

$$S_0' := \{\gamma' \in \Gamma' : \gamma' D \cap z^{-1}B(e,N) \neq \emptyset\},$$

$$S_t' := \{\gamma' \in \Gamma' : \gamma' D \subseteq z^{-1}C(xa) \text{ for some } x, a \text{ with } |xa| = N+1 \text{ and } \overline{\gamma'} D \not\subseteq z^{-1}C(xa)\}$$

$$\mathcal{S}' := S_0' \sqcup S_t'.$$

Then $\mathcal{S}'$ is a finite complete subtree (containing $B'(e,M)$), whose terminal vertices are $T(\mathcal{S}') = S_t'$ and can be characterized as follows

$$T(\mathcal{S}') = \{\gamma' = z^{-1}xy \in \Gamma' : |x| = N,\, |xa| = N+1,\, y \in P(a)\}.$$

Proof of Lemma 7.14. We shall prove a sequence of simple claims. Notice that since $|z| < N$, then for all $x \in \Gamma$ and $a \in A$ such that $|xa| = |x| + 1 = N+1$, xa does not belong to the geodesic between e and z and hence, according to Lemma 6.1, $z^{-1}C(xa) = C(z^{-1}xa)$.

Claim 1. If $\gamma' \in S_0'$, then $\overline{\gamma'} \in S_0'$ and hence the set S_0' is a subtree.

Proof: Let $v \in \gamma' D \cap z^{-1}B(e,N)$ be a vertex and let $x_0 = v, x_1, \dots, x_r = e$ be a sequence of vertices of the unique geodesic in $\mathcal{T}$ from $x_0 = v$ to $x_r = e$. By convexity of $z^{-1}B(e,N)$, $x_j \in z^{-1}B(e,N)$ for all $0 \leq j \leq r$. Since $\gamma' D$ is a subtree, the set $\{i : 0 \leq i \leq r, x_i \in \gamma' D\}$ is an interval, say $[0, i_0] \cap \mathbf{Z}$. Let $\gamma'' \in \Gamma'$ be (the unique element) such that $x_{i_0+1} \in \gamma'' D$. Then by construction $d(\gamma' D, \gamma'' D) = 1$ so that, by Lemma 7.2 (2), $\gamma'' = \overline{\gamma'}$ and $\overline{\gamma'}\, D \cap z^{-1}B(e,N) \neq \emptyset$, thus showing that $\overline{\gamma'} \in S_0'$.

Claim 2. If $\gamma' \in S_t'$, then $\overline{\gamma'} \in S_0'$ and hence the set $\mathcal{S}'$ is a subtree and $S_t' \subseteq T(\mathcal{S}')$.

Proof: Let $\gamma' \in S_t'$ and let $\gamma' D \subset z^{-1}C(xa)$ with $\overline{\gamma'} D \notin z^{-1}C(xa)$. Lemma 7.13 implies then immediately that $\overline{\gamma'} D \cap z^{-1}B(e,N) \neq \emptyset$ and hence $\overline{\gamma'} \in S_0'$.

Claim 3. The tree $\mathcal{S}'$ is complete and $S_t' = T(\mathcal{S}')$.

Proof: Let $\gamma' \in S_0'$ and let $a' \in A'$ so that $|\gamma' a'|' = |\gamma'|' + 1$. If $\gamma' a' \notin S_0'$, then, by Lemma 7.13, $\gamma' a' D \in z^{-1}C(xa)$ for some $|x| = N$ and $|xa| = N+1$. On the other hand $\overline{\gamma' a'} D = \gamma' D \notin z^{-1}C(xa)$ and hence $\gamma' \in S_t'$.

Claim 4. $T(\mathcal{S}') = \{\gamma' = z^{-1}xy \in \Gamma' : |x| = N,\, |xa| = N+1,\, y \in P(a)\}$.

Proof: By definition if $\gamma' \in S'_t$, then $\gamma' D \subseteq z^{-1}C(xa)$ and hence $\gamma' = z^{-1}xay$, for some $y \in \Gamma$. However, since we have also that $\overline{\gamma'} D \nsubseteq z^{-1}C(xa)$, then $z^{-1}x \in \overline{\gamma'} D$. Thus there exists $u \in D$ such that $\overline{\gamma'} = z^{-1}xu^{-1}$.

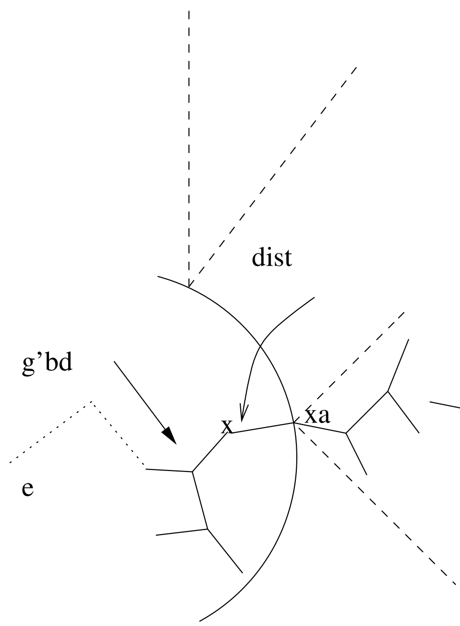

FIGURE 2: $\gamma' \in S'_t$ and $\overline{\gamma'} \in S'_0$ for $z = e$.

The assertion now follows by completing γ' with its last letter $c' \in A'$ in the reduced expression.

□

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