BOUNDED DIFFERENTIAL FORMS, GENERALIZED MILNOR–WOOD INEQUALITY AND AN APPLICATION TO DEFORMATION RIGIDITY

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Abstract. We establish sufficient conditions for a cohomology class of a discrete subgroup Γ of a connected semisimple Lie group with finite center to be representable by a bounded differential form on the quotient by Γ of the associated symmetric space; furthermore if $\rho : \Gamma \to \mathrm{PU}(1,q)$ is any representation of any discrete subgroup Γ of $\mathrm{SU}(1,p)$, we give an explicit closed bounded differential form on the quotient by Γ of complex hyperbolic space which is a representative for the pullback via ρ of the Kähler class of $\mathrm{PU}(1,q)$.

If G, G' are Lie groups of Hermitian type, we generalize to representations $\rho : \Gamma \to G'$ of lattices $\Gamma < G$ the invariant defined in [14] for which we establish a Milnor–Wood type inequality. As an application we study maximal representations into $\mathrm{PU}(1,q)$ of lattices in $\mathrm{SU}(1,1)$.

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Date: September 15, 2006.
A.I. was partially supported by FNS grant PP002-102765.

1. Introduction

The continuous cohomology $\mathrm{H}^\bullet_\mathrm{c}(G,\mathbb{R})$ of a topological group G is the cohomology of the complex $\big(\mathrm{C}(G^\bullet)^G, d^\bullet\big)$ of G-invariant continuous functions, while its bounded continuous cohomology $\mathrm{H}^\bullet_\mathrm{cb}(G,\mathbb{R})$ is the cohomology of the subcomplex $(\mathrm{C}_\mathrm{b}(G^\bullet)^G, d^\bullet)$ of G-invariant bounded continuous functions. The inclusion of the complex of bounded continuous functions into the one consisting of continuous functions gives rise to the comparison map

$$\mathsf{c}^\bullet_G : \mathrm{H}^\bullet_\mathrm{cb}(G,\mathbb{R}) \to \mathrm{H}^\bullet_\mathrm{c}(G,\mathbb{R})$$

which encodes subtle properties of G of algebraic and geometric nature, see [29], [1], [19], [44], [45], [20, § V.13], [13], (see also [7], [8], [35], [49], [27], [2], [26], [3], [28], [39], [40] in relation with the existence of quasimorphisms). We say that a continuous class on G is representable by a bounded continuous class if it is in the image of $\mathsf{c}^\bullet_G$.

When G is a connected semisimple Lie group with finite center and associated symmetric space $\mathcal{X}$ and $L < G$ is any closed subgroup, a useful tool in the study of the continuous cohomology of L is the van Est isomorphism, according to which $\mathrm{H}^\bullet_\mathrm{c}(L,\mathbb{R})$ is canonically isomorphic to the cohomology $\mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^L\big)$ of the complex $\big(\Omega^\bullet(\mathcal{X})^L, d^\bullet\big)$ of L-invariant smooth differential forms $\Omega^\bullet(\mathcal{X})$ on $\mathcal{X}$. For example, if $\Gamma < G$ is a torsionfree discrete subgroup, $\mathrm{H}^\bullet(\Gamma,\mathbb{R})$ is the de Rham cohomology $\mathrm{H}^\bullet_\mathrm{dR}(\Gamma\backslash\mathcal{X})$ of the manifold $\Gamma\backslash\mathcal{X}$. (Here and in the sequel we drop the subscript $_\mathrm{c}$ if the group is discrete.) For simplicity, in the introduction we restrict ourselves to this case, and we refer the reader to the body of the paper for the general statement in the case in which Γ is an arbitrary closed subgroup.

We do not know of an analogue of van Est theorem in the context of continuous bounded cohomology. This paper however explores a particular aspect of the comparison map and of the pullback, namely the relation between bounded continuous cohomology and the complex of, loosely speaking, invariant smooth differential forms with some boundedness condition. For instance, our first result gives us information on the differential forms that one can use to represent a class in the image of the comparison map.

Theorem 1. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group G with finite center and associated symmetric space $\mathcal{X}$. Any class in the image of the comparison map*

$$\mathsf{c}^\bullet_\Gamma : \mathrm{H}^\bullet_\mathrm{b}(\Gamma,\mathbb{R}) \to \mathrm{H}^\bullet(\Gamma,\mathbb{R}) \cong \mathrm{H}^\bullet_\mathrm{dR}(\Gamma\backslash\mathcal{X})$$

is representable by a closed form on $\Gamma\backslash\mathcal{X}$ which is bounded.

Here a form is bounded on $\Gamma\backslash\mathcal{X}$ if its supremum norm, computed using the Riemannian metric, is finite. In fact, this is only a particular case of the following more general result which describes some of the interplay between the comparison map and the pullback of a cohomology class via a homomorphism of a discrete group into a topological group (which in the case of Theorem 1 is the identity homomorphism).

THEOREM 2. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group G with finite center and associated symmetric space $\mathcal{X}$, and $\rho : \Gamma \to G'$ a homomorphism into a topological group G'. If $\alpha \in \mathrm{H}^n_{\mathrm{c}}(G', \mathbb{R})$ is representable by a continuous bounded class, then its pullback $\rho^{(n)}(\alpha) \in \mathrm{H}^n(\Gamma, \mathbb{R}) \cong \mathrm{H}^n_{\mathrm{dR}}(\Gamma\backslash\mathcal{X})$ is representable by a closed differential n-form on $\Gamma\backslash\mathcal{X}$ which is bounded.*

We shall see later that in the case in which G, G' are the connected components of the isometry groups of complex hyperbolic spaces and α is the Kähler class, the bounded closed 2-form in Theorem 2 can be given explicitly (see Theorem 5).

Even if G' is a connected Lie group, little is known about the surjectivity properties of the comparison map $\mathsf{c}^\bullet_{G'}$. However, as a direct consequence of a theorem of Gromov [36] which asserts that characteristic classes are bounded (see [9] for a resolution of singularities free proof), we have the following:

COROLLARY 3. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group with finite center G and associated symmetric space $\mathcal{X}$, and let $\rho : \Gamma \to G'$ be a homomorphism into a real algebraic group G'. If $\alpha \in \mathrm{H}^n_{\mathrm{c}}(G', \mathbb{R})$ comes from a characteristic class of a flat principal G'-bundle, then its pullback $\rho^{(n)}(\alpha) \in \mathrm{H}^n(\Gamma, \mathbb{R}) \cong \mathrm{H}^n_{\mathrm{dR}}(\Gamma\backslash\mathcal{X})$ is representable by a closed differential n-form on $\Gamma\backslash\mathcal{X}$ which is bounded.*

Notice that Theorem 1, and hence Theorem 2 and Corollary 3 are valid, with an appropriate formulation, for any closed subgroup $\Gamma < G$ (compare with Corollary 4.1 and Proposition 3.1).

Moreover, as a consequence of the surjectivity of the comparison map for Gromov hyperbolic groups ([36], [44], [45]) we have immediately:

COROLLARY 4. *Let $\Gamma < G$ be a torsionfree discrete subgroup of a connected semisimple Lie group with finite center G and associated symmetric space $\mathcal{X}$. Assume that Γ is finitely generated and word hyperbolic. Then for every $n \geq 2$, any class in $\mathrm{H}^n_{\mathrm{dR}}(\Gamma\backslash\mathcal{X})$ is representable by a closed differential n-form on $\Gamma\backslash\mathcal{X}$ which is bounded.*

If L is a connected semisimple group with finite center, one has full information about the comparison map in degree two,

$$\mathsf{c}_L^{(2)} : \mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(L,\mathbb{R}) \tag{1.1}$$

which is an isomorphism[1], [20]. This is the case we exploit, also because in this degree continuous cohomology is connected to a particularly fundamental geometric structure. Recall in fact that if $\mathcal{Y}$ is the symmetric space associated to L, the dimension of $\mathrm{H}^2_{\mathrm{c}}(L,\mathbb{R})$ is the number of irreducible factors of $\mathcal{Y}$ which are Hermitian symmetric and $\Omega^2(\mathcal{Y})^L$ is generated by the Kähler forms of the irreducible Hermitian factors of $\mathcal{Y}$. We say that a connected semisimple Lie group L with finite center is of *Hermitian type*[2] if $\mathcal{Y}$ is Hermitian symmetric; we denote by $\omega_{\mathcal{Y}}$ the Kähler form on $\mathcal{Y}$, by $\kappa_{\mathcal{Y}} \in \mathrm{H}^2_{\mathrm{c}}(L,\mathbb{R})$ the corresponding continuous class under the isomorphism $\mathrm{H}^2\big(\Omega^\bullet(\mathcal{Y})^L\big) \cong \mathrm{H}^2_{\mathrm{c}}(L,\mathbb{R})$ and by $\kappa^{\mathrm{b}}_{\mathcal{Y}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ its image under the isomorphism (1.1).

In the particular case of the complex hyperbolic spaces $\mathcal{H}^\ell_{\mathbb{C}}$, we can give explicitly the expression of the representative in Theorem 2. In fact in this case the multiple $\frac{1}{\pi}\kappa^{\mathrm{b}}_\ell$ of the bounded Kähler class κ^{b}_ℓ (which is here and in the following a shortcut for $\kappa^{\mathrm{b}}_{\mathcal{H}^\ell_{\mathbb{C}}}$) admits an explicit representative on $\partial\mathcal{H}^\ell_{\mathbb{C}}$ given by the Cartan cocycle

$$c_\ell : (\partial\mathcal{H}^\ell_{\mathbb{C}})^3 \to [-1,1],$$

(see § 6 for the definition and properties). Moreover, if $x \in \mathcal{H}^p_{\mathbb{C}}$ and ξ is a point in the boundary $\partial\mathcal{H}^p_{\mathbb{C}}$ of complex hyperbolic space, let $e^\xi(x) := e^{h\beta_\xi(0,x)}$, where h is the volume entropy of $\mathcal{H}^p_{\mathbb{C}}$, $\beta_\xi(0,x)$ is the Busemann function relative to a basepoint $0 \in \mathcal{H}^p_{\mathbb{C}}$, and μ_0 the $K = \mathrm{Stab}_{\mathrm{SU}(1,p)}(0)$-invariant probability measure on $\partial\mathcal{H}^p_{\mathbb{C}}$. Then we have:

THEOREM 5. *Let $\Gamma < \mathrm{SU}(1,p)$ be any torsionfree discrete subgroup, and let $\rho : \Gamma \to \mathrm{PU}(1,q)$ be a homomorphism with nonelementary image and associated Γ-equivariant measurable map $\varphi : \partial\mathcal{H}^p_{\mathbb{C}} \to \partial\mathcal{H}^q_{\mathbb{C}}$. The 2-form*

$$\int_{(\partial\mathcal{H}^p_{\mathbb{C}})^3} e^{\xi_0} \wedge de^{\xi_1} \wedge de^{\xi_2} c_q\big(\varphi(\xi_0),\varphi(\xi_1),\varphi(\xi_2)\big) d\mu_0(\xi_0) d\mu_0(\xi_1) d\mu_0(\xi_2)$$

is Γ-invariant, closed, bounded and represents $\frac{1}{\pi}\rho^{(2)}(\kappa_q) \in \mathrm{H}^2_{\mathrm{dR}}(\Gamma\backslash\mathcal{H}^p_{\mathbb{C}})$.

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[1] A similar statement holds, again in degree two, for any connected Lie group.

[2] In [16] a group of Hermitian type si required not to have compact factors, but this assumption is not necessary here.

As an application of the above results, we prove here a generalization of the Milnor–Wood inequality. Namely, to any representation of a torsionfree lattice $\Gamma < G$ into G', where G, G' are of Hermitian type, we associate a numerical invariant which we then prove to be bounded with a bound depending only on the rank of the symmetric spaces.

To define the aforementioned invariant, let G be of Hermitian type with associated symmetric space $\mathcal{X}$ and $\Gamma < G$ a torsionfree lattice; for $1 \leq p \leq \infty$ let $\mathrm{H}^\bullet_p(\Gamma\backslash\mathcal{X})$ denote the L^p-cohomology of $\Gamma\backslash\mathcal{X}$, which is the cohomology of the complex of smooth differential forms α on $\Gamma\backslash\mathcal{X}$ such that α and $d\alpha$ are in L^p. Inclusion in the complex of smooth differential forms gives thus a comparison map

$$i^\bullet_p : \mathrm{H}^\bullet_p(\Gamma\backslash\mathcal{X}) \to \mathrm{H}^\bullet_{\mathrm{dR}}(\Gamma\backslash\mathcal{X})\,.$$

Then we have:

Corollary 6. *Assume that G, G' are of Hermitian type, let $\Gamma < G$ be a torsionfree lattice, $\mathcal{X}$ the Hermitian symmetric space associated to G and $\rho : \Gamma \to G'$ a homomorphism. Then for every $1 \leq p \leq \infty$ there is a linear map*

$$\rho^{(2)}_p : \mathrm{H}^2_{\mathrm{c}}(G', \mathbb{R}) \to \mathrm{H}^2_p(\Gamma\backslash\mathcal{X})$$

such that the diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{c}}(G', \mathbb{R}) \arrow[rr, "\rho^{(2)}"] \arrow[dr, "\rho^{(2)}_p"'] & & \mathrm{H}^2(\Gamma, \mathbb{R}) \arrow[r, "\cong"] & \mathrm{H}^2_{\mathrm{dR}}(\Gamma\backslash\mathcal{X}) \\
& \mathrm{H}^2_p(\Gamma\backslash\mathcal{X}) \arrow[urr, "i^{(2)}_p"'] & &
\end{tikzcd}$$

commutes.

In the above situation – that is if $\Gamma < G$ is a lattice and $\mathcal{X}$ is Hermitian symmetric – the L^2-cohomology $\mathrm{H}^\bullet_2(\Gamma\backslash\mathcal{X})$ is reduced (*i. e.* Hausdorff) and finite dimensional in all degrees; it may hence be identified with the space of L^2-harmonic forms on $\Gamma\backslash\mathcal{X}$ and carries a natural scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$. The Kähler form $\omega_{\Gamma\backslash\mathcal{X}}$ is thus a distinguished element of $\mathrm{H}^2_2(\Gamma\backslash\mathcal{X})$. Given now a homomorphism $\rho : \Gamma \to G'$ and using Corollary 6, the invariant

$$i_\rho := \frac{\langle \rho^{(2)}_2(\kappa_{\mathcal{X}'}), \omega_{\Gamma\backslash\mathcal{X}} \rangle}{\langle \omega_{\Gamma\backslash\mathcal{X}}, \omega_{\Gamma\backslash\mathcal{X}} \rangle} \tag{1.2}$$

is well defined and finite. We have then finally the Milnor–Wood type inequality:

THEOREM 7. *Let G, G' be of Hermitian type with associated symmetric spaces $\mathcal{X}$ and $\mathcal{X}'$, let $\rho : \Gamma \to G'$ be a representation of a lattice in G with invariant i_ρ as in (1.2). Assume that $\mathcal{X}$ is irreducible and that the Hermitian metrics on $\mathcal{X}$ and $\mathcal{X}'$ are normalized so as to have minimal holomorphic sectional curvature -1. Then*

$$|i_\rho| \leq \frac{\operatorname{rk}\mathcal{X}'}{\operatorname{rk}\mathcal{X}}\,. \tag{1.3}$$

Special cases of the above theorem for invariants related to ours had been previously obtained, with restrictions on the target group and cocompactness conditions, by Milnor [43], Wood [52], Turaev [51], Toledo [50], Bradlow, Garcia-Prada and Gothen [6] and by Koziarz and Maubon [41]. In particular, if Γ is a torsionfree lattice in $\mathrm{PU}(1,1)$ so that $\Gamma\backslash\mathcal{X}$ is diffeomorphic to the interior of a compact oriented surface Σ, then i_ρ is, up to the multiple $\chi(\Sigma)$, equal to the Toledo invariant defined in [16, § 1]: notice however that this equality implies that i_ρ is independent of the hyperbolization on the interior of Σ, [16].

The study of *maximal* representations, that is representations such that the invariant i_ρ takes its maximum value $\operatorname{rk}\mathcal{X}'/\operatorname{rk}\mathcal{X}$, has been the subject of much research over the years ([30], [31], [29], [34], [42], [23], [32], [38], [6], [18], [41], [10], [15], [16], [17]). If $\Gamma < G$ is cocompact, then i_ρ is a characteristic number. If G is of rank one, that is if it is locally isomorphic to $\mathrm{SU}(1,p)$, then if $p \geq 2$, $\mathrm{H}^2_2(\Gamma\backslash\mathcal{H}^p_\mathbb{C})$ injects into $\mathrm{H}^2_{\mathrm{dR}}(\Gamma\backslash\mathcal{H}^p_\mathbb{C})$ ([54] and [48]), and hence once again i_ρ is a characteristic number. When G is locally isomorphic to $\mathrm{SU}(1,1)$ and $\Gamma < G$ is not cocompact, then $\mathrm{H}^2_2(\Gamma\backslash\mathcal{H}^1_\mathbb{C})$ is one-dimensional while $\mathrm{H}^2_{\mathrm{dR}}(\Gamma\backslash\mathcal{H}^1_\mathbb{C}) = 0$; this case has a different flavor as i_ρ is not a characteristic number, a fact which is reflected by the existence of nontrivial deformations of Γ in $\mathrm{PU}(1,2)$, [37].

For the rest of the paper we focus our attention to the case in which G is locally isomorphic to $\mathrm{SU}(1,1)$ and $\Gamma < G$ is any lattice.

THEOREM 8 ([11]). *Let $\Gamma < G$ be a lattice in a connected group locally isomorphic to* $\mathrm{SU}(1,1)$ *and let $\rho : \Gamma \to \mathrm{PU}(1,q)$ be a representation such that $|i_\rho| = 1$. Then $\rho(\Gamma)$ leaves a complex geodesic invariant.*

This was proven by Toledo [50] if Γ is a compact surface group. In the noncompact case a variant of Theorem 8 was obtained by Koziarz and Maubon in [41], with another definition of maximality which probably coincides with ours.

Thus Theorem 8 reduces the study of maximal representations into $\mathrm{PU}(1,q)$ to the case $q = 1$, for which we have the following:

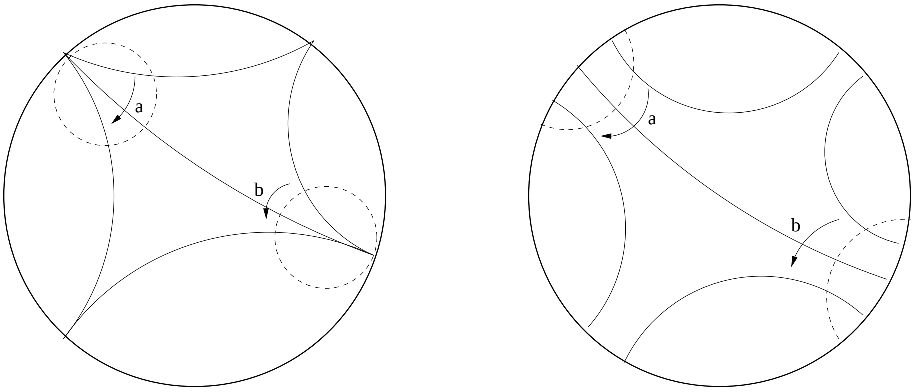

FIGURE 1. $\Gamma = \langle a, b \rangle$ and $\Lambda = \langle a', b' \rangle$

THEOREM 9 ([11]). *Let $\Gamma < G$ be a lattice, where $G = \mathrm{SU}(1,1)$ or $G = \mathrm{PU}(1,1)$ and let $\rho : \Gamma \to \mathrm{PU}(1,1)$ be a maximal representation. Then $\rho(\Gamma)$ is discrete and, modulo the center of Γ, ρ is injective. In fact, there is a continuous surjective map $f : \partial\mathcal{H}^1_{\mathbb{C}} \to \partial\mathcal{H}^1_{\mathbb{C}}$ such that:*

(1) *f is weakly order preserving;*
(2) *$f\big(\rho(\gamma)\xi\big) = \gamma f(\xi)$ for all $\gamma \in \Gamma$ and all $\xi \in \partial\mathcal{H}^1_{\mathbb{C}}$.*

Furthermore, if one of the following two assumptions is verified:

(i) *$\rho(\Gamma)$ is a lattice or*
(ii) *$\rho(\gamma)$ is a parabolic element if γ is a parabolic element,*

then f is a homeomorphism and $\rho(\Gamma)$ is a lattice.

Recall that, in the terminology of [38], a map $f : \partial\mathcal{H}^1_{\mathbb{C}} \to \partial\mathcal{H}^1_{\mathbb{C}}$ is *weakly order preserving* if whenever $\xi, \eta, \zeta \in \partial\mathcal{H}^1_{\mathbb{C}}$ are distinct points such that $f(\xi), f(\eta), f(\zeta) \in \partial\mathcal{H}^1_{\mathbb{C}}$ are also distinct, then the two triples have the same orientation.

EXAMPLE 10. We give an example that shows that the map f is not necessarily a homeomorphism. To this purpose, let us realize the free group on two generators in two different ways:

- Let $\Gamma = \langle a, b \rangle$ be the lattice in $\mathrm{PU}(1,1)$ generated by the parabolic elements a and b with quotient a thrice punctured sphere.
- Let $\Lambda = \langle a', b' \rangle$ be the convex cocompact group generated by the hyperbolic elements a' and b' – see Figure 1.

Let $\rho : \Gamma \to \Lambda$ be the representation defined by $\rho(a) = a'$ and $\rho(b) = b'$. Since Λ acts convex cocompactly on $\mathcal{H}^1_{\mathbb{C}}$, the orbit map $\Lambda \to \Lambda x$, for $x \in \mathcal{H}^1_{\mathbb{C}}$ is a quasi-isometry which extends to a homeomorphism $f_\Lambda : \partial\mathbb{F}_2 \to \mathcal{L}_\Lambda$, where $\mathbb{F}_2$ is the free group on two generators and $\mathcal{L}_\Lambda$ is the limit set of Λ in $\partial\mathcal{H}^1_{\mathbb{C}}$. Likewise, the orbit map $\Gamma \to \Gamma x$ extends to a continuous surjective map $f_\Gamma : \partial\mathbb{F}_2 \to \partial\mathcal{H}^1_\mathbb{C}$ which is one-to-one except for the cusps of Γ, where it is two-to-one. Then $f_\Gamma \circ f_\Lambda^{-1} : \mathcal{L}_\Lambda \to \partial\mathcal{H}^1_\mathbb{C}$ is also continuous, surjective and two-to-one on the cusps of Γ. By sending any interval in the complement of $\mathcal{L}_\Lambda$ in $\partial\mathcal{H}^1_\mathbb{C}$ to the image of its endpoints, we extend $f_\Gamma \circ f_\Lambda^{-1}$ to a map $f : \partial\mathcal{H}^1_\mathbb{C} \to \partial\mathcal{H}^1_\mathbb{C}$ such that

(1) f is weakly order preserving, and
(2) $f\big(\rho(\gamma)\xi\big) = \gamma f(\xi)$,

One can prove, using the results in [29], § 2.1 and § 5 that $i_\rho = 1$ (see Remark 5.4).

Finally we conclude with the following:

Corollary 11. *Any maximal representation* $\rho : \Gamma \to \mathrm{PU}(1,1)$ *of a torsionfree lattice* $\Gamma < \mathrm{PU}(1,1)$ *is induced by a diffeomorphism*

$$\Gamma\backslash\mathcal{H}^1_\mathbb{C} \to \rho(\Gamma)\backslash\mathcal{H}^1_\mathbb{C}\,. \tag{1.4}$$

Organization of the Paper: Theorem 1 is proven as Proposition 3.1, Theorem 2 is proven as Corollary 4.1, Theorem 5 is Proposition 6.2 Corollary 6 is proven as Corollary 4.2, Theorem 7 follows from Lemma 5.1 and Lemma 5.3, and Theorems 8 and 9 and Corollary 11 are proven in § 6.

Acknowledgments: The authors thank Theo Bühler and Anna Wienhard for detailed comments.

2. Preliminaries on Bounded Cohomology, Old and New: the Toledo Map and the Bounded Toledo Map

Let G be a locally compact group. The continuous bounded cohomology of G (with trivial coefficients) is the cohomology of the complex $\big(\mathrm{C_b}(G^\bullet)^G, d^\bullet\big)$ of the space of continuous bounded functions $G^{n+1} \to \mathbb{R}$ which are G-invariant with respect to the diagonal G-action on G^{n+1}. Notice that $\mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R})$ comes naturally equipped with a seminorm induced by the supremum norm on $\mathrm{C_b}(G^\bullet,\mathbb{R})$ and in some cases, as for instance in degree two, the seminorm is actually a norm.

Analogously to the case of the continuous cohomology, there are notions of *relatively injective* G-module and of *strong resolution* which serve for the homological algebra characterization of bounded continuous cohomology. For the precise definitions see [20] or [46], while for our purpose it will suffice to say that if (S,ν) is a regular measure G-space, then the G-module $\mathrm{L}^\infty_{\mathrm{alt}}(S)$ of L^∞ alternating functions on S is relatively injective if and only if the G-action on S is amenable in the sense of Zimmer, [53]. Moreover $\big(\mathrm{L}^\infty_{\mathrm{alt}}(S^\bullet), d^\bullet\big)$ is a strong resolution of $\mathbb{R}$ and hence the cohomology of the subcomplex of G-invariants

$$0 \longrightarrow \mathrm{L}^\infty_{\mathrm{alt}}(S)^G \longrightarrow \mathrm{L}^\infty_{\mathrm{alt}}(S^2)^G \longrightarrow \cdots \longrightarrow \mathrm{L}^\infty_{\mathrm{alt}}(S^n)^G \xrightarrow{d_n} \cdots ,$$

is canonically isomorphic to the bounded continuous cohomology of G.

2.1. **The Transfer Map in Bounded Continuous and Continuous Cohomology.** Let G be a locally compact second countable group and $L < G$ a closed subgroup. The injection $L \hookrightarrow G$ gives by contravariance the restriction map

$$\mathrm{r}^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(L, \mathbb{R})$$

in bounded cohomology. If we assume that $L\backslash G$ has a G-invariant probability measure μ, then the *transfer map*

$$\mathrm{T}^\bullet : \mathrm{C}_\mathrm{b}(G^\bullet)^L \to \mathrm{C}_\mathrm{b}(G^\bullet)^G ,$$

defined by integration

$$\mathrm{T}^{(n)} f(g_1, \dots, g_n) := \int_{L\backslash G} f(gg_1, \dots, gg_n) d\mu(\dot g) , \tag{2.1}$$

for all $(g_1, \dots, g_n) \in G^n$, induces in cohomology a left inverse of $\mathrm{r}^\bullet$ of norm one

$$\mathrm{T}^\bullet_\mathrm{b} : \mathrm{H}^\bullet_{\mathrm{cb}}(L, \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R}) ,$$

(see [46, Proposition 8.6.2, pp.106-107]).

Notice that an analogous construction in continuous cohomology fails in the case in which $L\backslash G$ carries a G-invariant probability measure μ but is not compact. For example, if $L = \Gamma < G$ is a nonuniform lattice, then there is in general no left inverse to the restriction in cohomology $\mathrm{H}^\bullet_\mathrm{c}(G, \mathbb{R}) \to \mathrm{H}^\bullet(\Gamma, \mathbb{R})$ as this map is often not injective. In fact, one can for instance consider the case in which $\mathcal{X} = G/K$ is an n-dimensional symmetric space of noncompact type: then $\mathrm{H}^n_\mathrm{c}(G, \mathbb{R}) = \Omega^n(\mathcal{X})^G$ is generated by the volume form and hence not zero, while if $\Gamma < G$ is any nonuniform torsionfree lattice, the cohomology $\mathrm{H}^n(\Gamma, \mathbb{R})$ vanishes as it is isomorphic to $\mathrm{H}^n_{\mathrm{dR}}(\Gamma\backslash\mathcal{X})$.

However, if $L\backslash G$ carries a finite invariant measure and is compact we can indeed define a transfer map in continuous cohomology. In fact, under these hypotheses, there is an obvious morphism of coefficient modules

$$\begin{aligned} m : \mathrm{L}^p(L\backslash G) \to & \quad \mathbb{R} \\ f \quad \mapsto & \int_{L\backslash G} f d\mu \,, \end{aligned}$$

and moreover, since $L\backslash G$ is compact, then $\mathrm{L}^p_{\mathrm{loc}}(L\backslash G) = \mathrm{L}^p(L\backslash G)$; one can hence compose the general induction map [4] valid for any closed subgroup

$$\imath^\bullet : \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{c}}\big(G, \mathrm{L}^p_{\mathrm{loc}}(L\backslash G)\big)$$

with the change of coefficients m in ordinary continuous cohomology to obtain a transfer map which is a left inverse to the restriction map and leads to a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) & \xrightarrow{\mathrm{c}^\bullet_L} & \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R}) \\ {\scriptstyle \mathrm{T}^\bullet_{\mathrm{b}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{T}^\bullet} \\ \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) & \xrightarrow{\mathrm{c}^\bullet_G} & \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R}) \end{array} \tag{2.2}$$

which is very useful in applications when it comes to identifying invariants in bounded cohomology in terms of ordinary cohomological invariants.

Before passing to the next subsection, we record here for later use that – although not really functorial since defined only on the subcomplex of L-invariants – the transfer map in continuous bounded cohomology can also be implemented on the complex of L^∞ alternating L-invariant functions on an amenable L-space. In fact, [46, Proposition 10.1.3] implies the following:

Lemma 2.1. *Let $L < G$ be a closed subgroup of a locally compact group G, and let (S,ν) be a regular amenable G-space. Let*

$$\mathrm{T}^\bullet_S : \big(\mathrm{L}^\infty_{\mathrm{alt}}(S^\bullet)^L, d^\bullet\big) \to \big(\mathrm{L}^\infty_{\mathrm{alt}}(S^\bullet)^G, d^\bullet\big) \tag{2.3}$$

be defined by

$$\mathrm{T}^{(n)}_S f(x_1,\dots,x_n) := \int_{L\backslash G} f(gx_1,\dots,gx_n)d\mu(g)\,, \tag{2.4}$$

for $(x_1,\dots,x_n)\in S^n$, and let

$$\mathrm{T}^\bullet_{S,\mathrm{b}} : \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R})$$

be the map obtained by the composition

$$
\begin{tikzcd}
\mathrm{H}^\bullet\big(\mathrm{L}^\infty_{\mathrm{alt}}(S^\bullet)^L, d^\bullet\big) \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}^\bullet\big(\mathrm{L}^\infty_{\mathrm{alt}}(S^\bullet)^G, d^\bullet\big) \arrow[d, "\cong"] \\
\mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \arrow[r, "\mathrm{T}^\bullet_{S,\mathrm{b}}"] & \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R})\,,
\end{tikzcd}
\tag{2.5}
$$

where the vertical arrows are the canonical isomorphisms in bounded cohomology extending the identity $\mathbb{R} \to \mathbb{R}$*, and the top horizontal arrow is the map in cohomology induced by* $\mathrm{T}^\bullet_S$ *in (2.3).*

Then $\mathrm{T}^\bullet_{S,\mathrm{b}} = \mathrm{T}^\bullet_{\mathrm{b}}$.

2.2. **The Toledo Map and the Bounded Toledo Map.** Let $L \leq G$ be a closed subgroup of a locally compact second countable group G such that on $L\backslash G$ there is a G-invariant probability measure, and let $\rho : L \to G'$ be a continuous homomorphism into a locally compact group G'. The composition of the pullback

$$\rho^\bullet_{\mathrm{b}} : \mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R})$$

with the transfer map $\mathrm{T}^\bullet_{\mathrm{b}}$ defined in (2.1) gives rise to the *bounded Toledo map*

$$\mathrm{T}^\bullet_{\mathrm{b}}(\rho) : \mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \tag{2.6}$$

which is the source of basic invariants of the homomorphism $\rho : L \to G'$.

A good part of this paper will be devoted to the interpretation and properties of a numerical invariant defined by this map in the case in which the cohomology spaces involved are one-dimensional (see § 5). To this purpose, remark that if $L\backslash G$ is in addition compact (for example, a uniform lattice) then we also have an analogous construction in ordinary cohomology. Namely, associated to the homomorphism $\rho : L \to G'$ we have the pullback

$$\rho^\bullet : \mathrm{H}^\bullet_{\mathrm{c}}(G',\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R})$$

which, composed with the transfer map $\mathrm{T}^\bullet$ defined above gives a map

$$\mathrm{T}^\bullet(\rho) : \mathrm{H}^\bullet_{\mathrm{c}}(G',\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R})$$

which we call the *Toledo map* and which has the property that the diagram

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \arrow[r, "\mathrm{c}^\bullet_{G'}"] \arrow[d, "\mathrm{T}^\bullet_{\mathrm{b}}(\rho)"'] & \mathrm{H}^\bullet_{\mathrm{c}}(G',\mathbb{R}) \arrow[d, "\mathrm{T}^\bullet(\rho)"] \\
\mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \arrow[r, "\mathrm{c}^\bullet_{G}"] & \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R})\,,
\end{tikzcd}
$$

where the horizontal arrows are comparison maps, commutes.

The interplay between these two maps is the basic ingredient in the interpretation of the above invariants in this paper for the cocompact case, as well as in [38], [18], [16] and [17]. In the finite volume case we will need to resort to a somewhat more elaborate version of the above diagram which can be developed when G is a connected semisimple Lie group – see (5.5) and which will encompass the above description.

3. A Factorization of the Comparison Map

The main point of this section is to provide, in the case of semisimple Lie groups, a substitute to the the missing arrow in

$$\begin{tikzcd} \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \arrow[r, "\mathrm{c}^\bullet_L"] \arrow[d, "\mathrm{T}^\bullet_{\mathrm{b}}"'] & \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R}) \arrow[d, dotted] \\ \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \arrow[r, "\mathrm{c}^\bullet_G"] & \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R}) \end{tikzcd} \tag{3.1}$$

if the subgroup $L \leq G$ is only of finite covolume.

Let G be a connected semisimple Lie group with finite center and $\mathcal{X}$ the associated symmetric space. Any closed subgroup $L \leq G$ acts properly on $\mathcal{X}$ and hence the complex

$$\mathbb{R} \longrightarrow \Omega^0(\mathcal{X}) \longrightarrow \dots \longrightarrow \Omega^k(\mathcal{X}) \longrightarrow \dots$$

of C^∞ differential forms on $\mathcal{X}$ with the usual exterior differential is a resolution by continuous injective L-modules (where injectivity now refers to the usual notion in continuous cohomology), from which one obtains a canonical isomorphism

$$\mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^L\big) \xrightarrow{\cong} \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R})$$

in cohomology, [47]. Let moreover $\big(\Omega^\bullet_\infty(\mathcal{X}), d^\bullet\big)$ denote the complex of smooth differential forms α on $\mathcal{X}$ such that the assignments $x \mapsto \|\alpha_x\|$ and $x \mapsto \|d\alpha_x\|$ are in $\mathrm{L}^\infty(\mathcal{X})$, and let $h(\mathcal{X})$ denote the volume entropy of $\mathcal{X}$, that is the rate of exponential growth of volume of geodesic balls in $\mathcal{X}$, [25]. Then we have:

Proposition 3.1. *Let G be a connected semisimple Lie group with finite center, $\mathcal{X}$ the associated symmetric space and let $L \leq G$ be any closed subgroup. Then there exists a map*

$$\delta^\bullet_{\infty,L} : \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \to \mathrm{H}^\bullet\big(\Omega^\bullet_\infty(\mathcal{X})^L\big)$$

such that the diagram

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_L"] \arrow[dr, "\delta^\bullet_{\infty,L}"'] & & \mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R}) & \mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^L\big) \arrow[l, "\cong"'] \\
& \mathrm{H}^\bullet\big(\Omega^\bullet_\infty(\mathcal{X})^L\big) \arrow[urr, "i^\bullet_{\infty,L}"'] & &
\end{tikzcd}
\tag{3.2}
$$

commutes, where $i^\bullet_{\infty,L}$ is the map induced in cohomology by the inclusion of complexes

$$i^\bullet_\infty : \Omega^\bullet_\infty(\mathcal{X}) \to \Omega^\bullet(\mathcal{X})\,.$$

Moreover, the norm of $\delta^{(k)}_{\infty,L}$ is bounded by $h(\mathcal{X})^k$.

Before proving the proposition, we want to push our result a little further in the case when $L = \Gamma < G$ is a lattice. In particular, we are going to see how the map $\delta^\bullet_{\infty,\Gamma}$ fits into a diagram where the transfer appears. If $1 \le p \le \infty$, let $\Omega^n_p(\mathcal{X})^\Gamma$ be the space of Γ-invariant smooth differential n-forms on $\mathcal{X}$ such that $x \mapsto \|\alpha_x\|$ and $x \mapsto \|d\alpha_x\|$ are in $\mathrm{L}^p(\Gamma\backslash\mathcal{X})$, and consider the complex[3] $\big(\Omega^\bullet_p(\mathcal{X})^\Gamma, d^\bullet\big)$. Let $\delta^\bullet_{p,\Gamma}$ be the map obtained by composing the map $\delta^\bullet_{\infty,\Gamma}$ in Proposition 3.1 with the map obtained by the inclusion of complexes

$$\Omega^\bullet_\infty(\mathcal{X})^\Gamma \to \Omega^\bullet_p(\mathcal{X})^\Gamma\,,$$

namely

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma,\mathbb{R}) \arrow[r, "\delta^\bullet_{\infty,\Gamma}"] \arrow[rr, bend left, "\delta^\bullet_{p,\Gamma}"] & \mathrm{H}^\bullet\big(\Omega^\bullet_\infty(\mathcal{X})^\Gamma\big) \arrow[r] & \mathrm{H}^\bullet\big(\Omega^\bullet_p(\mathcal{X})^\Gamma\big)\,.
\end{tikzcd}
$$

Also, since $\Omega(\mathcal{X})^G \subset \Omega_\infty(\mathcal{X})$ and $\Gamma\backslash\mathcal{X}$ is of finite volume, the restriction map

$$\Omega^\bullet(\mathcal{X})^G \to \Omega^\bullet_p(\mathcal{X})^\Gamma$$

is defined and admits a left inverse $j^\bullet_p$ defined by integration

$$j^\bullet_p\alpha = \int_{\Gamma\backslash G} (L^*_g\alpha)\,d\mu(\dot g)\,,$$

for $\alpha \in \Omega^\bullet_p(\mathcal{X})^\Gamma$ and where L_g is left translation by g. The following proposition gives an interesting diagram to be compared with (3.1)

[3]Notice that this is a rather misleading notation if $\mathcal{X}$ is not compact, because in this case only for $p = \infty$ one has that $\big(\Omega^\bullet_\infty(\mathcal{X})^\Gamma, d^\bullet\big)$ is the subcomplex of invariants of $\big(\Omega^\bullet_\infty(\mathcal{X}), d^\bullet\big)$.

Proposition 3.2. *Let G be a connected semisimple Lie group with finite center and associated symmetric space $\mathcal{X}$, and let $\Gamma < G$ be a lattice. The following diagram*

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma,\mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_\Gamma"] \arrow[drrr, "\delta^\bullet_{p,\Gamma}"'] \arrow[dd, "\mathrm{T}^\bullet_{\mathrm{b}}"'] & & \mathrm{H}^\bullet(\Gamma,\mathbb{R}) & & \mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^\Gamma\big) \arrow[ll, "\cong"'] \\
& & & \mathrm{H}^\bullet\big(\Omega^\bullet_p(\mathcal{X})^\Gamma\big) \arrow[ur, "i^\bullet_{p,\Gamma}"'] \arrow[dr, "j^\bullet_p"] & \\
\mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_G"] & & \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R}) & & \Omega^\bullet(\mathcal{X})^G \arrow[ll, "\cong"']
\end{tikzcd}
\tag{3.3}
$$

commutes for all $1 \leq p \leq \infty$.

3.1. **Proof of Proposition 3.1 and Proposition 3.2.** We start the proof by showing how to associate to an L^∞ alternating function c on $(\partial\mathcal{X})^{n+1}$ a differential n-form obtained by integrating, with respect to an appropriate density at infinity and weighted by the function c, a certain differential form constructed using the Busemann functions associated to n points at infinity.

So, let us consider on $\mathcal{X}$ the Riemannian metric obtained from the Killing form and let

$$B : \partial\mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

be the Busemann cocycle, where $\partial\mathcal{X}$ is the geodesic ray boundary of $\mathcal{X}$. Fix a basepoint $0 \in \mathcal{X}$ and let $K = \mathrm{Stab}_G(0)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition, $\mathfrak{a}^+ \subset \mathfrak{p}$ a positive Weyl chamber and $b \in \mathfrak{a}^+$ the vector predual to the sum of the positive roots associated to $\mathfrak{a}^+$. Then $h(\mathcal{X}) = \|b\|$. Let $\xi_b \in \partial\mathcal{X}$ be the point at infinity determined by b; let ν_0 be the unique K-invariant probability measure on $G\xi_b \subset \partial\mathcal{X}$. Then

$$d(g_*\nu_0)(\xi) = e^{-h(\mathcal{X})B_\xi(g0,0)} d\nu_0(\xi)\,. \tag{3.4}$$

For $\xi \in \partial\mathcal{X}$, let us define a C^∞ map by

$$
\begin{aligned}
e^\xi : \mathcal{X} &\longrightarrow \quad \mathbb{R} \\
x &\mapsto e^{-h(\mathcal{X})B_\xi(x,0)}\,.
\end{aligned}
\tag{3.5}
$$

Lemma 3.3. *Let G be a connected semisimple Lie group with finite center, and let $\mathcal{X}$ be its associated symmetric space with geodesic ray boundary $\partial\mathcal{X}$. For each $c \in \mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{n+1}, \nu_0^{n+1}\big)$, the differential form defined by*

$$\omega := \int_{(\partial\mathcal{X})^{n+1}} c(\xi_0,\dots,\xi_n)\, e^{\xi_0} \wedge de^{\xi_1} \wedge \dots \wedge de^{\xi_n}\, d\nu_0^{n+1}(\xi_0,\dots,\xi_n)\,. \tag{3.6}$$

is in $\Omega^n_\infty(\mathcal{X})$. Moreover the resulting map

$$\begin{aligned}\delta^\bullet_\infty : \mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1}\big) &\to \Omega^\bullet_\infty(\mathcal{X})\\ c \qquad\qquad &\longmapsto \quad \omega\end{aligned}$$

is a G-equivariant map of complexes, and

$$\|\delta^{(n)}_\infty\| \leq h(\mathcal{X})^n\,. \tag{3.7}$$

Proof. For $\xi \in \partial\mathcal{X}$, let $X_\xi(x)$ be the unit tangent vector at x pointing in the direction of ξ, and let $g_x(\,\cdot\,,\,\cdot\,)$ be the Riemannian metric on $\mathcal{X}$ at x. Since the gradient of the Busemann function $B_\xi(x,0)$ at x is $-X_\xi(x)$ [25], we have that for $v\in(T\mathcal{X})_x$, $(dB_\xi)_x(v) = -g_x\big(v, X_\xi(v)\big)$. Then

$$(de^\xi)_x(v) = h(\mathcal{X})g_x\big(v, X_\xi(x)\big)e^\xi(x)\,.$$

This implies that if $v_1,\dots,v_n$ are tangent vectors based at x, then

$$\begin{aligned}&|\omega_x(v_1,\dots,v_n)|\\ &\leq h(\mathcal{X})^n\int_{(\partial\mathcal{X})^{n+1}}\big|c(\xi_0,\xi_1,\dots,\xi_n)\big|e^{\xi_0}(x)\cdot\\ &\qquad\qquad\cdot\left(\prod_{i=1}^n\big|g_x\big(v_i, X_{\xi_i}(x)\big)\big|\,e^{\xi_i}(x)\right)d\nu_0^{n+1}(\xi_0,\dots,\xi_n)\\ &\leq h(\mathcal{X})^n\|c\|_\infty\left(\int_{\partial\mathcal{X}}e^{\xi_0}(x)d\nu_0(\xi_0)\right)\prod_{i=1}^n\left(\|v_i\|\int_{\partial\mathcal{X}}e^{\xi_i}(x)d\nu_0(\xi_i)\right)\,,\end{aligned}$$

where we used that $\big|g_x\big(v_i, X_{\xi_i}(x)\big)\big| \leq \|v_i\|$. But writing $x = g0$ and using that, as indicated in (3.4), $d(g_*\nu_0)$ is a probability measure, we get from (3.4) that for all $0\leq i\leq n$ and all $x\in\mathcal{X}$

$$\int_{\partial\mathcal{X}}e^{\xi_i}(x)d\nu_0(\xi_i) = 1\,, \tag{3.8}$$

which shows that

$$|\omega_x(v_1,\dots,v_n)| \leq h(\mathcal{X})^n\|c\|_\infty\prod_{i=1}^n\|v_i\|\,,$$

so that if

$$\delta^\bullet_\infty : \mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1}, \nu_0^{\bullet+1}\big) \to \Omega^\bullet(\mathcal{X})\,,$$

we have that

$$\|\delta^{(n)}_\infty c\| = \sup_{x\in\mathcal{X}}\ \sup_{\|v_1\|,\dots,\|v_n\|\leq 1}|\omega_x(v_1,\dots,v_n)| \leq h(\mathcal{X})^n\|c\|_\infty\,.$$

This proves (3.7) and the fact that the image $\delta^{(n)}_\infty(c)$ is a bounded form. Once we shall have proven that $\delta^{(n)}_\infty(dc) = d\delta^{(n-1)}_\infty(c)$, it will follow automatically that also $d\delta_\infty^{n-1}(c)$ is bounded and hence the image of $\delta_\infty^\bullet$ is in $\Omega_\infty^\bullet(\mathcal{X})$. To this purpose, let us compute for $c \in \mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^n, \nu_0^n\big)$

$$\begin{aligned}\delta_\infty^{(n)}(dc) = \int_{(\partial\mathcal{X})^{n+1}} & e^{\xi_0} \wedge de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} \cdot \\ & \cdot \left[\sum_{i=0}^n (-1)^i c(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_n)\right] d\nu_0^{n+1}(\xi_0, \dots, \xi_n)\,.\end{aligned}$$

For $i \geq 1$ the i-th term is

$$\begin{aligned}&(-1)^i \int_{(\partial\mathcal{X})^{n+1}} e^{\xi_0} \wedge de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} \cdot \\ &\qquad\qquad \cdot c(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_n) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\ &= -\, d\left(\int_{\partial\mathcal{X}} e^{\xi_i} d\nu_0(\xi_i)\right) \wedge \cdots = 0\end{aligned}$$

since by (3.8)

$$d\left(\int_{\partial\mathcal{X}} e^{\xi_i} d\nu_0(\xi_i)\right) = 0\,.$$

Thus

$$\begin{aligned}\delta_\infty^{(n)}(dc) &= \int_{(\partial\mathcal{X})^{n+1}} e^{\xi_0} \wedge de^{\xi_1} \wedge \cdots \wedge de^{\xi_n}\, c(\xi_1, \dots, \xi_n) d\nu_0^{n+1}(\xi_0, \dots, \xi_n) \\ &= \left[\int_{\partial\mathcal{X}} e^{\xi_0} d\nu_0(\xi_0)\right] \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} \cdot \\ &\qquad\qquad \cdot c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n) \\ &= \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge \cdots \wedge de^{\xi_n} c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n)\,.\end{aligned}$$

On the other hand

$$\delta_\infty^{(n-1)}(c) = \int_{(\partial\mathcal{X})^n} e^{\xi_1} \wedge de^{\xi_2} \wedge \cdots \wedge de^{\xi_n}\, c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n)\,,$$

so that by definition

$$\begin{aligned}d\delta_\infty^{(n-1)}(c) &= \int_{(\partial\mathcal{X})^n} de^{\xi_1} \wedge de^{\xi_2} \wedge \cdots \wedge de^{\xi_n}\, c(\xi_1, \dots, \xi_n) d\nu_0^n(\xi_1, \dots, \xi_n) \\ &= \delta_\infty^{(n)}(dc)\,.\end{aligned}$$

The G-equivariance of $\delta_\infty^\bullet$ follows from (3.4) and the cocycle property of the Busemann function $B_\xi(x, y)$, hence completing the proof. □

Proof of Proposition 3.1. This is a direct application of [46, Proposition 9.2.3]. Indeed, since the L-action on $(\partial\mathcal{X},\nu_0)$ is amenable, we have that $\big(\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1},\nu_0^{\bullet+1}\big)\big)$ is a strong resolution of $\mathbb{R}$ by relatively injective L-modules [20]; moreover, it is well known that, $(\Omega^\bullet(\mathcal{X}),d^\bullet)$ is a resolution of $\mathbb{R}$ by injective continuous L-modules, where in this case injectivity is meant in ordinary cohomology (see [47]), and $\Omega^\bullet(\mathcal{X})$ is as usual equipped with the C^∞-topology. Finally one checks on the formulas that the composition $i^\bullet_\infty\circ\delta^\bullet_\infty$, where $i^\bullet_\infty$ is the injection

$$i^\bullet_\infty:\Omega^\bullet_\infty(\mathcal{X})\to\Omega^\bullet(\mathcal{X})\,,$$

is a continuous L-morphism of complexes. The hypotheses of [46, Proposition 9.2.3] are hence verified and thus the map in cohomology

$$i^\bullet_{\infty,L}\circ\delta^\bullet_{\infty,L}:\mathrm{H}^\bullet\big(\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1},\nu_0^{\bullet+1}\big)^L\big)\to\mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^L\big)$$

realizes the canonical comparison map

$$\mathsf{c}^\bullet_L:\mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R})\to\mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R})$$

. □

Proof of Proposition 3.2. The proof of Proposition 3.1 remains valid verbatim for all $1\le p\le\infty$ to show the commutativity of the upper diagram, so it remains to show only the commutativity of the lower part. Notice moreover that since

$$i^\bullet_{p,G}:\Omega^\bullet_p(\mathcal{X})^G\to\Omega^\bullet(\mathcal{X})^G$$

is the identity, $\delta^\bullet_{p,G}$ realizes in cohomology the canonical comparison map. Furthermore, if P is the minimal parabolic in G stabilizing ξ_b and we identify $(\partial\mathcal{X},\nu_0)$ with $(G/P,\nu_0)$ as measure spaces, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1},\nu_0^{\bullet+1}\big)^\Gamma \arrow[r, "\delta^\bullet_{p,\Gamma}"] \arrow[d, "\mathrm{T}^{\bullet+1}_{\partial\mathcal{X}}"'] & \Omega^\bullet_p(\mathcal{X})^\Gamma \arrow[d, "j^\bullet_p"] \\
\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{X})^{\bullet+1},\nu_0^{\bullet+1}\big)^G \arrow[r, "\delta^\bullet_{p,G}"] & \Omega^\bullet(\mathcal{X})^G
\end{tikzcd},$$

is immediate, where $\mathrm{T}^\bullet_{\partial\mathcal{X}}$ is defined in (2.4). Then Lemma 2.1 completes the proof. □

4. A Factorization of the Pullback

Let L be a closed subgroup in a connected semisimple Lie group G with finite center and associated symmetric space $\mathcal{X}$, and let $\rho : L \to G'$ be a continuous homomorphism into a topological group G'. Combining the diagram in (3.2) with pullbacks in ordinary and bounded cohomology, we obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \arrow[r, "\mathsf{c}^\bullet_{G'}"] \arrow[d, "\rho^\bullet_{\mathrm{b}}"'] & \mathrm{H}^\bullet_{\mathrm{c}}(G',\mathbb{R}) \arrow[d, "\rho^\bullet"] & \\
\mathrm{H}^\bullet_{\mathrm{b}}(L,\mathbb{R}) \arrow[r, "\mathsf{c}^\bullet_{L}"] \arrow[dr, "\delta^\bullet_{\infty,L}"'] & \mathrm{H}^\bullet(L,\mathbb{R}) & \mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^L\big) \arrow[l, "\cong"'] \\
& \mathrm{H}^\bullet\big(\Omega^\bullet_\infty(\mathcal{X})^L\big) \arrow[ur, "i^\bullet_{\infty,L}"'] &
\end{tikzcd}
\tag{4.1}
$$

from which one immediately reads:

Corollary 4.1. *Let G' be a topological group, $L \le G$ any closed subgroup in a semisimple Lie group G with finite center and associated symmetric space $\mathcal{X}$, and $\rho : L \to G'$ a continuous homomorphism. If $\alpha \in \mathrm{H}^n_{\mathrm{c}}(G',\mathbb{R})$ is represented by a continuous bounded class, then $\rho^{(n)}(\alpha) \in \mathrm{H}^n(L,\mathbb{R})$ is representable by a L-invariant smooth closed differential n-form on $\mathcal{X}$ which is bounded.*

Analogously, if in addition $L = \Gamma < G$ is a lattice, then combining the top part of the diagram in (3.3) with pullbacks we obtain

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \arrow[r, "\mathsf{c}^\bullet_{G'}"] \arrow[d, "\rho^\bullet_{\mathrm{b}}"'] & \mathrm{H}^\bullet_{\mathrm{c}}(G',\mathbb{R}) \arrow[d, "\rho^\bullet"] & \\
\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma,\mathbb{R}) \arrow[r, "\mathsf{c}^\bullet_{\Gamma}"] \arrow[dr, "\delta^\bullet_{p,\Gamma}"'] & \mathrm{H}^\bullet(\Gamma,\mathbb{R}) & \mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^\Gamma\big) \arrow[l, "\cong"'] \\
& \mathrm{H}^\bullet\big(\Omega^\bullet_p(\mathcal{X})^\Gamma\big) \arrow[ur, "i^\bullet_{p,\Gamma}"'] &
\end{tikzcd}
\tag{4.2}
$$

In this section we shall mainly draw consequences from this, in especially relevant circumstances. For example, if G' also is a connected, semisimple Lie group with finite center, then in degree two the comparison map

$$\mathsf{c}^{(2)}_{G'} : \mathrm{H}^2_{\mathrm{cb}}(G',\mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(G',\mathbb{R})$$

is an isomorphism [20], and we may then compose $\big(\mathsf{c}_{G'}^{(2)}\big)^{-1}$ with $\rho_{\mathrm{b}}^{(2)}$ and $\delta_{p,\Gamma}^{(2)}$ to get a map

$$\rho_p^{(2)} := \rho_{\mathrm{b}}^{(2)} \circ \big(\mathsf{c}_{G'}^{(2)}\big)^{-1} : \mathrm{H}^2_{\mathrm{c}}(G',\mathbb{R}) \to \mathrm{H}^2\big(\Omega_p^\bullet(\mathcal{X})^\Gamma\big)\,, \tag{4.3}$$

for which the following holds:

Corollary 4.2. *If G, G' are connected semisimple Lie groups with finite center, $\mathcal{X}$ is the symmetric space associated to G and $\Gamma < G$ is a lattice, then the pullback via the homomorphism $\rho : \Gamma \to G'$ in ordinary cohomology and in degree two factors via L^p-cohomology*

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{c}}(G',\mathbb{R}) \arrow[d, "\rho^{(2)}"] \arrow[dd, bend right=60, "\rho_p^{(2)}"'] & \\
\mathrm{H}^2(\Gamma,\mathbb{R}) & \mathrm{H}^2\big(\Omega^\bullet(\mathcal{X})^\Gamma\big) \arrow[l, "\cong"'] \\
\mathrm{H}^2\big(\Omega_p^\bullet(\mathcal{X})^\Gamma\big) \arrow[ur, "i^{(2)}_{p,\Gamma}"'] &
\end{tikzcd}$$

Remark 4.3. (1) This is true for all closed subgroups $L < G$ in the case $p = \infty$.

(2) Notice that, so far, we have not used the commutativity of the lower part of the diagram in (3.3). This will be done in the following section, to identify a numerical invariant associated to a representation.

5. The Invariant and the Milnor–Wood Type Inequality

Let G be a connected, semisimple Lie group with finite center, and $\mathcal{X}$ the associated symmetric space. Assume that $\mathcal{X}$ is Hermitian symmetric, so that on $\mathcal{X}$ there exists a nonzero G-invariant (closed) differential 2-form, namely the Kähler form of the Hermitian metric, which we denote by $\omega_{\mathcal{X}} \in \Omega^2(\mathcal{X})^G$. Here and in the sequel, the Riemannian metric on $\mathcal{X}$ is normalized so as to have minimal holomorphic sectional curvature -1.

If $x \in \mathcal{X}$ is a reference point, and $\Delta(g_1x, g_2x, g_3x) \subset \mathcal{X}$ is a triangle with geodesic sides between the vertices g_1x, g_2x, g_3x, and arbitrarily C^1-filled, the function $c : G^3 \to \mathbb{R}$ defined by

$$c(g_1, g_2, g_3) := \int_{\Delta(g_1x, g_2x, g_3x)} \omega_{\mathcal{X}}$$

is a differentiable homogeneous G-invariant cocycle and defines the continuous class $\kappa_{\mathcal{X}} \in \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})$ corresponding to $\omega_{\mathcal{X}}$ by the van Est isomorphism $\mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R}) \simeq \Omega^2(\mathcal{X})^G$. Moreover, c is bounded ([24], [22]), and hence it defines a bounded continuous class $\kappa^{\mathrm{b}}_{\mathcal{X}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ which corresponds to $\kappa_{\mathcal{X}} \in \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})$ under the isomorphism

$$\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}) \cong \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})\,.$$

If moreover we assume that $\mathcal{X}$ is irreducible, then

$$\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}) \cong \mathbb{R}\cdot\kappa^{\mathrm{b}}_{\mathcal{X}}\,.$$

Let now $\rho:\Gamma\to G'$ be a homomorphism of a lattice $\Gamma < G$ into a connected semisimple Lie group G' with finite center and associated Hermitian symmetric space $\mathcal{X}'$ (not necessarily irreducible). The definition of the bounded Toledo map in § 2.1

$$\mathrm{T}^{(2)}_{\mathrm{b}}(\rho):\mathrm{H}^2_{\mathrm{cb}}(G',\mathbb{R})\to\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$$

leads to the definition of the *bounded Toledo invariant* $\mathrm{t_b}(\rho)$ by

$$\mathrm{T}^{(2)}_{\mathrm{b}}(\rho)(\kappa^{\mathrm{b}}_{\mathcal{X}'}) = \mathrm{t_b}(\rho)\kappa^{\mathrm{b}}_{\mathcal{X}}\,. \tag{5.1}$$

Then we have a Milnor–Wood type inequality:

LEMMA 5.1. *With the above notations,*

$$|\mathrm{t_b}(\rho)| \le \frac{\mathrm{rk}\,\mathcal{X}'}{\mathrm{rk}\,\mathcal{X}}\,.$$

Proof. If $\mathcal{Y}$ is any Hermitian symmetric space with metric normalized so as its minimal holomorphic sectional curvature is -1, then it follows from [24] and [22] that the Gromov norm of $\kappa^{\mathrm{b}}_{\mathcal{Y}}$ is

$$\|\kappa^{\mathrm{b}}_{\mathcal{Y}}\| = \pi\,\mathrm{rk}\,\mathcal{Y}\,.$$

This and the fact that $\mathrm{T}^{\bullet}_{\mathrm{b}}(\rho)$ is norm decreasing in bounded cohomology imply the assertion. □

The bounded Toledo invariant can now be nicely interpreted using the lower part of (3.3) in the case $p=2$. In fact, the space $\mathcal{X}$ being Hermitian symmetric, the L^2-cohomology spaces $\mathrm{H}^\bullet\big(\Omega^\bullet_2(\mathcal{X})^\Gamma\big)$ are reduced and finite dimensional, [5, § 3]. The following observation will be essential:

LEMMA 5.2. *Let $\mathcal{X}$ be a Hermitian symmetric space and Γ a lattice in the isometry group $G := \mathrm{Iso}(\mathcal{X})^\circ$. Then the map*

$$j^\bullet_2:\mathrm{H}^\bullet\big(\Omega^\bullet_2(\mathcal{X})^\Gamma\big)\to\mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^G\big)=\Omega^\bullet(\mathcal{X})^G$$

is the orthogonal projection, where we consider $\Omega^\bullet(\mathcal{X})^G$ as a subspace of $\mathrm{H}^\bullet\big(\Omega^\bullet_2(\mathcal{X})^\Gamma\big)$.

Proof. Denoting by $\langle\,\cdot\,,\,\cdot\,\rangle_x$ the scalar product on $\Lambda^\bullet(T_x\mathcal{X})^*$, the scalar product of two forms $\alpha,\beta\in\Omega_2^\bullet(\mathcal{X})^\Gamma$ is given by

$$\langle\alpha,\beta\rangle:=\int_{\Gamma\backslash\mathcal{X}}\langle\alpha_x,\beta_x\rangle_x dv(\dot{x})\,,\tag{5.2}$$

where dv is the volume measure on $\Gamma\backslash\mathcal{X}$; fixing $x_0\in\mathcal{X}$, and letting μ be the G-invariant probability measure on $\Gamma\backslash G$, (5.2) can be written as

$$\langle\alpha,\beta\rangle=\mathrm{vol}(\Gamma\backslash G)\int_{\Gamma\backslash G}\langle\alpha_{hx_0},\beta_{hx_0}\rangle_{hx_0}d\mu(\dot{h})\,.\tag{5.3}$$

Since we have identified $\mathrm{H}^\bullet\big(\Omega_2(\mathcal{X})^\Gamma\big)$ with the space of harmonic forms which are L^2 (modulo Γ), it suffices to show that

$$\big\langle j_2^\bullet(\alpha),\beta\big\rangle=\big\langle\alpha,j_2^\bullet(\beta)\big\rangle\,.$$

To this end we compute

$$\big\langle(L_g^*\alpha)_x,\beta_x\big\rangle_x=\langle\alpha_{gx}\circ\Lambda^\bullet d_xL_g,\beta_x\rangle_x=\big\langle\alpha_{gx},\beta_x\circ(\Lambda^\bullet d_xL_g)^{-1}\big\rangle_{gx}$$

and hence, using (5.3),

$$\begin{aligned}
&\langle j_2^\bullet(\alpha),\beta\rangle\\
&=\mathrm{vol}(\Gamma\backslash G)\int_{\Gamma\backslash G}\left(\int_{\Gamma\backslash G}\langle\alpha_{ghx_0},\beta_{hx_0}\circ(\Lambda^\bullet d_{hx_0}L_g)^{-1}\rangle_{ghx_0}d\mu(\dot{g})\right)d\mu(\dot{h})\\
&=\mathrm{vol}(\Gamma\backslash G)\int_{\Gamma\backslash G}\left(\int_{\Gamma\backslash G}\langle\alpha_{gx_0},\beta_{hx_0}\circ(\Lambda^\bullet d_{hx_0}L_{gh^{-1}})^{-1}\rangle_{gx_0}d\mu(\dot{g})\right)d\mu(\dot{h})\\
&=\mathrm{vol}(\Gamma\backslash G)\int_{\Gamma\backslash G}\left\langle\alpha_{gx_0},\int_{\Gamma\backslash G}\beta_{hx_0}\circ(\Lambda^\bullet d_{hx_0}L_{gh^{-1}})^{-1}d\mu(\dot{h})\right\rangle_{gx_0}d\mu(\dot{g})\,.
\end{aligned}$$

But $(\Lambda^\bullet d_{hx_0}L_{gh^{-1}})^{-1}=\Lambda^\bullet d_{gx_0}L_{hg^{-1}}$, so

$$\begin{aligned}
\int_{\Gamma\backslash G}\beta_{hx_0}\circ(\Lambda^\bullet d_{hx_0}L_{gh^{-1}})^{-1}d\mu(\dot{h})&=\int_{\Gamma\backslash G}\beta_{hx_0}\circ\Lambda^\bullet d_{gx_0}L_{hg^{-1}}d\mu(\dot{h})\\
&=\int_{\Gamma\backslash G}\beta_{hgx_0}\circ\Lambda^\bullet d_{gx_0}L_hd\mu(\dot{h})
\end{aligned}$$

and hence, using (5.3) and (5.2,

$$\big\langle j_2^\bullet(\alpha),\beta\big\rangle=\int_{\Gamma\backslash\mathcal{X}}\left\langle\alpha_x,\int_{\Gamma\backslash G}\beta_{hx}\circ\Lambda^\bullet d_xL_hd\mu(\dot{h})\right\rangle_x dv(\dot{x})=\big\langle\alpha,j_2^\bullet(\beta)\big\rangle$$

which shows that j_2 is self-adjoint. Being clearly a projection, this proves the lemma. □

If we assume that $\mathcal{X}$ is irreducible, then as a subspace of $\mathrm{H}^2\big(\Omega_2^\bullet(\mathcal{X})^\Gamma\big)$ the space $\Omega^2(\mathcal{X})^G = \mathbb{R}\,\omega_{\mathcal{X}}$ is identified with $\mathbb{R}\,\omega_{\Gamma\backslash\mathcal{X}}$, where $\omega_{\Gamma\backslash\mathcal{X}}$ is the Kähler form on $\Gamma\backslash\mathcal{X}$. With this we have that for $\alpha \in \mathrm{H}^2\big(\Omega_2^\bullet(\mathcal{X})^\Gamma\big)$,

$$j_2^{(2)}(\alpha) = \frac{\langle \alpha, \omega_{\Gamma\backslash\mathcal{X}}\rangle}{\langle \omega_{\Gamma\backslash\mathcal{X}}, \omega_{\Gamma\backslash\mathcal{X}}\rangle}\omega_{\Gamma\backslash\mathcal{X}}\,.$$

Define now

$$i_\rho := \frac{\langle \rho_2^{(2)}(\kappa_{\mathcal{X}'}), \omega_{\Gamma\backslash\mathcal{X}}\rangle}{\langle \omega_{\Gamma\backslash\mathcal{X}}, \omega_{\Gamma\backslash\mathcal{X}}\rangle}\,, \tag{5.4}$$

where $\rho_p^{(2)} : \mathrm{H}^2_{\mathrm{c}}(G', \mathbb{R}) \to \mathrm{H}^2\big(\Omega_p^\bullet(\mathcal{X})^\Gamma\big)$ is the map in (4.3). It finally follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G', \mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_{G'}"] \arrow[d, "\rho^\bullet_{\mathrm{b}}"'] \arrow[ddd, bend right=40, "\mathrm{T}^\bullet_{\mathrm{b}}(\rho)"'] & & \mathrm{H}^\bullet_{\mathrm{c}}(G', \mathbb{R}) \arrow[d, "\rho^\bullet"] & \\
\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma, \mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_\Gamma"] \arrow[ddd, phantom] \arrow[drr, "\delta^\bullet_{p,\Gamma}"'] \arrow[dd, "\mathrm{T}^\bullet_{\mathrm{b}}"'] & & \mathrm{H}^\bullet(\Gamma, \mathbb{R}) \arrow[r, "\cong"] & \mathrm{H}^\bullet\big(\Omega^\bullet(\mathcal{X})^\Gamma\big) \\
 & & \mathrm{H}^\bullet\big(\Omega_p^\bullet(\mathcal{X})^\Gamma\big) \arrow[ur, "i^\bullet_{p,\Gamma}"'] \arrow[dr, "j^\bullet_p"] & \\
\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R}) \arrow[rr, "\mathrm{c}^\bullet_G"] & & \mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R}) \arrow[r, "\cong"'] & \Omega^\bullet(\mathcal{X})^G\,.
\end{tikzcd} \tag{5.5}$$

in the special case of $p = 2$ and degree 2 and from Corollary 4.2 that:

Lemma 5.3. $i_\rho = \mathrm{t}_{\mathrm{b}}(\rho)$.

Theorem 7 then follows immediately from Lemma 5.3 together with Lemma 5.1.

As a further application of Lemma 5.3, we have the following:

Remark 5.4. Let $\Gamma = \langle a, b\rangle$ and $\Lambda = \langle a', b'\rangle$ be, as in Example 10, generated respectively by parabolic elements a, b and by hyperbolic elements a', b' in $\mathrm{PU}(1,1)$, and let $\rho : \Gamma \to \Lambda$ be the representation defined by $\rho(a) = a'$ and $\rho(b) = b'$. We shall prove that $i_\rho = 1$. In fact, let $\sigma : \Gamma \to \mathrm{PU}(1,1)$ be the identity representation. The properties (1) and (2) of the boundary map $f : \partial\mathcal{H}^1_{\mathbb{C}} \to \partial\mathcal{H}^1_{\mathbb{C}}$ in Example 10 say exactly that σ and $\rho : \Gamma \to \Lambda < \mathrm{PU}(1,1)$ are semiconjugate, so that

$$\rho_{\mathrm{b}}^{(2)}(\kappa_1^{\mathrm{b}}) = \sigma_{\mathrm{b}}^{(2)}(\kappa_1^{\mathrm{b}}) = \kappa_1^{\mathrm{b}}|_\Gamma\,,$$

where $\kappa_1^{\mathrm{b}}|_\Gamma$ is the restriction of the bounded Kähler class of G to Γ, [29]. Applying the transfer map to the above equation, we obtain

$$\mathrm{T}_{\mathrm{b}}^{(2)}(\rho)(\kappa_1^{\mathrm{b}}) = \mathrm{T}_{\mathrm{b}}^{(2)}(\kappa_1^{\mathrm{b}}|_\Gamma) = \kappa_1^{\mathrm{b}}\,,$$

which implies by (5.1) that $\mathrm{t}_{\mathrm{b}}(\rho) = 1$. Using Lemma 5.3 we conclude that $i_\rho = 1$.

6. Applications to Complex Hyperbolic Spaces and Maximal Representations

As mentioned already in the introduction, in the special case of complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^\ell$, the multiple $\frac{1}{\pi}\kappa_\ell^{\mathrm{b}}$ of the bounded Kähler class κ_ℓ^{b} admits an explicit representative on $\partial\mathcal{H}_{\mathbb{C}}^\ell$ given by the Cartan cocycle $c_\ell : (\partial\mathcal{H}_{\mathbb{C}}^\ell)^3 \to [-1,1]$, which is defined in terms of the Hermitian triple product of a triple of points in the underlying complex vector space V of dimension $\ell+1$ with a Hermitian form of signature $(1,\ell)$ whose cone of negative lines gives a model of complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^\ell$.

The very explicit form of the factorization of the comparison map between bounded and ordinary cohomology, together with the implementation of the pullback by boundary maps in [12] allows one to give explicit representatives of the class $\rho^{(2)}(\kappa_q)$ at least when $\mathcal{X}'$ is the complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^q$.

We start by recalling the following result, adapted to our case, which gives a canonical representative of the pullback in bounded cohomology.

Corollary 6.1 ([12, Corollary 2.2]). *Let G, G' be connected simple Lie groups with finite center and associated symmetric spaces $\mathcal{H}_{\mathbb{C}}^p$ and $\mathcal{H}_{\mathbb{C}}^q$ respectively, and let $L \leq G$ be any closed subgroup. Let $\rho : L \to G'$ be a homomorphism with nonelementary image and $\varphi : \partial\mathcal{H}_{\mathbb{C}}^p \to \partial\mathcal{H}_{\mathbb{C}}^q$ the associated L-equivariant measurable map. Then $\pi(c_q \circ \varphi) \in \mathrm{L}_{\mathrm{alt}}^\infty\big((\partial\mathcal{H}_{\mathbb{C}}^p)^3\big)^L$ is a cocycle which canonically represents $\rho_{\mathrm{b}}^{(2)}(\kappa_q^{\mathrm{b}}) \in \mathrm{H}_{\mathrm{b}}^2(L,\mathbb{R})$.*

Observe that the existence of such measurable map follows for instance from [21]. Let now, for $\xi \in \partial\mathcal{H}_{\mathbb{C}}^\ell$, e^ξ denote the exponential of the Busemann function defined in (3.5). Then we have:

Proposition 6.2. *Let G, G' be connected Lie groups with finite center and associated symmetric spaces $\mathcal{H}_{\mathbb{C}}^p$ and $\mathcal{H}_{\mathbb{C}}^q$ respectively, and let $L \leq G$ be any closed subgroup. Let $\rho : L \to G'$ be a homomorphism with* *nonelementary image and* $\varphi : \partial\mathcal{H}^p_{\mathbb{C}} \to \partial\mathcal{H}^q_{\mathbb{C}}$ *the associated* L*-equivariant measurable map. Then the differential* 2*-form*

$$\int_{(\partial\mathcal{H}^p_{\mathbb{C}})^3} c_q\big(\varphi(\xi_0),\varphi(\xi_1),\varphi(\xi_2)\big)e^{\xi_0}\wedge de^{\xi_1}\wedge de^{\xi_2}d\nu_0^3(\xi_0,\xi_1,\xi_2) \tag{6.1}$$

is a smooth L*-invariant bounded closed* 2*-form representing* $\rho^{(2)}(\kappa_q)\in \mathrm{H}^2(L,\mathbb{R})\cong \mathrm{H}^2\big(\Omega^\bullet(\mathcal{H}^p_{\mathbb{C}})^L\big)$.

Proof. By Corollary 6.1 and Lemma 3.3, (6.1) is a smooth differential 2-form in $\Omega^2_\infty(\mathcal{X})$ which is L-invariant and, by Proposition 3.1, it represents $\rho^{(2)}(\kappa_q)\in \mathrm{H}^2\big(\Omega^\bullet(\mathcal{H}^p_{\mathbb{C}})^L\big)$. □

The additional feature of the Cartan cocycle lies in the fact that it detects when three points in the boundary of hyperbolic space lie on a chain. Recall that a chain is the boundary of a complex geodesic, that is a totally geodesic holomorphically embedded copy of $\mathcal{H}^1_{\mathbb{C}}$. We refer the reader to [33] for the precise definitions, but we limit ourselves here to recall the following essential lemma:

Lemma 6.3. *The Cartan cocycle* $c_\ell : (\partial\mathcal{H}^\ell_{\mathbb{C}})^3 \to [-1,1]$ *is a strict* $\mathrm{SU}(1,\ell)$*-invariant Borel cocycle and* $|c_\ell(a,b,c)|=1$ *if and only if* a,b,c *are on a chain and pairwise distinct.*

Proof of Theorem 8. From Lemma 5.3, (5.1) and the definition of $\mathrm{T}_{\mathrm{b}}^{(2)}(\rho)$ in (2.6) we have that

$$i_\rho\kappa_1^{\mathrm{b}} = \mathrm{T}_{\mathrm{b}}^{(2)}\big(\rho_{\mathrm{b}}^{(2)}(\kappa_q^{\mathrm{b}})\big)\,. \tag{6.2}$$

Observe that $\rho(\Gamma)$ is not elementary. Indeed, otherwise $\rho(\Gamma)$ would be contained in a closed amenable subgroup in $\mathrm{PU}(1,q)$; the vanishing of the restriction of κ_q^{b} to such a subgroup would imply that $i_\rho=\mathrm{t_b}(\rho)=0$, contradicting the hypothesis that $i_\rho=1$.

Since $\partial\mathcal{H}^q_{\mathbb{C}}$ is an amenable $\mathrm{PU}(1,q)$-space, Lemma 2.1 with $S=\partial\mathcal{H}^q_{\mathbb{C}}$, Corollary 6.1 and (6.2) imply that

$$\int_{\Gamma\backslash\mathrm{SU}(1,1)} c_q\big(\varphi(g\xi),\varphi(g\eta),\varphi(g\zeta)\big)d\mu(\dot g) = i_\rho c_1(\xi,\eta,\zeta)$$

for almost every $(\xi,\eta,\zeta)\in\big(\partial\mathcal{H}^1_{\mathbb{C}}\big)^3$. Observe that we used here the fact that since Γ acts ergodically on $(\partial\mathcal{H}^1_{\mathbb{C}})^2$, then $\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathcal{H}^1_{\mathbb{C}})^2\big)^\Gamma=0$ and hence there are no coboundaries.

If $|i_\rho|=1$, since $|c_q|\le 1$, $|c_1|=1$ almost everywhere and μ is a probability measure, we have that

$$c_q\big(\varphi(\xi),\varphi(\eta),\varphi(\zeta)\big)=\pm c_1(\xi,\eta,\zeta) \tag{6.3}$$

for almost every $(\xi,\eta,\zeta)\in(\partial\mathcal{H}^1_{\mathbb{C}})^3$. Fix $\xi\neq\eta$ such that (6.3) holds for almost every $\zeta\in\partial\mathcal{H}^1_{\mathbb{C}}$. Then the essential image of φ is contained in the chain C determined by $\varphi(\xi)$ and $\varphi(\eta)$, from which readily follows that $\rho(\Gamma)$ leaves invariant the complex geodesic whose boundary is C. $\square$

Proof of Theorem 9 and Corollary 11. Let $\rho:\Gamma\to\mathrm{PU}(1,1)$ be a homomorphism with $i_\rho=1$ and let $\varphi:\mathcal{H}^1_{\mathbb{C}}\to\mathcal{H}^1_{\mathbb{C}}$ be the Γ-equivariant measurable map considered in the proof of Theorem 8. Then (6.3) holds with a positive sign and φ is weakly order preserving, so that [38, Proposition 5.5] implies that there exists a degree one monotone surjective continuous map

$$f:\partial\mathcal{H}^1_{\mathbb{C}}\to\partial\mathcal{H}^1_{\mathbb{C}}$$

such that $f(\rho(\gamma)x)=\gamma f(x)$ for all $\gamma\in\Gamma$ and all $x\in\partial\mathcal{H}^1_{\mathbb{C}}$. The surjectivity of f then implies that ρ is injective (modulo possibly the center of Γ), while its continuity that $\rho(\Gamma)$ is discrete.

According to [29], for every $x\in\partial\mathcal{H}^1_{\mathbb{C}}$, the inverse image $f^{-1}(x)$ is either a point or a connected component of $\partial\mathcal{H}^1_{\mathbb{C}}\setminus\mathcal{L}$, where $\mathcal{L}$ is the limit set of $\rho(\Gamma)$. This implies readily that

(6.4)
$$\gamma\text{ is parabolic}\Leftrightarrow\rho(\gamma)\text{ is}\begin{cases}\text{either parabolic}\\ \text{or hyperbolic, fixing the endpoints}\\ \quad\text{of a connected component of }\partial\mathcal{H}^1_{\mathbb{C}}\setminus\mathcal{L}.\end{cases}$$

Now $\rho(\Gamma)\backslash\mathcal{H}^1_{\mathbb{C}}$ is a complete hyperbolic surface of finite topological type, that is it has finite genus, finite number of expanding ends and finite number of cusps. If now $\rho(\gamma)$ is parabolic if γ is parabolic, there are no expanding ends and hence $\rho(\Gamma)$ is a lattice. In any case, if $\rho(\Gamma)$ is a lattice, it acts minimally on $\partial\mathcal{H}^1_{\mathbb{C}}$ and then f must be injective and hence a homeomorphism. This proves Theorem 9.

In order to prove Corollary 11, we observe that ρ is an isomorphism between $\Gamma=\pi_1(S)$ and $\Gamma':=\rho(\Gamma)=\pi_1(S')$, where $S:=\Gamma\backslash\partial\mathcal{H}^1_{\mathbb{C}}$ and $S':=\Gamma'\backslash\partial\mathcal{H}^1_{\mathbb{C}}$ are surfaces of finite topological type. Moreover, this isomorphism has the property – see (6.4) – that it sends boundary loops to boundary loops. It is hence induced by a diffeomorphism.

$\square$

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