

BOUNDED COHOMOLOGY THEORY AND APPLICATIONS TO DEFORMATION RIGIDITY IN COMPLEX HYPERBOLIC GEOMETRY

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ABSTRACT. We develop further basic tools in the theory of bounded continuous cohomology of locally compact groups; as such, this paper can be considered a sequel to [18], [39], and [11].

In particular:

- we establish sufficient conditions for a cohomology class of a discrete subgroup of a connected semisimple Lie group with finite center to be representable by a bounded differential form on the quotient of the associated symmetric space by Γ ; furthermore if $\rho : \Gamma \rightarrow \mathrm{PU}(q, 1)$ is any representation of any discrete subgroup Γ of $\mathrm{SU}(p, 1)$, we give an explicit bounded differential form on the quotient of complex hyperbolic space by Γ which is a representative for the pullback via ρ of the Kähler class of $\mathrm{PU}(q, 1)$;
- if G, G' are Lie groups of Hermitian type, we generalize to representations $\rho : \Gamma \rightarrow G'$ of lattices $\Gamma < G$ the invariant defined in [13] and we establish a Milnor–Wood type inequality.

As an application:

- we study maximal representations of a lattice in $\mathrm{SU}(1, 1)$ into $\mathrm{PU}(q, 1)$ for $q \geq 1$, and
- after establishing a measurable version of Cartan theorem [20] we prove a global rigidity result which was originally announced in [13] and [33] with a sketch of a proof using bounded cohomology techniques and then proven by Koziarz and Maubon in [36] using harmonic map techniques. As a corollary one obtains that a lattice in $\mathrm{SU}(p, 1)$ cannot be deformed nontrivially in $\mathrm{SU}(q, 1)$, $q \geq p$, if either $p \geq 2$ or the lattice is cocompact.

This generalizes to noncocompact lattices a theorem of Goldman and Millson, [29].

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1. INTRODUCTION

The continuous cohomology $H_c^\bullet(G, \mathbb{R})$ of a topological group G is the cohomology of the complex $(C(G^\bullet)^G, d^\bullet)$ of G -invariant continuous functions, while its bounded continuous cohomology $H_{cb}^\bullet(G, \mathbb{R})$ is the cohomology of the subcomplex $(C_b(G^\bullet)^G, d^\bullet)$ of G -invariant bounded continuous functions. The inclusion of the complex of bounded continuous functions into the one consisting of continuous functions gives rise to the comparison map

$$c_G^\bullet : H_{cb}^\bullet(G, \mathbb{R}) \rightarrow H_c^\bullet(G, \mathbb{R})$$

which encodes subtle properties of G of algebraic and geometric nature, see [2], [27], [38], [17], [18, § V.13], [12], (see also [7], [8], [30], [43], [3],

[25], [26], [24], [4], [35], [34] in relation with the existence of quasi-morphisms). We say that a continuous class on G is representable by a bounded continuous class if it is in the image of \mathfrak{c}_G^\bullet .

When G is a connected semisimple Lie group with finite center and associated symmetric space \mathcal{X} and $L < G$ is any closed subgroup, a useful tool in the study of the continuous cohomology of L is the van Est isomorphism, according to which $H_c^\bullet(L, \mathbb{R})$ is canonically isomorphic to the cohomology $H^\bullet(\Omega^\bullet(\mathcal{X})^L)$ of the complex $(\Omega^\bullet(\mathcal{X})^L, d^\bullet)$ of L -invariant smooth differential forms $\Omega^\bullet(\mathcal{X})$ on \mathcal{X} . For example, if $\Gamma < G$ is a torsionfree discrete subgroup, $H^\bullet(\Gamma, \mathbb{R})$ is the de Rham cohomology $H_{\text{dR}}^\bullet(\Gamma \backslash \mathcal{X})$ of the manifold $\Gamma \backslash \mathcal{X}$. (Here and in the sequel we drop the subscript c if the group is discrete.) For simplicity, in the introduction we restrict ourselves to this case, and we refer the reader to the body of the paper for the general statement in the case in which Γ is an arbitrary closed subgroup.

We do not know of an analogue of van Est theorem in the context of continuous bounded cohomology. This paper however explores a particular aspect of the comparison map and the pullback, namely the relation between bounded continuous cohomology and the complex of, loosely speaking, invariant smooth differential forms with some boundedness condition. For instance, our first result gives us information on the differential forms that one can use to represent a class in the image of the comparison map.

THEOREM 1. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group G with finite center, and $\rho : \Gamma \rightarrow G'$ a homomorphism into a topological group G' . If $\alpha \in H_c^n(G', \mathbb{R})$ is representable by a continuous bounded class, then its pullback $\rho^{(n)}(\alpha) \in H^n(\Gamma, \mathbb{R}) \cong H_{\text{dR}}^n(\Gamma \backslash \mathcal{X})$ is representable by a closed differential n -form on $\Gamma \backslash \mathcal{X}$ which is bounded.*

Here a form is bounded on $\Gamma \backslash \mathcal{X}$ if its supremum norm, computed using the Riemannian metric, is finite. We shall see later that in the case in which G, G' are the connected components of the isometry groups of complex hyperbolic spaces and α is the Kähler class, the bounded closed 2-form in Theorem 1 can be given explicitly (see Theorem 11).

In particular, by taking in the above theorem $\Gamma = G'$ and $\rho = Id$, we obtain:

COROLLARY 2. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group G with finite center and associated symmetric space \mathcal{X} . Any class in the image of the comparison map*

$$\mathfrak{c}_\Gamma^\bullet : H_b^\bullet(\Gamma, \mathbb{R}) \rightarrow H^\bullet(\Gamma, \mathbb{R}) \cong H_{\text{dR}}^\bullet(\Gamma \backslash \mathcal{X})$$

is representable by a closed form on $\Gamma \backslash \mathcal{X}$ which is bounded.

Even if G' is a connected Lie group, little is known about the surjectivity properties of the comparison map $\mathfrak{c}_{G'}^\bullet$. However, as a direct consequence of a theorem of Gromov [31] which asserts that characteristic classes are bounded (see [9] for a resolution of singularities free proof), we have the following:

COROLLARY 3. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group with finite center G and $\rho : \Gamma \rightarrow G'$ a homomorphism into a real algebraic group G' . If $\alpha \in H_c^n(G', \mathbb{R})$ comes from a characteristic class of a flat principal G' -bundle, then $\rho^{(n)}(\alpha) \in H^n(\Gamma, \mathbb{R}) \cong H_{\text{dR}}^n(\Gamma \backslash \mathcal{X})$ is representable by a closed differential n -form on $\Gamma \backslash \mathcal{X}$ which is bounded.*

Notice that Theorem 1, and hence Corollary 2 and Corollary 3 are valid, with an appropriate formulation, for any closed subgroup $\Gamma < G$ (compare with Corollary 4.4 and Proposition 4.1).

If L is a connected semisimple group with finite center, one has full information about the comparison map only in degree two, in which case

$$\mathfrak{c}_L^{(2)} : H_{\text{cb}}^2(L, \mathbb{R}) \rightarrow H_c^2(L, \mathbb{R})$$

is an isomorphism, [18]. This is the case we exploit, also because in this degree continuous cohomology admits a simple description. Recall in fact that if \mathcal{Y} is the symmetric space associated to L , the dimension of $H_c^2(L, \mathbb{R})$ is the number of irreducible factors of \mathcal{Y} of Hermitian type and $\Omega^2(\mathcal{Y})^L$ is generated by the Kähler forms of the irreducible Hermitian factors of \mathcal{Y} . We say that L is of *Hermitian type* if \mathcal{Y} is Hermitian symmetric and we denote by $\omega_{\mathcal{Y}}$ the Kähler form on \mathcal{Y} and by $\kappa_{\mathcal{Y}} \in H_c^2(L, \mathbb{R})$ the corresponding continuous class under the isomorphism $H_c^2(L, \mathbb{R}) \cong H^2(\Omega^\bullet(\mathcal{Y})^L)$.

Let G be of Hermitian type with associated symmetric space \mathcal{X} and $\Gamma < G$ a torsionfree lattice; for $1 \leq p \leq \infty$ let $H_p^\bullet(\Gamma \backslash \mathcal{X})$ denote the L^p -cohomology of $\Gamma \backslash \mathcal{X}$, which is the cohomology of the complex of smooth differential forms α on $\Gamma \backslash \mathcal{X}$ such that α and $d\alpha$ are in L^p . Inclusion in the complex of smooth differential forms gives thus a comparison map

$$i_p^\bullet : H_p^\bullet(\Gamma \backslash \mathcal{X}) \rightarrow H_{\text{dR}}^\bullet(\Gamma \backslash \mathcal{X}).$$

Then we have:

COROLLARY 4. *Assume that G, G' are of Hermitian type, let $\Gamma < G$ be a torsionfree lattice, \mathcal{X} the Hermitian symmetric space associated to G and $\rho : \Gamma \rightarrow G'$ a homomorphism. Then for every $1 \leq p \leq \infty$*

there is a linear map

$$\rho_p^{(2)} : H_c^2(G', \mathbb{R}) \rightarrow H_p^2(\Gamma \backslash \mathcal{X})$$

such that the diagram

$$\begin{array}{ccccc} H_c^2(G', \mathbb{R}) & \xrightarrow{\rho^{(2)}} & H^2(\Gamma, \mathbb{R}) & \xrightarrow{\cong} & H_{\text{dR}}^2(\Gamma \backslash \mathcal{X}) \\ & \searrow \rho_p^{(2)} & & \nearrow i_p^{(2)} & \\ & & H_p^2(\Gamma \backslash \mathcal{X}) & & \end{array}$$

commutes.

In the situation in which $\Gamma < G$ is a lattice and \mathcal{X} is Hermitian symmetric, the L^2 -cohomology $H_2^\bullet(\Gamma \backslash \mathcal{X})$ is reduced (*i. e.* Hausdorff) and finite dimensional in all degrees; it may hence be identified with the space of L^2 -harmonic forms on $\Gamma \backslash \mathcal{X}$ and carries a natural scalar product $\langle \cdot, \cdot \rangle$. The Kähler form $\omega_{\Gamma \backslash \mathcal{X}}$ is thus a distinguished element of $H_2^2(\Gamma \backslash \mathcal{X})$. Given now a homomorphism $\rho : \Gamma \rightarrow G'$ and using Corollary 4, the invariant

$$(1.1) \quad i_\rho := \frac{\langle \rho_2^{(2)}(\kappa_{\mathcal{X}'}) , \omega_{\Gamma \backslash \mathcal{X}} \rangle}{\langle \omega_{\Gamma \backslash \mathcal{X}} , \omega_{\Gamma \backslash \mathcal{X}} \rangle}$$

is well defined and finite. We have then the following Milnor–Wood type inequality:

THEOREM 5. *Let G, G' be of Hermitian type with associated symmetric spaces \mathcal{X} and \mathcal{X}' , let $\rho : \Gamma \rightarrow G$ be a representation of a lattice in G with invariant i_ρ as in (1.1). Assume that \mathcal{X} is irreducible and that the Hermitian metrics on \mathcal{X} and \mathcal{X}' are normalized so as to have minimal holomorphic sectional curvature -1 . Then*

$$(1.2) \quad |i_\rho| \leq \frac{\text{rk } \mathcal{X}'}{\text{rk } \mathcal{X}}.$$

Let us call a representation $\rho : \Gamma \rightarrow G'$ *maximal* when equality holds in (1.2). We assume this and we distinguish then the following cases:

$\text{rk } \mathcal{X} \geq 2$. Using Margulis' superrigidity [37], one sees that equality occurs if and only if there exists an equivariant isometric embedding

$$f : \mathcal{X} \rightarrow \mathcal{X}'$$

which is furthermore *tight*. Tightness is an analytic condition which singles out a certain type of isometric embeddings between Hermitian symmetric spaces, not necessarily holomorphic, and we refer to [45] and [15] for a systematic study of these.

rk $\mathcal{X} = 1$. In this case $\mathcal{X} = \mathcal{H}_{\mathbb{C}}^p$ is a complex hyperbolic space and we remark that one does not know to which extent lattices in $SU(p, 1)$ have superrigidity properties, at least when $p \geq 4$. We distinguish once again two cases.

The case in which $p = 1$ and \mathcal{X}' is a general Hermitian symmetric space is the object of an ongoing study in [14]. In this situation maximal representations lead to new interesting Kleinian groups in higher rank.

If of the other hand $p \geq 2$, we expect maximal representations to come from tight embeddings, but cannot rely on any general superrigidity result. In this paper, we study the case where $\text{rk } \mathcal{X}' = 1$ and we establish the following:

THEOREM 6. *Let $\Gamma < SU(p, 1)$ be a lattice and $\rho : \Gamma \rightarrow PU(q, 1)$ be a maximal representation. Assume that $p \geq 2$. Then there is an equivariant isometric embedding*

$$\varphi : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$$

which is holomorphic if $i_\rho = 1$ and antiholomorphic if $i_\rho = -1$.

V. Koziarz and J. Maubon gave in [36] a proof of Theorem 6 using harmonic map techniques. We refer to the introduction of their article for an excellent overview of the history and the context of the subject.

Concerning the case $p = 1$ we have:

THEOREM 7 ([10]). *Let $\Gamma < SU(1, 1)$ be a lattice and $\rho : \Gamma \rightarrow PU(q, 1)$ a representation such that $|i_\rho| = 1$. Then $\rho(\Gamma)$ leaves a complex geodesic invariant.*

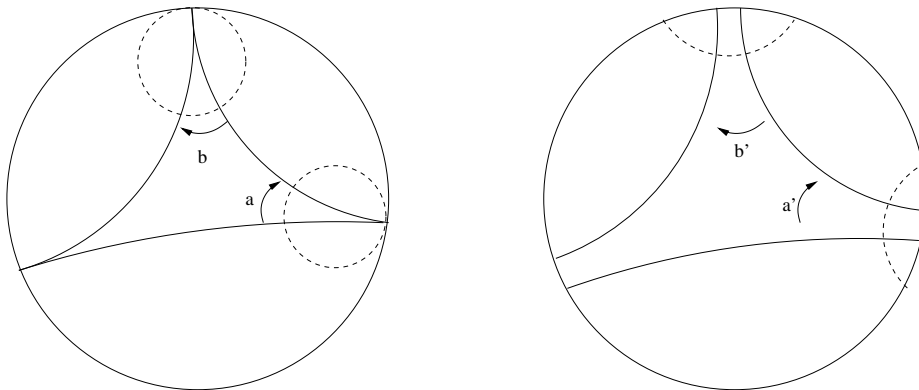
This was proven by Toledo [44] if Γ is a compact surface group. In the noncompact case a variant of Theorem 7 was obtained by Koziarz and Maubon in [36], with another definition of maximality which probably coincides with ours. Thus Theorem 7 reduces the study of maximal representations into $PU(q, 1)$ to the case $q = 1$, for which we have the following:

THEOREM 8 ([10]). *Let $\Gamma < SU(1, 1)$ be a lattice and $\rho : \Gamma \rightarrow PU(1, 1)$ a representation such that $|i_\rho| = 1$. Then $\rho(\Gamma)$ is discrete and, modulo the center of Γ , ρ is injective. In fact, there is a continuous surjective map $f : \partial\mathcal{H}_{\mathbb{C}}^1 \rightarrow \partial\mathcal{H}_{\mathbb{C}}^1$ such that:*

- (1) f is weakly order preserving;
- (2) $f(\rho(\gamma)\xi) = \gamma f(\xi)$ for all $\gamma \in \Gamma$ and all $\xi \in \partial\mathcal{H}_{\mathbb{C}}^1$.

Furthermore, if one of the following two assumptions is verified:

- (i) $\rho(\Gamma)$ is a lattice or
- (ii) $\rho(\gamma)$ is a parabolic element if γ is a parabolic element,

FIGURE 1. $\Gamma = \langle a, b \rangle$ and $\Lambda = \langle a', b' \rangle$

then f is a homeomorphism and $\rho(\Gamma)$ is a lattice.

Recall that, in the terminology of [33] a map $f : \partial\mathcal{H}_{\mathbb{C}}^1 \rightarrow \partial\mathcal{H}_{\mathbb{C}}^1$ is *weakly order preserving* if whenever $\xi, \eta, \zeta \in \partial\mathcal{H}_{\mathbb{C}}^1$ are distinct points such that $f(\xi), f(\eta), f(\zeta) \in \partial\mathcal{H}_{\mathbb{C}}^1$ are also distinct, then the two triples have the same orientation.

EXAMPLE 9. We give an example that shows that the map f is not necessarily a homeomorphism. To this purpose, let us realize the free group on two generators in two different ways:

- Let $\Gamma = \langle a, b \rangle$ be the lattice in $\text{PU}(1, 1)$ generated by the parabolic elements a and b with fundamental domain an ideal triangle.
- Let $\Lambda = \langle a', b' \rangle$ be the convex cocompact group generated by the hyperbolic elements a' and b' .

Let $\rho : \Gamma \rightarrow \Lambda$ be the representation defined by $\rho(a) = a'$ and $\rho(b) = b'$. Since Λ acts convex cocompactly on $\mathcal{H}_{\mathbb{C}}^1$, the orbit map $\Lambda \rightarrow \Lambda x$, for $x \in \mathcal{H}_{\mathbb{C}}^1$ is a quasi-isometry which extends to a homeomorphism $f_{\Lambda} : \partial\mathbb{F}_2 \rightarrow \mathcal{L}_{\Lambda}$, where \mathbb{F}_2 is the free group in two generators and \mathcal{L}_{Λ} is the limit set of Λ . Likewise, the orbit map $\Gamma \rightarrow \Gamma x$ extends to a continuous surjective map $f_{\Gamma} : \partial\mathbb{F}_2 \rightarrow \mathbb{S}^1$ which is one-to-one except for the cusps of Γ , where it is two-to-one. Then $f_{\Gamma} \circ f_{\Lambda}^{-1}$ extends to a map $f : \partial\mathcal{H}_{\mathbb{C}}^1 \rightarrow \partial\mathcal{H}_{\mathbb{C}}^1$ such that

- (1) f is weakly order preserving, and
- (2) $f(\rho(\gamma)\xi) = \gamma f(\xi)$,

One can prove, using the results in [27], § 3.3 and § 4.3 that $i_{\rho} = 1$.

We turn now to the description of the tools involved in the proof of Theorem 6. The ideal boundary $\partial\mathcal{H}_{\mathbb{C}}^{\ell}$ of complex hyperbolic ℓ -space $\mathcal{H}_{\mathbb{C}}^{\ell}$

carries a rich geometry whose “lines” are the *chains*. A chain in $\partial\mathcal{H}_{\mathbb{C}}^{\ell}$ is by definition the boundary of a complex geodesic in $\mathcal{H}_{\mathbb{C}}^{\ell}$; as such it is a circle equipped with a canonical orientation, and it is uniquely determined by any two points lying on it. The “geometry of chains” was first studied by E. Cartan who showed that, analogously to the Fundamental Theorem of Projective Geometry [1, Theorem 2.26], any automorphism of the incidence graph of the geometry of chains comes, for $\ell \geq 2$, from an isometry of $\mathcal{H}_{\mathbb{C}}^{\ell}$, [20]. Closely connected to this is Cartan’s *invariant angulaire*

$$c_{\ell} : (\partial\mathcal{H}_{\mathbb{C}}^{\ell})^3 \rightarrow [-1, 1]$$

introduced in the same paper [20], which is a full invariant for the $SU(\ell, 1)$ -action on triples of points in $\partial\mathcal{H}_{\mathbb{C}}^{\ell}$ assuming its maximum modulus (namely 1) exactly on triples of points lying on a chain.

Now let $\Gamma < SU(p, 1)$ be a lattice, $\rho : \Gamma \rightarrow PU(q, 1)$ a homomorphism with nonelementary image and let $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ be the Γ -equivariant measurable boundary map whose existence is recalled in § 4.4. A sizable portion of this paper is devoted to developing certain general tools in the theory of continuous bounded cohomology with appropriate coefficients, which will serve to establish a concrete link between our invariant i_{ρ} associated to a homomorphism $\rho : \Gamma \rightarrow PU(q, 1)$, the properties of the corresponding Γ -equivariant measurable boundary map $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$, and Cartan’s angular invariant. To illustrate this, let \mathcal{C}_p be the set of chains in $\partial\mathcal{H}_{\mathbb{C}}^p$, and for almost every chain $C \in \mathcal{C}_p$ (in a sense made precise in § 5), let us denote by φ_C the restriction of φ to C . Denoting by μ the $SU(p, 1)$ -invariant probability measure on $\Gamma \backslash SU(p, 1)$, we establish the following formula which gives a measure of how much the boundary map φ distorts a typical chain:

THEOREM 10. *Let $\Gamma < SU(p, 1)$ be a lattice, $\rho : \Gamma \rightarrow PU(q, 1)$ a homomorphism with nonelementary image and $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ be the associated Γ -equivariant measurable boundary map. For almost every chain $C \in \mathcal{C}_p$ and almost every triple $(\xi, \eta, \zeta) \in C^3$, we have*

$$\int_{\Gamma \backslash SU(p, 1)} c_q(\varphi_C(g\xi), \varphi_C(g\eta), \varphi_C(g\zeta)) d\mu(\dot{g}) = i_{\rho} c_p(\xi, \eta, \zeta),$$

where i_{ρ} is defined in (1.1), and c_q, c_p are the Cartan invariants.

In the same vein, if $\xi \in \partial\mathcal{H}_{\mathbb{C}}^p$ and $x \in \mathcal{H}_{\mathbb{C}}^p$, let $e^{\xi}(x) := e^{h\beta_{\xi}(0, x)}$, where h is the volume entropy of $\mathcal{H}_{\mathbb{C}}^p$, $\beta_{\xi}(0, x)$ is the Busemann function relative to a basepoint $0 \in \mathcal{H}_{\mathbb{C}}^p$, and μ_0 the $K = \text{Stab}_{SU(p, 1)}(0)$ -invariant probability measure on $\partial\mathcal{H}_{\mathbb{C}}^p$. The following is the more precise form of Theorem 1 announced above:

THEOREM 11. *Let $\Gamma < \mathrm{SU}(p, 1)$ be any torsionfree discrete subgroup, $\rho : \Gamma \rightarrow \mathrm{PU}(q, 1)$ a nonelementary homomorphism and let κ_q denote the Kähler class of $\mathrm{PU}(q, 1)$. The 2-form*

$$\int_{(\partial\mathcal{H}_{\mathbb{C}}^p)^3} e^{\xi_0} \wedge de^{\xi_1} \wedge de^{\xi_2} c_q(\varphi(\xi_0), \varphi(\xi_1), \varphi(\xi_2)) d\mu_0(\xi_0) d\mu_0(\xi_1) d\mu_0(\xi_2)$$

is Γ -invariant, closed, bounded and represents $\rho^{(2)}(\kappa_q) \in H_{\mathrm{dR}}^2(\Gamma \backslash \mathcal{H}_{\mathbb{C}}^p)$.

Our rigidity theorem on maximal representations follows from the formula in Theorem 10 and the following measurable analogue of Cartan's theorem, to which we alluded above:

THEOREM 12. *Let $n \geq 2$ and let $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ be a measurable map such that:*

- (i) *for almost every chain C and almost every triple (ξ, η, ζ) of distinct points on C , the triple $\varphi(\xi), \varphi(\eta), \varphi(\zeta)$ consists also of distinct points which lie on a chain and have the same orientation as (ξ, η, ζ) ;*
- (ii) *for almost every triple of points ξ, η, ζ not on a chain, $\varphi(\xi), \varphi(\eta), \varphi(\zeta)$ are also not on a chain.*

Then there is an isometric holomorphic embedding $F : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ such that ∂F coincides with φ almost everywhere.

A different way of looking at i_ρ as a *foliated Toledo number* was suggested to us by F. Labourie, and goes as follows. The space of configurations of points lying on chains can be seen as the space at infinity of the space of configurations of points lying on complex geodesics \mathcal{G}_p

$$\mathcal{GH}_{\mathbb{C}}^p = \left\{ (x, Y) : Y \text{ is a complex geodesic and } x \in Y \subset \mathcal{H}_{\mathbb{C}}^p \right\}$$

which is the total space of a foliation whose leaves are the fibers of the map

$$\begin{aligned} \pi_2 : \mathcal{GH}_{\mathbb{C}}^p &\rightarrow \mathcal{G}_p \\ (x, Y) &\mapsto Y \end{aligned}$$

which are transverse to the fibers of

$$\pi_1 : \mathcal{GH}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^p$$

which, incidentally, are compact. Given now $\Gamma < \mathrm{PU}(p, 1)$ a torsionfree lattice, since π_1 is Γ -equivariant, we get a foliated space $\pi_1 : \mathcal{GM} \rightarrow M$ lying above $M = \Gamma \backslash \mathcal{H}_{\mathbb{C}}^p$, where $\mathcal{GM} = \Gamma \backslash \mathcal{GH}_{\mathbb{C}}^p$ is foliated by complex geodesics. The restriction to the complex geodesics of the pullback $\pi_1^*(\omega_M)$ of the Kähler form ω_M of M , defines a tangential form $\omega_{\mathcal{GM}}$. If then $\rho : \Gamma \rightarrow \mathrm{PU}(q, 1)$ is a homomorphism and ω'_ρ is a bounded closed

representative of the class $\rho^{(2)}(\kappa_q) \in H_{\text{dR}}^2(M)$, then the tangential form Ω'_ρ , obtained by restricting $\pi^*(\omega'_\rho)$ to the leaves of the foliations, differs from $\omega_{\mathcal{G}M}$ by a bounded function, whose integral over $\mathcal{G}M$ gives i_ρ .

Application to Deformation Rigidity. If Γ is a discrete finitely generated group and L is a topological group, the space of homomorphisms $\text{Rep}(\Gamma, L)$ of Γ into L is topologized naturally as a closed subset of L^S , where S is a finite generating set of Γ . Let BL be the classifying space of continuous principal L -bundles, and $c \in H^\bullet(BL, \mathbb{R})$ a characteristic class. It is a standard observation that the map

$$\begin{aligned} \text{Rep}(\Gamma, L) &\rightarrow H^\bullet(\Gamma, \mathbb{R}) \\ \rho &\mapsto \rho_B^\bullet(c), \end{aligned}$$

where $\rho_B^\bullet : H^\bullet(BL, \mathbb{R}) \rightarrow H^\bullet(B\Gamma, \mathbb{R}) = H^\bullet(\Gamma, \mathbb{R})$ denotes the pullback, is constant on connected components of $\text{Rep}(\Gamma, L)$.

Assume now that L is irreducible of Hermitian type and let K be a maximal compact subgroup of L . It follows from the Iwasawa decomposition that BK is homotopic equivalent to BL , and by Chern-Weil theory $H^\bullet(K, \mathbb{R})$ is described by the K -invariant polynomials on the Lie algebra \mathfrak{k} of K . Since L is irreducible Hermitian, the center $Z(\mathfrak{k})$ is one dimensional and the orthogonal projection of \mathfrak{k} on $Z(\mathfrak{k})$ gives rise to an invariant linear form which, via Chern-Weil theory, gives rise to a class in $H^2(BK, \mathbb{R}) = H^2(BL, \mathbb{R})$. This class corresponds then via the natural homomorphism $H^2(BL, \mathbb{R}) \rightarrow H_c^2(L, \mathbb{R})$ to the Kähler class $\kappa_{\mathcal{Y}}$, and hence the commutativity of the diagram

$$\begin{array}{ccc} H^2(BL, \mathbb{R}) & \xrightarrow{\rho_B^{(2)}} & H^2(B\Gamma, \mathbb{R}) \\ \downarrow & & \downarrow = \\ H_c^2(L, \mathbb{R}) & \xrightarrow{\rho^{(2)}} & H^2(\Gamma, \mathbb{R}) \end{array}$$

implies that the map

$$\begin{aligned} \text{Rep}(\Gamma, L) &\rightarrow H^2(\Gamma, \mathbb{R}) \\ \rho &\mapsto \rho^{(2)}(\kappa_{\mathcal{Y}}), \end{aligned}$$

where \mathcal{Y} is the symmetric space associated to L , is constant on connected components of $\text{Rep}(\Gamma, L)$.

To turn to our immediate application, let us assume that $p \leq q$ and let $\rho_0 : \text{SU}(p, 1) \rightarrow \text{PU}(q, 1)$ be a *standard* homomorphism, that is a homomorphism associated to any isometric holomorphic embedding

$$F : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q.$$

Observe that any two such embeddings $\mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ are conjugate in $\mathrm{PU}(q, 1)$; moreover, the stabilizer in $\mathrm{PU}(q, 1)$ of the image of F is the almost direct product of the image $\rho_0(\mathrm{SU}(p, 1))$ and its centralizer $Z(\rho_0)$ in $\mathrm{PU}(q, 1)$, which is compact.

COROLLARY 13. *Let $\rho_0 : \mathrm{SU}(p, 1) \rightarrow \mathrm{PU}(q, 1)$ be a standard representation, let $\Gamma < \mathrm{SU}(p, 1)$ be a lattice and assume that $p \geq 2$. Then any representation $\rho : \Gamma \rightarrow \mathrm{PU}(q, 1)$ in the path connected component of $\rho_0|_{\Gamma}$ in the representation variety $\mathrm{Rep}(\Gamma, \mathrm{PU}(q, 1))$ is, modulo conjugation by $\mathrm{PU}(q, 1)$, of the form $\rho_0 \times \omega$, where ω is a homomorphism of Γ into the compact group $Z(\rho_0)$.*

REMARK 14. We recall that if $\Gamma < \mathrm{SU}(p, 1)$ is cocompact, this was proven by Goldman and Millson in [29]. On the other hand, Gusevskii and Parker found quasi-Fuchsian deformations of a noncocompact lattice $\Gamma < \mathrm{SU}(1, 1)$ into $\mathrm{PU}(2, 1)$, [32].

Organization of the Paper: Theorem 1 is proven as Corollary 4.4, Corollary 2 is proven as Proposition 4.1, Corollary 4 is proven as Corollary 4.5, Theorem 5 follows from Lemma 4.7 and Lemma 4.9, Theorems 6, 7 and 8 are proven in § 6, Theorem 10 is Theorem 4.12, Theorem 11 is Proposition 4.11 and finally Theorem 12 is proven as Theorem 5.1.

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2. PRELIMINARIES ON BOUNDED COHOMOLOGY AND HERMITIAN SYMMETRIC SPACES

Here we collect a few basic facts and definitions which will be useful in the sequel. Basic references are [39] and [28].

Let G be a locally compact group. A *coefficient G -module* is a Banach space E with an isometric G -action $\pi : G \rightarrow \mathrm{Iso}(E)$ contragredient to some separable continuous Banach G -module; in particular the G -action on E is continuous in the w^* -topology. Given a coefficient G -module E , the *bounded continuous cohomology with coefficients in E* is the cohomology of the complex $(C_b(G^\bullet, E)^G, d^\bullet)$ of the space of continuous bounded maps $G^{n+1} \rightarrow E$ which are G -invariant with respect to the G -action on $C_b(G^{n+1}, E)$ defined by

$$(gf)(x) := \pi(g)^{-1}f(gx),$$

for all $x \in G^{n+1}$ and $g \in G$. Basic examples of coefficient modules are continuous unitary representations on a separable Hilbert space \mathcal{H} – among which the trivial one with $\mathcal{H} = \mathbb{R}$ – and, more importantly in this paper, the space $L^\infty(G/H)$, where $H \leq G$ is a closed subgroup. Notice that $H_{\text{cb}}^n(G, E)$ comes naturally equipped with a seminorm induced by the supremum norm on $C_b(G^\bullet, E)$ and in some cases, as for instance if $n = 2$ and the coefficient module is separable, the seminorm is actually a norm.

There is a notion of *relatively injective* G -module which serves for the homological algebra characterization of cohomology. For the precise definition see [18] or [39], while for our purpose it will suffice to say that if (S, ν) is a regular measure G -space, then the G -module $L^\infty(S)$ is relatively injective if and only if the G -action on S is amenable in the sense of Zimmer, [47].

Let E be a coefficient G -module and (E_\bullet, d^\bullet) be a complex, where $E_0 = E$ is a coefficient G -module and E_n , for $n \geq 1$ are Banach G -modules. We say that (E_\bullet, d^\bullet) is a *strong resolution* of E if there is a sequence $h_n : E_n \rightarrow E_{n-1}$ of homotopy operators, such that

- (i) $\|h_n\| \leq 1$;
- (ii) h_n maps the subspace of G -continuous vectors $\mathcal{C}E_n$ into $\mathcal{C}E_{n-1}$.

Then, if the E_n , $n \geq 1$, are relatively injective, the cohomology of the subcomplex

$$0 \longrightarrow E^G \longrightarrow E_1^G \longrightarrow \dots \longrightarrow E_n^G \xrightarrow{d_n} \dots$$

is canonically isomorphic to the bounded continuous cohomology $H_{\text{cb}}^\bullet(G, E)$. The following lemma collects many functoriality statements needed in this paper, and is a small modification of a lemma in [39].

LEMMA 2.1. *Let G, G' be locally compact groups, $\rho : G \rightarrow G'$ a continuous homomorphism, E a G -coefficient module and F a G' -coefficient module. Let $\alpha : F \rightarrow E$ be a morphism of G -coefficient modules, where the G -module structure on F is via ρ . Let (E_\bullet) be a strong G -resolution of E by relatively injective G -modules, and let (F_\bullet) be a strong G' -resolution of F . Then any two extensions of the morphism α to a morphism of G -complexes induce the same map in cohomology*

$$H^\bullet(F_\bullet^{G'}) \rightarrow H^\bullet(E_\bullet^G).$$

Proof. By [39, Lemma 7.2.6] any two extensions of α are G -homotopic and hence induce the same map in cohomology

$$H^\bullet(F_\bullet^{\rho(G)}) \rightarrow H^\bullet(E_\bullet^G).$$

Moreover, the inclusion of complexes $F_{\bullet}^{G'} \subset F_{\bullet}^{\rho(G)}$ induces a unique map in cohomology

$$H^{\bullet}(F_{\bullet}^{G'}) \rightarrow H^{\bullet}(F_{\bullet}^{\rho(G)}),$$

hence proving the lemma. \square

Let now G be a connected, semisimple Lie group with finite center, and \mathcal{X} the associated symmetric space. Assume that \mathcal{X} is Hermitian symmetric, so that on \mathcal{X} there exists a nonzero G -invariant (closed) differential 2-form, namely the Kähler form of the Hermitian metric, which we denote by $\omega_{\mathcal{X}} \in \Omega^2(\mathcal{X})^G$. If $x \in \mathcal{X}$ is a reference point, and $\Delta(g_1x, g_2x, g_3x) \subset \mathcal{X}$ is a triangle with vertices g_1x, g_2x, g_3x , geodesic sides and arbitrarily C^1 -filled,

$$c(g_1, g_2, g_3) := \int_{\Delta(g_1x, g_2x, g_3x)} \omega_{\mathcal{X}}$$

is a differentiable homogeneous G -invariant cocycle and defines the continuous class $\kappa_{\mathcal{X}} \in H_c^2(G, \mathbb{R})$ corresponding to $\omega_{\mathcal{X}}$ by the van Est isomorphism $H_c^2(G, \mathbb{R}) \simeq \Omega^2(\mathcal{X})^G$. Moreover, c is bounded ([22], [21]), and hence it defines a bounded continuous class $\kappa_{\mathcal{X}}^b \in H_{cb}^2(G, \mathbb{R})$ which corresponds to $\kappa_{\mathcal{X}} \in H_c^2(G, \mathbb{R})$ under the isomorphism

$$H_{cb}^2(G, \mathbb{R}) \cong H_c^2(G, \mathbb{R}).$$

It follows again from [22] and [21] that if we normalize the metric on \mathcal{X} so as its minimal holomorphic sectional curvature is -1 , the Gromov norm of the class $\kappa_{\mathcal{X}}^b$ is

$$(2.1) \quad \|\kappa_{\mathcal{X}}^b\| = \pi \operatorname{rk} \mathcal{X}.$$

In the special case of complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^{\ell}$, the multiple $\frac{1}{\pi} \kappa_{\ell}^b$ of the bounded Kähler class κ_{ℓ}^b (which is here and in the following a shortcut for $\kappa_{\mathcal{H}_{\mathbb{C}}^{\ell}}^b$) admits an explicit representative on $\partial\mathcal{H}_{\mathbb{C}}^{\ell}$ given by the Cartan cocycle

$$c_{\ell} : (\partial\mathcal{H}_{\mathbb{C}}^{\ell})^3 \rightarrow [-1, 1],$$

which is defined in terms of the Hermitian triple product of a triple of points in the underlying complex vector space V of dimension $\ell + 1$ with a Hermitian form of signature $(\ell, 1)$ whose cone of negative lines gives a model of complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^{\ell}$. Any $(k + 1)$ -dimensional nondegenerate indefinite linear subspace $W \subset V$ gives rise to a k -plane, that is a totally geodesic holomorphically embedded isometric copy of $\mathcal{H}_{\mathbb{C}}^k$, whose boundary in $\partial\mathcal{H}_{\mathbb{C}}^{\ell}$ is called a k -chain. In particular, 1-chains (or chains) are the boundary of complex geodesics and play a fundamental role here. We refer the reader to [28] for

the precise definitions, and we limit ourselves to recall the following essential lemma:

LEMMA 2.2. *The Cartan cocycle $c_\ell : (\partial\mathcal{H}_\mathbb{C}^\ell)^3 \rightarrow [-1, 1]$ is a strict $SU(\ell, 1)$ -invariant Borel cocycle and $|c_\ell(a, b, c)| = 1$ if and only if a, b, c are on a chain and pairwise distinct.*

3. SOME COHOMOLOGICAL TOOLS FOR LOCALLY COMPACT GROUPS

In this section we develop some tools in bounded cohomology for locally compact groups and their closed subgroups which will be applied to our specific situation. More precisely, while the functorial machinery developed in [18], [39] and [11] applies in theory to general strong resolutions, in practice one ends up working mostly with spaces of functions on Cartesian products. In this section we deal with spaces of functions on fibered products (of homogeneous spaces), whose general framework would be that of complexes of functions on appropriate sequences (\mathcal{S}_n, ν_n) of (amenable) spaces which are analogues of simplicial sets in the category of measured spaces.

3.1. Cohomology with Coefficients: With the Use of Fibered Products. The invariants we consider in this paper are bounded classes with trivial coefficients; however applying a judicious change of coefficients – from \mathbb{R} to the L^∞ functions on a homogeneous space – we capture information which otherwise would be lost by the use of measurable maps (see the less cryptic Remark 4.14).

3.1.1. *Realization on Fibered Products.* The goal of this section is to define the fibered product of homogeneous spaces and prove that the complex of L^∞ functions on fibered products satisfy all properties necessary to be used to compute bounded cohomology. Observe that because of the projection in (3.1), we shall deal here with cohomology with coefficients.

Let G be a locally compact, second countable group and Q, H closed subgroups of G such that $Q \leq H$. We define the n -fold fibered product $(G/Q)_f^n$ of G/Q with respect to the canonical projection $p : G/Q \rightarrow G/H$ to be, for $n \geq 1$, the closed subset of $(G/Q)^n$ defined by

$$(G/Q)_f^n := \{(x_1, \dots, x_n) \in (G/Q)^n : p(x_1) = \dots = p(x_n)\},$$

and we set $(G/Q)_f^n = G/H$ if $n = 0$. The invariance of $(G/Q)_f^n$ for the diagonal G -action on $(G/Q)^n$ induces a G -equivariant projection

$$(3.1) \quad p_n : (G/Q)_f^n \rightarrow G/H$$

whose typical fiber is homeomorphic to $(H/Q)^n$.

A useful description of $(G/Q)_f^n$ as a quotient space may be obtained as follows. Considering H/Q as a subset of G/Q , the map

$$(3.2) \quad \begin{aligned} q_n : G \times (H/Q)^n &\rightarrow (G/Q)_f^n \\ (g, x_1, \dots, x_n) &\mapsto (gx_1, \dots, gx_n) \end{aligned}$$

is well defined, surjective, G -equivariant (with respect to the G -action on the first coordinate on $G \times (H/Q)^n$ and the product action on $(G/Q)_f^n$) and invariant under the right H -action on $G \times (H/Q)^n$ defined by

$$(3.3) \quad (g, x_1, \dots, x_n)h := (gh, h^{-1}x_1, \dots, h^{-1}x_n).$$

It is then easy to see that q_n induces a G -equivariant homeomorphism

$$(G \times (H/Q)^n)/H \rightarrow (G/Q)_f^n,$$

which hence realizes the fibered product $(G/Q)_f^n$ as a quotient space.

Let now μ and ν be Borel probability measures respectively on G and H/Q , such that μ is in the class of the Haar measure on G and ν is in the H -invariant measure class on H/Q . The pushforward $\nu_n = (q_n)_*(\mu \times \nu^n)$ of the probability measure $\mu \times \nu^n$ under q_n is then a Borel probability measure on $(G/Q)_f^n$ whose class is G -invariant and thus gives rise to Banach G -modules $L^\infty((G/Q)_f^n)$ and G -equivariant (norm) continuous maps

$$d_n : L^\infty((G/Q)_f^n) \rightarrow L^\infty((G/Q)_f^{n+1}), \text{ for } n \geq 0,$$

defined as follows:

- (i) $d_0 f(x) := f(p(x))$, for $f \in L^\infty(G/H)$, and
- (ii) $d_n f(x) = \sum_{i=1}^{n+1} (-1)^{i-1} f(p_{n,i}(x))$, for $f \in L^\infty((G/Q)_f^n)$ and $n \geq 1$,

where

$$(3.4) \quad p_{n,i} : (G/Q)_f^{n+1} \rightarrow (G/Q)_f^n$$

is obtained by leaving out the i -th coordinate. Observe that from the equality $(p_{n,i})_*(\nu_{n+1}) = \nu_n$, it follows that d_n is a well defined linear map between L^∞ spaces.

Then:

PROPOSITION 3.1. *Let $L \leq G$ be a closed subgroup.*

(1) *The complex*

$$0 \longrightarrow L^\infty(G/H) \longrightarrow \dots \longrightarrow L^\infty((G/Q)_f^n) \xrightarrow{d_n} L^\infty((G/Q)_f^{n+1}) \longrightarrow \dots$$

is a strong resolution of the coefficient L -module $L^\infty(G/H)$ by Banach L -modules.

(2) If Q is amenable and $n \geq 1$, then the G -action on $(G/Q)_f^n$ is amenable and $L^\infty((G/Q)_f^n)$ is a relatively injective Banach L -module.

Using [18, Theorem 2] (see also § 2), this implies immediately the following:

COROLLARY 3.2. *Assume that Q is amenable. Then the cohomology of the complex of L -invariants*

$$0 \longrightarrow L^\infty(G/Q)^L \longrightarrow L^\infty((G/Q)_f^2)^L \longrightarrow \dots$$

is canonically isomorphic to the bounded continuous cohomology $H_{\text{cb}}^\bullet(L, L^\infty(G/H))$ of L with coefficients in $L^\infty(G/H)$.

REMARK 3.3. Just like for the usual resolutions of L^∞ functions on the Cartesian product of copies of an amenable space (see § 2 or [18]), it is easy to see that the statements of Proposition 3.1 and of Corollary 3.2 hold verbatim if we consider instead the complex $(L_{\text{alt}}^\infty((G/Q)^\bullet), d^\bullet)$, where $L_{\text{alt}}^\infty((G/Q)_f^n)$ is the subspace consisting of functions in $L^\infty((G/Q)_f^n)$ which are alternating (observe that the symmetric group in n letters acts on $(G/Q)_f^n$).

Proof of Proposition 3.1(2). If $n \geq 1$ we have by definition the inclusion $(G/Q)_f^n \subset (G/Q)^n$ and hence there is a map of G -spaces

$$\pi : (G/Q)_f^n \rightarrow G/Q,$$

obtained by projection of the first component. Since $\pi_*(\nu_n) = \nu$, π realizes the measure G -space $(G/Q)_f^n$ as an extension of the measure G -space G/Q . If Q is amenable, the latter is an amenable G -space and hence the G -space $(G/Q)_f^n$ is also amenable [46]. Since L is a closed subgroup, $(G/Q)_f^n$ is also an amenable L -space [47, Theorem 4.3.5] and hence $L^\infty((G/Q)_f^n)$ is a relatively injective L -module, [18]. \square

The proof of Proposition 3.1(1) consists in the construction of appropriate contracting homotopy operators. Since it is rather long and technical, it will be given in the appendix at the end of this paper.

3.1.2. *An Implementation of the Transfer Map.* The point of this subsection is to see how the transfer map in [39] can be implemented, in a certain sense, on the resolution by L^∞ functions on the fibered product defined in § 3.1.1.

Let G be a locally compact second countable group and $L < G$ a closed subgroup. The injection $L \hookrightarrow G$ gives by contravariance the restriction map

$$r_{\mathbb{R}}^\bullet : H_{\text{cb}}^\bullet(G, \mathbb{R}) \rightarrow H_{\text{cb}}^\bullet(L, \mathbb{R})$$

in bounded cohomology. If we assume that $L \backslash G$ has a G -invariant probability measure μ , then the *transfer map*

$$T^\bullet : C_b(G^\bullet)^L \rightarrow C_b(G^\bullet)^G,$$

defined by integration

$$(3.5) \quad T^{(n)} f(g_1, \dots, g_n) := \int_{L \backslash G} f(gg_1, \dots, gg_n) d\mu(\dot{g}),$$

for all $(g_1, \dots, g_n) \in G^n$, induces in cohomology a left inverse of $r_{\mathbb{R}}^\bullet$ of norm one

$$T_b^\bullet : H_{\text{cb}}^\bullet(L, \mathbb{R}) \rightarrow H_{\text{cb}}^\bullet(G, \mathbb{R}),$$

(see [39, Proposition 8.6.2, pp.106-107]).

Notice that, when dealing with the transfer map, the functorial machinery recalled in § 2 does not apply directly, because T^\bullet is not a map of resolutions but is defined only on the subcomplex of invariant vectors.

Let now H, Q be closed subgroups of G such that $Q < H$. We assume that Q is amenable so that, by Proposition 3.1, the complex $(L^\infty((G/Q)_f^\bullet), d^\bullet)$ is a strong resolution of the coefficient module $L^\infty(G/H)$ by relatively injective L -modules. For $n \geq 1$, $f \in L^\infty((G/Q)_f^n)^L$, and $(x_1, \dots, x_n) \in (G/Q)_f^n$, let

$$(3.6) \quad (\tau_{G/Q}^{(n)} f)(x_1, \dots, x_n) := \int_{L \backslash G} f(gx_1, \dots, gx_n) d\mu(\dot{g}).$$

This defines a morphism of complexes

$$\tau_{G/Q}^\bullet : (L^\infty((G/Q)_f^\bullet)^L) \rightarrow (L^\infty((G/Q)_f^\bullet)^G)$$

and gives a left inverse to the inclusion

$$(L^\infty((G/Q)_f^\bullet)^G) \hookrightarrow (L^\infty((G/Q)_f^\bullet)^L).$$

The induced map in cohomology

$$\tau_{G/Q}^\bullet : H_b^\bullet(L, L^\infty(G/H)) \rightarrow H_{\text{cb}}^\bullet(G, L^\infty(G/H))$$

is thus a left inverse of the restriction map $r_{L^\infty(G/H)}^\bullet$.

LEMMA 3.4. *With the above notations, and for any amenable group Q , the diagram*

$$(3.7) \quad \begin{array}{ccc} H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{T_{\mathfrak{b}}^\bullet} & H_{\text{cb}}^\bullet(G, \mathbb{R}) \\ \theta_L^\bullet \downarrow & & \downarrow \theta_G^\bullet \\ H_{\text{cb}}^\bullet(L, L^\infty(G/H)) & \xrightarrow{\tau_{G/Q}^\bullet} & H_{\text{cb}}^\bullet(G, L^\infty(G/H)) \end{array}$$

commutes, where θ^\bullet is the canonical map induced in cohomology by the morphism of coefficients $\theta : \mathbb{R} \rightarrow L^\infty(G/H)$.

Observe that if in the above lemma we take $H = G$, then the fibered product $(G/Q)_f^n$ becomes the usual Cartesian product $(G/Q)^n$, and the cohomology of the complex of L -invariant $(L^\infty((G/Q)^\bullet)^L, d^\bullet)$ computes as usual the bounded cohomology of L with trivial coefficients. Hence we can record the following particular case of Lemma 3.4:

LEMMA 3.5. *With the above notations and for any amenable subgroup $Q \leq G$, let*

$$(3.8) \quad T_{G/Q}^\bullet : (L^\infty((G/Q)^\bullet)^L, d^\bullet) \rightarrow (L^\infty((G/Q)^\bullet)^G, d^\bullet)$$

be defined by

$$(3.9) \quad T_{G/Q}^{(n)} f(x_1, \dots, x_n) := \int_{L \backslash G} f(gx_1, \dots, gx_n) d\mu(g),$$

for $(x_1, \dots, x_n) \in (G/Q)_f^n$. Then the diagram

$$(3.10) \quad \begin{array}{ccc} H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{T_{\mathfrak{b}}^\bullet} & H_{\text{cb}}^\bullet(G, \mathbb{R}) \\ \cong \downarrow & & \downarrow \cong \\ H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{T_{G/Q}^\bullet} & H_{\text{cb}}^\bullet(G, \mathbb{R}) \end{array}$$

commutes, where the vertical arrows are the canonical isomorphisms in bounded cohomology extending the identity $\mathbb{R} \rightarrow \mathbb{R}$.

Proof of Lemma 3.4. Let G_f^n be the n -fold fibered product with respect to the projection $G \rightarrow G/H$. The restriction of continuous functions defined on G^n to the subspace $G_f^n \subset G^n$ induces a morphism of strong L -resolutions by L -injective modules

$$R^\bullet : C_{\text{b}}(G^\bullet) \rightarrow L^\infty(G_f^\bullet)$$

extending $\theta : \mathbb{R} \rightarrow L^\infty(G/H)$, so that the diagram

$$(3.11) \quad \begin{array}{ccc} C_b(G^n)^L & \xrightarrow{T^{(n)}} & C_b(G^n)^G \\ R_L^{(n)} \downarrow & & \downarrow R_G^{(n)} \\ L^\infty(G_f^n)^L & \xrightarrow{\tau_G^{(n)}} & L^\infty(G_f^n)^G \end{array}$$

commutes.

Likewise, the projection $\beta_n : G_f^n \rightarrow (G/Q)_f^n$, for $n \geq 1$, gives by pre-composition a morphism of strong L -resolutions by L -injective modules

$$\beta^\bullet : L^\infty((G/Q)_f^\bullet) \rightarrow L^\infty(G_f^\bullet)$$

extending the identity $L^\infty(G/H) \rightarrow L^\infty(G/H)$ and, as before, the diagram

$$(3.12) \quad \begin{array}{ccc} L^\infty(G_f^n)^L & \xrightarrow{\tau_G^{(n)}} & L^\infty(G_f^n)^G \\ \beta_L^{(n)} \uparrow & & \uparrow \beta_G^{(n)} \\ L^\infty((G/Q)_f^n)^L & \xrightarrow{\tau_{G/Q}^{(n)}} & L^\infty((G/Q)_f^n)^G, \end{array}$$

commutes.

The composition of the map induced in cohomology by R^\bullet with the inverse of the isomorphism induced by β^\bullet in cohomology realizes therefore the canonical map

$$(3.13) \quad \theta_L^\bullet : H_{\text{cb}}^\bullet(L, \mathbb{R}) \rightarrow H_{\text{cb}}^\bullet(L, L^\infty(G/H))$$

induced by the change of coefficient $\theta : \mathbb{R} \rightarrow L^\infty(G/H)$, [39, Proposition 8.1.1]. Hence the commutative diagrams induced in cohomology by (3.11) and (3.12) can be combined to obtain a diagram

$$\begin{array}{ccc} H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{T_b^\bullet} & H_{\text{cb}}^\bullet(G, \mathbb{R}) \\ \theta_L^\bullet \curvearrowleft \begin{array}{c} R_L^\bullet \downarrow \\ H_{\text{cb}}^\bullet(L, L^\infty(G/H)) \xrightarrow{\tau_G^\bullet} H_{\text{cb}}^\bullet(G, L^\infty(G/H)) \\ (\beta_L^\bullet)^{-1} \downarrow \cong \\ H_{\text{cb}}^\bullet(L, L^\infty(G/H)) \end{array} & & \begin{array}{c} \downarrow R_G^\bullet \\ H_{\text{cb}}^\bullet(G, L^\infty(G/H)) \\ \cong \downarrow (\beta_G^\bullet)^{-1} \\ H_{\text{cb}}^\bullet(G, L^\infty(G/H)) \end{array} \curvearrowright \theta_G^\bullet \\ & \xrightarrow{\tau_{G/Q}^\bullet} & H_{\text{cb}}^\bullet(G, L^\infty(G/H)) \end{array}$$

whose commutativity completes the proof. \square

3.1.3. *An Implementation of the Pullback.* In this section we shall use the results of [11] to implement the pullback in bounded cohomology followed by the change of coefficients, by using the resolution by L^∞ functions on the fibered product.

Let G' be a locally compact second countable group acting on a measurable space X . It is shown in [11, Proposition 2.1] that the complex $\mathcal{B}^\infty(X^\bullet)$ of bounded measurable functions is a strong resolution of \mathbb{R} . Not knowing whether the modules are relatively injective, we cannot conclude that the cohomology of this complex computes the bounded continuous cohomology of G' , however we can deduce the existence of a functorially defined map

$$\epsilon_X^\bullet : H^\bullet(\mathcal{B}^\infty(X^\bullet)^{G'}) \rightarrow H_{\text{cb}}^\bullet(G', \mathbb{R})$$

such that to any bounded measurable G' -invariant cocycle $c : X^{n+1} \rightarrow \mathbb{R}$ corresponds canonically a class $[c] \in H_{\text{cb}}^n(G', \mathbb{R})$, [11, Corollary 2.2].

Let now G be a locally compact second countable group, $L \leq G$ a closed subgroup acting measurably on X via a continuous homomorphism $\rho : L \rightarrow G'$, and let us assume that there exists an L -equivariant measurable map $\varphi : G/P \rightarrow X$, where $P < G$ is a closed subgroup. The main point of [11] is to show that the map φ can be used to implement the composition

$$H^\bullet(\mathcal{B}^\infty(X^\bullet)^{G'}) \xrightarrow{\epsilon_X^\bullet} H_{\text{cb}}^\bullet(G', \mathbb{R}) \xrightarrow{\rho_b^\bullet} H_{\text{cb}}^\bullet(L, \mathbb{R}).$$

More specifically, we recall here for later use that if $\kappa \in H_{\text{cb}}^n(G', \mathbb{R})$ is representable by a G' -invariant bounded strict measurable cocycle $c \in \mathcal{B}^\infty(X^{n+1})^{G'}$, then the image of the pullback $\rho_b^{(n)}(\kappa) \in H_{\text{cb}}^n(L, \mathbb{R})$ can be represented canonically by the cocycle in $L^\infty((G/P)^{n+1})^L$ defined by

$$(3.14) \quad (x_0, \dots, x_n) \mapsto c(\varphi(x_0), \dots, \varphi(x_n)).$$

The point of this section is to move one step further and to show how to represent canonically the composition of the above maps with the map θ_L^\bullet in (3.13).

To this purpose, let Q, H, P be closed subgroups of G such that $Q \leq H \cap P$, and let us consider the map

$$\begin{aligned} G \times H/Q &\rightarrow G/P \\ (g, xQ) &\mapsto gxP \end{aligned}$$

which, composed with φ , gives a measurable map $\tilde{\varphi} : G \times H/Q \rightarrow X$ which has the properties of being:

- (i) L -equivariant with respect to the action by left translations on the first variable: $\tilde{\varphi}(\gamma g, \dot{x}) = \rho(\gamma)\tilde{\varphi}(g, \dot{x})$ for all $\gamma \in L$ and a.e. $(g, \dot{x}) \in G \times H/Q$;
- (ii) H -invariant with respect to the right action considered in (3.3): $\tilde{\varphi}(gh^{-1}, h\dot{x}) = \tilde{\varphi}(g, \dot{x})$ for all $h \in H$ and all $(g, \dot{x}) \in G \times H/Q$.

For every $n \geq 1$, the measurable map

$$\begin{aligned} \tilde{\varphi}_f^n : G \times (H/Q)^n &\longrightarrow X^n \\ (g, \dot{x}_1, \dots, \dot{x}_n) &\mapsto (\tilde{\varphi}(g, \dot{x}_1), \dots, \tilde{\varphi}(g, \dot{x}_n)) \end{aligned}$$

gives, in view of (3.2), (i) and (ii), a measurable L -equivariant map $\varphi_f^n : (G/Q)_f^n \rightarrow X^n$ defined by the composition

$$(3.15) \quad \varphi_f^n : (G/Q)_f^n \xrightarrow{q_n^{-1}} (G \times (H/Q)^n)/H \xrightarrow{\tilde{\varphi}_f^n} X^n,$$

such that for every $1 \leq i \leq n+1$ the diagram

$$\begin{array}{ccc} (G/Q)_f^{n+1} & \xrightarrow{\varphi_f^{n+1}} & X^{n+1} \\ p_{n,i} \downarrow & & \downarrow \\ (G/Q)_f^n & \xrightarrow{\varphi_f^n} & X^n \end{array}$$

commutes, where $p_{n,i}$ was defined in (3.4) and the second vertical arrow is the map obtained by dropping the i -th coordinate. Precomposition by φ_f^n gives thus rise to a morphism of strong L -resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \dots & \longrightarrow & \mathcal{B}^\infty(X^n) & \longrightarrow & \dots \\ & & \downarrow & & & & \downarrow \varphi_f^{(n)} & & \\ 0 & \longrightarrow & L^\infty(G/H) & \longrightarrow & \dots & \longrightarrow & L^\infty((G/Q)_f^n) & \longrightarrow & \dots \end{array}$$

extending the inclusion $\mathbb{R} \hookrightarrow L^\infty(G/H)$. Let us denote by

$$(3.16) \quad \varphi_f^\bullet : H^\bullet(\mathcal{B}^\infty(X^\bullet)^{G'}) \rightarrow H_{\text{cb}}^\bullet(L, L^\infty(G/H))$$

the map obtained in cohomology.

PROPOSITION 3.6. *Assume that Q is amenable. Then the map φ_f^\bullet defined in (3.16) coincides with the composition*

$$H^\bullet(\mathcal{B}^\infty(X^\bullet)^{G'}) \xrightarrow{\epsilon_X^\bullet} H_{\text{cb}}^\bullet(G', \mathbb{R}) \xrightarrow{\rho_b^\bullet} H_{\text{cb}}^\bullet(L, \mathbb{R}) \xrightarrow{\theta_L^\bullet} H_{\text{cb}}^\bullet(L, L^\infty(G/H)).$$

Proof. By Proposition 3.1 $(L^\infty((G/Q)_f^\bullet), d^\bullet)$ is a strong resolution by relatively injective L -modules, so it is enough to apply Lemma 2.1 with $G = L$, $E = L^\infty(G/H)$, $F = \mathbb{R}$ the trivial coefficient G' -module, $F_\bullet = \mathcal{B}^\infty(X^\bullet)$, and $E_\bullet = (L^\infty(G/Q)_f^\bullet)$. \square

For further use we record the explicit reformulation of the above proposition:

COROLLARY 3.7. *Let G, G' be locally compact second countable groups, $L, H, Q, P \leq G$ closed subgroups with $Q \leq H \cap P$, and assume that Q is amenable. Let $\rho : L \rightarrow G'$ be a continuous homomorphism, X a measurable G' -space and assume that there is an L -equivariant measurable map $\varphi : G/P \rightarrow X$. Let $\kappa' \in H_{\text{cb}}^n(G', \mathbb{R})$ be a bounded cohomology class which admits as representative a bounded strict G' -invariant measurable cocycle $c' : X^{n+1} \rightarrow \mathbb{R}$. Then the class*

$$\theta_L^{(n)}(\rho_b^{(n)}(\kappa')) \in H_{\text{cb}}^n(L, L^\infty(G/H))$$

is represented by the L -invariant essentially bounded measurable cocycle

$$\tilde{c}' : (G/Q)_f^{n+1} \rightarrow \mathbb{R}$$

defined by

$$(3.17) \quad \tilde{c}'(x_0, x_1, \dots, x_n) := c'(\varphi_f^n(x_0, x_1, \dots, x_n)),$$

where φ_f^n is defined in (3.15).

In particular, if $L = G = G'$, $\rho = \text{Id}$ and $X = G/P$, then if the class $\kappa \in H_{\text{cb}}^n(G, \mathbb{R})$ admits as representative a bounded strict G -invariant Borel cocycle $c : (G/P)^{n+1} \rightarrow \mathbb{R}$, the class

$$\theta_G^{(n)}(\kappa) \in H_{\text{cb}}^n(G, L^\infty(G/H)).$$

is represented by the bounded strict G -invariant Borel cocycle

$$\tilde{c} : (G/Q)_f^{n+1} \rightarrow \mathbb{R}$$

defined by

$$(3.18) \quad \tilde{c}(x_1, \dots, x_{n+1}) := c(x_1P, \dots, x_{n+1}P).$$

3.2. A Formula. We apply now all the results obtained so far to prove the first instance of a formula deriving from a commutative diagram in cohomology. The main application of this formula will involve a numerical invariant attached to a representation, but in order to obtain this we need to know that some of the cohomology spaces involved are one-dimensional, like for instance if the groups G and G' are of Hermitian type and simple (see § 4.3). We want however to isolate here the preliminary result which is true for all locally compact second countable groups and which could be of independent interest.

PROPOSITION 3.8. *Let G and G' be locally compact second countable groups, let $L, H, Q, P \leq G$ be closed subgroups with $Q \leq H \cap P$, and assume $L \backslash G$ carries a G -invariant probability measure μ . Let $\rho : L \rightarrow G'$ be a continuous homomorphism, X a measurable G' -space and assume that there is an L -equivariant measurable map $\varphi : G/P \rightarrow X$. Let $\kappa' \in H_{\text{cb}}^2(G', \mathbb{R})$ and let $\kappa := T_b^{(2)}(\rho_b^{(2)}(\kappa')) \in H_{\text{cb}}^2(G, \mathbb{R})$. Let $c \in \mathcal{B}^\infty((G/P)^3)^G$ and $c' \in \mathcal{B}^\infty(X^3)^{G'}$ be alternating cocycles representing κ and κ' respectively, and let $\tilde{c} : (G/Q)_f^3 \rightarrow \mathbb{R}$ and $\tilde{c}' : (G/Q)_f^3 \rightarrow \mathbb{R}$ be the corresponding alternating cocycles defined respectively in (3.18) and (3.17).*

Assume that Q is amenable and that H acts ergodically on $H/Q \times H/Q$. Then, if φ_f^3 is the map defined in (3.15), we have

$$\int_{L \backslash G} \tilde{c}'(\varphi_f^3(gx_1, gx_2, gx_3)) d\mu(\dot{g}) = \tilde{c}(x_1, x_2, x_3)$$

for a. e. $(x_1, x_2, x_3) \in (G/Q)_f^3$.

Proof. The commutativity of the square in the following diagram (see Lemma 3.4)

$$\begin{array}{ccccccc} H^2(\mathcal{B}^\infty(X^3)^{G'}) & \xrightarrow{\omega_X^{(2)}} & H_{\text{cb}}^2(G', \mathbb{R}) & \xrightarrow{\rho_b^{(2)}} & H_{\text{cb}}^2(L, \mathbb{R}) & \xrightarrow{\theta_L^{(2)}} & H_{\text{cb}}^2(L, L^\infty(G/H)) \\ & & & & \downarrow T_b^{(2)} & & \downarrow \tau_{G/Q}^{(2)} \\ H^2(\mathcal{B}^\infty((G/P)^3)^G) & \xrightarrow{\omega_{G/P}^{(2)}} & H_{\text{cb}}^2(G, \mathbb{R}) & \xrightarrow{\theta_G^{(2)}} & H_{\text{cb}}^2(G, L^\infty(G/H)) & & \end{array}$$

applied to the class $\rho_b^{(2)}(\kappa') \in H_{\text{cb}}^2(L, \mathbb{R})$ reads

$$\tau_{G/Q}^{(2)}(\theta_L^{(2)}(\rho_b^{(2)}(\kappa'))) = \theta_G^{(2)}(T_b^{(2)}(\rho_b^{(2)}(\kappa'))) = \theta_G^{(2)}(\kappa).$$

Hence the representatives for the classes $\theta_G^{(2)}(\kappa)$ and $\theta_L^{(2)}(\rho_b^{(2)}(\kappa'))$ chosen according to Corollary 3.7 satisfy the relation

$$\tau_{G/Q}^{(2)}(\tilde{c}') = \tilde{c} + db,$$

where $b \in L^\infty((G/Q)_f^2)^G$, which, using the definition of $\tau_{G/Q}^{(2)}$ in (3.6) implies that

$$\int_{L \backslash G} \tilde{c}'(\varphi_f^3(gx_1, gx_2, gx_3)) d\mu(\dot{g}) = \tilde{c}(x_1, x_2, x_3) + db$$

for a. e. $(x_1, x_2, x_3) \in (G/Q)_f^3$. However G acts ergodically on $(G/Q)_f^2$ because it acts on the basis of the fibration $(G/Q)_f^2 \rightarrow G/H$ transitively with stabilizer H , which then by hypothesis acts ergodically on the typical fiber homeomorphic to $(H/Q)^2$. Hence $L^\infty((G/Q)_f^2)^G = \mathbb{R}$.

Thus db is constant and hence zero, being the difference of two alternating functions. \square

Notice that if $G = H$ and $P = Q$, the above formula takes a more familiar form, namely

COROLLARY 3.9. *Let $\kappa' \in H_{\text{cb}}^2(G', \mathbb{R})$ and $\kappa := T_{\text{b}}^{(2)}(\rho_{\text{b}}^{(2)}(\kappa'))$. With the same notation as in Corollary 3.7 assume that $L \backslash G$ carries a G -invariant probability measure μ . Let $c \in \mathcal{B}_{\text{alt}}^\infty((G/P)^3)^G$ and $c' \in \mathcal{B}_{\text{alt}}^\infty(X^3)^{G'}$ represent κ and κ' respectively and let $\tilde{c} : (G/P)^3 \rightarrow \mathbb{R}$ and $\tilde{c}' : (G/P)^3 \rightarrow \mathbb{R}$ be the corresponding alternating cocycles defined in (3.18) and (3.17).*

Assume that G acts ergodically on $G/P \times G/P$. Then

$$\int_{L \backslash G} c'(\varphi(x_1), \varphi(x_2), \varphi(x_3)) d\mu(\dot{g}) = c(x_1, x_2, x_3)$$

for a. e. $(x_1, x_2, x_3) \in (G/P)^3$.

3.3. The Toledo Map and the Bounded Toledo Map. We shall now see to which extent one can define a transfer map also in ordinary continuous cohomology and link it to the one previously defined in bounded cohomology. As we shall see, problems arise if the subgroup L is only of finite covolume and not cocompact.

Let L be a closed subgroup of a locally compact group G . Then we have a commutative diagram

$$(3.19) \quad \begin{array}{ccc} H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{c_L^\bullet} & H_c^\bullet(L, \mathbb{R}) \\ \downarrow \iota_{\text{b}}^\bullet & & \downarrow \iota^\bullet \\ H_{\text{cb}}^\bullet(G, L^\infty(L \backslash G)) & \xrightarrow{c_G^\bullet} & H_c^\bullet(G, L_{\text{loc}}^p(L \backslash G)) \end{array}$$

for $1 \leq p < \infty$, where ι^\bullet is the induction in ordinary continuous cohomology (see [5]), ι_{b}^\bullet is the isometric isomorphism defined by the induction in bounded continuous cohomology (see [18]) and the horizontal arrows are comparison maps. (Observe that for trivial coefficients the comparison map was recalled in the introduction; for a thorough discussion of this diagram see [39, § 10.1].) If $L \backslash G$ carries a G -invariant probability measure μ , we have, for $1 \leq p < \infty$, an obvious morphism

of coefficient modules

$$\begin{aligned} \mathfrak{m} : L^p(L \backslash G) &\rightarrow \mathbb{R} \\ f &\mapsto \int_{L \backslash G} f d\mu. \end{aligned}$$

If moreover $L \backslash G$ is compact, then $L^p_{\text{loc}}(L \backslash G) = L^p(L \backslash G)$, and composing ι with the change of coefficients \mathfrak{m} in continuous cohomology gives a transfer map in ordinary continuous cohomology

$$\mathbf{T}^\bullet : H_c^\bullet(L, \mathbb{R}) \rightarrow H_c^\bullet(G, \mathbb{R})$$

which is a left inverse to the restriction map and leads to a commutative diagram

$$(3.20) \quad \begin{array}{ccc} H_{\text{cb}}^\bullet(L, \mathbb{R}) & \xrightarrow{c_L^\bullet} & H_c^\bullet(L, \mathbb{R}) \\ \mathbf{T}_b^\bullet \downarrow & & \downarrow \mathbf{T}^\bullet \\ H_{\text{cb}}^\bullet(G, \mathbb{R}) & \xrightarrow{c_G^\bullet} & H_c^\bullet(G, \mathbb{R}) \end{array}$$

which is very useful in applications when it comes to identifying invariants in bounded cohomology in terms of ordinary cohomological invariants.

REMARK 3.10. The above fails if $L \backslash G$ is only of finite volume. For example, if $L = \Gamma < G$ is a nonuniform lattice, then there is in general no left inverse to the restriction in cohomology $H_c^\bullet(G, \mathbb{R}) \rightarrow H^\bullet(\Gamma, \mathbb{R})$ as this map is often not injective. In fact, one can for example consider the case in which $\mathcal{X} = G/K$ is an n -dimensional symmetric space of noncompact type: then $H_c^n(G, \mathbb{R}) = \Omega^n(\mathcal{X})^G$ is generated by the volume form and hence not zero, while if $\Gamma < G$ is any nonuniform torsionfree lattice, the cohomology $H^n(\Gamma, \mathbb{R})$ vanishes as it is isomorphic to $H_{\text{dR}}^n(\Gamma \backslash \mathcal{X})$.

The extent to which one can remedy this mishap by keeping track of the relation with bounded continuous cohomology is thus the object of § 4.1 in the context when G is a semisimple Lie group.

We apply here the above considerations to associate to a homomorphism a map in cohomology which will in some cases produce in addition a numerical invariant (see for instance the Hermitian case, § 4.3).

Let $L \leq G$ be a closed subgroup of a locally compact group G , and $\rho : L \rightarrow G'$ a continuous homomorphism into a locally compact group G' . The composition of the pullback

$$\rho_b^\bullet : H_{\text{cb}}^\bullet(G', \mathbb{R}) \rightarrow H_{\text{cb}}^\bullet(L, \mathbb{R})$$

with the transfer map T_b^\bullet defined in § 3.1.2 gives rise to the *bounded Toledo map*

$$(3.21) \quad T_b^\bullet(\rho) : H_{cb}^\bullet(G', \mathbb{R}) \rightarrow H_{cb}^\bullet(G, \mathbb{R})$$

which provides a basic invariant of the homomorphism $\rho : L \rightarrow G'$.

We remark once again that the bounded Toledo map is defined for *all* closed subgroups of G such that on $L \backslash G$ there is a G -invariant probability measure. If however $L \backslash G$ is in addition compact (for example, a uniform lattice) then we also have an analogous construction in ordinary cohomology. Namely, associated to the homomorphism $\rho : L \rightarrow G'$ we have a morphism

$$\rho^\bullet : H_c^\bullet(G', \mathbb{R}) \rightarrow H_c^\bullet(L, \mathbb{R})$$

which, composed with the transfer map T^\bullet defined in (3.3) gives a map

$$T^\bullet(\rho) : H_c^\bullet(G', \mathbb{R}) \rightarrow H_c^\bullet(G, \mathbb{R})$$

which we denote by the “Toledo map” and has the property that the diagram

$$\begin{array}{ccc} H_{cb}^\bullet(G', \mathbb{R}) & \xrightarrow{c_{G'}^\bullet} & H_c^\bullet(G', \mathbb{R}) \\ T_b^\bullet(\rho) \downarrow & & \downarrow T^\bullet(\rho) \\ H_{cb}^\bullet(G, \mathbb{R}) & \xrightarrow{c_G^\bullet} & H_c^\bullet(G, \mathbb{R}), \end{array}$$

where the horizontal arrows are comparison maps, commutes.

The interplay between these two maps is the basic ingredient in this paper for the cocompact case, as well as in [33], [16] and [14]. In the finite volume case we will need to resort to a somewhat more elaborate version of the above diagram which can be developed when G is a connected semisimple Lie group – see (4.15).

4. MORE COHOMOLOGICAL TOOLS FOR SEMISIMPLE LIE GROUPS AND APPLICATIONS

In this section we specialize the discussion to semisimple Lie groups (with finite center) and their closed subgroups. The immediate advantage will be the identification of the ordinary group cohomology with the cohomology of differential forms on symmetric spaces.

4.1. A Factorization of the Comparison Map. The main point of this section is to provide, in the case of semisimple Lie groups, a

substitute to the the missing arrow in

$$(4.1) \quad \begin{array}{ccc} \mathrm{H}_{\mathrm{cb}}^{\bullet}(L, \mathbb{R}) & \xrightarrow{c_L^{\bullet}} & \mathrm{H}_c^{\bullet}(L, \mathbb{R}) \\ \mathrm{T}_b^{\bullet} \downarrow & & \downarrow \text{dotted} \\ \mathrm{H}_{\mathrm{cb}}^{\bullet}(G, \mathbb{R}) & \xrightarrow{c_G^{\bullet}} & \mathrm{H}_c^{\bullet}(G, \mathbb{R}) \end{array}$$

if the subgroup $L \leq G$ is only of finite covolume.

Let G be a connected semisimple Lie group with finite center and \mathcal{X} the associated symmetric space. Any closed subgroup $L \leq G$ acts properly on \mathcal{X} and hence the complex

$$\mathbb{R} \longrightarrow \Omega^0(\mathcal{X}) \longrightarrow \dots \longrightarrow \Omega^k(\mathcal{X}) \longrightarrow \dots$$

of C^∞ differential forms on \mathcal{X} with the usual exterior differential is a resolution by continuous injective L -modules (where injectivity now refers to the usual notion in continuous cohomology), from which one obtains a canonical isomorphism

$$\mathrm{H}_c^{\bullet}(L, \mathbb{R}) \xrightarrow{\cong} \mathrm{H}^{\bullet}(\Omega^{\bullet}(\mathcal{X})^L)$$

in cohomology, [40]. Let moreover $(\Omega_{\infty}^{\bullet}(\mathcal{X}), d^{\bullet})$ denote the complex of smooth differential forms α on \mathcal{X} such that $x \mapsto \|\alpha_x\|$ and $x \mapsto \|d\alpha_x\|$ are in $L^\infty(\mathcal{X})$, and let $h(\mathcal{X})$ denote the volume entropy of \mathcal{X} , that is the rate of exponential growth of volume of geodesic balls in \mathcal{X} , [23]. Then

PROPOSITION 4.1. *Let G be a connected semisimple Lie group with finite center, \mathcal{X} the associated symmetric space and $L \leq G$ any closed subgroup. Then there exists a map*

$$\delta_{\infty, L}^{\bullet} : \mathrm{H}_{\mathrm{cb}}^{\bullet}(L, \mathbb{R}) \rightarrow \mathrm{H}^{\bullet}(\Omega_{\infty}^{\bullet}(\mathcal{X})^L)$$

such that the diagram

$$(4.2) \quad \begin{array}{ccccc} \mathrm{H}_{\mathrm{cb}}^{\bullet}(L, \mathbb{R}) & \xrightarrow{c_L^{\bullet}} & \mathrm{H}_c^{\bullet}(L, \mathbb{R}) & \xrightarrow{\cong} & \mathrm{H}^{\bullet}(\Omega^{\bullet}(\mathcal{X})^L) \\ & \searrow \delta_{\infty, L}^{\bullet} & & \nearrow i_{\infty, L}^{\bullet} & \\ & & \mathrm{H}^{\bullet}(\Omega_{\infty}^{\bullet}(\mathcal{X})^L) & & \end{array}$$

commutes, where $i_{\infty, L}^{\bullet}$ is the map induced in cohomology by the inclusion of complexes

$$i_{\infty}^{\bullet} : \Omega_{\infty}^{\bullet}(\mathcal{X}) \rightarrow \Omega^{\bullet}(\mathcal{X}).$$

Moreover, the norm of $\delta_{\infty, L}^{(k)}$ is bounded by $h(\mathcal{X})^k$.

Before proving the proposition, we want to push our result a little further in the case when $L = \Gamma < G$ is a lattice. In particular, we are going to see how the map $\delta_{\infty, \Gamma}^\bullet$ fits into a diagram where the transfer appears. If $1 \leq p \leq \infty$, let $\Omega_p^\bullet(\mathcal{X})^\Gamma$ be the space of Γ -invariant smooth differential n -forms on \mathcal{X} such that $x \mapsto \|\alpha_x\|$ and $x \mapsto \|d\alpha_x\|$ are in $L^p(\Gamma \backslash \mathcal{X})$, and consider the complex $(\Omega_p^\bullet(\mathcal{X})^\Gamma, d^\bullet)$. Incidentally, notice that this is a rather misleading notation if \mathcal{X} is not compact, because in this case only for $p = \infty$ one has that $(\Omega_\infty^\bullet(\mathcal{X})^\Gamma, d^\bullet)$ is the subcomplex of invariants of $(\Omega_\infty^\bullet(\mathcal{X}), d^\bullet)$. Let $\delta_{p, \Gamma}^\bullet$ be the map obtained by composing the map $\delta_{\infty, \Gamma}^\bullet$ in Proposition 4.1 with the map obtained by the inclusion of complexes

$$\Omega_\infty^\bullet(\mathcal{X})^\Gamma \rightarrow \Omega_p^\bullet(\mathcal{X})^\Gamma,$$

namely

$$\begin{array}{ccc} & \xrightarrow{\delta_{p, \Gamma}^\bullet} & \\ \text{H}_b^\bullet(\Gamma, \mathbb{R}) & \xrightarrow{\delta_{\infty, \Gamma}^\bullet} \text{H}^\bullet(\Omega_\infty^\bullet(\mathcal{X})^\Gamma) & \longrightarrow \text{H}^\bullet(\Omega_p^\bullet(\mathcal{X})^\Gamma). \end{array}$$

Also, since $\Omega(\mathcal{X})^G \subset \Omega_\infty(\mathcal{X})$ and $\Gamma \backslash \mathcal{X}$ is of finite volume, the restriction map

$$\Omega^\bullet(\mathcal{X})^G \rightarrow \Omega_p^\bullet(\mathcal{X})^\Gamma$$

is defined and admits a left inverse j_p^\bullet defined by integration

$$j_p^\bullet \alpha = \int_{\Gamma \backslash G} (L_g^* \alpha) d\mu(g),$$

for $\alpha \in \Omega_p^\bullet(\mathcal{X})^\Gamma$ and where L_g is left translation by g . The following proposition gives an interesting diagram to be compared with (4.1)

PROPOSITION 4.2. *Let G be a connected semisimple Lie group with finite center and associated symmetric space \mathcal{X} , and let $\Gamma < G$ be a lattice. The following diagram*

$$(4.3) \quad \begin{array}{ccccc} \text{H}_b^\bullet(\Gamma, \mathbb{R}) & \xrightarrow{c_\Gamma^\bullet} & \text{H}^\bullet(\Gamma, \mathbb{R}) & \xrightarrow{\cong} & \text{H}^\bullet(\Omega^\bullet(\mathcal{X})^\Gamma) \\ & \searrow \delta_{p, \Gamma}^\bullet & & \nearrow i_{p, \Gamma}^\bullet & \\ & & \text{H}^\bullet(\Omega_p^\bullet(\mathcal{X})^\Gamma) & & \\ & \downarrow T_b^\bullet & & \searrow j_p^\bullet & \\ \text{H}_{cb}^\bullet(G, \mathbb{R}) & \xrightarrow{c_G^\bullet} & \text{H}_c^\bullet(G, \mathbb{R}) & \xrightarrow{\cong} & \Omega^\bullet(\mathcal{X})^G \end{array}$$

commutes for all $1 \leq p \leq \infty$.

We start the proof by showing how to associate to an L^∞ function c on $(\partial\mathcal{X})^{n+1}$ a differential n -form obtained by integrating, with respect to an appropriate density at infinity and weighted by the function c , the differential form obtained from the Busemann vectors associated to n points at infinity.

So, let $\partial\mathcal{X}$ be the geodesic ray boundary of \mathcal{X} and

$$B : \partial\mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$$

the Busemann cocycle. Fix a basepoint $0 \in \mathcal{X}$ and let $K = \text{Stab}_G(0)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition, $\mathfrak{a}^+ \subset \mathfrak{p}$ a positive Weyl chamber and $b \in \mathfrak{a}^+$ the vector predual to the sum of the positive roots associated to \mathfrak{a}^+ . Then $h(\mathcal{X}) = \|b\|$. Let $\xi_b \in \partial\mathcal{X}$ be the point at infinity determined by b ; let ν_0 be the unique K -invariant probability measure on $G\xi_b \subset \partial\mathcal{X}$. Then

$$(4.4) \quad d(g_*\nu_0)(\xi) = e^{-hB_\xi(g^0,0)} d\nu_0(\xi).$$

For $\xi \in \partial\mathcal{X}$, let us define a C^∞ map by

$$(4.5) \quad \begin{aligned} e^\xi : \mathcal{X} &\rightarrow \mathbb{R} \\ x &\mapsto e^{-h(\mathcal{X})B_\xi(x,0)}. \end{aligned}$$

LEMMA 4.3. *Let G be a connected semisimple Lie group with finite center, and let \mathcal{X} be its associated symmetric space with geodesic ray boundary $\partial\mathcal{X}$. For each $c \in L^\infty((\partial\mathcal{X})^{n+1}, \nu_0^{n+1})$, the differential form defined by*

$$(4.6) \quad \omega := \int_{(\partial\mathcal{X})^{n+1}} c(\xi_0, \dots, \xi_n) e^{\xi_0} \wedge de^{\xi_1} \wedge \dots \wedge de^{\xi_n} d\nu_0^{n+1}(\xi_0, \dots, \xi_n).$$

is in $\Omega_\infty^n(\mathcal{X})$. Moreover the resulting map

$$\begin{aligned} \delta_\infty^\bullet : L^\infty((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1}) &\rightarrow \Omega_\infty^\bullet(\mathcal{X}) \\ c &\mapsto \omega \end{aligned}$$

is a G -equivariant map of complexes, and

$$(4.7) \quad \|\delta_\infty^{(n)}\| \leq h(\mathcal{X})^n.$$

Proof. For $\xi \in \partial\mathcal{X}$, let $X_\xi(x)$ be the unit tangent vector at x pointing in the direction of ξ , and let $g_x(\cdot, \cdot)$ be the Riemannian metric on \mathcal{X} at x . Since the gradient of the Busemann function $B_\xi(x, 0)$ at x is $-X_\xi(x)$ [23], we have that for $v \in (T\mathcal{X})_x$, $(dB_\xi)_x(v) = -g_x(v, X_\xi(x))$. Then

$$(de^\xi)_x(v) = h(\mathcal{X})g_x(v, X_\xi(x))e^\xi(x).$$

This implies that if v_1, \dots, v_n are tangent vectors based at x , then

$$\begin{aligned}
& |\omega_x(v_1, \dots, v_n)| \\
& \leq h(\mathcal{X})^n \int_{(\partial\mathcal{X})^{n+1}} |c(\xi_0, \xi_1, \dots, \xi_n)| e^{\xi_0(x)} \cdot \\
& \quad \cdot \left(\prod_{i=1}^n g_x(v_i, X_{\xi_i}(x)) e^{\xi_i(x)} \right) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\
& \leq h(\mathcal{X})^n \int_{(\partial\mathcal{X})^{n+1}} \|c\|_\infty e^{\xi_0(x)} \left(\prod_{i=1}^n \|v_i\| e^{\xi_i(x)} \right) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\
& = h(\mathcal{X})^n \|c\|_\infty \left(\int_{\partial\mathcal{X}} e^{\xi_0(x)} d\nu_0(\xi_0) \right) \prod_{i=1}^n \left(\|v_i\| \int_{\partial\mathcal{X}} e^{\xi_i(x)} d\nu_0(\xi_i) \right)
\end{aligned}$$

But writing $x = g0$ and using that, as indicated in (4.4), $d(g_*\nu_0)$ is a probability measure, we get that for all $0 \leq i \leq n$ and all $x \in \mathcal{X}$

$$(4.8) \quad \int_{\partial\mathcal{X}} e^{\xi_i(x)} d\nu_0(\xi_i) = 1,$$

which shows that

$$|\omega_x(v_1, \dots, v_n)| \leq h(\mathcal{X})^n \|c\|_\infty \prod_{i=1}^n \|v_i\|,$$

so that

$$\|\delta_\infty^{(n)} c\| = \sup_{x \in \mathcal{X}} \sup_{\|v_1\|, \dots, \|v_n\| \leq 1} |\omega_x(v_1, \dots, v_n)| \leq h(\mathcal{X})^n \|c\|_\infty.$$

This proves (4.7) and the fact that the image $\delta_\infty^{(n)}(c)$ is a bounded form. Once we shall have proven that $\delta_\infty^{(n)}(dc) = d\delta_\infty^{(n-1)}(c)$, it will follow automatically that also $d\delta_\infty^{n-1}(c)$ is bounded and hence the image of δ_∞^\bullet is in $\Omega_\infty^\bullet(\mathcal{X})$. To this purpose, let us compute for $c \in L^\infty((\partial\mathcal{X})^n, \nu_0^n)$

$$\begin{aligned}
\delta_\infty^{(n)}(dc) &= \int_{(\partial\mathcal{X})^{n+1}} e^{\xi_0} \wedge de^{\xi_1} \wedge \dots \wedge de^{\xi_n} \cdot \\
& \quad \cdot \left[\sum_{i=0}^n (-1)^i c(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_n) \right] d\nu_0^{n+1}(\xi_0, \dots, \xi_n).
\end{aligned}$$

For $i \geq 1$ the i -th term is

$$\begin{aligned} & (-1)^i \int_{(\partial\mathcal{X})^{n+1}} e^{\xi_0} \wedge de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} \\ & \qquad \qquad \qquad \cdot c(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_n) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\ & = -d \left(\int_{\partial\mathcal{X}} e^{\xi_i} d\nu_0(\xi_i) \right) \wedge \cdots = 0 \end{aligned}$$

since by (4.8)

$$d \left(\int_{\partial\mathcal{X}} e^{\xi_i} d\nu_0(\xi_i) \right) = 0.$$

Thus

$$\begin{aligned} \delta_\infty^{(n)}(dc) &= \int_{(\partial\mathcal{X})^{n+1}} e^{\xi_0} \wedge de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} c(\xi_1, \dots, \xi_n) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\ &= \left[\int_{\partial\mathcal{X}} e^{\xi_0} d\nu_0(\xi_0) \right] \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} \\ & \qquad \qquad \qquad \cdot c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n) \\ &= \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n). \end{aligned}$$

On the other hand

$$\delta_\infty^{(n-1)}(c) = \int_{(\partial\mathcal{X})^n} e^{\xi_1} \wedge de^{\xi_2} \wedge \cdots \wedge de^{\xi_n} c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n),$$

so that by definition

$$\begin{aligned} d\delta_\infty^{(n-1)}(c) &= \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge de^{\xi_2} \wedge \cdots \wedge de^{\xi_n} c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n) \\ &= \delta_\infty^{(n)}(dc). \end{aligned}$$

The G -equivariance of δ_∞^\bullet follows from (4.4) and the cocycle property of the Busemann function $B_\xi(x, y)$, hence completing the proof. \square

Proof of Proposition 4.1. This is a direct application of [39, Proposition 9.2.3]. Indeed, since the L -action on $(\partial\mathcal{X}, \nu_0)$ is amenable, we have that $(L^\infty((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1}))$ is a strong resolution of \mathbb{R} by relatively injective L -modules [18]; moreover, it is well known that, $(\Omega^\bullet(\mathcal{X}), d^\bullet)$ is a resolution of \mathbb{R} by injective continuous L -modules, where in this case injectivity is meant in ordinary cohomology (see [40]), and $\Omega^\bullet(\mathcal{X})$ is as usual equipped with the C^∞ -topology. Finally one checks on the formulas that the composition $i_\infty^\bullet \circ \delta_\infty^\bullet$, where i_∞^\bullet is the injection

$$i_\infty^\bullet : \Omega_\infty^\bullet(\mathcal{X}) \rightarrow \Omega^\bullet(\mathcal{X}),$$

is a continuous L -morphism of complexes. The hypotheses of [39, Proposition 9.2.3] are hence verified and thus

$$i_{\infty,L}^{\bullet} \circ \delta_{\infty,L}^{\bullet} : H^{\bullet}(L^{\infty}((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1})^L) \rightarrow H^{\bullet}(\Omega^{\bullet}(\mathcal{X})^L)$$

realizes the canonical comparison map

$$c_L^{\bullet} : H_{\text{cb}}^{\bullet}(L, \mathbb{R}) \rightarrow H_c^{\bullet}(L, \mathbb{R})$$

□

Proof of Proposition 4.2. The proof of Proposition 4.1 remains valid verbatim for all $1 \leq p \leq \infty$ to show the commutativity of the upper diagram, so it remains to show only the commutativity of the lower part. Notice moreover that since

$$i_{p,G}^{\bullet} : \Omega_p^{\bullet}(\mathcal{X})^G \rightarrow \Omega^{\bullet}(\mathcal{X})^G$$

is the identity, $\delta_{p,G}^{\bullet}$ realizes in cohomology the canonical comparison map. Furthermore, if P is the minimal parabolic in G stabilizing ξ_b and we identify $(\partial\mathcal{X}, \nu_0)$ with $(G/P, \nu_0)$ as measure spaces, the commutativity of the diagram

$$\begin{array}{ccc} L^{\infty}((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1})^{\Gamma} & \xrightarrow{\delta_{p,\Gamma}^{\bullet}} & \Omega_p^{\bullet}(\mathcal{X})^{\Gamma} \\ \text{T}_{G/P}^{\bullet+1} \downarrow & & \downarrow j_p^{\bullet} \\ L^{\infty}((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1})^G & \xrightarrow{\delta_{p,G}^{\bullet}} & \Omega^{\bullet}(\mathcal{X})^G \end{array}$$

is immediate, where $\text{T}_{G/P}^{\bullet}$ is defined in (3.9). The commutativity of the diagram (3.10) in Lemma 3.4 then completes the proof. □

4.2. A Factorization of the Pullback. Let L be a closed subgroup in a connected semisimple Lie group G with finite center and associated symmetric space \mathcal{X} , and let $\rho : L \rightarrow G'$ be a continuous homomorphism into a topological group G' . Combining the diagram in (4.2) with pullbacks in ordinary and bounded cohomology, we obtain the following

commutative diagram:

$$(4.9) \quad \begin{array}{ccccc} \mathrm{H}_{\mathrm{cb}}^{\bullet}(G', \mathbb{R}) & \xrightarrow{c_{G'}^{\bullet}} & \mathrm{H}_c^{\bullet}(G', \mathbb{R}) & & \\ \rho_b^{\bullet} \downarrow & & \downarrow \rho^{\bullet} & & \\ \mathrm{H}_b^{\bullet}(L, \mathbb{R}) & \xrightarrow{c_L^{\bullet}} & \mathrm{H}^{\bullet}(L, \mathbb{R}) & \xrightarrow{\cong} & \mathrm{H}^{\bullet}(\Omega^{\bullet}(\mathcal{X})^L) \\ & \searrow \delta_{\infty, L}^{\bullet} & & \nearrow i_{\infty, L}^{\bullet} & \\ & & \mathrm{H}^{\bullet}(\Omega_{\infty}^{\bullet}(\mathcal{X})^L) & & \end{array}$$

from which one immediately reads:

COROLLARY 4.4. *Let G' be a topological group, $L \leq G$ any closed subgroup in a semisimple Lie group G with finite center and associated symmetric space \mathcal{X} and $\rho : L \rightarrow G'$ a continuous homomorphism. If $\alpha \in \mathrm{H}_c^n(G', \mathbb{R})$ is represented by a continuous bounded class, then $\rho^{(n)}(\alpha) \in \mathrm{H}^n(L, \mathbb{R})$ is representable by a L -invariant smooth closed differential n -form on \mathcal{X} which is bounded.*

Analogously, if in addition $L = \Gamma < G$ is a lattice, then combining the top part of the diagram in (4.3) with pullbacks we obtain

$$(4.10) \quad \begin{array}{ccccc} \mathrm{H}_{\mathrm{cb}}^{\bullet}(G', \mathbb{R}) & \xrightarrow{c_{G'}^{\bullet}} & \mathrm{H}_c^{\bullet}(G', \mathbb{R}) & & \\ \rho_b^{\bullet} \downarrow & & \downarrow \rho^{\bullet} & & \\ \mathrm{H}_b^{\bullet}(\Gamma, \mathbb{R}) & \xrightarrow{c_{\Gamma}^{\bullet}} & \mathrm{H}^{\bullet}(\Gamma, \mathbb{R}) & \xrightarrow{\cong} & \mathrm{H}^{\bullet}(\Omega^{\bullet}(\mathcal{X})^{\Gamma}) \\ & \searrow \delta_{p, \Gamma}^{\bullet} & & \nearrow i_{p, \Gamma}^{\bullet} & \\ & & \mathrm{H}^{\bullet}(\Omega_p^{\bullet}(\mathcal{X})^{\Gamma}) & & \end{array}$$

In this section we shall mainly draw consequences from this, in especially relevant circumstances. For example, if G' also is a connected, semisimple Lie group with finite center, then in degree two the comparison map

$$c_{G'}^{(2)} : \mathrm{H}_{\mathrm{cb}}^2(G', \mathbb{R}) \rightarrow \mathrm{H}_c^2(G', \mathbb{R})$$

is an isomorphism [18], and we may then compose $(c_{G'}^{(2)})^{-1}$ with $\rho_b^{(2)}$ and $\delta_{p, \Gamma}^{(2)}$ to get a map, which we denote by $\rho_p^{(2)}$, for which the following holds:

COROLLARY 4.5. *If G, G' are connected semisimple Lie groups with finite center, \mathcal{X} is the symmetric space associated to G and $\Gamma < G$*

is a lattice, then the pullback via the homomorphism $\rho : \Gamma \rightarrow G'$ in ordinary cohomology and in degree two factors via L^p -cohomology

$$\begin{array}{ccc}
 & \mathrm{H}_c^2(G', \mathbb{R}) & \\
 & \downarrow \rho^{(2)} & \\
 \rho_p^{(2)} \swarrow & \mathrm{H}^2(\Gamma, \mathbb{R}) & \xrightarrow{\cong} \mathrm{H}^2(\Omega^\bullet(\mathcal{X})^\Gamma) \\
 & \searrow i_{p,\Gamma}^{(2)} & \\
 & \mathrm{H}^2(\Omega_p^\bullet(\mathcal{X})^\Gamma) &
 \end{array}$$

REMARK 4.6. (1) This is true for all closed subgroups $L < G$ in the case $p = \infty$.

(2) Notice that, so far, we have not used the commutativity of the lower part of the diagram in (4.3). This will be done in the following section, to identify a numerical invariant associated to a representation.

4.3. The Hermitian Case. Let \mathcal{X}' be the symmetric space associated to G' , assume that both \mathcal{X} and \mathcal{X}' are Hermitian symmetric, and that moreover \mathcal{X} is irreducible. Let $\kappa_{\mathcal{X}} \in \mathrm{H}_c^2(G, \mathbb{R})$ and $\kappa_{\mathcal{X}}^b \in \mathrm{H}_{\mathrm{cb}}^2(G, \mathbb{R})$ be the cohomology classes which corresponds to the Kähler form $\omega_{\mathcal{X}}$ on \mathcal{X} via the isomorphisms

$$\begin{aligned}
 \Omega^2(\mathcal{X})^G &\cong \mathrm{H}_c^2(G, \mathbb{R}) \cong \mathrm{H}_{\mathrm{cb}}^2(G, \mathbb{R}) \\
 \omega_{\mathcal{X}} &\leftrightarrow \kappa_{\mathcal{X}} \leftrightarrow \kappa_{\mathcal{X}}^b.
 \end{aligned}$$

The fact that $\mathrm{H}_{\mathrm{cb}}^2(G, \mathbb{R}) \cong \mathbb{R} \cdot \kappa_{\mathcal{X}}^b$ and the definition of

$$\mathrm{T}_b^{(2)}(\rho) : \mathrm{H}_{\mathrm{cb}}^2(G', \mathbb{R}) \rightarrow \mathrm{H}_{\mathrm{cb}}^2(G, \mathbb{R})$$

lead to the definition of the *bounded Toledo invariant* $t_b(\rho)$ by

$$(4.11) \quad \mathrm{T}_b^{(2)}(\rho)(\kappa_{\mathcal{X}'}^b) = t_b(\rho)\kappa_{\mathcal{X}}^b.$$

Then we have a Milnor–Wood type inequality:

LEMMA 4.7. *With the above notations,*

$$|t_b(\rho)| \leq \frac{\mathrm{rk} \mathcal{X}'}{\mathrm{rk} \mathcal{X}}.$$

Proof. This follows from the fact that for any Hermitian symmetric space \mathcal{Y} with the appropriate normalization of the metric (see § 2)

$$\|\kappa_{\mathcal{Y}}^b\| = \pi \mathrm{rk} \mathcal{Y},$$

and the fact that $\mathrm{T}_b^{(2)}(\rho)$ is norm decreasing in bounded cohomology. \square

The bounded Toledo invariant can now be nicely interpreted using the lower part of (4.3) in the case $p = 2$. In fact, the space \mathcal{X} being Hermitian symmetric, the L^2 -cohomology spaces $H^\bullet(\Omega_2^\bullet(\mathcal{X})^\Gamma)$ are reduced and finite dimensional, [6][§ 3]. The following observation will be essential:

LEMMA 4.8. *Let \mathcal{X} be a Hermitian symmetric space and Γ a lattice in the isometry group $G := \text{Iso}(\mathcal{X})^\circ$. Then the map*

$$j_2^\bullet : H^\bullet(\Omega_2^\bullet(\mathcal{X})^\Gamma) \rightarrow H^\bullet(\Omega^\bullet(\mathcal{X})^G) = \Omega^\bullet(\mathcal{X})^G$$

is the orthogonal projection.

Proof. Denoting by $\langle \cdot, \cdot \rangle_x$ the scalar product on $\Lambda^\bullet(T_x \mathcal{X})^*$, the scalar product of two forms $\alpha, \beta \in \Omega_2^\bullet(\mathcal{X})^\Gamma$ is given by

$$(4.12) \quad \langle \alpha, \beta \rangle := \int_{\Gamma \backslash \mathcal{X}} \langle \alpha_x, \beta_x \rangle_x dv(\dot{x}),$$

where dv is the volume measure on $\Gamma \backslash \mathcal{X}$; fixing $x_0 \in \mathcal{X}$, and letting μ be the G -invariant probability measure on $\Gamma \backslash G$, (4.12) can be written as

$$(4.13) \quad \langle \alpha, \beta \rangle = \text{vol}(\Gamma \backslash G) \int_{\Gamma \backslash G} \langle \alpha_{hx_0}, \beta_{hx_0} \rangle_{hx_0} d\mu(\dot{h}).$$

Since we have identified $H^\bullet(\Omega_2(\mathcal{X})^\Gamma)$ with the space of harmonic forms which are L^2 (modulo Γ), it suffices to show that

$$\langle j_2^\bullet(\alpha), \beta \rangle = \langle \alpha, j_2^\bullet(\beta) \rangle.$$

To this end we compute

$$\langle (L_g^* \alpha)_x, \beta_x \rangle_x = \langle \alpha_{gx} \circ \Lambda^\bullet d_x L_g, \beta_x \rangle_x = \langle \alpha_{gx}, \beta_x \circ (\Lambda^\bullet d_x L_g)^{-1} \rangle_{gx}$$

and hence, using (4.13),

$$\begin{aligned} & \langle j_2^\bullet(\alpha), \beta \rangle \\ &= \text{vol}(\Gamma \backslash G) \int_{\Gamma \backslash G} \left(\int_{\Gamma \backslash G} \langle \alpha_{ghx_0}, \beta_{hx_0} \circ (\Lambda^\bullet d_{hx_0} L_g)^{-1} \rangle_{ghx_0} d\mu(\dot{g}) \right) d\mu(\dot{h}) \\ &= \text{vol}(\Gamma \backslash G) \int_{\Gamma \backslash G} \left(\int_{\Gamma \backslash G} \langle \alpha_{gx_0}, \beta_{hx_0} \circ (\Lambda^\bullet d_{hx_0} L_{gh^{-1}})^{-1} \rangle_{gx_0} d\mu(\dot{g}) \right) d\mu(\dot{h}) \\ &= \text{vol}(\Gamma \backslash G) \int_{\Gamma \backslash G} \left\langle \alpha_{gx_0}, \int_{\Gamma \backslash G} \beta_{hx_0} \circ (\Lambda^\bullet d_{hx_0} L_{gh^{-1}})^{-1} d\mu(\dot{h}) \right\rangle_{gx_0} d\mu(\dot{g}). \end{aligned}$$

But $(\Lambda^\bullet d_{hx_0} L_{gh^{-1}})^{-1} = \Lambda^\bullet d_{gx_0} L_{hg^{-1}}$, so

$$\begin{aligned} \int_{\Gamma \backslash G} \beta_{hx_0} \circ (\Lambda^\bullet d_{hx_0} L_{gh^{-1}})^{-1} d\mu(\dot{h}) &= \int_{\Gamma \backslash G} \beta_{hx_0} \circ \Lambda^\bullet d_{gx_0} L_{hg^{-1}} d\mu(\dot{h}) \\ &= \int_{\Gamma \backslash G} \beta_{hgx_0} \circ \Lambda^\bullet d_{gx_0} L_h d\mu(\dot{h}) \end{aligned}$$

and hence, using (4.13) and (4.12),

$$\langle j_2^\bullet(\alpha), \beta \rangle = \int_{\Gamma \backslash \mathcal{X}} \left\langle \alpha_x, \int_{\Gamma \backslash G} \beta_{hx} \circ \Lambda^\bullet d_x L_h d\mu(\dot{h}) \right\rangle_x dv(x) = \langle \alpha, j_2^\bullet(\beta) \rangle$$

which shows that j_2 is self-adjoint. Being clearly a projection, this proves the lemma. \square

Applying the lemma in degree two, we have that for $\alpha \in H^2(\Omega_2^\bullet(\mathcal{X})^\Gamma)$,

$$j_2^{(2)}(\alpha) = \frac{\langle \alpha, \omega_{\mathcal{X}} \rangle}{\langle \omega_{\mathcal{X}}, \omega_{\mathcal{X}} \rangle} \omega_{\mathcal{X}},$$

where $\omega_{\mathcal{X}}$ is, as usual, the Kähler form on \mathcal{X} , which is a generator of $H^2(\Omega_2^\bullet(\mathcal{X})^G)$ since \mathcal{X} is assumed to be irreducible. Define now

$$(4.14) \quad i_\rho := \frac{\langle \rho_2^{(2)}(\kappa_{\mathcal{X}'}) , \omega_{\mathcal{X}} \rangle}{\langle \omega_{\mathcal{X}}, \omega_{\mathcal{X}} \rangle}.$$

It finally follows from the commutativity of the diagram

$$(4.15) \quad \begin{array}{ccccc} H_{\text{cb}}^\bullet(G', \mathbb{R}) & \xrightarrow{c_{G'}^\bullet} & H_c^\bullet(G', \mathbb{R}) & & \\ \downarrow \rho_b^\bullet & & \downarrow \rho^\bullet & & \\ H_b^\bullet(\Gamma, \mathbb{R}) & \xrightarrow{c_\Gamma^\bullet} & H^\bullet(\Gamma, \mathbb{R}) & \xrightarrow{\cong} & H^\bullet(\Omega^\bullet(\mathcal{X})^\Gamma) \\ & \searrow \delta_{p,\Gamma}^\bullet & \downarrow & \nearrow i_{p,\Gamma}^\bullet & \\ & & H^\bullet(\Omega_p^\bullet(\mathcal{X})^\Gamma) & & \\ \downarrow T_b^\bullet & & \downarrow j_p^\bullet & & \\ H_{\text{cb}}^\bullet(G, \mathbb{R}) & \xrightarrow{c_G^\bullet} & H_c^\bullet(G, \mathbb{R}) & \xrightarrow{\cong} & \Omega^\bullet(\mathcal{X})^G \end{array}$$

in the special case of $p = 2$ and degree 2 and from Corollary 4.5 that:

LEMMA 4.9. $i_\rho = t_b(\rho)$.

For our immediate applications, we draw the following conclusions when $\mathcal{X} = \mathcal{H}_{\mathbb{C}}^p$:

PROPOSITION 4.10. *Let G be a connected simple Lie group with finite center and associated symmetric space $\mathcal{H}_{\mathbb{C}}^p$ and let $\Gamma < G$ be a lattice. Let \mathcal{X}' be a Hermitian symmetric space, $G' := \text{Iso}(\mathcal{X}')^\circ$ and $\rho : \Gamma \rightarrow G'$ a representation. Then:*

- (1) $|i_\rho| \leq \text{rk } \mathcal{X}'$;
- (2) *if either $\Gamma \backslash \mathcal{H}_{\mathbb{C}}^p$ is compact or $n \geq 2$, then i_ρ is a characteristic number.*

Proof. (1) follows from Lemma 4.7 and Lemma 4.9.

(2) If $\Gamma \backslash \mathcal{H}_{\mathbb{C}}^p$ is compact we have evidently that

$$H^\bullet(\Omega_2^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma) = H^\bullet(\Omega^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma),$$

while if $p \geq 2$, we have that $H^2(\Omega_2^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma)$ injects into $H^2(\Omega^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma)$ [48]. In any case, by Corollary 4.5, i_ρ depends only upon $\rho^{(2)}(\kappa_{\mathcal{X}'}) \in H^2(\Omega^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma)$, which is well known to be a characteristic class (see § 1). Hence i_ρ is a characteristic number. \square

Actually, the injectivity of the map $H^2(\Omega_2^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma) \rightarrow H^2(\Omega^\bullet(\mathcal{H}_{\mathbb{C}}^p)^\Gamma)$ is proven in [48] in a more general framework but only for arithmetic lattices. In our specific case however the proof applies without the arithmeticity requirement [42].

4.4. The Formula, Once Again. The very explicit form of the factorization of the comparison map between bounded and ordinary cohomology, together with the implementation of the pullback by boundary maps in [11] allows one to give explicit representatives of the class $\rho^{(2)}(\kappa_q)$ at least when $\mathcal{X}' = \mathcal{H}_{\mathbb{C}}^q$.

To this end we assume that the homomorphism $\rho : L \rightarrow \text{Iso}(\mathcal{H}_{\mathbb{C}}^q)^\circ =: G'$ is nonelementary so that there exists a L -equivariant measurable map $\varphi : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ (see for example [19]). Let

$$c_q : (\partial\mathcal{H}_{\mathbb{C}}^q)^3 \rightarrow [-1, 1]$$

be the Cartan cocycle which is a representative of the Kähler class $\kappa_q^b \in H_{\text{cb}}^2(G', \mathbb{R})$, [12]. For $\xi \in \partial\mathcal{H}_{\mathbb{C}}^q$, let e^ξ denote the exponential of the Busemann function defined in (4.5). Then

PROPOSITION 4.11. *Let G be a connected simple Lie group with finite center and associated symmetric space $\mathcal{H}_{\mathbb{C}}^p$ and let $L \leq G$ be any closed subgroup. Then the differential 2-form*

$$(4.16) \quad \int_{(\partial\mathcal{H}_{\mathbb{C}}^p)^3} c_q(\varphi(\xi_0), \varphi(\xi_1), \varphi(\xi_2)) e^{\xi_0} \wedge de^{\xi_1} \wedge de^{\xi_2} d\nu_0^3(\xi_0, \xi_1, \xi_2)$$

is a smooth L -invariant bounded closed 2-form representing $\rho^{(2)}(\kappa_q) \in H^2(L, \mathbb{R}) \cong H^2(\Omega^\bullet(\mathcal{H}_\mathbb{C}^p)^L)$.

Proof. Since $c_q \in \mathcal{B}^\infty((\partial\mathcal{H}_\mathbb{C}^q)^3)^{G'}$ represents κ_q^b , by (3.14) the cocycle in $L^\infty((\partial\mathcal{H}_\mathbb{C}^p)^3)^L$

$$(\xi_0, \dots, \xi_n) \mapsto c(\varphi(\xi_0), \dots, \varphi(\xi_n))$$

represents canonically $\rho_b^{(2)}(\kappa_q^b) \in H_b^2(L, \mathbb{R})$. By Lemma 4.3, (4.16) is a smooth differential 2-form in $\Omega_\infty^2(\mathcal{X})$ which is L -invariant and, by Proposition 4.1, it represents $\rho^{(2)}(\kappa_q) \in H^2(\Omega^\bullet(\mathcal{H}_\mathbb{C}^p)^L)$. \square

Let us assume now that $L = \Gamma < \mathrm{SU}(p, 1)$ is a lattice and move to the main formula, which will be an implementation of § 3.2 in our concrete situation. Let \mathcal{C}_p be the set of all chains in $\partial\mathcal{H}_\mathbb{C}^p$ and, for any $k \geq 1$, let

$$\mathcal{C}_p^{\{k\}} := \{(C, \xi_1, \dots, \xi_k) : C \in \mathcal{C}_p, (\xi_1, \dots, \xi_k) \in C^k\}$$

be the space of configurations of k -tuples of points on a chain. Both \mathcal{C}_p and $\mathcal{C}_p^{\{1\}}$ are homogeneous spaces of $\mathrm{SU}(p, 1)$. In fact, the stabilizer H in G of a fixed chain $C_0 \in \mathcal{C}_p$ is also the stabilizer of a plane of signature $(1, 1)$ in $\mathrm{SU}(p, 1)$ and hence isomorphic to $\mathrm{S}(\mathrm{U}(1, 1) \times \mathrm{U}(p-1))$. Then $\mathrm{SU}(p, 1)$ acts transitively on \mathcal{C}_p (for example because it acts transitively on pairs of points in $\partial\mathcal{H}_\mathbb{C}^p$ and any two points in $\partial\mathcal{H}_\mathbb{C}^p$ determine uniquely a chain) and H acts transitively on C_0 , so that, if $Q = P \cap H$, where P is the stabilizer in $\mathrm{SU}(p, 1)$ of a fixed basepoint $\xi_0 \in C_0$, there are $\mathrm{SU}(p, 1)$ -equivariant (hence measure class preserving) diffeomorphisms

$$\begin{aligned} \mathrm{SU}(p, 1)/H &\rightarrow \mathcal{C}_p \\ gH &\mapsto gC_0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{SU}(p, 1)/Q &\rightarrow \mathcal{C}_p^{\{1\}} \\ gQ &\mapsto (gC_0, g\xi_0). \end{aligned}$$

Moreover, the projection $\pi : \mathcal{C}_p^{\{1\}} \rightarrow \mathcal{C}_p$ which associates to a point $(C, \xi) \in \mathcal{C}_p^{\{1\}}$ the chain $C \in \mathcal{C}_p$ is a $\mathrm{SU}(p, 1)$ -equivariant fibration, the space $\mathcal{C}_p^{\{k\}}$ appears then naturally as k -fold fibered product of $\mathcal{C}_p^{\{1\}}$ with respect to π , and for every $k \geq 1$, the map

$$(4.17) \quad \begin{aligned} (\mathrm{SU}(p, 1)/Q)_f^k &\rightarrow \mathcal{C}_p^{\{k\}} \\ (x_1Q, \dots, x_kQ) &\mapsto (gC_0, x_1\xi_0, \dots, x_k\xi_0) \end{aligned}$$

where $x_i H = gH$, $1 \leq i \leq k$, is a $SU(p, 1)$ -equivariant diffeomorphism which preserves the $SU(p, 1)$ -invariant Lebesgue measure class. Using Fubini's theorem, one has that for almost every $C \in \mathcal{C}_p$ the restriction

$$\varphi_C : C \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$$

of φ to C is measurable and for every $\gamma \in \Gamma$ and almost every $\xi \in C$

$$\varphi_{\gamma C}(\gamma\xi) = \rho(\gamma)\varphi_C(\xi).$$

This allows us to define

$$\begin{aligned} \varphi^{\{3\}} : \mathcal{C}_p^{\{3\}} &\rightarrow (\partial\mathcal{H}_{\mathbb{C}}^q)^3 \\ (C, \xi_1, \xi_2, \xi_3) &\mapsto (\varphi_C(\xi_1), \varphi_C(\xi_2), \varphi_C(\xi_3)). \end{aligned}$$

Then

THEOREM 4.12. *Let i_ρ be the invariant defined in (4.14). Then for almost every chain $C \in \mathcal{C}_p$ and almost every $(\xi_1, \xi_2, \xi_3) \in C^3$,*

$$\int_{L \backslash SU(p, 1)} c_q(\varphi^{\{3\}}(gC, g\xi_1, g\xi_2, g\xi_3)) d\mu(g) = i_\rho c_p(\xi_1, \xi_2, \xi_3),$$

where c_ℓ is the Cartan invariant and μ is the $SU(p, 1)$ -invariant probability measure on $\Gamma \backslash SU(p, 1)$.

COROLLARY 4.13. *Assume that $i_\rho = 1$. Then for almost every $C \in \mathcal{C}_p$ and almost every $(\xi_1, \xi_2, \xi_3) \in C^3$*

$$c_q(\varphi_C(\xi_1), \varphi_C(\xi_2), \varphi_C(\xi_3)) = c_p(\xi_1, \xi_2, \xi_3).$$

Proof of Theorem 4.12. Let $H, P, Q < SU(p, 1)$ such as in the above discussion. Since P is the stabilizer of a basepoint $\xi_0 \in \partial\mathcal{H}_{\mathbb{C}}^p$, it is a minimal parabolic subgroup and hence the closed subgroup Q is amenable. Moreover, H acts ergodically on $H/Q \times H/Q$ since in $H/Q \times H/Q$ there is an open H -orbit of full measure. We can hence apply Proposition 3.8 with $G = SU(p, 1)$, $G' = PU(q, 1)$ and $\kappa' = \kappa_q^b$. Moreover, by (4.11) and the definition of the bounded Toledo map, let $\kappa = T_b^{(2)}(\rho_b^{(2)}(\kappa_q^b)) = t_b(\rho)\kappa_p^b$, which in turns, by Lemma 4.9 implies that $\kappa = i_\rho \kappa_p^b$. Set $G/P = \partial\mathcal{H}_{\mathbb{C}}^p$, $c' = i_\rho c_p \in \mathcal{B}^\infty((\partial\mathcal{H}_{\mathbb{C}}^p)^3)^{SU(p, 1)}$, $X = \partial\mathcal{H}_{\mathbb{C}}^p$ and $c' = c_q \in \mathcal{B}^\infty((\partial\mathcal{H}_{\mathbb{C}}^p)^3)^{PU(q, 1)}$. Then the conclusion of the theorem is immediate if we observe that the identification in (4.17) transforms the map φ_f^3 defined in (3.15) into the map $\varphi^{\{3\}}$ defined above. \square

REMARK 4.14. It is now clear what is the essential use of the fibered product: the triples of points that lie on a chain form a set of measure zero in $(\partial\mathcal{H}_{\mathbb{C}}^p)^3$, and hence we would not have gained any information

on these configuration of points by the direct use of the more familiar formula as in Corollary 3.9.

5. THE MEASURABLE CARTAN THEOREM

The goal of this section is to prove the following measurable version of a theorem of E. Cartan.

THEOREM 5.1. *Let $p \geq 2$ and let $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ be a measurable map such that:*

- (i) *for almost every chain C and almost every triple (ξ, η, ζ) of distinct points on C , the triple $\varphi(\xi), \varphi(\eta), \varphi(\zeta)$ consists also of distinct points which lie on a chain and have the same orientation as (ξ, η, ζ) ;*
- (ii) *for almost every triple of points ξ, η, ζ not on a chain, $\varphi(\xi), \varphi(\eta), \varphi(\zeta)$ are also not on a chain.*

Then there is an isometric holomorphic embedding $F : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ such that ∂F coincides with φ almost everywhere.

The proof goes as follows. We first show by induction that the statement of the theorem for a fixed p follows from the analogous statement in one lower dimension, provided $p \geq 3$; this leaves us to show the statement for $p = 2$. The next step is to show that if $p = 2$ any map $\varphi : \partial\mathcal{H}_{\mathbb{C}}^2 \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ satisfying the hypotheses of Theorem 5.1 takes values also in $\partial\mathcal{H}_{\mathbb{C}}^2$; this will be achieved by an appropriate convex hull argument. The last step is hence to show the assertion for $p = q = 2$, for which we need a careful modification of Cartan's argument for point-wise defined maps.

5.1. Reduction to the case $p = 2$. If $k \leq p$, let \mathcal{P}_k denote the set of k -planes (see the end of § 2) and, if $x \in \mathcal{H}_{\mathbb{C}}^p$, $\mathcal{P}_k(x)$ the subset of \mathcal{P}_k of k -planes through x .

We now let $p \geq 3$, we assume that the theorem holds for $p-1$ and we want to show that then it holds for p . Let us start by observing that a simple verification using Fubini's theorem applied to the configuration spaces

$$\{(X, C) : X \in \mathcal{P}_{p-1}, C \in \mathcal{C}_p, C \subset \partial X\}$$

and

$$\{(X, \xi_1, \xi_2, \xi_3) : X \in \mathcal{P}_{p-1}, \text{ and } \xi_1, \xi_2, \xi_3 \in \partial X\}$$

shows that, for almost every $X \in \mathcal{P}_{p-1}$, the restriction $\varphi|_{\partial X}$ of φ to ∂X is measurable and satisfies the hypotheses of Theorem 5.1.

Applying the induction hypothesis we get for almost every $X \in \mathcal{P}_{p-1}$ an isometric holomorphic embedding

$$F_X : X \rightarrow \mathcal{H}_{\mathbb{C}}^q$$

such that $\partial F_X = \varphi|_{\partial X}$ almost everywhere. Thus the set

$$\left\{ (x, X) : X \in \mathcal{P}_{p-1}(x) \text{ and there is } F_X : X \rightarrow \mathcal{H}_{\mathbb{C}}^q \text{ as above} \right. \\ \left. \text{with } \partial F_X = \varphi|_{\partial X} \text{ almost everywhere} \right\}$$

is of full measure in the configuration space

$$\{(x, X) : X \in \mathcal{P}_{p-1}(x)\},$$

and we may define for almost every $x \in \mathcal{H}_{\mathbb{C}}^p$ and almost every $X \in \mathcal{P}_{p-1}(x)$ the function

$$f(x, X) := F_X(x).$$

Using again Fubini's theorem, one checks that for almost every $X_1, X_2 \in \mathcal{P}_{p-1}(x)$,

$$\partial F_{X_1}|_{\partial X_1 \cap \partial X_2} = \varphi|_{\partial X_1 \cap \partial X_2} = \partial F_{X_2}|_{\partial X_1 \cap \partial X_2}$$

and, since $p \geq 3$,

$$\partial X_1 \cap \partial X_2 = \partial(X_1 \cap X_2) \neq \emptyset,$$

which implies that $f(x, X_1) = f(x, X_2)$. Thus $f(x, X)$ is almost everywhere independent of $X \in \mathcal{P}_{p-1}(x)$ and gives rise to a well defined map $f : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ which by construction preserves the distances of almost every pair of points. It is then not difficult to see that f coincides almost everywhere with an isometric embedding $\mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$. This, together with the fact that $\partial f = \varphi$ preserves the orientation on chains, implies that the embedding must be holomorphic. \square

5.2. Reduction to the case $p = q = 2$. Denote by $(\partial \mathcal{H}_{\mathbb{C}}^p)^{(k)}$ the subset of full measure in $(\partial \mathcal{H}_{\mathbb{C}}^p)^k$ consisting of k -tuples of distinct points in $\partial \mathcal{H}_{\mathbb{C}}^p$. Recall that any two distinct chains are either disjoint or intersect in a point, and hence every pair of distinct points $(\xi, \eta) \in (\partial \mathcal{H}_{\mathbb{C}}^p)^{(2)}$ determines a unique chain $C(\xi, \eta)$.

LEMMA 5.2. *Let $\varphi : \partial \mathcal{H}_{\mathbb{C}}^p \rightarrow \partial \mathcal{H}_{\mathbb{C}}^q$ be a measurable map satisfying the hypothesis (i) of Theorem 5.1 and let $c_\ell : (\partial \mathcal{H}_{\mathbb{C}}^\ell)^3 \rightarrow [-1, 1]$ be the Cartan cocycle. Then:*

- (1) *for almost every $(\xi_1, \xi_2) \in (\partial \mathcal{H}_{\mathbb{C}}^p)^{(2)}$, we have that $\varphi(\xi_1) \neq \varphi(\xi_2)$, and*

(2) for almost every $\xi_3 \in C(\xi_1, \xi_2)$, we have that

$$\varphi(\xi_3) \in C(\varphi(\xi_1), \varphi(\xi_2))$$

and

$$c_q(\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)) = c_p(\xi_1, \xi_2, \xi_3).$$

As a consequence we have:

COROLLARY 5.3. *Let, as above, $\varphi : \partial\mathcal{H}_\mathbb{C}^p \rightarrow \partial\mathcal{H}_\mathbb{C}^q$ be a measurable map satisfying the hypothesis (i) of Theorem 5.1 and let c_ℓ be the Cartan cocycle. Then there is a measurable map*

$$(5.1) \quad \Phi : \mathcal{C}_p \rightarrow \mathcal{C}_q$$

such that

$$(5.2) \quad \Phi(C(\xi_1, \xi_2)) = C(\varphi(\xi_1), \varphi(\xi_2))$$

for almost every $(\xi_1, \xi_2) \in (\partial\mathcal{H}_\mathbb{C}^p)^2$.

Proof of Lemma 5.2. Consider the measure class preserving bijection

$$\begin{aligned} (\partial\mathcal{H}_\mathbb{C}^p)^{(2)} &\rightarrow \{(C, \xi_1, \xi_2) : C \in \mathcal{C}_p, \xi_1, \xi_2 \in C, \xi_1 \neq \xi_2\} \\ (\xi_1, \xi_2) &\mapsto (C(\xi_1, \xi_2), \xi_1, \xi_2). \end{aligned}$$

Then Theorem 5.1(i) implies by Fubini that for almost every $C \in \mathcal{C}_p$, for almost every $(\xi_1, \xi_2) \in C^{(2)}$ and for almost every $\xi_3 \in C$ we have

$$(5.3) \quad c_q(\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)) = c_p(\xi_1, \xi_2, \xi_3)$$

which, using the above bijection, is equivalent to the fact that for almost every $(\xi_1, \xi_2) \in (\partial\mathcal{H}_\mathbb{C}^p)^{(2)}$ and for almost every $\xi_3 \in C(\xi_1, \xi_2)$, (5.3) holds, which shows that $\varphi(\xi_1) \neq \varphi(\xi_2)$ and that (2) holds. \square

Proof of Corollary 5.3. It is clear that if $C \in \mathcal{C}_p$ is such that for almost every $(\xi_1, \xi_2) \in C^{(2)}$ and for almost every $\xi_3 \in C$ (5.3) holds, then in particular if $(\xi_1, \xi_2) \in C^{(2)}$ and $(\eta_1, \eta_2) \in C^{(2)}$ are such that

$$\begin{aligned} c_q(\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)) &= c_p(\xi_1, \xi_2, \xi_3) \\ c_q(\varphi(\eta_1), \varphi(\eta_2), \varphi(\eta_3)) &= c_p(\eta_1, \eta_2, \eta_3) \end{aligned}$$

then $\varphi(\xi_1) \neq \varphi(\xi_2)$, $\varphi(\eta_1) \neq \varphi(\eta_2)$ and $C(\varphi(\xi_1), \varphi(\xi_2)) \cap C(\varphi(\eta_1), \varphi(\eta_2))$ contains the essential image of $\varphi|_C$. Since this cannot be reduced to a point, we have that $C(\varphi(\xi_1), \varphi(\xi_2)) = C(\varphi(\eta_1), \varphi(\eta_2))$, which leads to the map

$$\begin{aligned} \Phi : \quad \mathcal{C}_p &\rightarrow \mathcal{C}_q \\ (C(\xi_1, \xi_2)) &\mapsto C(\varphi(\xi_1), \varphi(\xi_2)) \end{aligned}$$

which is then well defined and satisfies (5.2). \square

Now choose a Borel map

$$\begin{aligned} (\partial\mathcal{H}_{\mathbb{C}}^p)^{(2)} &\rightarrow \mathcal{M}^1(\partial\mathcal{H}_{\mathbb{C}}^p) \\ (\xi_1, \xi_2) &\mapsto \mu_{(\xi_1, \xi_2)}, \end{aligned}$$

such that $\mu_{(\xi_1, \xi_2)}$ is in the Lebesgue measure class of $C(\xi_1, \xi_2)$. Let $d\lambda$ be the ‘‘round measure’’ on $\partial\mathcal{H}_{\mathbb{C}}^p$ and consider the map

$$\begin{aligned} M : (\partial\mathcal{H}_{\mathbb{C}}^p)^{(3)} &\rightarrow \mathcal{M}^1(\partial\mathcal{H}_{\mathbb{C}}^p \times \partial\mathcal{H}_{\mathbb{C}}^p) \\ (\xi_1, \xi_2, \xi_3) &\mapsto \mu_{(\xi_1, \xi_2)} \otimes \mu_{(\xi_1, \xi_3)}. \end{aligned}$$

LEMMA 5.4. *The measure on $\partial\mathcal{H}_{\mathbb{C}}^p \times \partial\mathcal{H}_{\mathbb{C}}^p$ defined by*

$$\int_{(\partial\mathcal{H}_{\mathbb{C}}^p)^{(3)}} (\mu_{(\xi_1, \xi_2)} \otimes \mu_{(\xi_1, \xi_3)}(f)) d\lambda^3(\xi_1, \xi_2, \xi_3)$$

for $f \in C(\partial\mathcal{H}_{\mathbb{C}}^p \times \partial\mathcal{H}_{\mathbb{C}}^p)$, is equivalent to λ^2 .

Proof. This is obvious. \square

LEMMA 5.5. *For almost every $(\xi_1, \xi_2, \xi_3) \in (\partial\mathcal{H}_{\mathbb{C}}^p)^{(3)}$, for almost every $(a, b) \in C(\xi_1, \xi_2) \times C(\xi_1, \xi_3)$ and for almost every $c \in C(a, b)$, we have that:*

- $\varphi(a) \in C(\varphi(\xi_1), \varphi(\xi_2))$;
- $\varphi(b) \in C(\varphi(\xi_1), \varphi(\xi_3))$;
- $\varphi(c) \in C(\varphi(a), \varphi(b))$.

Proof. This follows from repeated applications of Lemma 5.2(2) and Lemma 5.4. \square

COROLLARY 5.6. *Assume that in Theorem 5.1 we have $p = 2$. Then the essential image of φ is contained in a 2-chain.*

Proof. Fix (ξ_1, ξ_2, ξ_3) not on a chain, for which Lemma 5.5 holds. Let $E \subset \partial\mathcal{H}_{\mathbb{C}}^2$ be the set of $c \in \partial\mathcal{H}_{\mathbb{C}}^2$ such that there are $(a, b) \in C(\xi_1, \xi_2) \times C(\xi_1, \xi_3)$ with $c \in C(a, b)$ and Lemma 5.5 holds for a, b, c . Then $E \subset \partial\mathcal{H}_{\mathbb{C}}^2$ is of full measure. Moreover,

- $\varphi(c)$ is in the \mathbb{C} -linear span of $\varphi(a)$ and $\varphi(b)$,
- $\varphi(a)$ is in the \mathbb{C} -linear span of $\varphi(\xi_1)$ and $\varphi(\xi_2)$, and
- $\varphi(b)$ is in the \mathbb{C} -linear span of $\varphi(\xi_1)$ and $\varphi(\xi_3)$,

so that for all $c \in E$, $\varphi(c)$ is in the 2-chain determined by the 3-dimensional space $\varphi(\xi_1) \oplus \varphi(\xi_2) \oplus \varphi(\xi_3)$. \square

5.3. **The Case $p = q = 2$.** So we are reduced to show Theorem 5.1 for $p = q = 2$. The essential point of the proof will be the following:

PROPOSITION 5.7. *Let $g : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable map such that for almost every circle $S \subset \mathbb{C}$, there is a circle $\Gamma(S) \subset \mathbb{C}$ such that:*

- (i) *for almost every $z \in S$, $g(z) \in \Gamma(S)$;*
- (ii) *for almost every $z_1, z_2, z_3 \in S$ distinct, $g(z_1), g(z_2), g(z_3) \in \Gamma(S)$ are distinct and in the same cyclic order;*
- (iii) *for almost every $z \in \mathbb{C}$ the set*

$$\{(S_1, S_2) : z \in S_i, i = 1, 2, \Gamma(S_1) = \Gamma(S_2)\}$$

is of measure zero.

Then g coincides almost everywhere with an affine map $z \mapsto \lambda z + c$, where $\lambda \in \mathbb{C}^\times$ and $c \in \mathbb{C}$.

Proof. By Fubini's theorem, the set E of $z \in \mathbb{C}$ such that (i), (ii) and (iii) hold for almost every circle through z is of full measure in \mathbb{C} ; fixing $z \in E$ and composing with an affine map, we may assume that $g(z) = z$. Conjugating g with an inversion $i : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ through a circle with center z , we get a map $f : \mathbb{C} \rightarrow \mathbb{C}$ which induces a map $F : \mathcal{D} \rightarrow \mathcal{D}$ on the set \mathcal{D} of affine \mathbb{R} -lines in \mathbb{C} satisfying the following properties:

- (ii)' *for almost every $d \in \mathcal{D}$, there is $F(d) \in \mathcal{D}$ such that for almost every $A, B, C \in d$ distinct, $f(A), f(B), f(C)$ are distinct and lie on $F(d)$;*
- (iii)' *for almost every point $w \in \mathbb{C}$, the set*

$$\{(d_1, d_2) \in \mathcal{D}(w) : F(d_1) = F(d_2)\}$$

is of measure zero. Here $\mathcal{D}(w)$ is the set of lines through w .

Notice that in (ii)' one could have stated also a condition corresponding to the preservation of the cyclic ordering. However this is never used for the maps f and F themselves and comes into play only at the very end of the proof when we return to the map g , so that we chose to ignore it.

LEMMA 5.8. *Let*

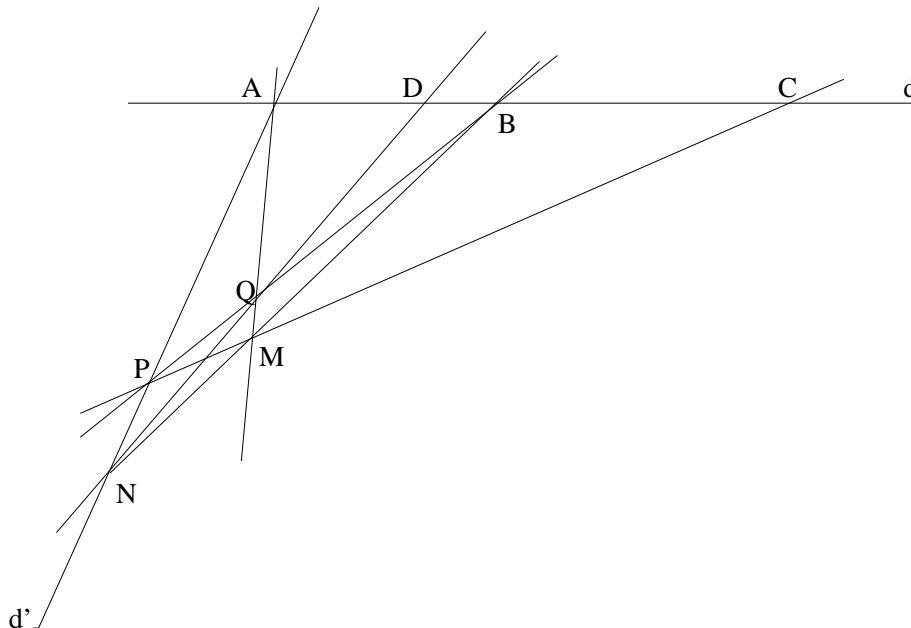
$$\mathcal{K} = \{(d, A, B, C) : d \in \mathcal{D}, \text{ and } A, B, C \text{ are distinct points on } d\}.$$

Then for almost every $(d, A, B, C) \in \mathcal{K}$:

- (i) *we have*

$$(F(d), f(A), f(B), f(C)) \in \mathcal{K},$$

and

FIGURE 2. *La méthode du quadrilatère complet*

(ii) f preserves the property for 4-tuples of points on a line to be in anharmonic position: namely, if $[\cdot, \cdot, \cdot, \cdot]$ denotes the crossratio, and if $D \in d$ and $D' \in F(d)$ are points such that $[A, B, C, D] = -1$ and $[f(A), f(B), f(C), D'] = -1$, then

$$f(D) = D'.$$

Proof. Let

$$\mathcal{K}_0 = \{(d, A); d \in \mathcal{D}, A \in d\}$$

and let

$$\mathcal{K}_1 = \{(d', d, A, B, C, M) : (d, A, B, C) \in \mathcal{K}, (d', A) \in \mathcal{K}_0 \\ \text{and } d' \neq d, M \notin d' \cup d\}.$$

First we observe that properties (ii)' and (iii)' together with a repeated use of Fubini's theorem imply that there is a subset $E \subset \mathcal{K}_1$ of full measure such that for all $(d', d, A, B, C, M) \in E$, then

$$(F(d'), F(d), f(A), f(B), f(C), f(M)) \in \mathcal{K}_1.$$

Now to every $(d', d, A, B, C, M) \in \mathcal{K}_1$ associate the picture in Figure 2 and the following seven maps $m_i : \mathcal{K}_1 \rightarrow \mathcal{K}$ given by

$$\begin{aligned}
m_1 &: (d', d, A, B, C, M) \mapsto (\langle C, M \rangle, C, M, P) \\
m_2 &: (d', d, A, B, C, M) \mapsto (\langle P, B \rangle, P, Q, B) \\
m_3 &: (d', d, A, B, C, M) \mapsto (\langle A, M \rangle, A, Q, M) \\
m_4 &: (d', d, A, B, C, M) \mapsto (\langle B, M \rangle, B, M, N) \\
m_5 &: (d', d, A, B, C, M) \mapsto (d', A, P, N) \\
m_6 &: (d', d, A, B, C, M) \mapsto (\langle N, Q \rangle, N, Q, D) \\
m_7 &: (d', d, A, B, C, M) \mapsto (d, A, B, D).
\end{aligned}$$

One verifies easily that for every set $T \subset \mathcal{K}$ of full measure, $m_i^{-1}(T) \subset \mathcal{K}_1$ is of full measure for $1 \leq i \leq 7$. So let $T \subset \mathcal{K}$ be the subset consisting of (d, A, B, C) such that $(f(A), f(B), f(C))$ are pairwise distinct and pass through $F(d)$, that is $(F(d), f(A), f(B), f(C)) \in \mathcal{K}$. Then T is of full measure, and hence the set

$$E' = \{e \in E : m_i(e) \in T, 1 \leq i \leq 7\}$$

is of full measure. But it follows then from *la méthode du quadrilatère complet* [20] that if $(d', d, A, B, C, M) \in E'$, and

$$\begin{aligned}
D &\in d \text{ with } [A, B, C, D] = -1 \\
D' &\in F(d) \text{ with } [f(A), f(B), f(C), D'] = -1
\end{aligned}$$

then $f(D) = D'$. □

Continuation of the proof of Proposition 5.7. Applying this argument to every $z \in E$, we conclude that our original map g has the property that

$$\begin{aligned}
(5.4) \quad &\text{for a. e. } z_1, z_2, z_3, z_4 \in \mathbb{C} \text{ such that } [z_1, z_2, z_3, z_4] = -1, \\
&\text{we have that } [g(z_1), g(z_2), g(z_3), g(z_4)] = -1.
\end{aligned}$$

Fix now any z_4 such that (5.4) holds and consider the composition $\tilde{g} := i_2 \circ g \circ i_1$, where i_1, i_2 are inversions with $i_1(\infty) = z_4$ and $i_2(g(z_4)) = \infty$. Then, for almost every $(z_1, z_2) \in \mathbb{C}^2$, we have that

$$2\tilde{g}\left(\frac{z_1 + z_2}{2}\right) = \tilde{g}(z_1) + \tilde{g}(z_2),$$

which implies that \tilde{g} coincides almost everywhere with an \mathbb{R} -affine transformation of \mathbb{C} . But since \tilde{g} sends circles to circles, it is either \mathbb{C} -affine or $\overline{\mathbb{C}}$ -affine, so that g is either a homography or an antihomography. But then, using that g has to preserve cyclic order on circles, one concludes that g is \mathbb{C} -affine. □

Proof of Theorem 5.1. Let $\xi \in \partial\mathcal{H}_{\mathbb{C}}^2$, P the stabilizer of ξ in $SU(2, 1)$ and N its unipotent radical. Then $\partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\}$ is a principal homogeneous space for N and the orbits of the center $Z(N)$ of N correspond to the chains through ξ . Fixing an identification of $N/Z(N)$ with the additive group \mathbb{C} leads to a quotient map

$$\pi_{\xi} : \partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\} \rightarrow \mathbb{C}$$

whose fibers are the chains through ξ (with ξ removed). One can then fix an identification

$$P/Z(N) \xrightarrow{\cong} \text{Aff}(\mathbb{C})$$

such that for the corresponding quotient homomorphism

$$\omega_{\xi} : P \rightarrow \text{Aff}(\mathbb{C}),$$

π_{ξ} is equivariant with respect to ω_{ξ} . We recall here from [28] the following two facts:

- (*) For every chain $C \subset \partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\}$, $\pi_{\xi}|_C$ is injective with image a circle in \mathbb{C} ;
- (**) for every circle $S \subset \mathbb{C}$ and any $s \in \partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\}$ with $\pi_{\xi}(s) \in S$, there is a (unique) chain C through s such that $\pi_{\xi}(C) = S$.

Let us denote by $\mathcal{C}_p(\xi)$ the set of chains through the point $\xi \in \partial\mathcal{H}_{\mathbb{C}}^2$ and let now $\varphi : \partial\mathcal{H}_{\mathbb{C}}^2 \rightarrow \partial\mathcal{H}_{\mathbb{C}}^2$ be a measurable map satisfying the hypotheses of Theorem 5.1. Let $\Phi : \mathcal{C}_2 \rightarrow \mathcal{C}_2$ be the map induced almost everywhere on the set of chains defined in (5.1), and $E \subset \partial\mathcal{H}_{\mathbb{C}}^2$ the subset of full measure such that for every $\xi \in E$ and almost every $C \in \mathcal{C}_2(\xi)$, also $\Phi(C) \in \mathcal{C}_2(\varphi(\xi))$. Fix $\xi \in E$; composing with an element from $SU(2, 1)$, we may assume that $\varphi(\xi) = \xi$. Then $\Phi : \mathcal{C}_2(\xi) \rightarrow \mathcal{C}_2(\xi)$ induces a measurable map $g_{\xi} : \mathbb{C} \rightarrow \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} \partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\} & \xrightarrow{\varphi} & \partial\mathcal{H}_{\mathbb{C}}^2 \setminus \{\xi\} \\ \pi_{\xi} \downarrow & & \downarrow \pi_{\xi} \\ \mathbb{C} & \xrightarrow{g_{\xi}} & \mathbb{C} \end{array}$$

commutes and g_{ξ} satisfies the assumptions of Proposition 5.7, hence coincides almost everywhere with an element of $\text{Aff}(\mathbb{C})$. Thus composing φ with an element $h \in P$ such that $\omega_{\xi}(h) = g_{\xi}$ almost everywhere, we may assume that $g_{\xi} = \text{Id}$ almost everywhere. That is, for almost every $C \in \mathcal{C}_2(\xi)$ and almost every $\xi \in C$, $\varphi(\xi) \in C$. Now pick such a C and $\eta \in C \cap E$. Composing with an element from $Z(N)$, we may assume that $\varphi(\eta) = \eta$. But then the map $g_{\eta} : \mathbb{C} \rightarrow \mathbb{C}$ fixes $\pi_{\eta}(\xi)$, leaves all circles through $\pi_{\eta}(\xi)$ invariant and coincides with an affine map almost everywhere. Hence $g_{\eta} = \text{Id}_{\mathbb{C}}$ almost everywhere. Then for

almost every chain $C \in \mathcal{C}_2(\xi)$, π_η being injective on C and $g_\eta = Id_{\mathbb{C}}$, $\varphi|_C = Id_C$ and since those chains foliate a set of $\partial\mathcal{H}_{\mathbb{C}}^2$ of full measure, we conclude that φ coincides almost everywhere with the identity. \square

6. PROOF OF THEOREMS 6, 7 AND 8

Proof of Theorem 6. Assume that $i_\rho = t_b(\rho) = 1$ so that $\rho(\Gamma) \subset \text{PU}(q, 1)$ is nonelementary and let $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \rightarrow \partial\mathcal{H}_{\mathbb{C}}^q$ be a Γ -equivariant measurable map. Corollary 4.13 implies that φ satisfies condition (i) of Theorem 5.1. Under the hypothesis that $p \geq 2$, we show now that it satisfies also condition (ii). Assume that (ii) fails. As before, let $\mathcal{C}_p(\xi)$ denote the set of chains through a point $\xi \in \partial\mathcal{H}_{\mathbb{C}}^p$ and $\Phi : \mathcal{C}_p \rightarrow \mathcal{C}_q$ the measurable map defined in (5.1). Then for a set of positive measure of ξ 's, the set

$$\{(C_1, C_2) \in \mathcal{C}_p(\xi) \times \mathcal{C}_p(\xi) : \Phi(C_1) = \Phi(C_2)\}$$

is of positive measure. In particular, for some $\xi \in \partial\mathcal{H}_{\mathbb{C}}^p$ and $C_1 \in \mathcal{C}_p(\xi)$, the set of $C_2 \in \mathcal{C}_p(\xi)$ with $\Phi(C_1) = \Phi(C_2)$ is of positive measure. Now if $E \subset \mathcal{C}_p(\xi)$ is a set of positive measure, then the convex hull of $\cup_{C \in E} C$ is of full measure in $\partial\mathcal{H}_{\mathbb{C}}^p$, which implies that the essential image $\text{EssIm}(\varphi)$ of φ is contained in the chain $\Phi(C_1)$ and hence that $\rho(\Gamma)$ stabilizes a complex geodesic. We may thus assume that $q = 1$.

CLAIM. Let $\mathcal{L}_\rho \subset \partial\mathcal{H}_{\mathbb{C}}^1 = S^1$ be the limit set of $\rho(\Gamma)$. Then $\mathcal{L}_\rho = S^1$.

Proof. Observe first that since $\rho(\Gamma) \subset \text{PU}(1, 1)$ is nonelementary, it is Zariski dense, hence either discrete or dense in $\text{PU}(1, 1)$. If now $\mathcal{L}_\rho \neq S^1$, then $\rho(\Gamma)$ is discrete and finitely generated, and hence contains a normal torsionfree subgroup Λ of finite index. Since $\mathcal{L}_\rho = \mathcal{L}_\Lambda \neq S^1$, the quotient $\Lambda \backslash \mathcal{H}_{\mathbb{C}}^1$ is a noncompact surface, hence Λ is a free group, which implies $H^2(\rho(\Gamma), \mathbb{R}) = H^2(\Lambda, \mathbb{R}) = 0$, and hence $\rho^{(2)}(\kappa_1) = 0$ in $H^2(\Gamma, \mathbb{R})$. Since however $i_\rho = 1$, clearly $\rho^{(2)}(\kappa_1) \neq 0$, which is a contradiction. This shows that $\mathcal{L}_\rho = S^1$. \square

CLAIM. With the above hypotheses $\ker \rho$ is finite and Γ is cocompact.

Proof. Let us start by observing that if $I \subset S^1$ is any interval with nonvoid interior, then $\varphi^{-1}(I)$ contains, up to a null set, an open subset of $\partial\mathcal{H}_{\mathbb{C}}^p$.

Indeed, since $\text{EssIm}(\varphi) = S^1$, then $\varphi^{-1}(I)$ is of positive measure; moreover, for any $a \neq b$ in $\varphi^{-1}(I)$ such that $[\varphi(a), \varphi(b)] \subset I$, the interval

$$\{z \in \partial\mathcal{H}_{\mathbb{C}}^p : c_p(a, z, b) = 1\}$$

belongs to $\varphi^{-1}(I)$ (up to measure zero), which implies easily the assertion.

To prove the claim, assume that $N := \ker \rho < \Gamma$ is infinite. Being discrete, its limit set in $\partial\mathcal{H}_{\mathbb{C}}^p$ is nonvoid, hence equals $\partial\mathcal{H}_{\mathbb{C}}^p$, which implies that N acts minimally on $\partial\mathcal{H}_{\mathbb{C}}^p$. Pick any interval $I \subset S^1$ with nonvoid interior and let $\mathcal{O} \subset \partial\mathcal{H}_{\mathbb{C}}^p$ be an open set such that \mathcal{O} is included in $\varphi^{-1}(I)$ up to a set of measure zero. Then φ being N -invariant, and N acting minimally on $\partial\mathcal{H}_{\mathbb{C}}^p$, we have that $\partial\mathcal{H}_{\mathbb{C}}^p = \cup_{n \in N} n\mathcal{O}$ is contained in $\varphi^{-1}(I)$, up to measure zero. But since I was arbitrary, this is a contradiction. This shows that $\ker \rho$ is finite. If Γ were not cocompact, then – since $p \geq 2$ – it would contain an integer Heisenberg group which would be sent, almost injectively, into $\text{PU}(1, 1)$. Since this is impossible, it follows that Γ is cocompact. \square

Thus Γ and $\rho(\Gamma)$ are commensurable, and hence their virtual cohomological dimensions coincide; thus $\rho(\Gamma)$ has virtual cohomological dimension 4 and hence cannot be discrete in $\text{PU}(1, 1)$. Being Zariski dense, $\rho(\Gamma)$ is therefore dense in $\text{PU}(1, 1)$. Passing to a subgroup of finite index of Γ , we may in addition assume that Γ is torsionfree and ρ is injective. Since the set of elliptic elements is open in $\text{PU}(1, 1)$, we may pick $\gamma \neq Id$, with $\rho(\gamma)$ elliptic. Since Γ is cocompact and torsionfree, then γ is necessarily hyperbolic. Now pick a pair of open intervals $\emptyset \neq I \subset I'$ such that the complement of I' is of nonvoid interior. Let $\mathcal{O} \subset \partial\mathcal{H}_{\mathbb{C}}^p$ be nonvoid open subset such that $\mathcal{O} \subset \varphi^{-1}(I)$ up to a set of measure zero. Conjugating by an element of Γ , we may assume that the repulsive fixed point of γ is in \mathcal{O} . Let now $\{n_k\}_{k \in \mathbb{N}}$ be a divergent sequence of integers such that $\lim_{k \rightarrow \infty} \rho(\gamma)^{n_k} = Id$ in $\text{PU}(1, 1)$; we may assume that $\rho(\gamma)^{n_k} I \subset I'$ for all $k \geq 1$. Then $\cup_{k \geq 1} \gamma^{n_k} \mathcal{O} = \partial\mathcal{H}_{\mathbb{C}}^p \setminus \{\xi\}$, where ξ is the attractive fixed point, and hence $\varphi^{-1}(I')$ equals $\partial\mathcal{H}_{\mathbb{C}}^p$ up to a set of measure zero. Since I' was arbitrary, this is a contradiction.

Thus we finally conclude that φ satisfies also Theorem 5.1(ii), and hence there is a unique embedding $F : \mathcal{H}_{\mathbb{C}}^p \rightarrow \mathcal{H}_{\mathbb{C}}^q$ which is isometric and holomorphic such that $\partial F = \varphi$ almost everywhere. The uniqueness then implies that F is Γ -equivariant. \square

Proof of Theorem 7. By Corollary 4.13, we have that

$$(6.1) \quad c_q(\varphi(\xi), \varphi(\eta), \varphi(\zeta)) = c_1(\xi, \eta, \zeta)$$

for almost every $(\xi, \eta, \zeta) \in (\partial\mathcal{H}_{\mathbb{C}}^1)^3$. Fix $\xi \neq \eta$ such that (6.1) holds for almost every $\zeta \in \partial\mathcal{H}_{\mathbb{C}}^1$. Then the essential image of φ is contained in the chain C determined by $\varphi(\xi)$ and $\varphi(\eta)$, from which readily follows that $\rho(\Gamma)$ leaves invariant the complex geodesic whose boundary is C . \square

Proof of Theorem 8. Let now $\rho : \Gamma \rightarrow \mathrm{PU}(1, 1)$ be a homomorphism with $|i_\rho| = 1$ and, identifying $\partial\mathcal{H}_\mathbb{C}^1 = \mathbb{S}^1$, let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the Γ -equivariant measurable map. Then (6.1) holds and φ is weakly order preserving, so that [33, Proposition 5.5] implies that there exists a degree one monotone surjective continuous map $F : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $f(\rho(\gamma)x) = \gamma f(x)$ for all $\gamma \in \Gamma$ and all $x \in \mathbb{S}^1$. The surjectivity of f then implies that ρ is injective, while its continuity that $\rho(\Gamma)$ is discrete.

Let now $\mathcal{L} \subset \mathbb{S}^1$ be the limit set of $\rho(\Gamma)$. To complete the proof we shall see that if the image under ρ of a parabolic element is also parabolic, then $\mathcal{L} = \mathbb{S}^1$ and hence $\rho(\Gamma)$ is a lattice. The proof will then be concluded by observing that if $\rho(\Gamma)$ is a lattice, then it acts minimally on \mathbb{S}^1 and hence f is also injective.

Let us suppose by contradiction that $\mathcal{L} \subsetneq \mathbb{S}^1$. Then, since $\rho(\Gamma) \backslash \mathcal{H}_\mathbb{C}^1$ is a complete hyperbolic surface topologically of finite type, for any connected component I of $\mathbb{S}^1 \setminus \mathcal{L}$, one has that

$$\mathrm{Stab}_{\rho(\Gamma)}(I) = \langle \rho(\gamma) \rangle,$$

where $\rho(\gamma) \in \Gamma$ is the hyperbolic element whose fixed points are the endpoints a, b of the interval \bar{I} . Since f is a semiconjugacy in the sense of Ghys [27], the set $f(\bar{I})$ is reduced to a point, say $f(\bar{I}) = \{\xi\}$. Clearly $\gamma\xi = \xi$ and we shall show that this is the only fixed point of γ hence γ is parabolic contradicting the hypothesis. Let us suppose that there exists $\eta \in \mathbb{S}^1$ with $\gamma\eta = \eta$ and, since f is surjective, let $x \in \mathbb{S}^1$ such that $f(x) = \eta$. Without loss of generality let us assume that $x \neq b$ (otherwise $\xi = \eta$ and we are done) and that $\rho(\gamma)^n x \rightarrow a$. Then

$$\eta = \gamma^n \eta = f(\rho(\gamma)^n x) \rightarrow f(a) = \xi,$$

and hence $\xi = \eta$ showing that γ is parabolic. \square

APPENDIX A. PROOF OF PROPOSITION 3.1

For the proof of Proposition 3.1(1) we need to show the existence of norm one contracting homotopy operators from $L^\infty((G/Q)_f^{n+1})$ to $L^\infty((G/Q)_f^n)$ sending L -continuous vectors into L -continuous vectors.

To this purpose we use the map q_n which identifies the complex of Banach G -modules $(L^\infty(G/Q)_f^\bullet)$ with the subcomplex $(L^\infty(G \times (H/Q)^\bullet)^H)$ of H -invariant vectors of the complex $(L^\infty(G \times (H/Q)^\bullet))$, where now the differential d_n is given by

$$d_n f(g, x_1, \dots, x_n) = \sum_{i=0}^n (-1)^i f(g, x_1, \dots, \hat{x}_i, \dots, x_n),$$

and we show more generally that:

LEMMA A.1. *For every $n \geq 0$ there are linear maps*

$$h_n : L^\infty(G \times (H/Q)^{n+1}) \rightarrow L^\infty(G \times (H/Q)^n)$$

such that:

- (1) h_n is norm-decreasing and H -equivariant;
- (2) for any closed subgroup $L < G$, the map h_n sends L -continuous vectors into L -continuous vectors, and
- (3) for every $n \geq 1$ we have the identity

$$h_n d_n + d_{n-1} h_{n-1} = Id.$$

The Lemma A.1 and the remarks preceding it imply then Proposition 3.1.

The construction of the homotopy operator in Lemma A.1 requires the following two lemmas, the first of which showing that the measure ν on H/Q can be chosen to satisfy certain regularity properties, and the second constructing an appropriate Bruhat function for $H < G$.

Let dh and $d\xi$ be the left invariant Haar measures on H and Q .

LEMMA A.2. *There is an everywhere positive continuous function $q : H \rightarrow \mathbb{R}^+$ and Borel probability measure ν on H/Q such that*

$$\int_{H/Q} d\nu(x) \int_Q f(x\xi) d\xi = \int_H f(h) q(h) dh,$$

for every $f \in C_{00}(H)$.

Proof. Let $q_1 : H \rightarrow \mathbb{R}^+$ be an everywhere positive continuous function satisfying

$$q_1(x\eta) = q_1(x) \frac{\Delta_Q(\eta)}{\Delta_H(\eta)}, \forall \eta \in Q, x \in H,$$

where Δ_Q, Δ_H are the respective modular functions (see [41]), and let ν_1 be the corresponding positive Radon measure on H/Q such that the above formula holds. Then choose $q_2 : H/Q \rightarrow \mathbb{R}^+$ continuous and everywhere positive, such that $q_2 d\nu_1$ is a probability measure. Then the lemma holds with $q = q_1 q_2$ and $\nu = q_2 \otimes \nu_1$. \square

A direct computation shows that

$$(A.1) \quad \int_{H/Q} f(y^{-1}x) d\nu(x) = \int_{H/Q} f(x) \lambda_y(x) d\nu(x),$$

where $\lambda_y(x) = q(yx)/q(x)$, for all $f \in C_{00}(H/Q)$ and $h \in H$. In particular, the class of ν is H -invariant since λ_y is continuous and everywhere positive on H/Q .

LEMMA A.3. *There exists a function $\beta : G \rightarrow \mathbb{R}^+$ such that*

- (1) *for every compact set $K \subset G$, β coincides on KH with a continuous function with compact support;*
- (2) *$\int_H \beta(gh)dh = 1$ for all $g \in G$, and*
- (3) *$\lim_{g_0 \rightarrow e} \sup_{g \in G} \int_H |\beta(g_0gh) - \beta(gh)|dh = 0$*

Proof. Let β_0 be any function satisfying (1) and (2) (see [41]) and let $f \in C_{00}(G)$ be any nonnegative function normalized so that

$$\int_G f(x)d_r x = 1,$$

where $d_r x$ is a right invariant Haar measure on G . Define

$$\beta(g) = \int_G f(gx^{-1})\beta_0(x)d_r x, \quad g \in G.$$

It is easy to verify that also β satisfies (1) and (2), and, moreover, it satisfies (3) as well. In fact, we have that for all $g_0, g \in G, h \in H$

$$\beta(g_0gh) - \beta(gh) = \int_G (f(g_0gx^{-1}) - f(gx^{-1}))\beta_0(xh)d_r x,$$

which implies, taking into account that $\int_G \beta_0(xh)dh = 1$ and the invariance of $d_r x$, that

$$\int_H |\beta(g_0gh) - \beta(gh)|dh \leq \int_G |f(g_0x^{-1}) - f(x^{-1})|d_r x,$$

so that

$$\lim_{g_0 \rightarrow e} \sup_{g \in G} \int_H |\beta(g_0gh) - \beta(gh)|dh \leq \lim_{g_0 \rightarrow e} \int_G |f(g_0x^{-1}) - f(x^{-1})|d_r x = 0.$$

□

Proof of Lemma A.1. Let ν be as in Lemma A.2 and β as in Lemma A.3. define a function

$$\psi : G \times H/Q \rightarrow \mathbb{R}^+$$

by

$$\psi(g, x) := \int_H \beta(gh)\lambda_{h^{-1}}(x)dh,$$

where $\lambda_h(x)$ is as in (A.1). The following properties are then direct verifications:

- (1) $\psi(gh^{-1}, hx)\lambda_h(x) = \psi(g, x)$ for all $g \in G, h \in H$ and $x \in H/Q$;
- (2) $\int_{H/Q} \psi(g, x)d\nu(x) = 1$, for all $g \in G$;
- (3) $\psi \geq 0$ and is continuous.

This being, define for $n \geq 0$ and $f \in L^\infty(G \times (H/Q)^{n+1})$:

$$h_n f(g, x_1, \dots, x_n) = \int_{H/Q} \psi(g, x) f(g, x_1, \dots, x_n, x) d\nu(x).$$

Then, $h_n f \in L^\infty(G \times (H/Q)^n)$ and (2) implies that $\|h_n f\|_\infty \leq \|f\|_\infty$. The fact that h_n is an H -equivariant homotopy operator is a formal consequence of (1) and (2).

Finally, let $L < G$ be a closed subgroup and $f \in L^\infty(G \times (H/Q)^{n+1})$ an L -continuous vector, that is

$$\lim_{l \rightarrow e} \|\theta(l)f - f\|_\infty = 0,$$

where

$$(\theta(l)f)(g, x_1, \dots, x_{n+1}) = f(lg, x_1, \dots, x_n).$$

Then

$$\begin{aligned} & h_n f(lg, x_1, \dots, x_n) - h_n f(g, x_1, \dots, x_n) \\ &= \int_{H/Q} \psi(lg, x) (f(lg, x_1, \dots, x_n, x) - f(g, x_1, \dots, x_n, x)) d\nu(x) \\ &+ \int_{H/Q} (\psi(lg, x) - \psi(g, x)) f(g, x_1, \dots, x_n, x) d\nu(x). \end{aligned}$$

The first term is bounded by $\|\theta(l)f - f\|_\infty$ taking into account (2), while the second is bounded by $\|f\|_\infty \int_{H/Q} (\psi(lg, x) - \psi(g, x)) d\nu(x)$.

Now

$$\psi(lg, x) - \psi(g, x) = \int_H (\beta(lgh) - \beta(gh)) \lambda_{h^{-1}}(x) dh,$$

which, taking into account that $\int_{H/Q} \lambda_{h^{-1}}(x) d\nu(x) = 1$, implies that

$$\int_{H/Q} |\psi(lg, x) - \psi(g, x)| d\nu(x) \leq \int_{H/Q} |\beta(lgh) - \beta(gh)| dh.$$

Thus

$$\begin{aligned} \|\theta(l)h_n f - h_n f\|_\infty &\leq \|\theta(l)f - f\|_\infty \\ &+ \|f\|_\infty \sup_{g \in G} \int_H |\beta(lgh) - \beta(gh)| dh \end{aligned}$$

which, using Lemma A.3, implies that

$$\lim_{l \rightarrow e} \|\theta(l)h_n f - h_n f\|_\infty = 0$$

and shows that $h_n f$ is an L -continuous vector. \square

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