INVARIANT GEOMETRIC STRUCTURES: A NON-LINEAR EXTENSION OF THE BOREL DENSITY THEOREM

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1. Introduction. Let M be a connected smooth n-dimensional manifold and let H be a subgroup of $\mathrm{GL}(n,\mathbb{R})$. An H-structure $P \to M$ is a reduction to H of the frame bundle of M, or equivalently a principal H-bundle contained in the frame bundle. Basic examples of H-structures are a volume density on M (where $H = \mathrm{SL}'(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \colon \ |\det A| = 1\}$), an orientation ($H = \mathrm{GL}'(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \ \det A > 0\}$), a Riemannian metric ($H = \mathrm{O}(n,\mathbb{R})$), a Lorentz metric ($H = \mathrm{O}(1,n-1)$). If we consider also higher order H-structures, that is reductions of the higher order frame bundles (i.e. $H \subset \mathrm{GL}(n,\mathbb{R})^{(k)}$, where $\mathrm{GL}(n,\mathbb{R})^{(k)}$ is the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing the origin), a torsionfree affine connection on M can also be viewed as an H-structure, where in this case $H = \mathrm{GL}(n,\mathbb{R})$. The concept of H-structure serves therefore as a unifying framework within which to discuss many diverse geometric questions. In particular, this is the point of view from which we can try to answer some questions about the symmetries of geometric objects, by trying to understand the automorphism groups of the corresponding H-structure. Recall that the automorphism group of an H-structure $P \to M$, $\mathrm{Aut}(P)$ is defined as the subgroup of $\mathrm{Diff}(M)$ whose corresponding action on the frame bundle preserves P. Among the many examples of results that can be proven from this perspective we mention only the following.

Theorem 1.1 [Z5], [Z8]**.** *Let $P \to M$ be an H-structure defining a volume density on a compact n-manifold, where H is a real algebraic group. Let G be a connected semisimple group with finite center and no compact factors acting (faithfully) by diffeomorphisms on M preserving the H-structure. Then there is a Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$ such that the representation $\mathfrak{g} \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{sl}(n,\mathbb{R})$ contains $\mathrm{ad}_{\mathfrak{g}}$ as a direct summand.*

Consider for instance the case of a compact Lorentz manifold. Then it can be proven [Z5] that, if G is simple and non-compact, G is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$

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and, more generally, that if G is just semisimple then the connected component of the identity in $\mathrm{Aut}(P)$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})\times K$, where K is a compact Lie group. Note also that if we relax the hypothesis on the existence of a finite invariant measure we can only conclude that $\mathbb{R}\text{-rank}(G)\leqslant\mathbb{R}\text{-rank}(H)$, [Z8].

For further examples of these sorts of theorems investigating the automorphism group of an H-structure on a compact manifold, see [Gr], [Z5], [Z6], [Z7], [Z8]. On the other hand not much is known about the automorphism group of an H-structure on a non-compact manifold.

Notice that the general tenor of these theorems has been to determine when a group G can be a subgroup of $\mathrm{Aut}(P)$ for a given H-structure $P\to M$. To extend these results, a fairly natural approach (inspired somewhat by the Mostow-Margulis rigidity theory) would be to consider the following picture. Suppose Γ is a lattice in G, (that is, a discrete subgroup such that the quotient G/Γ carries a finite G-invariant measure) and that there is a homomorphism $\Gamma\to\mathrm{Aut}(P)\hookrightarrow\mathrm{Diff}(M)$ which is the restriction of a homomorphism $G\to\mathrm{Diff}(M)$; is it then true that the image of G under this homomorphism is contained in $\mathrm{Aut}(P)$ as well? The main result of this paper provides an affirmative answer, under certain additional hypotheses.

Theorem 3.2. *Let G be a connected semisimple Lie group with finite center and no compact factors and let $\Gamma\subset G$ be an irreducible lattice. If H is a real algebraic group and there are no locally closed G-orbits with compact stabilizer, then every Γ-invariant H-structure on M is G-invariant.*

Remark. As will be clear in the proof, we in fact merely need the set $M'=\{m\in M \text{ s.t. } \mathrm{Stab}_G(m) \text{ is compact or } G\cdot m \text{ is locally closed }\}$ to be conull in M. Or, even more precisely, it suffices to assume that M' intersects every non-empty open set in a set of positive measure.

Recall that a lattice $\Gamma\subset G$ is irreducible if it is dense when projected onto G/N, where N is any normal subgroup of positive dimension (i.e. N is not central); in other words we want to eliminate situations in which $\Gamma=\Gamma_1\times\Gamma_2\subset G_1\times G_2=G$. However, we can relax the assumption of the irreducibility of Γ, in which case we need to ask, for all the factors G_i of G, that there are no locally closed G_i-orbits in M with compact stabilizer.

Notice this result holds in great generality with regards to the manifold M; in particular we do not require it either to be compact or to carry a finite invariant measure. In the case in which we assume M is compact, our result together with Theorem 1.1 provides evidence for a conjecture formulated by Zimmer, [Z7]. Suppose that $H\subseteq\mathrm{SL}(n,\mathbb{R})$ and each factor in G has $\mathbb{R}$-rank at least 2. Then it has been conjectured that either there is a Lie algebra embedding $\mathfrak{g}\hookrightarrow\mathfrak{h}$ or there is a smooth Γ-invariant Riemannian metric on M. Partial results have been proven in this direction for particular classes of actions, however Theorems 1.1 and 3.2 together provide some support for the conjecture in the case in which the Γ-action extends to a G-action. In fact in this situation, our theorem implies that the H-structure is G-invariant as well, in which case Theorem 1.1 asserts the existence of the conjectured Lie algebra embedding.

Now we want to illustrate how, viewed from a slightly different perspective, Theorem 3.2 provides a generalization of the Borel Density Theorem. Let π be a rational representation of G on some finite dimensional vector space V; if π is rational Borel proved, [B], that every Γ-invariant line in V is G-invariant (equivalently Γ is Zariski dense in G). Analogously, if π is infinite dimensional and unitary, Moore proved, [Mo], that every Γ-invariant vector in V is necessarily G-invariant (equivalently, the restriction to Γ of an ergodic G-action with finite invariant measure is ergodic; see [Mo] and [Z2] for the corresponding result for arbitrary ergodic actions). This infinite dimensional invariance is far from being true for more general (non-unitary) representations, although some results in this direction can be obtained; more precisely, in a recent paper Wigner proved, [Wg], that every Γ-invariant analytic vector of an infinite dimensional representation is G-invariant. However, this result cannot be applied in many geometric situations since, as Wigner communicated to the author, even analytic sections of vector bundles are not necessarily analytic vectors of infinite dimensional representations. Results involving some particular infinite dimensional representations arising from purely geometric considerations can be obtained as a particular case of the following theorem, which is a slightly more general version of Theorem 3.2.

Theorem 3.4. *Let $P \to M$ be any principal H-bundle on which G acts by automorphisms. Suppose X consists of the real points of a variety defined over $\mathbb{R}$ on which H acts algebraically and let $E \to M$ be the bundle with fiber X associated to P. If every G-orbit in M with compact stabilizer is not locally closed, then every Γ-invariant section of E is G-invariant.*

Notice that if M is a point this theorem reduces to the Borel Density Theorem. On the other hand, if X is a finite dimensional vector space and the H-action on X is linear, (hence E is a vector bundle), then the space of sections of E is an infinite dimensional vector space on which G acts linearly, so that Theorem 3.4 provides an infinite dimensional analogue of the Borel Density Theorem. Furthermore, if π is an admissible, finitely generated representation of G on a Banach space V, Theorem 3.4 implies that every Γ-invariant C^∞-vector in V is G-invariant. In fact, we can continuously and equivariantly embed the space of C^∞-vectors of such a representation π in the space of C^∞-vectors of a representation induced from a finite dimensional representation σ on X of the minimal parabolic subgroup P, that is in the space of C^∞-sections of the bundle over $M = G/P$ with fiber X, associated to $G \to G/P$, [Wa].

We want to remark here how none of the hypotheses of these theorems can be eliminated. It is possible to show, in fact, that for suitable M there are reductions to Γ of a principal G-bundle on M, which are Γ-invariant but not G-invariant, hence showing that Theorem 3.2 need not hold if the group H is not algebraic (see §3). Moreover, the hypothesis on the orbits in M is also necessary, at least in the case in which M is not compact; in fact if G acts on itself by translations, every non-trivial vector field on G/Γ can be lifted to a Γ-invariant vector field on G which is not G-invariant (here of course the stabilizer of the only orbit is the identity). However, if the manifold M is compact, the issue is definitely more delicate and worthy of further study. More precisely, it can be proven (see §5) that in the case of $\mathrm{SL}(2,\mathbb{R})$ acting on the 2-sphere by fractional linear transformations (notice the presence of the two hemispherical orbits which are locally closed and have the orthogonal group $\mathrm{SO}(2,\mathbb{R})$ as stabilizer), there exist continuous vector fields on S^2 which are Γ-invariant but not G-invariant, but no such example can be smooth. This leads to the natural question of whether or not Theorem 3.4 might be true without the assumption on the orbits in M, if we restrict our attention only to smooth sections of $E \to M$.

The proofs of these theorems involve arguments of ergodic theory, algebraic groups, differential geometry and representation theory. Of these we want to mention in this introduction only a new ergodic theoretical result which, besides being of interest by itself, is the basic ingredient in the proof of some other results on smooth orbit equivalence which will appear in a forthcoming paper, [I].

Recall that if $P \to M$ is any principal H-bundle on which G acts by principal bundle automorphisms, G-invariant reductions to $H_1 \subset H$ are in one-to-one correspondence with G-invariant sections of the bundle $P/H_1 \to M$. Then, if G is ergodic on M and H is algebraic, there exists a unique (up to conjugacy) smallest algebraic subgroup $H_1 \subset H$ such that there is a measurable G-invariant reduction of P to H_1. The conjugacy class of H_1 is called the algebraic hull of the action of G on $P \to M$; if $G \subset \mathrm{Diff}(M)$, the algebraic hull of the action of G on M is defined to be the algebraic hull of the action of G by automorphisms of the frame bundle on M. Then the proof of our results mentioned above consists essentially of applying the following theorem to study the H-invariance of sections of bundles over M which have as fibers subspaces of projective spaces.

Theorem 3.6. *Let G and Γ be as above. Let G be acting by automorphisms on a principal H-bundle $P \to M$ and suppose that there are no locally closed G-orbits with compact stabilizer. Then (on each ergodic component) the algebraic hulls of the actions of G and Γ on $P \to M$ are the same.*

It should be remarked here that this result about the algebraic hull of the restriction to Γ is not equivalent to ergodicity of the restriction to Γ. In fact it is possible to provide examples of ergodic actions of $\mathbb{R}$ which are still ergodic when restricted to $\mathbb{Z}$, but such that the algebraic hulls are different (see §4).

The work described in this paper is part of the results contained in my Ph.D. thesis at The University of Chicago. I am very grateful to my advisor, Robert Zimmer, for countless useful conversations and for his constant help; I want also to thank Ralph Spatzier who suggested an easier and more general proof of a result in §5.

2. Ergodic measures and probability measures. Let G, H be locally compact second countable groups. Recall that, as in [Z6], we call a topological space Y tame if it is T_0 and it has a countable basis for the topology: we call the continuous action of H on Y tame if both Y and Y/H are tame. Moreover we say that S is an ergodic G-space if there is a quasi-invariant measure on S and every G-invariant set is either null or conull with respect to this measure. It is well known that there are only two types of ergodic G-spaces: essentially transitive spaces, i.e. spaces where there is only a conull orbit, and properly ergodic spaces, i.e. spaces where every orbit is a null set.

Suppose that G acts on a manifold M with quasi-invariant measure μ: then there exist a separable metrizable space (E,ν) and a measurable map $\varphi\colon M \to E$ with the following properties: each $\varphi^{-1}(e)$ is G-invariant and there are ergodic measures μ_e supported on $\varphi^{-1}(e)$ such that, for any measurable set $A \subseteq M$, the map $e \mapsto \mu_e(A \cap \varphi^{-1}(e))$ is a measurable map and $\mu(A) = \int_{e\in E} \mu_e(A\cap\varphi^{-1}(e))d\nu(e)$.

Any set $\varphi^{-1}(e)$ is an ergodic component and the decomposition is unique up to sets of measure zero. Notice that if the action is ergodic E reduces to a point and if the action is tame E is just the set of orbits. For more detailed treatment see [Vr] for the case of a finite invariant measure and [E], [R] for the general case.

Now we shall illustrate some results about the action of a real algebraic group $H \subseteq \mathrm{GL}(n, \mathbb{R})$ on the space of probability measures on H/H_1 where $H_1 \subseteq H$ is a real algebraic subgroup. It is well known, by a theorem of Chevalley, that under these hypotheses, there is a rational representation defined over $\mathbb{R}$ and a point $x \in \mathbb{P}^{n-1}(\mathbb{R})$ such that H_1 is the stabilizer of x in H, [Ch]. In other words we can identify H/H_1 with an orbit in $\mathbb{P}^{n-1}(\mathbb{R})$. If we denote by $M(\mathbb{P}^{n-1})$ the space of probability measures on $\mathbb{P}^{n-1}$ and by $i\colon H/H_1 \to \mathbb{P}^{n-1}$ the above identification, we can define $M(H/H_1) = \{i^*\mu = \mu \circ i \mid \mu(\mathbb{P}^{n-1} - i(H/H_1)) = 0, \text{ where } \mu \in M(\mathbb{P}^{n-1})\}$; then there is an injection $i_*\colon M(H/H_1) \to M(\mathbb{P}^{n-1})$ so that $M(H/H_1)$ can be identified with a subset of $M(\mathbb{P}^{n-1})$. If we give $M(\mathbb{P}^{n-1})$ the w-$\star$-topology as the dual of the space of continuous functions on $\mathbb{P}^{n-1}$ we have that the H-action on $M(\mathbb{P}^{n-1})$ is continuous, tame and has real algebraic stabilizers [Z4]: then $M(H/H_1)$, with the induced topology as a subset of $M(\mathbb{P}^{n-1})$, is a separable metrizable continuous H-space with real algebraic stabilizers. The tameness of the H-action follows from the next result.

Proposition 2.1. *The H-orbits in $M(H/H_1)$ are locally closed.*

Proof. Let $\mu \in M(H/H_1) \subseteq M(\mathbb{P}^{n-1})$: then, by [Z4, Theorem 3.2.6] we have that the $\mathrm{PGL}(n, \mathbb{R})$-orbit of μ is locally closed in $M(\mathbb{P}^{n-1})$; moreover there is a homeomorphism $\mathrm{PGL}(n, \mathbb{R}) \cdot \mu \simeq \mathrm{PGL}(n, \mathbb{R})/\mathrm{PGL}(n, \mathbb{R})_\mu$, where $\mathrm{PGL}(n, \mathbb{R})_\mu$ is the stabilizer of μ in $\mathrm{PGL}(n, \mathbb{R})$. But $\mathrm{PGL}(n, \mathbb{R})_\mu$ is an algebraic group [Z4, Theorem 3.2.4] hence $\mathrm{PGL}(n, \mathbb{R}) \cdot \mu$ is a variety and the H-action on it is tame: thus $H \cdot \mu$ is locally closed in $\mathrm{PGL}(n, \mathbb{R}) \cdot \mu$ and in $M(\mathbb{P}^{n-1})$. But $H \cdot \mu \subseteq M(H/H_1)$ hence $H \cdot \mu$ is locally closed in $M(H/H_1)$ with the induced topology. □

3. Proof of the main theorems. We start this section recalling the notion of cocycle and of algebraic hull of a cocycle. Let G act by diffeomorphisms on a manifold M with a quasi-invariant measure μ: then G acts on the frame bundle $P(M) \to M$ by bundle automorphisms. Since there are always measurable sections, (as one can always measurably glue together local sections, [A], [VN]), we can measurably trivialize the bundle $P(M) \to M$ and write $P(M) \simeq M \times \mathrm{GL}(n, \mathbb{R})$. Then the G-action on the trivialized $P(M)$ is given by $g(m, h) = (gm, h\alpha(m, g))$ where $\alpha\colon M \times G \to \mathrm{GL}(n, \mathbb{R})$ is a measurable cocycle. Remember that if S is a G-space with a quasi-invariant measure we say that two cocycles $\alpha, \beta\colon S \times G \to H$ are equivalent, and we write $\alpha \sim \beta$ if there is a measurable map $\phi\colon S \to H$ such that for each $g \in G$, $\phi(gs)\alpha(s, g)\phi(g)^{-1} = \beta(s, g)$ for almost every $s \in S$. Although different measurable sections lead to different measurable trivializations and hence, usually, to different cocycles, it is easy to check that these cocycles are equivalent. Given an H-space Y with a quasi-invariant measure and a cocycle $\alpha\colon S \times G \to H$, we define a G-action on the space $F(S, Y)$ of measurable functions from S to Y by $(g \cdot \psi)(s) = \alpha(s, g)^{-1}\psi(gs)$ a.e. $s \in S$, where $\psi \in F(S, Y)$: then a function $\psi \in F(S, Y)$ is called α-invariant if for every $g \in G$, we have $g \cdot \psi = \psi$ a.e., in other words if $\alpha(s, g)\psi(s) = \psi(gs)$ for a.e. s. If H_0 is a closed subgroup of H and $Y = H/H_0$ then there is such an α-invariant function if and only if there is a cocycle β equivalent to α with values in H_0, (see [Z4, 4.2.18]). Notice that if $H = \mathrm{GL}(n, \mathbb{R})$, and S is a manifold this corresponds to the existence of a measurable G-invariant H_0-structure on S.

The following is a well known result which will be used later, [Z4, Lemma 5.2.11].

Lemma 3.1 (Cocycle reduction lemma). *Let S be an ergodic G-space, Y a space on which H acts continuously and tamely, and let $\alpha\colon S\times G\to H$ be a cocycle. Suppose that there is an α-invariant function $\varphi\colon S\to Y$. Then there exists $y\in Y$ such that $\alpha\sim\beta$ where $\beta(S\times G)\subseteq H_y=Stab_H(y)$. Moreover $\varphi(s)\in H\cdot y$ a.e. s.*

If $\alpha\colon S\times G\to H$ is a cocycle of an ergodic G-action into an algebraic group H, then there is an algebraic subgroup $L\subseteq H$ such that α is equivalent to a cocycle taking values in L, but α is not equivalent to a cocycle taking values in any proper algebraic subgroup of L. The group L is unique up to conjugacy in H and the conjugacy class of L is called the *algebraic hull* of α. Note that if $\alpha\sim\beta$ with $\beta(S\times G)\subseteq J$ and J is algebraic, then J contains a conjugate of L, [Z4, 9.2]; in this case, with a minor abuse of terminology, we shall say that J contains the algebraic hull of α.

Theorem 3.2. *Let G be a connected semisimple Lie group with finite center and no compact factors and let $\Gamma\subset G$ be an irreducible lattice. If H is a real algebraic group, let $P\to M$ be an H-structure preserved by G and suppose that there are no locally closed G-orbits with compact stabilizer. Then, if $H_1\subset H$ is algebraic, every Γ-invariant reduction of $P\to M$ to H_1 is G-invariant.*

Proof. After the measurable trivialization $P\simeq M\times H$, a G-invariant reduction of an H-structure to H_1, where $H_1\subset H$ is a real algebraic subgroup, corresponds to an α-invariant section of $M\times H/H_1\to M$, namely to an α-invariant function f in $F(M,H/H_1)$. Then saying that f is α_Γ-invariant, where $\alpha_\Gamma=\alpha|_{M\times\Gamma}$, amounts to saying (possibly changing the trivialization) that $\alpha_\Gamma(M\times\Gamma)\subseteq H_1$. Hence, by taking the decomposition of M in ergodic components, it will be enough to prove the following:

Theorem 3.3. *Let G_i be connected, non-compact, simple Lie groups with finite center, $G=\prod G_i$, Γ an irreducible lattice and S an ergodic G-space not essentially isomorphic (as a Borel G-space) to G/G_0 with G_0 compact. Let H_1 be an $\mathbb{R}$-subgroup of a real algebraic group H and $\alpha\colon S\times G\to H$ be a cocycle such that the algebraic hull of $\alpha_\Gamma(S\times\Gamma)$ is contained in H_1. Then every α_Γ-invariant function $f\in F(S,H/H_1)$ is α-invariant.*

Remark. Recall that under these hypotheses on G, Γ, S, the restriction to Γ of the G-action is well known to be ergodic [Mo], [Z2]; one of the many ways we use this result is when we consider the algebraic hull of α_Γ which would not otherwise make sense.

Before proceeding with the proof of Theorem 3.3, we want to remark how this result immediately implies also the next theorem.

Theorem 3.4. *Let $P\to M$ be any principal H-bundle on which G acts by automorphisms. Suppose X consists of the real points of a variety defined over $\mathbb{R}$ on which H acts algebraically and let $E\to M$ be the bundle with fiber X associated to P. If every G-orbit in M with compact stabilizer is not locally closed, then every Γ-invariant section of E is G-invariant.*

Proof of 3.4 from 3.3. After the measurable trivialization $E\simeq M\times X$ corresponding to the cocycle α, a Γ-invariant section of E corresponds to an α_Γ-invariant function

$f \in F(M, X)$. Let S be an ergodic component of M. By the Remark above the restriction of the action to Γ is still ergodic and by hypothesis the H-action on X is tame; hence the function $f|_S : S \to X$ takes values in some H-orbit, say $H \cdot x_s$ for some $x_s \in X$, (Lemma 3.1). Then if H_{x_s} is the stabilizer of $x_s \in S$, and hence H_{x_s} is a real algebraic subgroup of H, we can consider $f|_S$ as a function in $F(S, H/H_{x_s})$. Then, possibly choosing a different trivialization, the α_Γ-invariance of $f|_S$ is equivalent to saying that $\alpha_\Gamma(S \times \Gamma) \subset H_{x_s}$, so that an application of Theorem 3.3 will provide the end of the proof. □

Proof of 3.3. We recalled already the existence of a representation π of H into $\mathrm{GL}(n, \mathbb{R})$ such that H_1 stabilizes some $x \in \mathbb{P}^{n-1}(\mathbb{R})$. If $i\colon H/H_1 \to \mathbb{P}^{n-1}$ is the immersion induced by π, $i([h]) = \pi(\theta([h]))x$, where $\theta\colon H/H_1 \to H$ is a measurable section of the canonical projection, let us define $\bar{f} = i \circ f \in F(S, \mathbb{P}^{n-1}(\mathbb{R}))$ and $\beta = \pi \circ \alpha\colon S \times G \to \mathrm{GL}(n, \mathbb{R})$. Then, writing β_Γ for $\beta|_{M\times\Gamma}$, we have that for a.e. s

$$\begin{aligned}\beta_\Gamma(s,\gamma)\bar{f}(s) &= \pi(\alpha_\Gamma(s,\gamma))i(f(s)) = \pi(\alpha_\Gamma(s,\gamma))\pi(\theta(f(s)))x = \\ &= \pi(\alpha_\Gamma(s,\gamma)\theta(f(s))) = \pi(\theta(f(\gamma s)))x = i(f(\gamma s)) = \bar{f}(\gamma s)\end{aligned}$$

that is the α_Γ-invariance of f implies the β_Γ-invariance of $\bar{f}$. If we could prove that $\bar{f}$ is hence β-invariant we would be done: in fact for every $g \in G$, a.e. $s \in S$

$$\pi(\theta(f(gs)))x = \bar{f}(gs) = \beta(s,g)\bar{f}(s) = \pi(\alpha(s,g))\pi(\theta(f(s)))x = \pi(\alpha(s,g)\theta(f(s)))x$$

that is $\pi(\theta(f(gs))^{-1}\alpha(s,g)\theta(f(s)))$ stabilizes x: hence $\theta(f(gs))^{-1}\alpha(s,g)\theta(f(s)) \in H_1$ or equivalently $f\colon S \to H/H_1$ is α-invariant. The proof will then be complete if we show the following:

Theorem 3.5. *Let G_i,G,Γ,S be as above and let $\beta\colon S\times G \to GL(n,\mathbb{R})$ be a cocycle such that the algebraic hull of $\beta_\Gamma(S\times\Gamma)$ is contained in $T_1 = \begin{pmatrix} \mathbb{R}^\times & * \\ 0 & GL(n-1,\mathbb{R}) \end{pmatrix}$. Then every β_Γ-invariant function in $F(S, \mathbb{P}^{n-1}(\mathbb{R}))$ is also β-invariant.*

The proof of this result will depend strongly on the next theorem, the proof of which will take the entire next section.

Theorem 3.6. *Let G,Γ,S be as in Theorem 3.3 and let α be an $S \times G$ cocycle into a real algebraic group H. Then the algebraic hulls of α and α_Γ are the same.*

Proof of Theorem 3.5. By Theorem 3.6 it follows that there is a cocycle β_1 equivalent to β such that $\beta_1(S \times G) \subseteq T_1$. This implies that the set of β-invariant functions is not empty, $\phi^{-1}f$ being β-invariant, where $f \in F(S, \mathbb{P}^{n-1}(\mathbb{R}))$ is any β_Γ-invariant function and ϕ is the measurable function which implements the equivalence between β and β_1. We want to show that f itself is β-invariant.

Let us define $F_G(S, \mathbb{P}^{n-1}(\mathbb{R})) = \mathrm{Span}_{F(S,\mathbb{R})}\{f_1, \dots, f_m\}$ where we chose the $\{f_j\}_{j=1}^m \subset F(S, \mathbb{P}^{n-1}(\mathbb{R}))$, $1 \leqslant m \leqslant n$, to be a maximal set of β-invariant measurable functions which are a.e. linearly independent, whose existence is guaranteed by ergodicity of G on S (the set on which given β-invariant measurable functions are linearly independent is invariant, hence null or conull). We shall prove first that $F_G(S, \mathbb{P}^{n-1}(\mathbb{R}))$ contains all the β_Γ-invariant functions in $F(S, \mathbb{P}^{n-1}(\mathbb{R}))$, whereupon a closer observation of the action of the cocycle β in a simpler form will give the desired result. To this purpose, remember that the space $F(S,\mathbb{P}^{n-1}(\mathbb{R}))$ is nothing but the space of sections of a bundle over S with fiber $\mathbb{P}^{n-1}(\mathbb{R})$, so that the G-invariance of a Γ-invariant section will not depend on the trivialization; in term of cocycles this means that if β, $\tilde\beta$ are equivalent cocycles, then every β_Γ-invariant function is β-invariant if and only if every $\tilde\beta_\Gamma$-invariant function is $\tilde\beta$-invariant.

Let us choose f_j to be the j-th axis in a coordinate system in $\mathbb{R}^n$; this implies that the algebraic hull of $\beta(S\times G)$ is contained in $T_m = \begin{pmatrix} A_m & * \\ 0 & \mathrm{GL}(n-m,\mathbb{R}) \end{pmatrix}$, where $A_m \subset \mathrm{GL}(m,\mathbb{R})$ is the group of diagonal matrices. We claim that every $f \in F(S,\mathbb{P}^{n-1}(\mathbb{R}))$ which is β_Γ-invariant is actually contained in $F_G(S,\mathbb{P}^{n-1}(\mathbb{R}))$: in other words the set $\{f_1,\dots,f_m\}$ is maximal not only among the β-invariant functions, but also among the β_Γ-invariant ones. For, if this were not true, i.e. if on a set of positive measure (hence on a conull set, again by ergodicity) we had that $f_1,\dots,f_m,f$ were linearly independent, then the algebraic hull of β_Γ and hence of $\beta(S\times G)$, by Theorem 3.6, would be contained in $T_{m'}\subset T_m$ with $m'>m$, contradicting the hypothesis that the set $\{f_1,\dots,f_m\}$ is maximal. We can hence restrict our attention to the action of the cocycle β on $F_G(S,\mathbb{P}^{n-1}(\mathbb{R}))$.

We can assume, after rearranging the elements $f_1,\dots,f_m$ of the basis and passing (if necessary) to an equivalent cocycle, that

$$\beta(S\times G)\subset \begin{pmatrix} \beta^1 I_{m_1} & \dots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta^p I_{m_p} & * \\ 0 & \dots & 0 & \mathrm{GL}(n-m,\mathbb{R}) \end{pmatrix}$$

where $\beta^j: S\times G\to\mathbb{R}^\times$ is a one-dimensional cocycle, I_{m_j} is the identity matrix in dimension m_j, $j=1,\dots,p$, $\sum_{j=1}^p m_j = m$ and β^j is not equivalent to β^i if $j\neq i$. Let $[V_j]$, $j=1,\dots,p$, be the image in $\mathbb{P}^{n-1}(\mathbb{R})$ of the subspace of $\mathbb{R}^n$ corresponding to the action via β^j. We claim that if $f\in F_G(S,\mathbb{P}^{n-1}(\mathbb{R}))$ is β_Γ-invariant, then its image must lie in only one of the $[V_j]$ for a.e. $s\in S$. Let us assume this for the moment and proceed with the proof. Let us suppose, for simplicity of notation that $j=1$ and let $f(s)=[c_1(s),\dots,c_{m_1}(s),0,\dots,0]\in[V_1]$ be β_Γ-invariant; if f is not almost everywhere equal to the zero function, by ergodicity of Γ at least one of the coordinate functions of f, say $c_1(s)$, is different from zero for every $s\in S_0$, where S_0 is a conull set. Since on each $[V_j]$ the β-action is just multiplication by scalars, the β_Γ-invariance of f gives

$$\begin{aligned}[c_1(s),\dots,c_{m_1}(s),0,\dots,0] &= [\beta^1_\Gamma(s,\gamma)c_1(s),\dots,\beta^1_\Gamma(s,\gamma)c_{m_1}(s),0,\dots,0] = \\ = \beta_\Gamma(s,\gamma)f(s) &= f(\gamma s) = [c_1(\gamma s),\dots,c_{m_1}(\gamma s),0,\dots,0].\end{aligned}$$

Hence the functions $q_i(s)=c_i(s)/c_1(s)$, $i=1,\dots,m_1$, defined almost everywhere on S_0 are essentially Γ-invariant and ergodicity of Γ implies that the q_i's are essentially constant, $q_i(s)=q_i$ a.e. s. Then

$$\begin{aligned} f(s) &= [c_1(s),\dots,c_{m_1}(s),0,\dots,0] = \\ &= [c_1(s),q_2c_1(s),\dots,q_{m_1}c_1(s),0,\dots,0] = [1,q_2,\dots,q_{m_1},0,\dots,0]\end{aligned}$$

hence f is β-invariant.

The proof will then be complete if we show that $f(s) \in [V_1]$ for $s \in S'$ where $S' \subset S$ is conull. We shall prove in fact that if $f(S') \subset [V_1] \cup [V_i]$ and neither $f(S') \subset [V_1]$ nor $f(S') \subset [V_i]$ then $\beta^1 \sim \beta^i$, which is a contradiction. For simplicity of notation and without loss of generality, let us assume that $i = 2$ and that both $[V_1]$ and $[V_2]$ are one-dimensional, i.e. that $m_1 = m_2 = 1$. Let $f(s) = [c(s), d(s), 0, \dots, 0]$ be β_Γ-invariant and such that $[c(s), 0, 0, \dots, 0] \in [V_1]$, $[0, d(s), 0, \dots, 0] \in [V_2]$ for $s \in S'$; assume that both $c(s)$ and $d(s)$ are different from zero almost everywhere. Then

$$[c(\gamma s), d(\gamma s), 0, \dots, 0] = f(\gamma s) = \beta_\Gamma(s,\gamma) f(s) = [\beta^1_\Gamma(s,\gamma)c(s), \beta^2_\Gamma(s,\gamma)d(s), 0, \dots, 0]$$

hence

$$\frac{\beta^1_\Gamma(s,\gamma)c(s)}{\beta^2_\Gamma(s,\gamma)d(s)} = \frac{c(\gamma s)}{d(\gamma s)}$$

or else $\varphi^{-1}(\gamma s)\beta^1_\Gamma(s,\gamma)(\beta^2_\Gamma(s,\gamma))^{-1}\varphi(s) = 1$, where $\varphi\colon S \to \mathbb{R}^\times$ is defined as $\varphi(s) = c(s)d(s)^{-1}$. Thus the cocycle $\beta^1_\Gamma(\beta^2_\Gamma)^{-1}\colon S \times \Gamma \to \mathbb{R}^\times$ is equivalent to the trivial one and, by our Theorem 3.6 on algebraic hulls, this must be the case also for the cocycle $\beta^1(\beta^2)^{-1}$ relative to the G-action. This amounts to saying that the cocycles β^1 and β^2 are equivalent, completing the proof. □

Remark. If H is not algebraic Theorem 3.2 need not hold: for example let G, G' be simple non-compact algebraic groups and let G be embedded in G' so as to act ergodically on $G'/\Gamma' = M$ where Γ' is a lattice in G'. Assume for simplicity that G has trivial center. The tangent bundle to the orbits in M is smoothly equivalent to $M \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, and the action of G by automorphisms is given by $g \cdot (m, v) = (g \cdot m, \mathrm{Ad}(g)v)$. Hence, since $G \simeq \mathrm{Ad}(G)$, there is a natural reduction P to G of the frame bundle $M \times \mathrm{GL}(\mathfrak{g})$ associated to $M \times \mathfrak{g}$; since for every $g, g_0 \in G$ we have $g_0 \cdot (m, g) = (g_0 \cdot m, g_0 g)$, then any closed subgroup Γ of G gives a Γ-invariant reduction of P to Γ. In particular suppose Γ is a lattice, hence not algebraic: then if such a reduction to Γ were also G-invariant this would imply the existence of a G-map $\varphi\colon G'/\Gamma' \to G/\Gamma$ which is measure preserving (by the uniqueness of the G-invariant measure on G/Γ) and hence, [Wt], of a continuous surjective homomorphism $\sigma\colon G' \to G$, which is impossible if G' has dimension higher than G, since G' is simple.

4. Proof of theorem 3.6.

Recall that if (S, μ) is a G-space, (T, ν) is an H-space and $\alpha\colon S \times G \to H$ is a cocycle, we can define a G-action on $(S \times T, \mu \times \nu)$ by $g(s,t) = (gs, \alpha(s,g)t)$. If $S = M$ is a manifold, $P \to M$ is a principal H-bundle, this is nothing but the action of $G \subset Aut(P)$ on the bundle over M associated to P with fiber T, after its trivialization. $S \times T$ with this action will be denoted by $S \times_\alpha T$: on $S \times_\alpha T$ there is also an H-action $(s,t) \cdot h = (s, t \cdot h)$ and this commutes with the G-action. If $G = H$ and $\alpha(s,g) = g$ then this reduces to the usual product action and if α and β are equivalent cocycles then $S \times_\alpha T$ and $S \times_\beta T$ are isomorphic G-spaces. Moreover if S is a transitive G-space, i.e. $S = G/G_0$, then there is a homomorphism $\sigma_\alpha\colon G_0 \to H$ corresponding to the cocycle α, which is defined by $\sigma_\alpha(g) = \alpha([e], g)$, for every $g \in G_0$: conversely if $\sigma\colon G_0 \to H$ is a measurable homomorphism and $\theta\colon G/G_0 \to G$ is a measurable section of the canonical projection with $\theta([e]) = e$ then the cocycle $\alpha\colon S \times G \to H$ defined by $\alpha(x, g) = \sigma(\theta(x)g\theta(xg)^{-1})$ is such that

$\sigma_\alpha = \sigma$: in fact, there is a bijection between equivalence classes of cocycles and conjugacy classes of homomorphisms [Z4, Proposition 4.2.13].

Proposition 4.1. *Let $G = \prod G_i$, where G_i are connected, non compact, simple Lie groups with finite center, Γ be an irreducible lattice in G and let S be an ergodic G-space such that the restriction to Γ of the G-action is still ergodic. Let $\beta\colon S\times G \to K$ be a cocycle into a compact algebraic group such that the algebraic hull of β is K. If the G-action on $S \times_\beta K$ is ergodic, the Γ-action on $S \times_\beta K$ is ergodic as well.*

Proof. It is enough to show that if $S \times_\beta K \simeq G/G_1$, then G_1 is not compact. Then in this case the result will follow from [Mo], and, in the case in which $S \times_\beta K$ is a properly ergodic G-space, from [Z2]. If $S \times_\beta K$ is an essentially transitive G-space, then this is the case also for S. Moreover, since Γ is ergodic on S, we have that $S \simeq G/G_0$ where G_0 is not compact. Let $([e], k) \in S\times_\beta K$ with $\mathrm{Stab}_G(([e], k)) = G_1$; then

$$\begin{aligned} G_1 = \mathrm{Stab}_G(([e], k)) &= \{g \in G \mid ([g], \beta([e], g)k) = ([e], k))\} = \\ &= \{g_0 \in G_0 \mid \beta([e], g_0) = \sigma_\beta(g_0) = e\} = Ker(\sigma_\beta). \end{aligned}$$

Since the action of G on $S \times_\beta K$ is transitive, the homomorphism $\sigma_\beta(G_0)\colon G_0 \to K$ is surjective, so that $K \simeq G_0/Ker(\sigma_\beta) = G_0/G_1$: if G_1 were compact, then the compactness of G_0/G_1 would imply that G_0 is also compact. $\square$

If $\alpha\colon S \times G \to H$ is a cocycle, recall that the Mackey range of α is defined as the H-action on the space of G-ergodic components of $S \times_\alpha H$ ([Ma], [Z1]): if S is an ergodic G-space then the Mackey range of α is always an ergodic H-space and it is an invariant of the equivalence class of α. The proof of the following can be found in [Z1].

Theorem 4.2. *α is equivalent to a cocycle taking values in a closed subgroup $H' \subseteq H$ if and only if there exists an H-map $\varphi\colon E_\alpha \to H/H'$ where E_α is the Mackey range of the cocycle α.*

Corollary 4.3. *Let $\beta\colon S \times G \to K$ be a cocycle where S is an ergodic G-space and K is a compact real algebraic group. Then the algebraic hull of β is K if and only if $S \times_\beta K$ is an ergodic G-space.*

Proof. $S \times_\beta K$ is an ergodic G-space if and only if the Mackey range of β is just one point. Moreover, since any compact real linear group is algebraic, saying that β is equivalent to a cocycle into a closed subgroup $K' \subseteq K$ is equivalent to say that the algebraic hull of β is contained in K'. Hence we can rephrase the statement of Theorem 4.2 as the equivalence between the existence of a K-map $\varphi\colon E_\beta \to K/K'$ and the fact that K' is the algebraic hull of β. Suppose that E_β is just one point $\{x\}$ and let $K' \subseteq K$ be the algebraic hull of β. Then for every $k \in K$ we have that $\varphi(x) = \varphi(kx) = k\varphi(x)$, i.e. $K = \mathrm{Stab}_K(\varphi(x)) = K'$. Conversely since K is compact and E_β is an ergodic K-space, we have that the K-action on E_β is essentially transitive [Z4, Proposition 2.1.10, Corollary 2.1.13], i.e., up to set of measure zero, there exists a K-map $\varphi\colon E_\beta \to K/K_0$ where $K_0 \subseteq K$ is a closed subgroup. But if E_β is not just one point then the algebraic hull of β is $K_0 \subsetneq K$. $\square$

Recall that with G, Γ, S as above, the ergodicity of Γ on S implies the ergodicity of G on $S \times G/\Gamma$ [Z4, Proposition 2.2.2].

If $\alpha\colon S \times G \to H$, let us define a cocycle $\tilde{\alpha}\colon S \times G/\Gamma \times G \to H$ by $\tilde{\alpha}(s, [x], g) = \alpha(s, g)$. Let us denote by $Z(S \times G; H)$ the set of cocycles on $S \times G$ with values in H and by $H^1(S \times G; H)$ the set of equivalence classes of these cocycles.

Proposition 4.4. *There is a bijection $\Phi\colon H^1(S \times G/\Gamma \times G; H) \to H^1(S \times \Gamma; H)$ such that $\Phi^{-1}(\alpha_\Gamma) = \tilde{\alpha}$ where $\alpha \in H^1(S \times G; H)$. Moreover the correspondence preserves the algebraic hulls.*

Proof. The first part of the proposition is in [Z3, Proposition 2.2]: we shall use the same notations and recall only what is necessary to prove the second assertion. If $p\colon G \to G/\Gamma$ is the canonical projection, let us choose a Borel section $\theta\colon G/\Gamma \to G$ such that $\theta(p(g))g \in \Gamma$, for every $g \in G$. Let $[\alpha] \in H^1(S \times \Gamma; H)$ and $[\beta] \in H^1(S \times G/\Gamma \times G; H)$ such that $\Phi([\beta]) = [\alpha]$ and $\Psi([\alpha]) = [\beta]$, where $\Psi = \Phi^{-1}$. Let A and B be the algebraic hulls of $[\alpha]$ and $[\beta]$ respectively and let $\alpha \in [\alpha]$ be such that $\alpha(S \times \Gamma) \subseteq A$. Let us define a map $\Psi_c\colon Z(S \times \Gamma; H) \to Z(S \times G/\Gamma \times G; H)$ by $\Psi_c(\alpha) = \beta$ where

$$\beta(s, x, g) = \Psi_c(\alpha)(s, x, g) = \alpha(\theta(x)s, \theta(gx)g\theta(x)^{-1}).$$

Equivalent cocycles are mapped into equivalent cocycles hence we have the induced map $\Psi\colon H^1(S \times \Gamma; H) \to H^1(S \times G/\Gamma \times G; H)$. Moreover it follows that $\beta(S \times G/\Gamma \times G) \subseteq A$, i.e. $hBh^{-1} \subseteq A$ for some $h \in H$. Suppose $hBh^{-1} \subsetneq A$ and let $\beta' \in [\beta]$ such that $\beta'(S \times G/\Gamma \times G) \subseteq B$. Fix $y = g_1[e] \in G/\Gamma$ and define $\Phi_c\colon Z(S \times G/\Gamma \times G; H) \to Z(S \times \Gamma; H)$, $\Phi_c(\beta') = \alpha'$ by

$$\alpha'(s, \gamma) = \Phi_c(\beta')(s, \gamma) = \beta'(g_1 s, y, g_1 \gamma g_1^{-1}) \in B \subsetneq h^{-1}Ah$$

(when we pass to the equivalence classes of cocycles this definition turns out to be independent of the choice of $y \in G/\Gamma$). Then it follows that $\alpha'(S \times \Gamma) \subseteq B \subsetneq h^{-1}Ah$, i.e. $h\alpha'(S \times \Gamma)h^{-1} \subsetneq A$ and this is impossible since $h\alpha' h^{-1} \sim \alpha$ and A is the algebraic hull of α. □

Remark. As clear from the proof, the result of Proposition 4.4 does not depend on the fact that Γ is a lattice in G. The bijection exists for every closed subgroup; however, if we want to have the correspondence of the algebraic hulls, we need to restrict our attention to subgroups whose action is still ergodic.

The following result is straightforward from the definitions and we shall omit the proof.

Lemma 4.5. *Let $\alpha\colon S \times G \to H$ be a cocycle with algebraic hull H, $H' \subseteq H$ an algebraic normal subgroup, $p\colon H \to H/H' = K$ the canonical projection and $\beta = p \circ \alpha\colon S \times G \to K$. Then the algebraic hull of β is K.*

Now we are ready to prove Theorem 3.6.

Proof. Let H be the algebraic hull of α and $H_1 \subseteq H$ the algebraic hull of α_Γ. Suppose that we know the existence of a finite H-invariant measure on H/H_1 and let us proceed with this assumption. By the Borel Density Theorem, [D], there is an algebraic subgroup H' of H_1 which is cocompact and normal in H, i.e. $H/H' = K$ is a compact group. Let us consider the cocycle $\beta\colon S \times G \to K$ defined by $\beta = p \circ \alpha$ where $p\colon H \to H/H'$ is the canonical projection. By Lemma 4.5, K is the algebraic hull of β hence the G action on $S \times_\beta K$ is ergodic (Corollary 4.2);

then the restriction to Γ of the G-action on $S \times_\beta K$ is still ergodic (Proposition 4.1), hence the algebraic hull of β_Γ is $K = H/H'$ (again Corollary 4.3). Since $H' \subseteq H_1$, and H_1 is the algebraic hull of α_Γ, if we suppose $H_1 \subsetneq H$, then we would have that the algebraic hull of β_Γ would be $p(H_1) = (H_1 H')/H' = H_1/H' \subsetneq H/H' = K$ (Lemma 4.5). Hence $H_1 = H$.

Now we go back to the proof of the existence of a finite H-invariant measure on H/H_1. From Proposition 4.4 we have, corresponding to α_Γ, a cocycle $\tilde{\alpha}\colon S \times G/\Gamma \times G \to H$, whose algebraic hull is again H_1. Then there is an $\tilde{\alpha}$-invariant map $\varphi\colon S \times G/\Gamma \to H/H_1$, i.e. for every $g \in G$,

$$\varphi(gs, gx) = \varphi(g(s,x)) = \tilde{\alpha}(s,x,g)\varphi(s,x) = \alpha(s,g)\varphi(s,x)$$

for a.e. s, a.e. x. For every $s \in S$ we can define a map $\varphi_s\colon G/\Gamma \to H/H_1$ by $\varphi_s(x) = \varphi(s,x)$ so that the condition on the $\tilde{\alpha}$-invariance becomes $\varphi_{gs}(gx) = \alpha(s,g)\varphi_s(x)$. If μ is the finite invariant measure on G/Γ, define a map $\Phi\colon S \to M(H/H_1)$ by $\Phi(s) = (\varphi_s)_*\mu = \nu_s$. We claim that Φ is an α-invariant map, i.e. for every $g \in G$, $\Phi(gs) = \alpha(s,g)\Phi(s)$,a.e. $s \in S$. Let $A \subseteq H/H_1$: then by the $\tilde{\alpha}$-invariance of φ and the G-invariance of μ we have

$$\begin{aligned}
\Phi(gs)(A) = \mu(\varphi_{gs}^{-1}(A)) &= \mu(\{x \in G/\Gamma | \varphi(gs,x) \in A\}) = \\
&= \mu(\{gy \in G/\Gamma \mid \alpha(s,g)\varphi_s(y) \in A\}) = \\
&= \mu(\{y \in G/\Gamma \mid \varphi_s(y) \in \alpha(s,g)^{-1}A\}) = \\
&= \mu(\varphi_s^{-1}(\alpha(s,g)^{-1}A)) = \\
&= \Phi(s)(\alpha(s,g)^{-1}A) = \\
&= (\alpha(s,g)\Phi(s))(A)
\end{aligned}$$

which proves the α-invariance of Φ. By the remarks on $M(H/H_1)$, Proposition 2.1 and the Cocycle Reduction Lemma, α is equivalent to a cocycle taking values in the stabilizer H_ν of a measure ν in $M(H/H_1)$. Since the stabilizer of any measure in $M(H/H_1)$ is algebraic, we know that H_ν is an algebraic group which cannot be a proper subgroup of H, since the algebraic hull of α is H. Therefore $H_\nu = H$ and this means that ν is an H-invariant probability measure on H/H_1. □

Remark. Before concluding this section we want to give an example to point out how this theorem about algebraic hulls is not equivalent to ergodicity.

Let M be a manifold on which $\mathbb{R}$ acts ergodically and such that the restriction of the action to $\mathbb{Z}$ is still ergodic. Let us consider the cocycle $\alpha\colon M \times \mathbb{R} \to S^1$ defined as $\alpha(m,t) = \pi(t)$, where $\pi\colon \mathbb{R} \to S^1$ is $\pi(t) = e^{2\pi i t}$; then the algebraic hull of α_Γ is just the identity in S^1. Now form the induced $\mathbb{R}$-space $M \times_\alpha S^1$; ergodicity of $\mathbb{Z}$ on M implies ergodicity of $\mathbb{R}$ on $M \times_\alpha \mathbb{R}/\mathbb{Z} = M \times_\alpha S^1$, and Corollary 4.3 shows that the algebraic hull of α is S^1.

5. Invariant vector fields on the 2-sphere. We shall give now some examples to better illustrate the hypothesis on the orbits in M. First of all notice that, at least in the case in which M is not compact, such a hypothesis is essential. In fact if G acts on itself by translations, the lifting to G of any non-constant vector field on G/Γ is Γ-invariant but not G-invariant.

In the case in which M is compact the question is a little more delicate. Let $G = SL(2,\mathbb{R})$ act on $M = \mathbb{R}^2 \cup \{\infty\} \simeq S^2$ by fractional linear transformations and let Γ be any lattice in G. We shall prove that even though the only continuous G-invariant vector field on S^2 is the trivial one, it is possible to construct non-trivial continuous Γ-invariant vector fields on S^2; however, there are no such vector fields (i.e. Γ-invariant) which are also smooth and therefore one might make the audacious conjecture that if the manifold is compact and we restrict our attention to smooth structures, the hypothesis on the orbits can be eliminated. For the $SL(2,\mathbb{R})$-action on the 2-sphere it is well known that there are three orbits, all of which are locally closed: the x-axis $\mathbb{R}' = \mathbb{R} \cup \{\infty\} \simeq SL(2,\mathbb{R})/P$ (where P is the group of the upper triangular matrices) and the two half-planes, diffeomorphic to $SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

Let $X\colon M \to TM$ be a Γ-invariant vector field: then its restriction to the extended x-axis must be zero. In fact, by Theorem 3.4, $X|_{\mathbb{R}'}$ must be G-invariant, P not being compact: moreover if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and we write $X_{(x_0,y_0)} = a_1(x_0,y_0)\left.\dfrac{\partial}{\partial x}\right|_{(x_0,y_0)} + a_2(x_0,y_0)\left.\dfrac{\partial}{\partial y}\right|_{(x_0,y_0)}$, then the G-invariance of X, $dgX_{(x,0)} = X_{g(x,0)}$, applied to the functions $f_i(x_1,x_2) = x_i$ for $i = 1,2$, gives $a_i\left(\dfrac{ax+b}{cx+d}, 0\right) = a_i(x,0)\dfrac{1}{(cx+d)^2}$: when $x = 0$ we have $a_i\left(\dfrac{b}{d}, 0\right) = \dfrac{1}{d^2}a_i(0,0)$, hence, by transitivity, $a_i(x,0) = 0$ for every $x \in \mathbb{R}'$, $i = 1,2$.

Notice that there are no non-trivial G-invariant vector fields on the upper half-plane $\mathcal{H}$: for, since $SO(2,\mathbb{R}) = \mathrm{Stab}_G(0,1)$ acts on $TM_{(0,1)}$ by rotation, every G-invariant vector field must be zero at $(0,1)$ and hence at every point in the orbit. On the other hand, we can construct a continuous Γ-invariant vector field on $\mathcal{H}$ that can be extended continuously to the boundary. Namely if F is a fixed fundamental domain for Γ in G acting on $\mathcal{H}$, given any $(x_0',y_0') \in \mathcal{H}$ there exist $\gamma \in \Gamma$ and $(x_0,y_0) \in F$ such that $\gamma(x_0,y_0) = (x_0',y_0')$. Then if X is a continuous vector field compactly supported in F which is zero on the boundary of F, we can extend X to a continuous Γ-invariant vector field on $\mathcal{H}$ defining $X_{(x_0',y_0')} = X_{\gamma(x_0,y_0)} = d\gamma X_{(x_0,y_0)}$: note that this is well defined since (x_0,y_0) and γ are uniquely determined unless (x_0',y_0') is on the boundary of some fundamental domain in which case we set $X_{(x_0',y_0')} = 0$. To see that X can be extended continuously to the boundary let ω_z^0 be the usual metric in $\mathbb{R}^2 \simeq \mathbb{C}$ and $\omega_z = \dfrac{1}{Im^2(z)}\omega_z^0$ be the hyperbolic metric which is well known to be G-invariant. If $\{z_n\} \in \mathcal{H}$ is any sequence then there exists $\{\gamma_n\} \in \Gamma$ and $\{f_n\} \in F$ such that $z_n = \gamma_n f_n$. By Γ-invariance of X and G-invariance of the hyperbolic metric we have

$$\begin{aligned}
\|X_{z_n}\|^2 = \omega_{z_n}^0(X_{z_n}, X_{z_n}) &= Im^2(z_n)\omega_{z_n}(X_{z_n}, X_{z_n}) = \\
&= Im^2(z_n)\omega_{\gamma_n f_n}(X_{\gamma_n f_n}, X_{\gamma_n f_n}) = \\
&= Im^2(z_n)\omega_{\gamma_n f_n}(d\gamma_n X_{f_n}, d\gamma_n X_{f_n}) = \\
&= Im^2(z_n)\omega_{f_n}(X_{f_n}, X_{f_n})
\end{aligned}$$

If $Im(z_n) \to 0$ then $\|X_{z_n}\|^2 \to 0$, provided that $\omega_{f_n}(X_{f_n}, X_{f_n}) = \dfrac{1}{Im^2(f_n)}\|X_{f_n}\|^2$

is bounded, which is true because X is continuous and compactly supported in F.

We shall prove now that if $X_{(x,y)} = a_1(x,y)\dfrac{\partial}{\partial x} + a_2(x,y)\dfrac{\partial}{\partial y}$ is a smooth Γ-invariant vector field on S^2 then it must be trivial. The argument will consist in showing first that $\left.\dfrac{\partial a_i}{\partial y}\right|_{(x,0)} = 0$ for every $(x,0) \in \mathbb{R}^2 \cup \{\infty\}$; then using this result together with some estimate for the coefficients a_i's, expressed in their Taylor expansion in the vertical direction, we shall prove that actually $a_i(x,y) = 0$ for every $(x,y) \in \mathbb{R}^2 \cup \{\infty\}$, $i = 1,2$. First of all notice that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ then

$$dg_{(x,0)} = \begin{pmatrix} \dfrac{1}{(cx+d)^2} & 0 \\ 0 & \dfrac{1}{(cx+d)^2} \end{pmatrix}$$

so that we can identify $dg_{(x,0)}$ with the map $dg_{(x,0)} \colon \mathbb{R} \to \mathbb{R}$, defined as $dg_{(x,0)} = (cx+d)^{-2}$. Then a straighforward computation shows that for every $\gamma \in \Gamma$ we have $\dfrac{\partial a_i(\gamma(x,0))}{\partial y} = d\gamma_{(x,0)}\dfrac{\partial a_i(x,0)}{\partial y}$, i.e. $\dfrac{\partial a_i}{\partial y}$'s are Γ-invariant. Let us consider the set $D \subset \mathbb{R} \times \{0\}$ of points which are fixed by some hyperbolic element $\gamma \in \Gamma$. Then if $(x_0,0) \in D$ with $\gamma(x_0,0) = (x_0,0)$, the above equation becomes $\dfrac{\partial a_i(x_0,0)}{\partial y} = d\gamma_{(x_0,0)}\dfrac{\partial a_i(x_0,0)}{\partial y}$; hence it will suffice to prove that $d\gamma_{(x_0,0)} \neq 1$, so that $\dfrac{\partial a_i(x_0,0)}{\partial y} = 0$. In fact, since $\Gamma \subset G$ is a lattice, then D is dense in $\mathbb{R}' \times \{0\}$, [K], so that, for $i = 1,2$, $\left.\dfrac{\partial a_i}{\partial y}\right|_{\mathbb{R}' \times \{0\}} \equiv 0$ by continuity.

Lemma 5.1. *Let $\gamma \in \Gamma$ be a hyperbolic element. Then there are only two points $(x_1,0)$, $(x_2,0) \in \mathbb{R}' \times \{0\}$ such that $d\gamma_{(x_1,0)} = d\gamma_{(x_2,0)} = 1$, neither of which is a γ-fixed point.*

Proof. A direct computation would prove the Lemma; however, we prefer to proceed in the following way. Let $I(\gamma)$ be the isometric circle of the transformation γ. Then the only points in $\mathbb{R}'$ where $d\gamma_{(x,0)} = 1$ are the points where $I(\gamma)$ intersects the extended x-axis and we claim that these are not γ-fixed points. In fact γ maps $I(\gamma)$ into $I(\gamma^{-1})$ and, since γ is hyperbolic, $I(\gamma)$ and $I(\gamma^{-1})$ do not intersect. $\square$

Now, to complete our proof, let us fix a point $(x,0) \in \mathbb{R}^2$; then for h in some interval $0 < h < \epsilon$ we have $a_i(x,h) = a_i(x,0) + \left.\dfrac{\partial a_i}{\partial y}\right|_{(x,0)} \cdot h + \dfrac{1}{2}\left.\dfrac{\partial^2 a_i}{\partial y^2}\right|_{(x,h')} \cdot h^2 = \dfrac{1}{2}\left.\dfrac{\partial^2 a_i}{\partial y^2}\right|_{(x,h')} \cdot h^2$, where $0 < h' < h$, for $i = 1,2$. Let I be an interval containing x: since X is smooth there exist $M_i = \sup\{M_{i,x} : x \in I\}$ where $M_{i,x} = \sup\left\{\dfrac{1}{2}\left|\left.\dfrac{\partial^2 a_i}{\partial y^2}\right|_{(x,h')}\right| : 0 < h' < h\right\}$, hence $|a_i(x,h)| \leqslant M_i h^2$ for every $x \in I$, $0 < h < \epsilon$, $i = 1,2$. Since the hyperbolic metric is G-invariant, for every $(x,y) \in \mathcal{H}$ we have where $K = \omega_{(x,y)}(X,X)$ is constant on the upper half plane. But, by the estimate above on a_i's, this implies that $h^2 K \leqslant (M_1^2 + M_2^2)h^4 = Mh^4$ and hence $K \leqslant Mh^2$ for every $0 < h < \epsilon$; it follows that $K = 0$, that is $X \equiv 0$.

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