A NEW FAMILY OF REPRESENTATIONS OF VIRTUALLY FREE GROUPS

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Abstract. We construct a new family of irreducible unitary representations of a finitely generated virtually free group Λ. We prove furthermore a general result concerning representations of Gromov hyperbolic groups that are weakly contained in the regular representation, thus implying that all the representations in this family can be realized on the boundary of Λ. As a corollary, we obtain an analogue of Herz majorization principle.

1. Introduction

Free groups are ubiquitous in mathematics and their representation theory has been widely studied. However, a finitely generated free group is not type I, and consequently there is no hope of explicitly parameterizing its unitary dual, that is the space of equivalence classes of its irreducible unitary representations. In fact, to construct a unitary representation of Γ it is only necessary to fix a Hilbert space $\mathcal{H}$ and to choose a unitary operator for each generator. A "random" choice will yield an irreducible representation. Likewise, because the free group is not type I, the direct integral decomposition of a given unitary representation into irreducibles is generally speaking not unique. Much work on the representation theory of the free group has concentrated on constructing specific interesting irreducible representations, on proving them to be unitary, and on studying their common properties.

If we restrict our attention to those representations that are weakly contained in the regular representation the situation drastically changes. For brevity we shall say that a representation is *tempered* if it is weakly contained in the regular representation. Using the fact that the reduced C^* algebra of Γ is simple ([Pow75]), one can prove (see [KS01]

Date: June 11, 2012.

1991 *Mathematics Subject Classification.* Primary; 22D10, 43A65. Secondary: 15A48, 22E45, 22E40.

Key words and phrases. free group, Gromov hyperbolic group, irreducible unitary representation, boundary realization, cross product, Herz majorization principle.

A. I. was partial supported by the Swiss National Science Foundation project 2000021-127016/2; M. G. K. and T. S. were partially supported by PRIN.

or Theorem 4.3) that a tempered representation $(\pi, \mathcal{H})$ can always be realized as a subrepresentation of a *boundary representation* (see §2.1 for the definition). This implies in particular that the Hilbert space $\mathcal{H}$ can be chosen to be a subspace of a direct integral of a measurable field of Hilbert spaces $\mathcal{H}_{\partial\Gamma} = \int_{\partial\Gamma}^{\oplus} \mathcal{H}_x d\mu(x)$ over the boundary $\partial\Gamma$ of Γ for a suitable quasi-invariant measure μ which depends on the representation.

In 2004, a large family of irreducible unitary tempered representations of the free group, the so-called *multiplicative* representations, was introduced [KS04]. Although these representations have a very concrete and seemingly elementary definition, this family is large enough to include all known specific irreducible tempered representations constructed using the action of Γ on its Cayley graph.

In [IKS] we extended the class in [KS04] to include also representations that are obtained with a similar procedure as in [KS04] but are only *finitely reducible*. This has the advantage that this enlarged class of representations, called the class $\mathbf{Mult}(\Gamma)$, is now stable under many natural operations, such as the restriction to a subgroup and the induction to a free supergroup (see [IKS]). Moreover, although the construction presented in [KS04] seems to depend on the choice of generators, the class $\mathbf{Mult}(\Gamma)$ is independent on that choice and in fact it is invariant under the action of $\mathrm{Aut}(\Gamma)$. This fact is not true for example for the restriction to the free group in two generators of the spherical series of the group of automorphisms of the homogeneous tree of valency four. (See Remark 2.3(2) for more on the irreducibility of these representations.)

In this paper, in analogy with the case of the free group, we define a new class of representations for virtually free groups. These groups include for example $\mathrm{PSL}(2, \mathbf{Z}) \cong \mathbf{Z}_2 * \mathbf{Z}_3$, whose commutator subgroup is a torsion-free surface group and whose abelianization $\mathrm{PSL}(2, \mathbf{Z})_{\mathrm{ab}} \cong \mathbf{Z}_2 \times \mathbf{Z}_3$ has order six. Furthermore, virtually free groups are Gromov hyperbolic and can be realized as fundamental groups of finite graph of finite groups, [KPS73].

We define a class $\mathbf{Mult}(\Lambda)$ of unitary representations of a finitely generated virtually free group Λ by inducing a representation of the class $\mathbf{Mult}(\Gamma)$ from a (in fact, any) free subgroup Γ of finite index (see § 3) . For these classes of representations we prove the following

Theorem 1. *Let Λ be a virtually free group.*

(1) *The classes $\mathbf{Mult}(\Lambda)$ and $\mathbf{Mult}_{irr}(\Lambda)$ are non-empty and $\mathrm{Aut}(\Lambda)$-invariant (Corollary 3.5).*

(2) *The representations in the class* $\mathbf{Mult}(\Lambda)$ *are weakly contained in the regular representation (Corollary 3.6).*
(3) *The representations of the class* $\mathbf{Mult}(\Lambda)$ *are subrepresentation of cocycle representations, that is representations of the cross product* $\Lambda \ltimes \mathcal{C}(\partial\Lambda)$ *(Corollary 4.5).*

As we mentioned earlier, the representations of the class $\mathbf{Mult}(\Gamma)$ encompass all tempered representations of the free group Γ that arise from the embedding of Γ into the group of automorphisms of its Cayley graph (with respect to some set of generators). On the other hand, to the authors' knowledge, we are not aware of other realizations of any of the representations in the class $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$ of a virtually free group Λ. Constructions similar to ours (in the cocompact case) can be found for example in [BM11], where the authors show the irreducibility of the quasi-regular representation of a compact surface group $\pi_1(\Sigma)$ on the geodesic boundary[1] $\partial\Sigma$ with respect to the Patterson–Sullivan measure. Likewise, in [BdlH97], the authors show that if $H < L$ are discrete groups such that $H = \mathrm{Comm}_L(H)$, then the induction to L of any finite dimensional irreducible representation of H remains irreducible. None of these results seem to have a nonempty intersection with our construction.

The last item in the above theorem follows from the following result that was already known for free groups and we record here for a general Gromov hyperbolic group (see Theorem 4.3):

Theorem 2. *Let G be a torsion free not almost cyclic Gromov hyperbolic group. Then every tempered representation of G is a subrepresentation of a cocycle representation with respect to some quasi-invariant measure.*

If the representation is irreducible, the measure can be taken to be ergodic.

Conversely, every cocycle representation is tempered.

As a consequence of this result we prove the following analogue of Herz majorization principle:

Corollary 1. *Let $(\pi, \mathcal{H})$ be a tempered representation of a torsion free not almost cyclic Gromov hyperbolic group G and let v be any vector in $\mathcal{H}$. Then there exists a quasi-invariant measure μ on ∂G and a positive*

[1] One word of warning for the reader: what the authors in [BM11] call "boundary representation" is not what is referred to with the same terminology in this paper, but what we call "quasi-regular" representation in Definition 4.2.

function $f \in L^2(\partial G, d\mu)$ *with* $\|f\|_2 = \|v\|_{\mathcal{H}}$ *such that*

$$|\langle \pi(x)v, v\rangle| \leq |\langle \rho(x)f, f\rangle|$$

where ρ *is the quasi-regular representation on* $L^2(\partial G, d\mu)$.

The measure on ∂G must however depend on the tempered representation, thus implying that a Harish-Chandra function cannot exist (see Remark 4.9) and exhibiting one more instance of the fact that Gromov hyperbolic group behave morally as rank one groups.

We remark that the above construction relies not only upon the stability properties of the class $\mathbf{Mult}(\Gamma)$ of a free group Γ (which were proven in [IKS]), but also of the non-obvious corresponding properties of the extension of multiplicative representations to boundary representations (see for example Theorem B.1).

The structure of the paper is as follows. In § 2 we recall the definition of boundary representation of a free group – and, more generally, of a Gromov hyperbolic group – and the construction of the boundary multiplicative representations of the free group; we recall moreover from [IKS] the stability properties of the class of representations of the free group obtained from matrix systems with an inner product. In § 3 we define the classes $\mathbf{Mult}(\Lambda)$ and $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$ of representations of a finitely generated virtually free group Λ obtained by induction from any free subgroup of finite index and we show both that $\mathbf{Mult}(\Lambda)$ and $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$ are $\mathrm{Aut}(\Lambda)$-invariant and that these representations are tempered. In § 4 we prove that (irreducible) tempered representations of a Gromov hyperbolic group G are subrepresentations of cocycle representations with respect to an (ergodic) measure and we deduce the analogue of Herz majorization principle (Corollary 4.7). In the Appendix B we prove the essential stability results for multiplicative boundary representations that are not proven in [IKS].

Acknowledgments: The first named author thanks the Institute Mittag-Leffler in Stockholm for their warm hospitality in the last phase of the preparation of this paper. Likewise, the second named author is grateful to the Forschungsinstitut für Mathematik at ETH, Zürich for its hospitality. Finally, we thank the referee for many useful comments.

2. Preliminaries

2.1. **Boundary Representations.**

Definition 2.1. Let G be any discrete group, $\mathcal{A}$ be a commutative C^*-algebra and $\lambda : G \to \mathrm{Aut}(\mathcal{A})$ a homomorphism of G into the group of isometric automorphisms of $\mathcal{A}$. A *covariant representation* of $(G, \mathcal{A})$ on a Hilbert space $\mathcal{H}$ is a triple $(\pi, \alpha, \mathcal{H})$ where

- $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G;
- $\alpha : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a C^*-representation of $\mathcal{A}$ in the space of bounded linear operators on $\mathcal{H}$;
- for all $\gamma \in G$ and $A \in \mathcal{A}$

$$\pi(\gamma)\alpha(A)\pi(\gamma^{-1}) = \alpha\big(\lambda(\gamma)A\big) .$$

Two covariant representations $(\pi, \alpha, \mathcal{H})$ and $(\rho, \beta, \mathcal{L})$ of G and $\mathcal{A}$ are *equivalent* if there exists a unitary operator $J : \mathcal{H} \to \mathcal{L}$, such that for all $\gamma \in G$ and all $A \in \mathcal{A}$,

$$\rho(\gamma)\, J = J\, \pi(\gamma) \qquad \text{and} \qquad \beta(A)\, J = J\, \alpha(A) .$$

If K is any compact metrizable space on which G acts continuously and by isometries, the space of complex valued functions $\mathcal{C}(K)$ is a C^*-algebra naturally endowed with a continuous isometric action of G, $\lambda : G \to \mathrm{Aut}\,\big(\mathcal{C}(K)\big)$, defined by

$$\lambda(\gamma)F(k) := F(\gamma^{-1}k) ,$$

for all $F \in \mathcal{C}(K)$, $\gamma \in G$ and $k \in K$.

In the case in which G is a Gromov hyperbolic group, the space K can be taken to be the boundary of the Cayley graph associated to a fixed generating system, which we denote by ∂G. For the sake of the reader, we recall the definition of ∂G in the Appendix A. We mention here only that ∂G is a compact metrizable space with the G-action defined by $(\gamma, \omega) \mapsto \gamma^{-1}\omega$ and that different generating sets correspond to homeomorphic boundaries.

Definition 2.2. A *boundary representation* of a hyperbolic group G on $\mathcal{H}$ is a covariant representation $(\pi, \alpha, \mathcal{H})$ of $\big(G, \mathcal{C}(\partial G)\big)$.

The reader who is familiar with crossed-product C^*-algebras will recognize that a boundary representation is nothing but a representation of the crossed product C^*-algebra $G \ltimes \mathcal{C}(\partial G)$ (see § 4).

General Gromov hyperbolic groups will be considered again in their full generality in § 4, while in the rest of this section we will consider only free groups.

2.2. **Boundary Multiplicative Representations of the Free Group.** We begin with the definition of *multiplicative representation* in the context of finitely generated free groups, referring to [KS04] for details and proofs.

Let $\mathbb{F}_A$ be a free group with a finite symmetric set of free generators A. A *matrix system* (V_a, H_{ba}), is an assignment of a complex vector space $a \mapsto V_a$ for every generator $a \in A$ and a linear map $H_{ba} : V_a \to V_b$, for every $a, b \in A$, such that $H_{ba} = 0$ whenever $ba = e$. An *invariant subsystem* (W_a, H_{ba}) of the matrix system (V_a, H_{ba}) is an assignment of vector subspaces $a \mapsto W_a \subseteq V_a$ such that $H_{ba}W_a \subset W_b$ for all $a, b \in A$. If (W_a, H_{ba}) is an invariant subsystem of (V_a, H_{ba}), the *quotient system* $(\widetilde{V}_a, \widetilde{H}_{ba})$ is the assignment $a \mapsto \widetilde{V}_a := V_a/W_a$ such that $\widetilde{H}_{ba}\widetilde{v}_a = \widetilde{H_{ba}v_a}$, for any representative v_a of $\widetilde{v}_a \in \widetilde{V}_a$. A system (V_a, H_{ba}) is called *irreducible* if it is nonzero and admits no nontrivial invariant subsystems.

We endow $\mathbb{F}_A$ with the word metric $d(x, e) := |x|$ with respect to the generating set A. We say that a function

$$f : \mathbb{F}_A \to \bigsqcup_{a \in A} V_a$$

is *multiplicative* if there exists $N \geq 0$, depending only on f, such that for all x with $|x| \geq N$

$$\begin{aligned} &f(x) \in V_a && \text{if } x = x'a \text{ is reduced} \\ &f(xb) = H_{ba}f(x) && \text{if } x = x'a \text{ is reduced and } ba \neq e\,. \end{aligned} \tag{2.1}$$

We denote by $\mathcal{H}^\infty(V_a, H_{ba})$ (or by $\mathcal{H}^\infty$ if no confusion arises) the quotient of the space of multiplicative functions with respect to the equivalence relation according to which two multiplicative functions are equivalent if they differ only on finitely many words.

If for every $a \in A$ the V_a's are equipped with a positive definite sesquilinear form B_a and if these forms satisfy the *compatibility condition*

$$B_a(v_a, v_a) = \sum_{b \in A} B_b(H_{ba}v_a, H_{ba}v_a) \tag{2.2}$$

for all $v_a \in V_a$, then

$$\langle f_1, f_2 \rangle := \sum_{|x|=N} \sum_{\substack{a \\ |xa|=|x|+1}} B_a\big(f_1(xa), f_2(xa)\big) \tag{2.3}$$

defines an inner product on $\mathcal{H}^\infty$, where N should be taken to be large enough that both f_1 and f_2 satisfy (2.1) outside the ball of radius N. We remark that, up to a normalization, every matrix system

(V_a, H_{ba}) admits a compatible tuple $(B_a)_{a\in A}$ of positive semidefinite forms. When the matrix system is irreducible, then each B_a is strictly definite positive and, up to multiple scalars, it is also unique. Whether the system is irreducible or not, the triple (V_a, H_{ba}, B_a) will be called a *matrix system with inner product*. We can hence define a representation of $\mathbb{F}_A$ on $\mathcal{H}^\infty(V_a, H_{ba})$ by

$$\big(\pi(x)f\big)(y) := f(x^{-1}y)\,,$$

which can be proved to be unitary. If $\mathcal{H}(V_a, H_{ba}, B_a)$ is the completion of $\mathcal{H}^\infty(V_a, H_{ba})$ with respect to the inner product in (2.3), then π extends to a unitary representation on $\mathcal{H}(V_a, H_{ba}, B_a)$, which we called *multiplicative*.

The next step is to show that multiplicative representations extend in a natural way to boundary representations of the free group.

The boundary $\partial\mathbb{F}_A$ of a free group $\mathbb{F}_A$ consists of the set of infinite reduced words, with the topology defined by the basis

$$\partial\mathbb{F}_A(x) := \{\omega \in \partial\mathbb{F}_A : \text{ the reduced word for } \omega \text{ starts with } x\}\,,$$

for all $x \in \mathbb{F}_A$, $x \neq e$. The sets $\partial\mathbb{F}_A(x)$ are both open and closed in $\partial\mathbb{F}_A$ and $\partial\mathbb{F}_A$ is a compact (as well as Hausdorff, perfect, separable, and totally disconnected) space. In order to extend a given unitary representation $(\pi, \mathcal{H})$ of $\mathbb{F}_A$ to a boundary representation, we need to define an algebra C^*-homomorphism $\alpha : \mathcal{C}(\partial\mathbb{F}_A) \to \mathcal{L}(\mathcal{H})$ satisfying

$$\pi(x)\alpha(F)\pi(x^{-1}) = \alpha\big(\lambda(x)F\big)\,, \tag{2.4}$$

for any $x \in \mathbb{F}_A$ and $F \in \mathcal{C}(\partial\mathbb{F}_A)$.

For every $x \in \mathbb{F}_A$, $x \neq e$, let $\mathbf{1}_{\partial\mathbb{F}_A(x)}$ denote the characteristic function of $\partial\mathbb{F}_A(x)$. Since the subalgebra spanned by the functions $\{\mathbf{1}_{\partial\mathbb{F}_A(x)}\}_{x\in\mathbb{F}_A}$ is a dense C^*-subalgebra of $\mathcal{C}(\partial\mathbb{F}_A)$, it is sufficient to define $\alpha_\pi(\mathbf{1}_{\partial\mathbb{F}_A(x)})$ for every x, and in fact on the dense subspace $\mathcal{H}^\infty \subset \mathcal{H}$. Denote by $\mathbf{1}_{\mathbb{F}_A(x)}$ the characteristic function of the cone

$$\mathbb{F}_A(x) := \{y \in \mathbb{F}_A : \text{ the reduced word for } y \text{ starts with } x\} \tag{2.5}$$

and define $\alpha_\pi(\mathbf{1}_{\partial\mathbb{F}_A(x)}) : \mathcal{H}^\infty \to \mathcal{H}^\infty$ by setting

$$\big(\alpha_\pi(\mathbf{1}_{\partial\mathbb{F}_A(x)})f\big)(y) := \mathbf{1}_{\mathbb{F}_A(x)}(y)f(y) = \begin{cases} f(y) & \text{if } y \in \mathbb{F}_A(x) \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.6}$$

A routine calculation shows that (2.4) is verified and hence every multiplicative representation $(\pi, \mathcal{H})$ gives rise in a natural way to a boundary representation $(\pi, \alpha_\pi, \mathcal{H})$ of $\mathbb{F}_A$.

Remark 2.3. (1) When a boundary representation is considered as a representation of $\mathbb{F}_A$ it is always weakly contained in the regular representation. This follows from general considerations since $\mathbb{F}_A$ acts amenably on $\partial\mathbb{F}_A$; a two pages proof specifically for the case of the free group can be found in [KS96, § 2].

(2) In [KS04] it is shown that multiplicative representations built from an irreducible system are irreducible as boundary representations, (that is as representations of the cross-product $\mathbb{F}_A \ltimes \mathcal{C}(\partial\mathbb{F}_A)$, see § 4) while, as representations of $\mathbb{F}_A$, they are either irreducible or, in some special cases, are sum of two irreducible nonequivalent representations.

2.3. **Stability Properties of Multiplicative Boundary Representations.** The definition of multiplicative representation seems to depend on the generating set A that we have fixed. We shall see that the dependence is only apparent, as soon as we allow general (not only irreducible) matrix systems. The advantage of considering general matrix systems is that the new class of representations so obtained is closed under change of generators, restriction and induction. The price to pay is not so high, as the following result shows:

Theorem 2.4 ([IKS])**.** *If π is a representation constructed from a matrix system with inner product (V_a, H_{ba}, B_a), then π decomposes as an orthogonal direct sum with respect to $(B_a)_{a\in A}$ of a finite number of representations defined from irreducible matrix systems and the same is true when π is considered as a boundary representation.*

We proceed now to infer further properties of multiplicative boundary representations of $\mathbb{F}_A$.

Theorem 2.5 ([IKS])**.** *Let $\mathbb{F}_A$ be a group freely generated by the symmetric set A and let $\big(\pi, \alpha_\pi, \mathcal{H}(V_a, H_{ba}, B_a)\big)$ be a multiplicative boundary representation constructed from a matrix system with inner products $(V_a, H_{ba}, B_a)_{a\in A}$. If A' is another symmetric set of free generators such that $\mathbb{F}_A \cong \mathbb{F}_{A'}$, then there exists a multiplicative boundary representation $\big(\pi', \alpha_{\pi'}, \mathcal{H}(V_s, H_{ts}, B_s)\big)$ constructed from a matrix system with inner products $(V_s, H_{ts}, B_s)_{s\in A'}$, such that $\big(\pi, \alpha_\pi, \mathcal{H}(V_a, H_{ba}, B_a)\big)$ appears as a subrepresentation of $\big(\pi', \alpha_{\pi'}, \mathcal{H}(V_s, H_{ts}, B_s)\big)$.*

We can therefore denote a free group by Γ without any explicit dependence on a free generating set.

We warn the reader that there is no guarantee that changing generators will preserve the irreducibility of the system: in [IKS] it is shown that a representation of the principal series for the free group can be realized as a multiplicative representation from an irreducible matrix system, but, once the simplest nontrivial change of generator is performed, it arises from a quotient of a reducible matrix system.

Theorem 2.6 ([IKS]). *Let $\Gamma_0 \leq \Gamma$ be a subgroup of finite index in the free group Γ. Then:*

(1) *the restriction to Γ_0 of a multiplicative boundary representation $(\pi, \alpha_\pi, \mathcal{H})$ of Γ is a multiplicative boundary representation of Γ_0;*
(2) *if $(\pi', \alpha'_{\pi'}, \mathcal{H})$ is a multiplicative boundary representation of Γ_0, then the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi')$ is a boundary multiplicative representation of Γ.*

Strictly speaking, the theorems stated in this section are proved in [IKS] when all the representations involved are considered only as representations of the free group rather than as boundary representations. The extension of these results to the case of boundary representations is, in most of the cases, a straightforward verification. The one that is a bit more involved is the proof of Theorem 2.6(2): since it uses heavily the notations and the techniques of [IKS], we defer it to the appendix of this paper.

The above theorems lead to the following:

Definition 2.7. Let Γ be a finitely generated free group. A representation $\rho : \Gamma \to \mathcal{U}(H)$ is in the class $\mathbf{Mult}(\Gamma)$ if there exist a symmetric set A of free generators, a matrix system with inner product (V_a, H_{ba}, B_a), a dense subspace $M \subset \mathcal{H}$ and a unitary operator $J : \mathcal{H} \to \mathcal{H}(V_a, H_{ba}, B_a)$ such that

(1) J is an isomorphism between M and $\mathcal{H}^\infty(V_a, H_{ba}, B_a)$, and
(2) for all $m \in M$ and $x \in \Gamma$, $J\big(\rho(x)m\big) = \pi(x)(Jm)$, where π is the multiplicative representation constructed from (V_a, H_{ba}, B_a).

3. The Classes $\mathbf{Mult}(\Lambda)$ and $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$

Definition 3.1. We say that a representation π of a virtually free group Λ belongs to the class $\mathbf{Mult}(\Lambda)$ if there exists a finite index free subgroup $\Gamma \leq \Lambda$ and a representation π' in the class $\mathbf{Mult}(\Gamma)$ such that π is a component of $\mathrm{Ind}_\Gamma^\Lambda(\pi')$,

$$\mathbf{Mult}(\Lambda) := \big\{\pi \in \Lambda \,:\, \exists\, \pi' \in \mathbf{Mult}(\Gamma) \text{ for some free subgroup } \Gamma \leq \Lambda \\ \text{of finite index such that } \pi \leq \mathrm{Ind}_\Gamma^\Lambda(\pi')\big\}\,.$$

The next proposition shows that the subgroup Γ, that *a priori* depends on the representation π, can in fact be chosen uniformly for all representations.

Proposition 3.2. *Let Λ be a virtually free group and $\Gamma_0 < \Lambda$ any finite index free subgroup. Then any representation $\pi \in \mathbf{Mult}(\Lambda)$ is a subrepresentation of $\mathrm{Ind}_{\Gamma_0}^{\Lambda}(\pi')$ for some $\pi' \in \mathbf{Mult}(\Gamma_0)$.*

Proof. Let Γ_i, $i = 0, 1$, be free subgroups of finite index in Λ and denote by $\mathbf{Mult}_i(\Lambda)$ the corresponding class of subrepresentations induced from multiplicative representations of Γ_i. It is enough to show that $\mathbf{Mult}_0(\Lambda) = \mathbf{Mult}_1(\Lambda)$.

The stabilizer of the pair $\Gamma_0 \times \Gamma_1 \in \Lambda/\Gamma_0 \times \Lambda/\Gamma_1$ for the diagonal action of Λ is $\Gamma_0 \cap \Gamma_1$. Hence $\Lambda/\Gamma_0 \cap \Gamma_1$, as well as $\Gamma_0/\Gamma_0 \cap \Gamma_1$ and $\Gamma_1/\Gamma_0 \cap \Gamma_1$, are finite. Assume now that $\pi \in \mathbf{Mult}_0(\Lambda)$. By definition there exists a representation π_0 of Γ_0 such that π is a component of $\mathrm{Ind}_{\Gamma_0}^{\Lambda}(\pi_0)$. By general properties of induction (see for example [Mac76]), we have that

$$\begin{aligned}\pi &\leq \mathrm{Ind}_{\Gamma_0}^{\Lambda}(\pi_0) \leq \mathrm{Ind}_{\Gamma_0}^{\Lambda}\left(\mathrm{Ind}_{\Gamma_0\cap\Gamma_1}^{\Gamma_0}(\pi_0|_{\Gamma_0\cap\Gamma_1})\right) \\ &= \mathrm{Ind}_{\Gamma_0\cap\Gamma_1}^{\Lambda}(\pi_0|_{\Gamma_0\cap\Gamma_1}) = \mathrm{Ind}_{\Gamma_1}^{\Lambda}\left(\mathrm{Ind}_{\Gamma_0\cap\Gamma_1}^{\Gamma_1}(\pi_0|_{\Gamma_0\cap\Gamma_1})\right).\end{aligned}$$

By Theorem 2.6(1) we know that $\pi_0|_{\Gamma_0\cap\Gamma_1} \in \mathbf{Mult}(\Gamma_0 \cap \Gamma_1)$ and hence, by Theorem 2.6(2), $\mathrm{Ind}_{\Gamma_0\cap\Gamma_1}^{\Gamma_1}(\pi_0|_{\Gamma_0\cap\Gamma_1}) \in \mathbf{Mult}(\Gamma_1)$. It follows that $\pi \in \mathbf{Mult}_1(\Lambda)$ and, by symmetry, $\mathbf{Mult}_0(\Lambda) = \mathbf{Mult}_1(\Lambda)$. □

The representation in the class $\mathbf{Mult}(\Lambda)$ are not necessarily irreducible, but are however finitely reducible, as the following proposition shows:

Proposition 3.3. *Let Λ_0 be a subgroup of finite index of a group Λ and let $\pi : \Lambda_0 \to \mathcal{U}(\mathcal{H})$ be an irreducible representation. Then $\left(\mathrm{Ind}_{\Lambda_0}^{\Lambda}(\pi), \mathrm{Ind}_{\Lambda_0}^{\Lambda}(\mathcal{H})\right)$ is a finite sum of irreducible representations.*

Proof. Let us set $\rho := \mathrm{Ind}_{\Lambda_0}^{\Lambda}(\pi)$ and $\mathcal{L} := \mathrm{Ind}_{\Lambda_0}^{\Lambda}(\mathcal{H})$. Recall that

$$\mathcal{L} := \left\{ f : \Lambda \to \mathcal{H} : \ \pi(\gamma_0)f(\gamma) = f(\gamma{\gamma_0}^{-1}), \text{ for all } \gamma_0 \in \Lambda_0, \gamma \in \Lambda \right\}$$

on which Λ acts by

$$\big(\rho(\gamma)f\big)(\eta) := f(\gamma^{-1}\eta)$$

for all $\eta, \gamma \in \Lambda$. The fact that Λ_0 is of finite index in Λ, namely $\Lambda = \sqcup_{u\in D} u\Lambda_0$, where D is a finite set of representatives, induces a finite decomposition

$$\mathcal{L} = \bigoplus_{u\in D} \mathcal{L}_u\,, \tag{3.1}$$

where

$$\mathcal{L}_u := \left\{ f \in \mathcal{L} : , \mathrm{supp}(f) \subset u\Lambda_0 \right\}.$$

It is immediate to verify that for all $\eta \in \Lambda$ and $u \in D$, one has that $\rho(\eta)\mathcal{L}_u \subseteq \mathcal{L}_{\eta u}$ and hence

$$\rho(u\gamma_0 u^{-1})\mathcal{L}_u \subseteq \mathcal{L}_u$$

for all $\gamma_0 \in \Lambda_0$. Moreover for all $u \in D$, the evaluation operator

$$\begin{aligned} E_u : \mathcal{L}_u &\to \mathcal{H} \\ f &\mapsto f(u) \end{aligned}$$

is a unitary isomorphism with the property that

$$\pi(\gamma_0)\, E_u = E_u\, \rho(u\gamma_0 u^{-1})\,,$$

for all $\gamma_0 \in \Lambda_0$ and $u \in D$. In other words, E_u is an intertwining operator between $(\pi, \mathcal{H})$ and $(\rho|_{u\Lambda_0 u^{-1}}, \mathcal{L}_u)$. Since $(\pi, \mathcal{H})$ is irreducible, $(\rho|_{u\Lambda_0 u^{-1}}, \mathcal{L}_u)$ is irreducible as well.

Let now $T : \mathcal{L} \to \mathcal{L}$ be an intertwining operator for ρ. If $p_u : \mathcal{L} \to \mathcal{L}_u$ is the orthogonal projection, then, for all $u, v \in D$, $p_v\, T\, p_u$ intertwines $(\rho|_{(v\Lambda_0 v^{-1})\cap(u\Lambda_0 u^{-1})}, \mathcal{L}_v)$ and $(\rho|_{(v\Lambda_0 v^{-1})\cap(u\Lambda_0 u^{-1})}, \mathcal{L}_u)$. Since $(v\Lambda_0 v^{-1}) \cap (u\Lambda_0 u^{-1})$ is of finite index both in $v\Lambda_0 v^{-1}$ and in $u\Lambda_0 u^{-1}$, each of the above representations is finitely reducible, [Pog75, Corollary 2]. Hence the space of intertwining operators between $(\rho|_{(v\Lambda_0 v^{-1})\cap(u\Lambda_0 u^{-1})}, \mathcal{L}_v)$ and $(\rho|_{(v\Lambda_0 v^{-1})\cap(u\Lambda_0 u^{-1})}, \mathcal{L}_u)$ is finite dimensional, which forces the space of intertwining operators of $(\rho, \mathcal{L})$ to be finite dimensional as well. □

Definition 3.4. We say that a representation π of Λ belongs to the class $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$ if there exists a finite index free subgroup $\Gamma \leq \Lambda$ and a representation π' in the class $\mathbf{Mult}(\Gamma)$ such that π is an irreducible component of $\mathrm{Ind}_\Gamma^\Lambda(\pi')$,

$$\begin{aligned} \mathbf{Mult}_{\mathrm{irr}}(\Lambda) := \big\{\pi \in \Lambda \,:\, \exists\, \pi' \in \mathbf{Mult}(\Gamma) \text{ for some free subgroup } \Gamma \leq \Lambda \\ \text{of finite index such that } \pi \leq \mathrm{Ind}_\Gamma^\Lambda(\pi') \text{ and } \pi \text{ is irreducible}\big\}\,. \end{aligned}$$

The fact that this class is not empty follows from Proposition 3.3 and the fact that, by Theorem 2.4, any representation in the class $\mathbf{Mult}(\Gamma)$ is a finite sum of irreducible representations in the same class.

Corollary 3.5. *For a finitely generated virtually free group Λ the classes $\mathbf{Mult}(\Lambda)$ and $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$ are $\mathrm{Aut}(\Lambda)$-invariant.*

Proof. Let $\alpha \in \mathrm{Aut}(\Lambda)$, let $\Gamma < \Lambda$ be a free subgroup of finite index and let $\pi \in \mathbf{Mult}(\Gamma)$. For $\gamma \in \alpha(\Gamma)$ set $\pi^\alpha(\gamma) := \pi(\alpha^{-1}\gamma)$. An easy verification shows that

$$\mathrm{Ind}_{\alpha(\Gamma)}^\Lambda(\pi^\alpha) \simeq \mathrm{Ind}_\Gamma^\Lambda\big(\pi\big) \circ \alpha\,.$$

The fact that $\pi^\alpha \in \mathbf{Mult}(\alpha(\Gamma))$ ([IKS]) and Proposition 3.2 show the assertion. □

We may then conclude:

Corollary 3.6. *The representations of a finitely generated virtually free group Λ in the class* $\mathbf{Mult}(\Lambda)$ *(and hence* $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$*) are weakly contained in the regular representations.*

Proof. Since representations in the class **Mult** of a free group are weakly contained in the regular representation [KS96], the continuity of the induction map ensures that every representation in the class $\mathbf{Mult}(\Lambda)$ is weakly contained in the regular representation of Λ. □

4. Tempered Representations of Gromov Hyperbolic Groups

In this section we prove further properties of the representations in the class $\mathbf{Mult}(\Lambda)$, namely that they can be extended to boundary representations (Theorem 4.3). This will follow from general arguments in operator algebras which hold for a Gromov hyperbolic group and do not depend on the particular construction of the class $\mathbf{Mult}(\Lambda)$, but rather only on the fact that the representations in the class $\mathbf{Mult}(\Lambda)$ are tempered. In this section G is a Gromov hyperbolic group.

We saw already that boundary representations are associated with the action of G on its boundary ∂G and we mentioned that they are in fact representations of the crossed product $G \ltimes \mathcal{C}(\partial G)$. We recall here the definitions that will be needed for the proof of the next theorem and at the same time clarify the above assertions.

Let $\mathcal{A}$ be a C^*-algebra and let us denote by $\mathcal{A}[G]$ the space of finitely supported functions $G \to \mathcal{A}$,

$$\mathcal{A}[G] := \left\{ \sum_i^{<\infty} \zeta_i \delta_{\gamma_i} : \zeta_i \in \mathcal{A}, \gamma_i \in G \right\},$$

where δ_γ is the Kronecker function at $\gamma \in G$. If G acts on $\mathcal{A}$ by isometric automorphisms $\lambda : G \to \mathrm{Aut}(\mathcal{A})$, we endow $\mathcal{A}[G]$ with a C^*-algebra structure as follows. Define the sum of two elements of $\mathcal{A}[G]$ in the obvious way (as $\mathcal{A}$-valued functions on G) and let

$$(\zeta_1 \delta_{\gamma_1}) \cdot (\zeta_2 \delta_{\gamma_2}) := \big(\zeta_1 \lambda(\gamma_1) \zeta_2\big) \delta_{\gamma_1 \gamma_2} . \tag{4.1}$$

Use the distributive law to extend (4.1) to a product on $\mathcal{A}[G]$. Finally set

$$(\zeta \delta_\gamma)^* := \lambda(\gamma^{-1}) \zeta^* \delta_{\gamma^{-1}} .$$

In order to define a norm on $\mathcal{A}[G]$, take any covariant representation $(\pi, \alpha, \mathcal{H})$ of $(G, \mathcal{A})$ and for $f = \sum_i \zeta_i \delta_{\gamma_i} \in \mathcal{A}[G]$ define the operator

$$(\pi \ltimes \alpha)(f) := \sum_i \alpha(\zeta_i)\pi(\gamma_i)\,.$$

Define now the *universal norm*

$$\|f\| := \sup \|(\pi \ltimes \alpha)(f)\|_{\mathcal{H}}\,, \tag{4.2}$$

where the supremum is taken over all covariant representations $(\pi, \alpha, \mathcal{H})$ of G. The completion of $\mathcal{A}[G]$ with respect to the above norm is the *(full) crossed product C^*-algebra $G \ltimes \mathcal{A}$*.

Given a C^*-representation α of $\mathcal{A}$ on $\mathcal{H}$, one can always get a covariant representation $(\tilde{\lambda}, \tilde{\alpha})$ of $(G, \mathcal{A})$ on $\ell^2(G) \otimes \mathcal{H}$ by setting

$$\begin{aligned}\big(\tilde{\alpha}(\zeta)\xi\big)(\gamma) &:= \alpha(\lambda(\gamma^{-1})\zeta)\xi(\gamma)\\ \big(\tilde{\lambda}(\gamma')\xi\big)(\gamma) &:= \xi({\gamma'}^{-1}\gamma)\,,\end{aligned}$$

for all $\zeta \in \mathcal{A}$, $\gamma, \gamma' \in G$ and $\xi \in \ell^2(G) \otimes \mathcal{H}$. We remark, for further purposes, that $\tilde{\lambda}$ consists of d copies of the regular representation π_{reg} of G, where d is the Hilbert dimension of $\mathcal{H}$. The completion of $\mathcal{A}[G]$ with respect to the *reduced norm*

$$\|f\|_{\mathrm{red}} := \sup_{\alpha} \|(\tilde{\lambda} \ltimes \tilde{\alpha})(f)\|_{\ell^2(G)\otimes\mathcal{H}}\,,$$

where the supremum in (4.2) is taken only over those covariant representations of the form $(\tilde{\lambda}, \tilde{\alpha})$, is the *reduced crossed product C^*-algebra $G \ltimes_{\mathrm{red}} \mathcal{A}$*.

Example 4.1. The examples of this construction relevant to our purposes are the following:

- $\mathcal{A} = \mathbf{C}$ is the C^*-algebra of complex numbers with the trivial G-action; in this case $G \ltimes \mathbf{C}$ is called the *group C^*-algebra*, denoted by $C^*(G)$, and $G \ltimes_{\mathrm{red}} \mathbf{C}$ is called the *reduced group C^*-algebra*, denoted by $C^*_{\mathrm{red}}(G)$.
- $\mathcal{A} = \mathcal{C}(\partial G)$ is the C^*-algebra of continuous functions on the boundary ∂G of G.

We conclude this discussion by exhibiting a universal construction for representations of the cross product $G \ltimes \mathcal{C}(\partial G)$ [Tak03, Chapter X, Theorem 3.8] . Such representations are also called *cocycle representations* (see for instance the papers of C. Anantharaman [AD03] and of C. Anatharaman and J. Renault [AR01]) and also appear in the context of measured semidirect product groupoids (see [Ren80]).

Let X be standard Borel space equipped with a G-quasi-invariant positive measure μ. We assume here that G is acting on the left by measurable bijections. Let $P(\omega,\gamma) := \frac{d\mu(\gamma^{-1}\omega)}{d\mu(\omega)}$ denote the Radon–Nikodym cocycle of the G-action and let $\omega \to \mathcal{H}_\omega$ be a Borel field of Hilbert spaces. Denote by $\mathcal{H} := \int_X^\oplus \mathcal{H}_\omega d\mu(\omega)$ the direct integral. For ω_1 and ω_2 in X, denote by $\mathrm{Iso}(\mathcal{H}_{\omega_1}, \mathcal{H}_{\omega_2})$ the space of all isometries from $\mathcal{H}_{\omega_1}$ to $\mathcal{H}_{\omega_2}$. A unitary Borel cocycle is a map $A : (\omega,\gamma) \in X \times G \to A(\omega,\gamma) \in \mathrm{Iso}(\mathcal{H}_{\gamma^{-1}\omega}, \mathcal{H}_\omega)$ such that

- $A(\omega,\gamma_1\gamma_2) = A(\omega,\gamma_1)A({\gamma_1}^{-1}\omega,\gamma_2)$ [a.e.μ], and
- the map $\omega \to \big\langle f(\omega), A(\omega,\gamma)g(\gamma^{-1}\omega)\big\rangle$ is measurable for every pair of elements f, $g \in \mathcal{H}$ and every $\gamma \in G$.

We define a unitary representation π on $\mathcal{H}$ by

$$\big(\pi(\gamma)f\big)(\omega) := P^{\frac{1}{2}}(\omega,\gamma)A(\omega,\gamma)f(\gamma^{-1}\omega)\,. \tag{4.3}$$

Definition 4.2.
- If $\omega \mapsto \mathbf{C}$ is the trivial field of Hilbert spaces and A is the trivial cocycle, then the representation π in (4.3) is called the *quasi-regular representation* on $L^2(X,d\mu)$.
- If $X = \partial G$, the representation π in (4.3) is called *cocycle representation.*

We can now prove the following:

Theorem 4.3. *Let G be a torsion free not almost cyclic Gromov hyperbolic group. Then every tempered representation of G is a subrepresentation of a cocycle representation with respect to some quasi-invariant measure.*

If the representation is irreducible, the measure can be taken to be ergodic and hence the dimension of $\mathcal{H}_\omega$ is almost everywhere constant.

Conversely, every cocycle representation is tempered.

Proof. The inclusion $\mathbf{C} \hookrightarrow \mathcal{C}(\partial G)$ defined by $\zeta \mapsto \zeta \mathbf{1}_{\partial G}$, where $\mathbf{1}_{\partial G} \in \mathcal{C}(\partial G)$ denotes the function identically one on ∂G, induces a map $\phi : \mathbf{C}[G] \to \mathcal{C}(\partial G)[G]$ defined by

$$\phi\bigg(\sum_i \zeta_i \delta_{\gamma_i}\bigg) := \sum_i \zeta_i \mathbf{1}_{\partial G}\delta_{\gamma_i}\,.$$

It is immediate to verify that ϕ is continuous with respect to the reduced norm on both sides: in fact, since $\alpha(\mathbf{1}_{\partial G})$ is the identity operator, then

$$\begin{aligned}\Big\|\phi\Big(\sum_i \zeta_i\delta_{\gamma_i}\Big)\Big\|_{\mathrm{red}} &= \sup_\alpha \Big\|(\tilde\lambda \ltimes \tilde\alpha)\Big(\sum_i \zeta_i \mathbf{1}_{\partial G}\delta_{\gamma_i}\Big)\Big\|_{\ell^2(G)\otimes\mathcal{H}}\\ &= \Big\|\tilde\lambda\Big(\sum_i \zeta_i\delta_{\gamma_i}\Big)\Big\|_{\ell^2(G)\otimes\mathcal{H}}\\ &= \Big\|\pi_{\mathrm{reg}}\Big(\sum_i \zeta_i\delta_{\gamma_i}\Big)\Big\|_{\ell^2(G)}\\ &= \Big\|\sum_i \zeta_i\delta_{\gamma_i}\Big\|_{\mathrm{red}}.\end{aligned}$$

Since the reduced C^*-algebra of G is simple [Pow75] (see also [BCdlH94] concerning lattices in semisimple Lie groups) the extension of the above map $\overline{\phi}$ is actually an inclusion

$$\overline{\phi} : C^*_{\mathrm{red}}(G) \hookrightarrow G \ltimes_{\mathrm{red}} \mathcal{C}(\partial G)\ . \tag{4.4}$$

Moreover, since the action of G on ∂G is amenable (see [Ada94] or the more recent [Kai04]), the reduced crossed product and the full crossed product coincide (see [AD02, Theorem 5.3])) and hence we have

$$\overline{\phi} : C^*_{\mathrm{red}}(G) \hookrightarrow G \ltimes \mathcal{C}(\partial G)\,. \tag{4.5}$$

Assume now that π is tempered, that is there is an inequality of operator norms $\|\pi(f)\| \leq \|\pi_{\mathrm{reg}}(f)\|$ for $f \in \mathcal{A}[G]$, where $\mathcal{A} = \mathcal{C}(\partial G)$, [Fel60]. Then, by continuity, π extends to a representation of $C^*_{\mathrm{red}}(G)$. By standard arguments involving the Hahn–Banach Theorem (see [Dix64, Lemma 2.10.1]) one can see that π can be extended to a representation $\pi^{\partial G}$ of $G \ltimes \mathcal{C}(\partial G)$. The Hilbert space $\mathcal{H}_{\partial G}$ of the extended representation includes the original Hilbert space $\mathcal{H}$, but it may be strictly larger. By [Tak03, Chapter X, Theorem 3.8 and Theorem 3.15] the representations of the full crossed product are exactly the cocycle representations for some quasi-invariant measure μ on ∂G and some field of Hilbert spaces $\omega \to \mathcal{H}_\omega$.

The same argument in [Dix64, Lemma 2.10.1] shows that if π is irreducible one can require the extension $\pi^{\partial G}$ to be also irreducible. Since $\pi^{\partial G}$ is irreducible, the corresponding measure μ is ergodic and, since the map $\omega \mapsto \dim(\mathcal{H}_\omega)$ is measurable and G-invariant, the dimension of the Hilbert spaces $\mathcal{H}_\omega$ is constant $[a.e.\mu]$.

Finally, since cocycle representations are exactly the representations of the full crossed-product C^*-algebra and since the action of G on ∂G is amenable ([Ada94]), we have that the restriction of a representation of $G \ltimes \mathcal{C}(\partial G)$ to G is weakly contained in the regular representation (see [Kuh94]). $\square$

Remark 4.4. The existence of the map (4.4), and hence of the inclusion (4.5), is independent of the representation π and depends only on the compactness of ∂G and the amenability of the G-action.

Corollary 4.5. *Let Λ be a finitely generated virtually free group and let π be a representation in the class $\mathbf{Mult}_{\mathrm{irr}}(\Lambda)$. Then π is a subrepresentation of a cocycle representation with respect to a quasi-invariant ergodic measure μ on $\partial\Lambda$.*

Proof. Theorem 4.3 and Corollary 3.6. $\square$

Remark 4.6. Theorem 4.3 states that every tempered representation $(\pi, \mathcal{H})$ of a Gromov hyperbolic group G admits at least one extension to an irreducible representation $(\pi^{\partial G}, \mathcal{H}_{\partial G})$ of $G \ltimes_{\mathrm{red}} \mathcal{C}(\partial G)$. We call such an extension a *boundary realization for* π. We say moreover that a boundary realization is *perfect* if one can take $\mathcal{H}_{\partial G} = \mathcal{H}$.

Even if there is no *a priori* reason for $(\pi^{\partial G}, \mathcal{H}_{\partial G})$ to be unique, we have noticed that this is the case when $G = \Gamma$ is a free group and π is a representation of the class $\mathbf{Mult}(\Gamma)$ whose matrix coefficients are sufficently "big" in the sense of [KS01]. In fact for all irreducible tempered representations $(\pi, \mathcal{H})$ of the free group known so far, there are only three possibilities:

- π admits only one boundary realization which is perfect. In this case the irreducible representation $(\pi^{\partial\Gamma}, \mathcal{H})$ of $\Gamma \ltimes \mathcal{C}(\partial\Gamma)$ remains irreducible also when restricted to Γ; in this case we say that π satisfies *monotony*.
- π admits only one boundary realization which is not perfect, so that the inclusion $\mathcal{H} \hookrightarrow \mathcal{H}_{\partial\Gamma}$ is proper. In this case the representation $(\pi^{\partial\Gamma}, \mathcal{H}_{\partial\Gamma})$ is irreducible as representation of $\Gamma \ltimes_{\mathrm{red}} \mathcal{C}(\partial\Gamma)$, but, when restricted to Γ, it splits into the sum of two irreducible inequivalent representations; we say that π satisfies *oddity*.
- π admits exactly two perfect boundary realizations, no other boundary realization is perfect and any other (not perfect) boundary realization can be obtained as a linear combination of these two perfect ones; in this last case we say that π satisfies *duplicity*.

We believe that those are the only three possibilities for any tempered representation of a free group, but we can prove it so far only for representations of the class $\mathbf{Mult}(\Gamma)$, [KS01]. We think that the same problem is well posed also for a Gromov hyperbolic group and perhaps passing to a more general class of groups will give a better understanding of this phenomenon.

As a consequence of Theorem 4.3 we can state an analogue of Herz majorization principle for Gromov hyperbolic groups.

Corollary 4.7. *Let $(\pi, \mathcal{H})$ be a tempered representation of a torsion free Gromov hyperbolic group G that is not almost cyclic and let v be any vector in $\mathcal{H}$. Then there exists a quasi-invariant (not necessarily ergodic) measure μ on ∂G and a positive function $f \in L^2(\partial G, d\mu)$ with $\|f\|_2 = \|v\|_{\mathcal{H}}$ such that*

$$|\langle \pi(x)v, v\rangle| \leq |\langle \rho(x)f, f\rangle| \,, \tag{4.6}$$

where ρ is the quasi-regular representation on $L^2(\partial G, d\mu)$.

Proof. By Theorem 4.3, there exists a quasi-invariant measure μ on ∂G and a realization $\mathcal{H} = \int_{\partial G}^{\oplus} \mathcal{H}_\omega d\mu(\omega)$ of $\mathcal{H}$ as a direct integral of Hilbert spaces, such that

$$(\pi(x)v)(\omega) = P^{\frac{1}{2}}(\omega, x)A(\omega, x)v(x^{-1}\omega) \,,$$

where $v \in \mathcal{H}$, $P(\omega, x)$ is the Radon–Nikodym derivative of μ with respect to the G action and $A(\omega, x)$ is some unitary Borel cocycle .

Fix now $v \in \mathcal{H}$ and, for all $\omega \in \partial G$, define

$$f(\omega) := \|v(\omega)\|_\omega \,. \tag{4.7}$$

Then $f \in L^2(\partial G, d\mu)$ and $\|f\|_{L^2(\partial G, d\mu)} = \|v\|_{\mathcal{H}}$.

Moreover

$$\begin{aligned} |\langle \pi(x)v, v\rangle| &= \left| \int_{\partial G} \langle A(\omega, x)v(x^{-1}\omega), v(\omega)\rangle_\omega P^{\frac{1}{2}}(\omega, x) d\mu \right| \\ &\leq \int_{\partial G} \|v(x^{-1}\omega)\|_{x^{-1}\omega} \|v(\omega)\|_\omega P^{\frac{1}{2}}(\omega, x) d\mu \\ &= \langle \rho(x)f, f\rangle \,, \end{aligned}$$

where ρ is the quasi-regular representation on $L^2(\partial G, d\mu)$. □

Remark 4.8. If we take $v \in \mathcal{H}$ such that $\|v\|_{\mathcal{H}} = 1$, we may replace μ with a quasi-invariant probability measure in the same measure class and use in (4.7) the constant function $f := \mathbf{1}_{\partial G}$ identically one on ∂G, to get the following majorization:

$$|\langle \pi(x)v, v\rangle| \leq \langle \rho(x)\mathbf{1}_{\partial G}, \mathbf{1}_{\partial G}\rangle \,.$$

Remark 4.9. If H is a semisimple Lie group with finite center and maximal compact K, there exists a unique K-invariant probability measure μ on the maximal Furstenberg boundary H/P, for P a minimal parabolic. In this case the quasi-regular representation ρ on $L^2(H/P, d\mu)$ plays a very important role, namely the Harish-Chandra function $\Xi(x) = \langle \rho(x)\mathbf{1}_{H/P}, \mathbf{1}_{H/P}\rangle$ dominates all spherical functions associated with tempered unitary representations. If H has property (T) one can push this further by exhibiting a positive definite function Ψ which dominates all positive definite non-constant spherical functions on H. R. Howe and E. C. Tan constructed in their book [HT92, Chapter V] such a function Ψ from Ξ for $SL(n, \mathbf{R})$, for $n \geq 3$, while the more recent paper of H.Oh [Oh02] treats the general case.

We remark that the measure μ of Corollary 4.7 must depend on π, making our case much more similar to $SL(2, \mathbf{R})$, for which it is impossible to bound an arbitrary matrix coefficient (that is, not only spherical functions) in terms of Ξ. To see this, let G be as in Theorem 4.3. For any $w \in G$ we have that $\langle w\rangle \cong \mathbf{Z}$. Let $\pi_{\mathbf{Z}}$ be the representation induced from the trivial character of $\mathbf{Z}$ and $\mathbf{1}_{[\mathbf{Z}]}$ the characteristic function of the coset $[\mathbf{Z}]$. If there were to exist a fixed measure μ such that (4.6) holds for every tempered π, one would have

$$\langle \rho(w)\mathbf{1}_{\partial G}, \mathbf{1}_{\partial G}\rangle \geq \langle \pi_{\mathbf{Z}}(w)\mathbf{1}_{[\mathbf{Z}]}, \mathbf{1}_{[\mathbf{Z}]}\rangle = 1 \ .$$

Since $w \in G$ is arbitrary, this would then imply that $\langle \rho(\,\cdot\,)\mathbf{1}_{\partial G}, \mathbf{1}_{\partial G}\rangle \equiv 1$ identically on G. If G is not amenable this is impossible, since ρ is weakly contained in the regular representation.

Appendix A. Boundaries

Fix a generator system S for a Gromov hyperbolic group G and denote by X its Cayley graph with respect to S. Then X is a hyperbolic geodesic space with respect to the word metric d_X.

Fix, once and for all, a base point $p \in X$. A sequence of points $\{x_j \in X\}$ is said to *tend to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_p = +\infty \ , \tag{A.1}$$

where $(x|y)_p$ is the Gromov product defined by

$$(x|y)_p := \frac{1}{2}\big\{d_X(x,p) + d_X(y,p) - d_X(x,y)\big\} \ ,$$

for all $x, y, p \in X$. It can be proved that (A.1) does not depend on the choice of the basepoint p. Denote by S_∞ the set of all sequences in X

tending to infinity. Two sequences $\{x_j\}$ and $\{y_j\}$ in S_∞ are *equivalent* if

$$\lim_{j\to\infty}(x_j|y_j)_p = +\infty\,.$$

It can be proved that this is a true equivalence relation. The *boundary at infinity* ∂X of X is the set of all equivalence classes of sequences tending to infinity. When a sequence $\{x_j\}$ represents a class $\omega \in \partial X$, we say that x_j converges to ω.

Another notion of boundary of a hyperbolic group can be given as follows. A *geodesic ray* is an isometric embedding $r : [0, +\infty) \to X$ of $\mathbf{R}^+$ into X. Given a geodesic ray, there exists a unique $r(\infty) \in \partial X$ such that $r(t_j)$ converges to $r(\infty)$ for every sequence of real points $\{t_j\}$ going to $+\infty$.

Denote by R_p the set of all geodesic rays starting at p ($r(0) = p$). Two rays r and r' are equivalent ($r \sim r'$) if

$$d_X\big(r(t), r'(t)\big) \text{ is bounded as } t \to \infty\,.$$

The quotient set $R_p/\sim$ is called the *visual boundary* of X and it can be proved that it does not depend on the choice of p. Since R_p has a topology derived from the uniform convergence on compact intervals of geodesic rays, we endow $R_p/\sim$ with its quotient topology. Since G is finitely generated (see [dlHG90]) every closed ball is finite (hence compact!) and so X is a *proper* geodesic space. By [Ohs02, Proposition 2.64] the *visual boundary* is compact and coincides with the boundary at infinity defined above, so that we shall denote by ∂G any of these two boundaries. The action of G on X extends in an obvious way to an action on ∂G.

Appendix B. Proof of Theorem 2.6

As mentioned in § 2.3, the proof is just a straightforward verification that however uses heavily all the operators and objects defined in [IKS]. We will hence show here only the following result; the other assertions of Theorem 2.6 are left to the reader.

Theorem B.1. *Let $\Gamma_0 \le \Gamma$ be a subgroup of finite index in the free group Γ. If $(\pi_0, \mathcal{H}_0) \in \mathbf{Mult}(\Gamma_0)$ is a boundary representation of Γ_0, then the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi_0)$ is a boundary representation of Γ in the class $\mathbf{Mult}(\Gamma)$.*

We start by recalling some objects that were defined in [IKS] and that will be needed in the proof.

Let $\mathcal{T}$ be the Cayley graph of the free group Γ with respect to a symmetric set of free generators A, and let $\Gamma_0 \le \Gamma$ be a subgroup. $\mathcal{T}$

is a tree in which we fix an origin e and which is a metric space with the word distance with respect to the generating set A. It is always possible to choose a fundamental domain D for the action of Γ_0 on $\mathcal{T}$ having the following properties:

- D is a subtree containing e
- Γ_0 is generated by the set
$$A' := \left\{a'_j \in \Gamma_0 : \, d(D, a'_j D) = 1\right\}.$$

For any generator $a \in A$, define the set
$$P(a) := (D^{-1} \cdot A') \cap \Gamma(a)\,,$$
where $D^{-1} = \left\{u^{-1} : \, u \in D\right\}$ and
$$\Gamma(a) := \{\gamma \in \Gamma \, : \, \text{the reduced word for } \gamma \text{ starts with } a\}\,.$$

If $(V_{a'}, H_{b'a'}, B_{a'})$ is a matrix system with inner products for Γ_0 and π_0 is a representation in the class $\mathbf{Mult}(\Gamma_0)$ acting on the Hilbert space $\mathcal{H}_0 := \mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})$, we consider the induced representation $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\pi_0)$ on $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\mathcal{H}_0)$. If $f \in \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\mathcal{H}_0^\infty)$, the assignment $f \mapsto \tilde{f}$, where $\tilde{f}(x) := f(u)(h)$ for $x = uh$, with $h \in \Gamma_0$ and $u \in D$, provides the identification
$$\begin{aligned}\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\mathcal{H}_0^\infty) \cong \Big\{\tilde{f} : D \cdot \Gamma_0 \to \coprod_{a' \in A'} V_{a'} \, : \, \pi_0(h)\tilde{f}(g) = \tilde{f}(gh^{-1}),\\ \text{for all } h \in \Gamma_0, g \in \Gamma \text{ and } \tilde{f} \text{ is multiplicative as a function of } \Gamma_0\Big\}\,.\end{aligned}$$

In [IKS], $\left(\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\pi_0), \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\mathcal{H}_0)\right)$ is proved to be equivalent to a multiplicative representation $(\pi, \mathcal{H})$ on $\mathcal{H} := \mathcal{H}(V_a, H_{ba}, B_a)$. The spaces V_a are indexed on pairs (u, c') corresponding to elements $u \in D$ and $c' \in A'$ such that $u^{-1}c' \in P(a)$ while the H_{ba} are block matrices that will perform three kinds of operations on a vector $w_a \in V_a$ with coordinates $w_a = (w_a)_{u,c'}$. We give for completeness the explicit expressions for V_a, H_{ba} and B_a, but only the V_a will be used in the sequel:
$$V_a := \bigoplus_{u \in D} \bigoplus_{\substack{c' \in A' \\ u^{-1}c' \in P(a)}} V_{c'}\,,$$

$$(H_{ba} w_a)_{v,d'} := \begin{cases} (w_a)_{va^{-1},d'} & \text{if } va^{-1} \in D \\ H_{d'c'}(w_a)_{u,c'} & \text{if } va^{-1} \notin D \text{ and } a^{-1}v = u^{-1}c' \\ 0 & \text{otherwise}\,, \end{cases}$$

and
$$(B_a)_{u,c'} := B_{c'} \qquad \text{where } u^{-1}c' \in P(a)\,.$$

What will be important instead is the explicit form of the intertwining operator

$$J : \mathrm{Ind}_{\Gamma_0}^{\Gamma}\big(\mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})\big) \to \mathcal{H}(V_a, H_{ba}, B_a)$$

defined as

$$Jf(xa) := \bigoplus_{(u,c')} f(xu^{-1})(c') \tag{B.1}$$

on the dense subspace $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\big(\mathcal{H}^\infty(V_{a'}, H_{b'a'}, B_{a'})\big)$ of $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\big(\mathcal{H}(V_{a'}, H_{b'a'}, B_{a'})\big)$.

Proof of Theorem B.1. We assume the result of Theorem 2.4, namely the independence of the generating set.

Let $\Gamma_0 < \Gamma$ be a finite index free subgroup, $(\pi_0, \mathcal{H}_0) \in \mathbf{Mult}(\Gamma_0)$ a boundary representation of Γ_0 and let us consider as in (2.6) be the associated boundary representation $(\pi_0, \alpha_{\pi_0}, \mathcal{H}_0)$ of Γ_0.

Let λ denote the isometric action both of Γ on $\mathcal{C}(\partial\Gamma)$ and of Γ_0 on $\mathcal{C}(\partial\Gamma_0)$. If $\varphi : \partial\Gamma_0 \to \partial\Gamma$ is the boundary homeomorphism, every function $F \in \mathcal{C}(\partial\Gamma)$ defines a function $F \circ \varphi \in \mathcal{C}(\partial\Gamma_0)$, and hence, if we define (with quite a dose of pedantry...) an action α of $\mathcal{C}(\partial\Gamma)$ on $\mathcal{H}_0$ by

$$\alpha(F) := \alpha_{\pi_0}(F \circ \varphi)\,, \tag{B.2}$$

it is straightforward to verify that $(\pi_0, \alpha, \mathcal{H}_0)$ is a covariant representation of $\big(\Gamma_0, \mathcal{C}(\partial\Gamma)\big)$.

Let us now consider the induced representation $\pi_{\mathrm{ind}} := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi_0)$ on $\mathcal{H}_{\mathrm{ind}} := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathcal{H}_0)$. Define an action of $\mathcal{C}(\partial\Gamma)$ on $\mathcal{H}_{\mathrm{ind}}$ by setting

$$\big(\Pi(F)f\big)(x) := \alpha\big(\lambda(x^{-1})F\big)f(x)\,, \tag{B.3}$$

for $f \in \mathcal{H}_{\mathrm{ind}}$, $F \in \mathcal{C}(\partial\Gamma)$ and $x \in \Gamma$. We have to check that, under the above assumptions, $\Pi(F)f$ is still in $\mathcal{H}_{\mathrm{ind}}$, namely

$$\big(\Pi(F)f\big)(x\gamma) = \pi_0(\gamma^{-1})\big(\Pi(F)f\big)(x)\,. \tag{B.4}$$

But this is straightforward as, by using (B.3), the covariance of $(\pi_0, \alpha, \mathcal{H}_0)$ and (B.4), we verify that

$$\begin{aligned}
\Pi(F)f(x\gamma) =& \alpha\big(\lambda(x\gamma)^{-1}F\big)f(x\gamma)\\
=& \alpha\big(\lambda(\gamma^{-1})\lambda(x^{-1})F\big)f(x\gamma)\\
=& \pi_0(\gamma^{-1})\alpha\big(\lambda(x^{-1})F\big)\pi_0(\gamma)f(x\gamma)\\
=& \pi_0(\gamma^{-1})\alpha\big(\lambda(x^{-1})F\big)f(x)\\
=& \pi_0(\gamma^{-1})\big(\Pi(F)f\big)(x)\,.
\end{aligned}$$

In the same way one proves that

$$\pi_{\mathrm{ind}}(\gamma)\Pi(F)\pi_{\mathrm{ind}}(\gamma^{-1}) = \Pi\big(\lambda(\gamma)F\big)$$

for all $x \in \Gamma$ and $F \in \mathcal{C}(\partial\Gamma)$, thus showing that $(\pi_{\mathrm{ind}}, \Pi, \mathcal{H}_{\mathrm{ind}})$ is a boundary representation of Γ.

What is left to be shown is that if J is the operator that intertwines $(\pi_{\mathrm{ind}}, \mathcal{H}_{\mathrm{ind}})$ and a multiplicative representation $(\pi, \mathcal{H})$ of Γ, then J intertwines also $\Pi : \mathcal{C}(\partial\Gamma) \to \mathcal{L}(\mathcal{H}_{\mathrm{ind}})$ and $\alpha_\pi : \mathcal{C}(\partial\Gamma) \to \mathcal{L}(\mathcal{H})$ (defined respectively in (B.3) and (2.6)), namely that

$$J\,\Pi(F) = \alpha_\pi(F)\,J\,, \tag{B.5}$$

for all $F \in \mathcal{C}(\partial\Gamma)$. It will be indeed enough to verify that for all $f \in \mathcal{H}^\infty$ and $F \in \mathcal{C}(\partial\Gamma)$

$$J\big(\Pi(F)f\big) = \alpha_\pi(F)J(f)\,.$$

Using the direct sum decomposition in (3.1), we assume first that f is supported on the coset Γ_0 and that $F = \mathbf{1}_{\partial\Gamma(y)}$ for some $y \in \Gamma$. By definition of J in (B.1) , we have

$$J\big(\Pi(\mathbf{1}_{\partial\Gamma(y)})f\big)(xa) = \sum_{u^{-1}c' \in P(a)} \big(\Pi(\mathbf{1}_{\partial\Gamma(y)})f\big)(xu^{-1})(c')\,, \tag{B.6}$$

where by (B.3)

$$\big(\Pi(\mathbf{1}_{\partial\Gamma(y)})f\big)(xu^{-1}) = \alpha\big(\lambda(ux^{-1})^{-1}\mathbf{1}_{\partial\Gamma(y)}\big)f(xu^{-1})\,.$$

Since f is supported on Γ_0 the right hand side of (B.6) is zero unless $xu^{-1} = \gamma \in \Gamma_0$: for these x and u, by using the definition of f and the covariance property of $(\pi_0, \alpha, \mathcal{H}_0)$, we have

$$\begin{aligned}\alpha\big(\lambda(ux^{-1})^{-1}\mathbf{1}_{\partial\Gamma(y)}\big)f(xu^{-1}) &= \alpha\big(\lambda(\gamma^{-1})\mathbf{1}_{\partial\Gamma(y)}\big)\pi_0(\gamma^{-1})f(e)\\ &= \pi_0(\gamma^{-1})\alpha(\mathbf{1}_{\partial\Gamma(y)})f(e) = \alpha(\mathbf{1}_{\partial\Gamma(y)})f(\gamma)\,.\end{aligned}$$

Substituting the result of these last two computations in (B.6) we obtain

$$\begin{aligned}J\big(\Pi(\mathbf{1}_{\partial\Gamma(y)})f\big)(xa) &= \sum_{\substack{x=\gamma u\\ u^{-1}c' \in P(a)}} \big(\alpha(\mathbf{1}_{\partial\Gamma(y)})f(\gamma)\big)(c')\\ &= \sum_{\substack{x=\gamma u\\ u^{-1}c' \in P(a)}} \alpha(\mathbf{1}_{\partial\Gamma(y)})\tilde{f}(\gamma c') = \sum_{\substack{x=\gamma u\\ u^{-1}c' \in P(a)\\ \gamma c' \in \Gamma(y)}} \tilde{f}(\gamma c')\,,\end{aligned}$$

where in the last equality we used the definition of α in (B.2) (and hence of α_{π_0} in (2.6)). We may assume that $|x| > |y|$, so that $xa \in \Gamma(y)$ if and only if $xu^{-1}c' = \gamma c' \in \Gamma(y)$; hence, by (2.6),

$$\begin{aligned}\alpha_\pi(\mathbf{1}_{\partial\Gamma(y)})J(f)(xa) &= \mathbf{1}_{\Gamma(y)}(xa)J(f)(xa) \\ &= \sum_{\substack{x=\gamma u \\ u^{-1}c'\in P(a) \\ \gamma c'\in\Gamma(y)}} \tilde{f}(\gamma c') = J\big(\Pi(\mathbf{1}_{\partial\Gamma(y)})f\big)(xa)\,,\end{aligned}$$

which proves (B.5) for all f supported on Γ_0.

Finally, if f is supported on $u\Gamma_0$ for some $u \in D$ then, applying (B.5) to $\pi_{\mathrm{ind}}(u^{-1})f$ (which is supported on Γ_0) and using both the covariance of $(\pi_{\mathrm{ind}}, \Pi, \mathcal{H}_{\mathrm{ind}})$ and of $(\pi, \alpha_\pi, \mathcal{H})$ and the fact that J intertwines $(\pi_{\mathrm{ind}}, \mathcal{H}_{\mathrm{ind}})$ and $(\pi, \mathcal{H})$, one can easily verify (B.5) in general. □

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