

Bounded Cohomology, Boundary Maps, and Rigidity of Representations into $\text{Homeo}_+(\mathbf{S}^1)$ and $\text{SU}(1, n)$

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Abstract We define, associated to a given a representation $\pi : \Gamma \rightarrow H$ of a finitely generated group into a topological group, invariants defined in terms of bounded cohomology classes. In the case $H = \text{SU}(1, n)$ we illustrate, among others and without proof, rigidity results which generalize a theorem of Goldman and Millson ([14]). In the case $H = \text{Homeo}_+(\mathbf{S}^1)$, the group of orientation preserving homeomorphisms of the circle, we give a new complete proof of a rigidity result of Matsumoto ([17]), stating that any two representations with maximal Euler number are semiconjugate.

The methods used rely on the homological approach to continuous bounded cohomology developed in [5] and [1].

1 Introduction

A systematic theory of continuous bounded cohomology for locally compact groups using homological methods has been recently developed by Burger and Monod [5], and has proven to have far reaching and very diverse applications in rigidity theory. In this paper we shall give a few examples to illustrate how this theory can be used to obtain rigidity results for actions of finitely generated groups.

We shall define invariants associated to a representation $\pi : \Gamma \rightarrow H$, where Γ is a finitely generated group and H is a topological group, via the interplay between the pull-backs of bounded cohomology classes and of ordinary cohomology classes of H . It should be noted that the idea of considering what information one can obtain by looking at cohomology classes which admit a bounded representative as bounded cohomology classes is certainly not new (see, for example [10] or [17]). The point that we are making here is that extra information can be obtained by considering a functorial approach to (continuous) bounded cohomology.

We shall specialize the discussion to two particular cases: in the first one, $H = \text{SU}(1, n)$, and we shall simply illustrate, without proofs and in a section completely independent of the rest of the paper, some new results we can obtain; in the second, H is the group $\text{Homeo}_+(\mathbf{S}^1)$ of orientation preserving

homeomorphisms of the circle, and we shall give a new complete proof of some known results of Matsumoto, Milnor–Woods and Goldman. Roughly speaking, we shall see that in all cases the most information can be obtained when the invariants associated to the representation $\pi : \Gamma \rightarrow H$ are either maximal or zero.

1.1 $H = \mathrm{SU}(1, n)$

In the first result we want to describe, information will be obtained from the vanishing of an appropriate cohomology class, defined in terms of the Kähler form ω_m on the m -dimensional complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^m$, as follows. For any singular C^1 -simplex $\sigma : \Delta^2 \rightarrow \mathcal{H}_{\mathbb{C}}^m$, let us consider a C^1 -simplex σ^* obtained by replacing each side of σ with the corresponding geodesic segment and whose interior has been arbitrarily filled. Then, because of Stokes theorem,

$$k_b(\sigma) = \int_{\sigma^*} \omega_m \quad (1)$$

is independent of the choice of the interior of σ^* ; restricting k_b to simplices with geodesic sides and with vertices g_1x, g_2x, g_3x , where $x \in \mathcal{H}_{\mathbb{C}}^m$ is a base point and $g_i \in \mathrm{SU}(1, m)$, gives rise to a continuous bounded cocycle on $\mathrm{SU}(1, m)^3$ and defines a bounded cohomology class in $H_{\mathrm{b}}^2(\mathrm{SU}(1, n), \mathbb{R})$ which we also denote by k_b . Note that this is just a particular case of a more general construction due to Dupont, [6].

If $\pi : \Gamma \rightarrow \mathrm{SU}(1, n)$ is a homomorphism of a finitely generated group Γ , we can pull back k_b in bounded cohomology and obtain a bounded class $\pi^*(k_b) \in H_{\mathrm{b}}^2(\Gamma, \mathbb{R})$. Then vanishing of $\pi^*(k_b)$ is equivalent to the image $\pi(\Gamma)$ being small in an appropriate sense, namely:

Theorem 1.1 ([3]). *The bounded class $\pi^*(k_b)$ vanishes if and only if either $\pi(\Gamma)$ fixes a point in the boundary of $\mathcal{H}_{\mathbb{C}}^n$, or $\pi(\Gamma)$ leaves a totally real subspace of $\mathcal{H}_{\mathbb{C}}^n$ invariant.*

Corollary 1.2 ([3]). *Let Γ be a finitely generated group and $\pi : \Gamma \rightarrow \mathrm{SU}(1, n)$ a homomorphism with Zariski dense image. Then $\pi^*(k_b) \in H_{\mathrm{b}}^2(\Gamma, \mathbb{R})$ does not vanish.*

Notice that (1) defines also a singular bounded cohomology class $k_M \in H_{\mathrm{s}, \mathrm{b}}^2(M)$ for any quotient M of $\mathcal{H}_{\mathbb{C}}^m$. As a geometric counterpart of Theorem 1.1 we obtain:

Corollary 1.3 ([3]). *Let $M = \Gamma \backslash \mathcal{H}_{\mathbb{C}}^m$ be a compact arithmetic manifold, let V be a compact manifold and $f : V \rightarrow M$ a continuous map. Let $k_M \in H_{\mathrm{s}, \mathrm{b}}^2(M)$ be the singular bounded class defined by the Kähler form on M . Then $f^*(k_M) \in H_{\mathrm{s}, \mathrm{b}}^2(V)$ vanishes if and only if there exists a compact, totally real immersed submanifold $T \subset M$ such that f is homotopic to a map with image in T .*

If on the other hand we specialize Γ to be a lattice in $SU(1, m)$, then we can obtain information exactly from the opposite situation, that is, roughly speaking, from the maximality of the invariant. To illustrate this, recall that Goldman and Millson proved in [14] that any cocompact lattice $\Gamma < SU(1, m)$ has no non-trivial deformations in $SU(1, n)$, for $n > m$. On the other hand, Gusevskii and Parker constructed in [15] examples of non-cocompact lattices in $SU(1, 1)$ which have quasi-Fuchsian deformations in $SU(1, 2)$.

Recently, we extended the result in [14] to non-cocompact lattices in $SU(1, m)$ with $m > 1$, using bounded cohomology techniques. Namely, let $M = \Gamma \backslash \mathcal{H}_{\mathbb{C}}^m$ be a finite volume complex hyperbolic manifold and assume that either $m \geq 2$ or M is compact. In either case, the L^2 -cohomology $H_{(2)}^2(M)$ of M injects into $H_{\text{dR}}^2(M) \simeq H^2(\Gamma, \mathbb{R})$ (see [24]). Then if $\pi : \Gamma \rightarrow SU(1, n)$ is a representation, ω_M is the Kähler class on M and $\langle \cdot, \cdot \rangle$ is the standard inner product in $H_{(2)}^2(M)$, we prove the crucial fact that the pull-back $\pi^*(\omega_n)$ is in $H_{(2)}^2(M)$, so that it makes sense to consider the scalar product $\langle \pi^*(\omega_n), \omega_M \rangle$.

Theorem 1.4 ([2]). *Let $m \geq 2$. Then*

$$I_\pi := \left| \frac{\langle \pi^*(\omega_n), \omega_M \rangle}{\langle \omega_M, \omega_M \rangle} \right| \leq 1,$$

and equality holds if and only if π is equivariant with respect to an isometric embedding $\mathcal{H}_{\mathbb{C}}^m \hookrightarrow \mathcal{H}_{\mathbb{C}}^n$.

By purely topological methods one can see that I_π is constant on connected components of the representation variety $\text{Rep}(\Gamma, SU(1, n))$. Hence we can conclude the following:

Corollary 1.5 ([2]). *There are no non-trivial deformations of Γ in $SU(1, n)$.*

Note that the case treated by Gusevskii and Parker falls nicely out of this method of proof, as in that case $H^2(M, \mathbb{R}) = 0$.

1.2 $H = \text{Homeo}_+(\mathbf{S}^1)$

We illustrate our method by tackling a problem in which H is not necessarily a linear group. Here, as in Theorem 1.4, we shall obtain most information from the maximality of the invariant. As mentioned before, the results are not new, although additional information about the structure of Γ -equivariant boundary maps and semiconjugacies derive naturally from our proofs.

Let Γ be a discrete group and $\pi : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ a homomorphism. If $e \in H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbb{Z})$ is the Euler class (see Sect. 2), then $\pi^*(e) \in H^2(\Gamma, \mathbb{Z})$ is a class which represents the obstruction to lifting the Γ -action on \mathbf{S}^1 to its universal cover \mathbb{R} . However, since the class e has a bounded representative, we may follow Ghys and consider it as a

bounded class $e^b \in H_b^2(\text{Homeo}_+(\mathbf{S}^1), \mathbb{Z})$ with $\pi^*(e^b) \in H_b^2(\Gamma, \mathbb{Z})$ its pull-back in bounded cohomology. That this could be advantageous is illustrated in first instance by the fact that $\pi^*(e)$ is an invariant of semiconjugacy (see Sect. 5 for the definition), but it is only $\pi^*(e^b)$ which is a complete invariant (see [10, Proposition 5.2 and Théorème A, respectively]). In particular, when $\Gamma = \mathbb{Z}$, we have that $H^2(\mathbb{Z}, \mathbb{Z}) = 0$ (hence $\pi^*(e) = 0$), while $H_b^2(\mathbb{Z}, \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ and $\pi^*(e^b)$ represents the rotation number of the homeomorphism $\pi(1)$ ([10, Théorème A (3)]).

Now let Σ_g be a compact oriented surface of genus $g \geq 2$ and let Γ_g be its fundamental group. Then, associated to any homomorphism $\pi : \Gamma_g \rightarrow \text{Homeo}_+(\mathbf{S}^1)$, one can define another invariant. To this purpose, observe that, since Σ_g is a $K(\Gamma_g, 1)$ manifold, we have that $H^2(\Gamma_g, \mathbb{Z}) \simeq H^2(\Sigma_g, \mathbb{Z})$. Then the Euler number $eu(\pi)$ of the homomorphism π is defined as

$$eu(\pi) = \langle \pi^*(e), [\Sigma_g] \rangle ,$$

where $[\Sigma_g]$ is the fundamental class in $H_2(\Sigma_g, \mathbb{Z})$ and $\langle \cdot, \cdot \rangle : H^2(\Sigma_g, \mathbb{Z}) \times H_2(\Sigma_g, \mathbb{Z}) \rightarrow \mathbb{Z}$ is the usual pairing. Notice that to any hyperbolization of Σ_g corresponds a homomorphism $\pi_0 : \Gamma_g \rightarrow \text{PSL}(2, \mathbb{R}) < \text{Homeo}_+(\mathbf{S}^1)$ (to which we shall refer as *standard*) which has Euler number $eu(\pi_0)$ equal to the Euler characteristic $\chi(\Sigma_g)$ (see Corollary 3.5 for instance). For a general representation we prove the Milnor–Wood inequality:

Theorem 1.6 ([18], [22]). $|eu(\pi)| \leq |\chi(\Sigma_g)|$.

Note that all of the above cohomology classes can be viewed as cohomology classes with real coefficients. In doing so, some information is lost (which however could be recovered with appropriate long exact sequences, [9]), but now to our avail comes the homological approach to bounded continuous cohomology with real coefficients developed in [5]. The proof that we shall give of the above inequality exploits this approach and is an immediate consequence of the following:

Proposition 1.7. *Let $\pi : \Gamma_g \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ be a homomorphism. Let $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ be any Γ_g -equivariant measurable map taking values in the space $\mathcal{M}(\mathbf{S}^1)$ of probability measures on \mathbf{S}^1 , where the action on the domain is via π_0 and the one on the target via π . Then for almost every $(x, y, z) \in (\mathbf{S}^1)^3$, with respect to the product of Lebesgue measures, we have*

$$\int_{\pi_0(\Gamma_g) \backslash \text{PSL}(2, \mathbb{R})} \varphi(gx) \otimes \varphi(gy) \otimes \varphi(gz)(c) d\mu(g) = \frac{eu(\pi)}{\chi(\Sigma_g)} c(x, y, z) , \quad (2)$$

where $c : (\mathbf{S}^1)^3 \rightarrow \mathbb{Z}$ is the orientation cocycle given by

$$c(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } (x_1, x_2, x_3) \text{ are positively oriented} \\ -1 & \text{if } (x_1, x_2, x_3) \text{ are negatively oriented} \\ 0 & \text{otherwise .} \end{cases} \quad (3)$$

and μ is the $\mathrm{PSL}(2, \mathbb{R})$ -invariant probability measure on $\pi_0(\Gamma_g) \backslash \mathrm{PSL}(2, \mathbb{R})$ for a standard representation π_0 of Γ_g .

In fact, if $(x, y, z) \in (\mathbf{S}^1)^3$ is a triple of distinct points, (2) reads

$$\left| \int_{\pi_0(\Gamma_g) \backslash \mathrm{PSL}(2, \mathbb{R})} \varphi(gx) \otimes \varphi(gy) \otimes \varphi(gz)(c) d\mu(g) \right| = \left| \frac{eu(\pi)}{\chi(\Sigma_g)} \right|,$$

from which Theorem 1.6 follows immediately, by observing that φ takes values in $\mathcal{M}(\mathbf{S}^1)$, that $|c| \leq 1$ and hence $|\varphi(gx) \otimes \varphi(gy) \otimes \varphi(gz)(c)| \leq 1$, and finally that μ is a probability measure as well.

Note moreover that the orientation cocycle is closely related to the bounded Euler class e^b , in that c is boundedly cohomologous to $-2e^b$ (see Lemma 2.1).

Then, with a completely elementary argument, we can prove a result of Matsumoto:

Theorem 1.8 ([17]). *If π_0 and π are representations such that π_0 is standard and $|eu(\pi)|$ is maximal, there exists a continuous surjective semiconjugacy from π to π_0 (possibly after conjugation with an orientation reversing homeomorphism of \mathbf{S}^1). In particular, π is injective and its image in $\mathrm{Homeo}(\mathbf{S}^1)$ is discrete in the compact-open topology.*

As we shall see, an essential step in the proof is the following:

Proposition 1.9. *Let $\pi : \Gamma_g \rightarrow \mathrm{Homeo}_+(\mathbf{S}^1)$ be a homomorphism with maximal Euler number. Then every Γ_g -equivariant measurable map $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ takes values in the space of Dirac masses.*

As a straightforward consequence of the proof of Theorem 1.8, we have:

Corollary 1.10 ([13]). *Any homomorphism $\pi : \Gamma_g \rightarrow \mathrm{PSL}(2, \mathbb{R})$ with maximal Euler number gives an hyperbolization of Σ_g .*

We would like to point out that a formula completely analogous to that in Proposition 1.7 is the basic stepping stone in the proof of Theorem 1.4. Then, even in that situation, in the case when the invariant that appears there is maximal, an in-depth analysis of the properties of an appropriate boundary map will complete the proof.

We want to conclude the Introduction by illustrating a further consequence of our method. Margulis has recently proven a conjecture of Ghys which can be viewed as a substitute of the Tits alternative for the full group $\mathrm{Homeo}(\mathbf{S}^1)$ of homeomorphisms of the circle (for which it is known that the Tits alternative fails to be true, [12]). In fact, he proved that if G is a subgroup of $\mathrm{Homeo}(\mathbf{S}^1)$, either G contains a non-commutative free subgroup or there exists a G -invariant probability measure on \mathbf{S}^1 , [16].

Corollary 1.11. *Let $\pi : \Gamma_g \rightarrow \mathrm{Homeo}_+(\mathbf{S}^1)$ be as above, such that $eu(\pi) \neq 0$. Then $\pi(\Gamma_g)$ contains a non-commutative free subgroup.*

The paper is organized as follows: in Sect. 2 we recall the definition of the Euler class and give its explicit relation with the orientation cocycle; in Sect. 3 we give the cohomological proof of Proposition 1.7; in Sect. 4 we prove some properties of T_g -equivariant maps into the probability measures on \mathbf{S}^1 which can be deduced from (2) and the maximality of the Euler number; in Sect. 5 we give the definition of semiconjugacy, and prove Theorem 1.8 (called Corollary 5.8 there) and Corollary 1.10.

2 The Euler class and the orientation cocycle

Let $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ be the universal covering of the group $\text{Homeo}_+(\mathbf{S}^1)$, that is the group of homeomorphisms of the real line which commute with the integer translation $T : \mathbb{R} \rightarrow \mathbb{R}$, $T(x) = x + 1$. Given any homeomorphism $f \in \text{Homeo}_+(\mathbf{S}^1)$, there is a canonical lift $\tilde{f} \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ defined by requiring that $\tilde{f}(0) \in [0, 1)$. Hence there exists a map $\varepsilon : (\text{Homeo}_+(\mathbf{S}^1))^2 \rightarrow \mathbb{Z}$ such that

$$\widetilde{f \circ g} \circ T^{\varepsilon(f,g)} = \tilde{f} \circ \tilde{g},$$

for all $f, g \in \text{Homeo}_+(\mathbf{S}^1)$. It is easy to verify that ε defines a 2-cocycle whose cohomology class e is the classical Euler class $e \in H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbb{Z})$ associated to the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \xrightarrow{p} \text{Homeo}_+(\mathbf{S}^1) \longrightarrow 0.$$

Notice that the class e is independent of the choice of the section of the projection p . However, since the cocycle ε takes only values 0 and 1, this particular choice of section allows us to identify a bounded representative for e , that is a representative of $e^b \in H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbb{Z})$.

For our purpose, it will be convenient to think of e as represented by a homogeneous cocycle (see Theorem 3.1 and Proposition 3.2). Let $c : (\mathbf{S}^1)^3 \rightarrow \mathbb{Z}$ be the map defined in (3). It is easy to verify that c is a Borel measurable $\text{Homeo}_+(\mathbf{S}^1)$ -invariant homogeneous cocycle. Moreover, if $1 \in \mathbf{S}^1$ is the image of the origin in \mathbb{R} under the canonical projection $\mathbb{R} \rightarrow \mathbf{S}^1$, we have the following:

Lemma 2.1. *Let $f, g \in \text{Homeo}_+(\mathbf{S}^1)$. Then*

$$\varepsilon(f, g) = -\frac{1}{2}c(1, f(1), f(g(1))) + \frac{1}{2} + \frac{1}{2}(\delta_1(f(g(1))) - \delta_1(f(1)) - \delta_1(g(1))).$$

Proof. Let $\varepsilon(f, g) = -\frac{1}{2}c(1, f(1), f(g(1))) + \frac{1}{2} + \frac{1}{2}\psi(f, g)$. We first claim that

$$\text{if } 1 \notin \{f(1), g(1), f(g(1))\} \text{ then } \psi(f, g) = 0. \quad (4)$$

To see this, let us assume first that $\varepsilon(f, g) = 0$, so that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$. Since by assumption $0 < \tilde{g}(0)$, we deduce that $\tilde{f}(0) < \tilde{f}(\tilde{g}(0)) = \widetilde{f \circ g}(0) < 1$. Hence $c(1, f(1), f(g(1))) = 1$, which implies that $\psi(f, g) = 0$.

If on the other hand $\varepsilon(f, g) = 1$, then $\widetilde{f \circ g}(x) + 1 = \tilde{f}(\tilde{g}(x))$ for every $x \in \mathbb{R}$. Then,

$$1 < 1 + \widetilde{f \circ g}(0) = \tilde{f}(\tilde{g}(0)) < \tilde{f}(1) = 1 + \tilde{f}(0),$$

so that $0 < \widetilde{f \circ g}(0) < \tilde{f}(0)$. Hence $c(1, f(1), f(g(1))) = -1$, which again implies that $\psi(f, g) = 0$.

It thus follows from (4) that

$$\psi(f, g) = \alpha\delta_1(f(1)) + \beta\delta_1(g(1)) + \gamma\delta_1(f(g(1))),$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Before proceeding, observe that

$$\text{if either } f(1) = 1 \text{ or } g(1) = 1 \text{ then } \varepsilon(f, g) = 0. \quad (5)$$

In fact, if for example $f(1) = 1$, so that $\tilde{f}(1) = 1$, we have that $\tilde{f}(\tilde{g}(0)) < \tilde{f}(1) = 1$. Analogously, if $g(1) = 1$, so that $\tilde{g}(0) = 0$, we have that $\tilde{f}(\tilde{g}(0)) < \tilde{f}(0) < 1$. In either cases we have that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

Having established (5), the proof will be complete once we identify the constants α, β, γ , task for which we have to consider three distinct cases all arising from situations in which $c(1, f(1), f(g(1))) = 0$.

- $f(1) = 1$ and $g(1) \neq 1$, so that $f(g(1)) \neq 1$. From (5) we have that $\alpha = -1$;
 - $f(1) \neq 1$ and $g(1) = 1$, so that $f(g(1)) \neq 1$. As before, from (5) we deduce that $\beta = -1$;
 - $f(1) = g(1) = 1$, so that $f(g(1)) = 1$. Once again (5) implies that $\gamma = 1$.
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3 The Proof of the “Formula”

We recall here the relevant facts in bounded cohomology from [5] and [1], to which we refer for details.

Let G be a locally compact group. The continuous bounded cohomology groups $H_{\text{cb}}^*(G)$ are defined as the cohomology groups of the complex

$$0 \longrightarrow L^\infty(G)^G \xrightarrow{d} L^\infty(G^2)^G \xrightarrow{d} \dots \quad (6)$$

with the usual homogeneous coboundary operator. Unfortunately, in general the complex in (6) is rather intractable, but to our avail comes the characterization of amenable actions in terms of (relatively injective modules in) continuous bounded cohomology. Rather than giving here the definition of amenable action, we refer the reader to [23, Ch. 4] and give instead some

examples and properties that will be useful in what follows. Roughly speaking, the concept of amenable action is the fibered version of that of amenable group, and, as such, it involves fixed point properties. The basic example of amenable action is given by any homogeneous action with amenable stabilizer. So, for instance, if G is a connected simple Lie group and $P < G$ is a minimal parabolic subgroup (hence amenable), G/P is the prototype of a regular amenable G -space. In particular, the action of $\mathrm{PSL}(2, \mathbb{R})$ on \mathbf{S}^1 is amenable. Moreover, amenability of an action is preserved by restricting the action to closed subgroups. In particular, any discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ will also act amenably on \mathbf{S}^1 . The crucial role of amenable actions in continuous bounded cohomology is highlighted in the following:

Theorem 3.1 ([5, Theorem 2 and Sect. 1.6]). *If G is a locally compact second countable group and (B, ν) is a regular amenable G -space, then there is a canonical isometric isomorphism between the continuous bounded cohomology $H_{\mathrm{cb}}^*(G)$ and the cohomology of the complex*

$$0 \longrightarrow L^\infty(B)^G \xrightarrow{d} L^\infty(B^2)^G \xrightarrow{d} L^\infty(B^3)^G \xrightarrow{d} \dots$$

of the G -invariant bounded measurable cochains on B^n , or of the subcomplex

$$0 \longrightarrow L_{\mathrm{alt}}^\infty(B)^G \xrightarrow{d} L_{\mathrm{alt}}^\infty(B^2)^G \xrightarrow{d} L_{\mathrm{alt}}^\infty(B^3)^G \xrightarrow{d} \dots$$

of alternating cochains.

Now recall that any continuous homomorphism of locally compact second countable or discrete groups $\pi : \Gamma \rightarrow G_2$ induces a map in bounded continuous cohomology $\pi^* : H_{\mathrm{cb}}^*(G_2) \rightarrow H_{\mathrm{cb}}^*(\Gamma)$ which is defined in a completely functorial way (see [5, Sect. 1.5] or [1, Sect. 2]). However, whenever we are dealing with bounded cohomology classes which can be represented by bounded Borel measurable strict invariant cocycles on boundaries – as it is the case for many classes of “geometric” nature, among which certainly the Euler class – the pullback π^* can be realized in a completely canonical way via boundary maps. Recall that a measurable map $\varphi : B \rightarrow X$ from a Γ -space with an invariant measure class ν to a G_2 -space X is Γ -equivariant if $\varphi(gb) = \pi(g)\varphi(b)$ for all $g \in \Gamma$ and ν -a.e. $b \in B$.

Proposition 3.2 ([1]). *Let $\pi : \Gamma \rightarrow G_2$ be any homomorphism of discrete groups, Y a separable compact metrizable continuous G_2 -space, (B, ν) an amenable regular Γ -space, and $\varphi : B \rightarrow \mathcal{M}^1(Y)$ any measurable Γ -equivariant map. Let $c : Y^{n+1} \rightarrow E$ be a Borel measurable, G_2 -invariant, bounded cocycle, and $[c] \in H_{\mathrm{b}}^n(G_2)$ the associated cohomology class. Then*

$$(b_1, \dots, b_{n+1}) \mapsto \varphi(b_1) \otimes \dots \otimes \varphi(b_{n+1})(c)$$

defines an element in $L^\infty(B^{n+1})$ which represents the class $\pi^{(n)}([c]) \in H_{\mathrm{b}}^n(\Gamma)$.

Let $\pi_0 : \Gamma \rightarrow G$ be a homomorphism with closed image from a discrete group Γ to a unimodular locally compact second countable group G . Let (B, ν) be a regular G -space with a quasi-invariant measure (not necessarily amenable), and let $F(B^n)$ be a space of functions on B^n . We use the notation $t : F(B^n)^{\pi_0(\Gamma)} \rightarrow F(B^n)^G$ to indicate the map

$$t\psi(y_1, \dots, y_n) = \int_{\pi_0(\Gamma) \backslash G} \psi(gy_1, \dots, gy_n) d\mu(g), \quad (7)$$

where μ is the unique G -invariant probability measure on $\pi_0(\Gamma) \backslash G$. We are going to use the above notation in the case in which $F(B^n)$ is either the space $L^\infty(B^n)$ or the space $C(G^n)$ of continuous functions on the group G itself. The proof of the following lemma is a completely straightforward verification.

Lemma 3.3. *Let $\pi_0 : \Gamma \rightarrow G$ and (B, ν) be as above, and let $\kappa : L^\infty(B^n) \rightarrow C(G^n)$ be the map defined by*

$$\kappa\psi(g_1, \dots, g_n) = \int_{B^n} \psi(g_1 y_1, \dots, g_n y_n) d\nu(y_1) \dots d\nu(y_n). \quad (8)$$

Then for all $n \geq 0$ the diagram

$$\begin{array}{ccc} L^\infty(B^n)^{\pi_0(\Gamma)} & \xrightarrow{\kappa} & C(G^n)^{\pi_0(\Gamma)} \\ \downarrow t & & \downarrow t \\ L^\infty(B^n)^G & \xrightarrow{\kappa} & C(G^n)^G \end{array} \quad (9)$$

commutes. \square

Note now that if (B, ν) is in particular an amenable regular G -space, then by Theorem 3.1 the resolutions $L^\infty(B^*)^{\pi_0(\Gamma)}$ and $L^\infty(B^*)^G$ can be used to compute respectively the bounded cohomology of Γ and the bounded continuous cohomology of G . We denote by t_b^* the map induced by t in cohomology. Moreover, the continuous cohomology of G can be computed as the cohomology of the complex

$$0 \longrightarrow C(G)^G \longrightarrow C(G^2)^G \longrightarrow \dots,$$

and the cohomology of Γ as the cohomology of the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow C(G)^\Gamma \longrightarrow C(G^2)^\Gamma \longrightarrow \dots$$

(since G is a locally compact space on which Γ acts properly discontinuously and with paracompact quotient, see [19]), both with the usual homogeneous coboundary operator. Let t^* be the map induced in cohomology by t and note that the map defined in (8) induces the natural comparison map from bounded (continuous) cohomology to ordinary (continuous) cohomology. Then we have:

Corollary 3.4. *Let $\pi_0 : \Gamma \rightarrow G$ be as in Lemma 3.3 and let $\pi : \Gamma \rightarrow G_2$ be a homomorphism of discrete groups. Then, with the above definitions, the diagram*

$$\begin{array}{ccc} \mathrm{H}_b^*(G_2) & \xrightarrow{\kappa} & \mathrm{H}^*(G_2) \\ \pi^* \downarrow & & \downarrow \pi^* \\ \mathrm{H}_b^*(\Gamma) & \xrightarrow{\kappa} & \mathrm{H}^*(\Gamma) \\ t_b^* \downarrow & & \downarrow t^* \\ \mathrm{H}_{cb}^*(G) & \xrightarrow{\kappa} & \mathrm{H}_c^*(G) \end{array} \quad (10)$$

commutes.

Proof. The commutativity of the bottom square follows from Lemma 3.3 and the observations following it. The commutativity of the upper square diagram follows from standard homological properties in ordinary cohomology for which we refer the reader to [5, Sect. 1.5]. \square

Now we are ready to prove the formula in (2).

Proof of Proposition 1.7. Let $\Gamma = \Gamma_g$, $G = \mathrm{PSL}(2, \mathbb{R})$, $\pi_0 : \Gamma \rightarrow G$ a standard representation and $B = \mathbf{S}^1$ with the Lebesgue measure. Let G_2 be the group of orientation preserving homeomorphisms of the circle with the discrete topology. In this setting we shall use the commutativity of the diagram in (10) in degree 2. We redisplay it here adding some information that soon will be justified.

$$\begin{array}{ccccc} \mathrm{H}_b^2(G_2) & \xrightarrow{\kappa} & \mathrm{H}^2(G_2) & & (11) \\ \pi^{(2)} \downarrow & & \downarrow \pi^{(2)} & & \\ \mathrm{H}_b^2(\Gamma) & \xrightarrow{\kappa} & \mathrm{H}^2(\Gamma) & \xrightarrow{\cong} & \mathrm{H}_{dR}^2(\mathcal{H}_{\mathbb{R}}^2/\pi_0(\Gamma)) = \mathbb{R} \cdot \omega_0 \\ t_b^{(2)} \downarrow & & \downarrow t^{(2)} & & \uparrow \mathrm{Res}^{(2)} \\ \mathbb{R} \cdot c = \mathrm{H}_{cb}^2(G) & \xrightarrow{\cong} & \mathrm{H}_c^2(G) & \xrightarrow{=} & \Omega^2(\mathcal{H}_{\mathbb{R}}^2)^G = \mathbb{R} \cdot \omega \end{array}$$

Let us first of all make the essential observation that since $G = \mathrm{PSL}(2, \mathbb{R})$, both $\mathrm{H}_{cb}^2(G)$ and $\mathrm{H}_c^2(G)$ are one-dimensional. This can be achieved in a variety of ways, using any two of the following facts:

1. $\mathrm{H}_{cb}^2(G) = \{\alpha \in L_{alt}^\infty((\mathbf{S}^1)^3)^G : d\alpha = 0\}$, where $L_{alt}^\infty((\mathbf{S}^1)^3)$ is the space of L^∞ alternating functions on $(\mathbf{S}^1)^3$, [5]. This can be easily seen to follow from Theorem 3.1 by observing that because of the ergodicity of G on $\mathbf{S}^1 \times \mathbf{S}^1$, there are no L^∞ alternating G -invariant functions on $(\mathbf{S}^1)^2$. In particular, since the orientation cocycle defined in (3) is the unique, up to constants, measurable alternating G -invariant bounded cocycle on $(\mathbf{S}^1)^3$, then $\mathrm{H}_{cb}^2(G) = \mathbb{R} \cdot c$;

2. $H_c^2(G) = \Omega^2(\mathcal{H}_{\mathbb{R}}^2)^G = \mathbb{R} \cdot \omega$, where the first equality is due to Van Est [21], with $\Omega^2(\mathcal{H}_{\mathbb{R}}^2)$ the space of G -invariant two-forms on the upper half space, and where ω is the G -invariant volume form on $\mathcal{H}_{\mathbb{R}}^2$;
3. The comparison map $\kappa : H_{\text{cb}}^2(G) \rightarrow H_c^2(G)$ is an isomorphism [4] sending the orientation cocycle c to the multiple ω/π of the volume form, [7].

Let $-c/2 \in L^\infty((\mathbf{S}^1)^3)$ be the cocycle representing the Euler class $e_{\mathbb{R}}^b \in H_b^2(G_2)$ (see Lemma 2.1). We repeat once more that c is a bounded Borel measurable G_2 -invariant cocycle to emphasize that we can apply Proposition 3.2; from this, from (7) with $F(B^3) = L^\infty((\mathbf{S}^1)^3)$, and the invariance properties of φ and c , we deduce that on the left hand side of the diagram, $t_b^{(2)} \pi^{(2)}(e_{\mathbb{R}}^b) \in H_{\text{cb}}^2(G)$ is represented by the cocycle

$$(x_1, x_2, x_3) \mapsto -\frac{1}{2} \int_{\pi_0(\Gamma) \backslash G} \varphi(x_1) \otimes \varphi(x_2) \otimes \varphi(x_3)(c) d\mu(g) ;$$

since $H_{\text{cb}}^2(G) = \mathbb{R} \cdot c$, there exists $\rho \in \mathbb{R}$ such that for almost every $(x_1, x_2, x_3) \in (\mathbf{S}^1)^3$, we have

$$\int_{\pi_0(\Gamma) \backslash G} \varphi(x_1) \otimes \varphi(x_2) \otimes \varphi(x_3)(c) d\mu(g) = \rho c(x_1, x_2, x_3) .$$

To conclude the proof of the desired formula we need to identify ρ , which will be done using the information obtained from the right hand side of the diagram.

To start, observe the following:

Claim. The map $t^{(2)} : H^2(\Gamma) \rightarrow H_c^2(G)$ is an isomorphism with inverse given by the restriction map $\text{Res}^{(2)}$.

To see this, since $t \circ \text{Res} = \text{Id}$, it is enough to observe that $H^2(\Gamma)$ and $H_c^2(G)$ are one-dimensional. This follows from the fact that if ω_0 is the projection of ω to $\mathcal{H}_{\mathbb{R}}^2/\pi_0(\Gamma)$, then $H^2(\Gamma) \simeq H_{\text{dR}}^2(\mathcal{H}_{\mathbb{R}}^2/\pi_0(\Gamma)) = \mathbb{R} \cdot \omega_0$.

Denote now by $e_{\mathbb{R}}$ the image in $H^2(G_2)$ of $e_{\mathbb{R}}^b \in H_b^2(G_2)$. It follows then from the commutativity of the diagram (11) and the above Claim, that in $H^2(\Gamma)$

$$\pi^{(2)}(e_{\mathbb{R}}) = \text{Res}^{(2)}(-\kappa(\rho \cdot c)) = \text{Res}^{(2)}\left(-\frac{\rho}{2\pi}\omega\right) = -\frac{\rho}{2\pi}\omega_0 .$$

Summarizing, we have

$$\begin{array}{ccc}
-\frac{1}{2}c \in e_{\mathbb{R}}^b & \xrightarrow{\kappa} & e_{\mathbb{R}} \\
\pi^{(2)} \downarrow & & \downarrow \pi^{(2)} \\
(x_1, x_2, x_3) \mapsto & & \\
-\frac{1}{2}\varphi(x_1) \otimes \varphi(x_2) \otimes \varphi(x_3)(c) & \xrightarrow{\kappa} & \pi^{(2)}(e_{\mathbb{R}}) = \text{Res}^{(2)}\left(-\frac{\rho}{2\pi}\omega\right) \\
t_b^{(2)} \downarrow & & \uparrow \text{(2) Res} \\
(x_1, x_2, x_3) \mapsto & & \\
-\frac{1}{2}\int \varphi(x_1) \otimes \varphi(x_2) \otimes \varphi(x_3)(c)d\mu(g) & \xrightarrow{\kappa} & -\frac{\rho}{2\pi}\omega \\
= -\frac{\rho}{2}c(x_1, x_2, x_3) & &
\end{array}$$

As observed already, since Σ_g is a $K(\Gamma, 1)$ -manifold, we have that $H^2(\Gamma) \simeq H^2(\Sigma_g)$. If $[\Sigma_g] \in H_2(\Sigma_g)$ is the fundamental class, the above formula, the definition of Euler number and Gauss–Bonnet’s theorem imply that

$$\begin{aligned}
eu(\pi) &= \langle \pi^{(2)}(e_{\mathbb{R}}), [\Sigma_g] \rangle = -\frac{\rho}{2\pi} \langle \omega_0, [\Sigma_g] \rangle = -\frac{\rho}{2\pi} \int_{\Sigma_g} \omega_0 \\
&= -\frac{\rho}{2\pi} 4\pi(g-1) = -2\rho(g-1) = \rho\chi(\Sigma_g),
\end{aligned}$$

from which $\rho = eu(\pi)/\chi(\Sigma_g)$, as desired. \square

It is worthwhile noticing that part of the usefulness of the formula (2) lies in the fact that it holds for every Γ -equivariant map, as illustrated for instance by the proof of Lemma 4.2 in the next section, or by the following well known:

Corollary 3.5. *If π_0 is a standard representation, then $|eu(\pi_0)| = |\chi(\Sigma_g)|$.*

Proof. It follows immediately from (2) and the observation that one can take φ to be the identity. \square

4 Γ -Equivariant Measurable Maps into $\mathcal{M}(\mathbf{S}^1)$

Recall that a group Γ acts doubly ergodically (or mixing) on a standard measure space B with an invariant measure class ν if its diagonal action on the product space $(B \times B, \nu \times \nu)$ is ergodic. The following result is due to Ghys.

Proposition 4.1 ([11, Proposition 4.2]). *Let Γ be a group acting doubly ergodically on a standard measure space B with a Γ -invariant measure class ν . Let $\pi : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ be a homomorphism and $\varphi : B \rightarrow \mathcal{M}(\mathbf{S}^1)$ a Γ -equivariant map. Then either:*

1. φ is essentially constant and hence there exists a $\pi(\Gamma)$ -invariant probability measure on \mathbf{S}^1 , or;
2. for almost every $x \in \mathbf{S}^1$, $\varphi(x)$ has atoms and hence there exists an integer $k \geq 1$ and a Γ -equivariant measurable map $\varphi_{(k)} : B \rightarrow S_k^1$, where Γ acts diagonally via π on the space S_k^1 of subsets of \mathbf{S}^1 consisting of k points.

The existence of such a Γ -equivariant map follows from the amenability of the action of Γ on B , [8]. In turn, the existence of such an amenable doubly ergodic Γ -space B for any finitely generated group Γ was proven by Burger and Monod in [5].

The proof of Proposition 4.1 consists of exploiting the two cases when for almost every $x \in \mathbf{S}^1$, either (1) $\varphi(x)$ has no atoms, or (2) $\varphi(x)$ has atoms. In the first case, one can first show that the map $x \mapsto \text{supp } \varphi(x)$ is itself essentially constant and then that φ is essentially constant. In the latter case, the Γ -equivariance of φ and the ergodicity of the Γ -action on \mathbf{S}^1 gives a “stratification” of the atomic part of the image of φ and allows to define a new map φ' whose image consists of atoms of appropriate size.

Note that the two conclusions of Proposition 4.1 are not mutually exclusive, as one could certainly have an essentially constant map $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}_k^1$. However, the next Lemma shows that under the hypothesis that the Euler number of the homomorphism is not zero, then the first conclusion of Proposition 4.1 cannot be satisfied. Then, under the stronger condition that the Euler number is maximal, we shall prove that a map which we know already has values into k -tuples of points in \mathbf{S}^1 has actually values into \mathbf{S}^1 (Lemma 4.3), and furthermore that this is true for any Γ -equivariant map (Corollary 4.4).

For the remainder of this section, $\pi : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ will be a homomorphism of a surface group $\Gamma = \pi_1(\Sigma_g)$.

Lemma 4.2. *If $eu(\pi) \neq 0$, then for any Γ -equivariant map $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$, $\varphi(x)$ is purely atomic for almost every $x \in \mathbf{S}^1$.*

Proof. We begin with the following:

Claim. There is no $\pi(\Gamma)$ -invariant probability measure on \mathbf{S}^1 .

By contradiction denote by m such a $\pi(\Gamma)$ -invariant measure, so that the map $\psi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ defined by $\psi(x) = m$ will be Γ -equivariant. Then, if $(x_1, x_2, x_3) \in (\mathbf{S}^1)^3$ is a triple with pairwise distinct coordinates, and since the Euler cocycle is alternating, (2) implies that

$$\begin{aligned} & \int_{\pi_0(\Gamma) \backslash G} \psi(gx_1) \otimes \psi(gx_2) \otimes \psi(gx_3)(c) d\mu(g) \\ &= \int_{(\mathbf{S}^1)^3} c(x_1, x_2, x_3) dm(x_1) dm(x_2) dm(x_3) = 0, \end{aligned}$$

contradicting the fact that the Euler number is non-zero.

Now let $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ be any Γ -equivariant map and let $\varphi(x) = \varphi_{\text{cont}}(x) + \varphi_{\text{atom}}(x)$ be its decomposition into the atomic part and the continuous part. Since φ is Γ -equivariant, so are also φ_{cont} and φ_{atom} . However, since $\varphi_{\text{cont}}(x)$ has not atoms for almost every $x \in \mathbf{S}^1$, Proposition 4.1 implies that φ_{cont} must be $\pi(\Gamma)$ -invariant, which contradicts the above Claim. Hence, for almost every $x \in \mathbf{S}^1$, $\varphi(x) = \varphi_{\text{atom}}(x)$. \square

Proof of Corollary 1.11. It is immediate from the above Claim and [16]. \square

Lemma 4.3. *Assume now that $|eu(\pi)| = |\chi(\Sigma_g)|$. If $\varphi : \mathbf{S}^1 \rightarrow S_k^1$ is a Γ -equivariant map, then $k = 1$.*

We postpone the proof to first show the following:

Corollary 4.4. *Let $\pi : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ be a homomorphism with maximal Euler number. Then every Γ -equivariant measurable map $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ takes values in the Dirac measures, and can hence be identified with a map $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$.*

Proof. The argument is the same as in the proof of Proposition 4.1 (2).

From Lemma 4.2 we know that $\varphi(x) = \varphi_{\text{atom}}(x)$ for almost all $x \in \mathbf{S}^1$. The idea is to look at the stratification of the image of $\varphi(x)$ given by the size of the atoms and then conclude that, on each stratum we can have only one atom (Lemma 4.3).

For $\alpha \in (0, 1]$, and $m \in \mathcal{M}(\mathbf{S}^1)$, let $N(m, \alpha) = |\{\xi \in \mathbf{S}^1 : m(\{\xi\}) \geq \alpha\}| < \infty$. The map $N : \mathbf{S}^1 \rightarrow \mathbb{N}$ defined by $N(x) = N(\varphi(x), \alpha)$ is $\pi_0(\Gamma)$ -invariant and hence, by ergodicity of the $\pi_0(\Gamma)$ -action on \mathbf{S}^1 , essentially constant, say $N(\varphi(x), \alpha) = N_\alpha$ for almost all $x \in \mathbf{S}^1$.

Since $\varphi(x)$ is atomic, the set $A = \{\alpha \in (0, 1] : N_\alpha \neq 0\}$ is not empty. If $\alpha_0 \in A$, then for all $\alpha \leq \alpha_0$ we have $N_\alpha \geq N_{\alpha_0}$. For any such α define $\varphi_\alpha : \mathbf{S}^1 \rightarrow \mathbf{S}_{N_\alpha}^1$ to be the map which associates to a point $x \in \mathbf{S}^1$ the N_α atoms of $\varphi(x)$ of measure greater than or equal to α . It follows from Lemma 4.3 that $N_\alpha = 1$ for every $\alpha \leq \alpha_0$, hence proving the assertion. \square

We begin the proof of Lemma 4.3 with the following easy observation which exploits the fact that, given four points on \mathbf{S}^1 , the orientation cocycle gives information on their mutual ordering.

Lemma 4.5. *Let X, Y, Z be finite subsets of \mathbf{S}^1 which are pairwise disjoint and such that for all $x \in X$, $y \in Y$ and $z \in Z$, $c(x, y, z) = 1$. Then $X \cup Y$ is contained in a connected component of $\mathbf{S}^1 \setminus Z$.*

Proof. The assert is trivially true if $|Z| = 1$, so we may as well assume that $|Z| \geq 2$.

For any pair of points $z_1, z_2 \in Z$, and for any $x \in X$ and $y \in Y$, the cycle relation for c applied to (z_1, x, y, z_2) reads

$$c(x, y, z_2) - c(z_1, y, z_2) + c(z_1, x, z_2) - c(z_1, x, y) = 0,$$

from which, since by hypothesis $c(x, y, z_2) = c(x, y, z_1)$, we deduce that $c(z_1, x, z_2) = c(z_1, y, z_2)$, or, equivalently, that x, y are in the same connected component of $\mathbf{S}^1 \setminus \{z_1, z_2\}$. Since this holds for all $x \in X$ and $y \in Y$, the Lemma is proven. \square

Proof of Lemma 4.3. From Proposition 4.1 and the fact that $\varphi(x)$ is purely atomic for almost every $x \in \mathbf{S}^1$, it follows that for almost every $x_1, x_2, x_3 \in \mathbf{S}^1$ we have

$$c(x_1, x_2, x_3) = c(\xi_1, \xi_2, \xi_3) \text{ for all } \xi_i \in \text{supp } \varphi(x_i). \quad (12)$$

By Fubini's theorem, there exists $x_3 \in \mathbf{S}^1$ and $E \subset \mathbf{S}^1 \setminus \{x_3\}$ of full measure, such that for all $x_1, x_2 \in E$ with $x_1 \neq x_2$, (12) holds.

Observe moreover that for all $x_1, x_2 \in E$ with $x_1 \neq x_2$, we have $\varphi(x_i) \cap \varphi(x_j) = \emptyset$ for $1 \leq i \neq j \leq 3$. In fact, if this were not the case and $\xi \in \varphi(x_i) \cap \varphi(x_j)$, then for any $\xi_\ell \in \varphi(x_\ell)$ (where $\{i, j, \ell\} = \{1, 2, 3\}$), we would have $1 = |c(x_i, x_j, x_\ell)| = |c(\xi, \xi, \xi_\ell)| = 0$.

Now we can apply Lemma 4.5 to the sets $\varphi(x_1), \varphi(x_2)$ and $\varphi(x_3)$ to deduce that there exists a connected component I of $\mathbf{S}^1 \setminus \varphi(x_3)$ which contains $\varphi(x)$ for all $x \in E$. Note that I is an interval in \mathbf{S}^1 which inherits a total ordering from the orientation on \mathbf{S}^1 : since for all $x \in E$, $\varphi(x) \subset I$, let $I_x \subset I$ be the smallest interval containing $\varphi(x)$. Notice that if $|\varphi(x)| \geq 2$ the corresponding intervals have positive Lebesgue measure. Since by the previous paragraph the I_x 's are disjoint, and since $\lambda(I) < \infty$, it follows that $|\varphi(x)| \geq 2$ only for a countable number of $x \in \mathbf{S}^1$. Hence for almost every $x \in \mathbf{S}^1$, $|\varphi(x)| = 1$. \square

5 Semiconjugacy and the Proofs of Matsumoto's and Goldman's Theorems

Definition 5.1 ([10]). A map $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is a *degree one monotone map* if there exists $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ such that:

- $\tilde{\varphi}$ covers φ ;
- $\tilde{\varphi}(t+1) = \tilde{\varphi}(t) + 1$ for all $t \in \mathbb{R}$;
- $\tilde{\varphi}$ is monotone.

Remark 5.2. In general a degree one monotone map φ need not be continuous or injective. However, given any degree one monotone map $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$, one can define a map $\varphi^* : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ which is the “inverse” of φ in a sense which will be clear later. Namely, let $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ be the lift of φ such that $\tilde{\varphi}(x+1) = \tilde{\varphi}(x) + 1$. Define a map $\tilde{\varphi}^* : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\tilde{\varphi}^*(s) = \sup\{t \in \mathbb{R} : \tilde{\varphi}(t) \leq s\}.$$

Then $\tilde{\varphi}^*$ is monotone, satisfies $\tilde{\varphi}^*(s+1) = \tilde{\varphi}^*(s) + 1$ and descends to a map $\varphi^* : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ which is degree one monotone (see [10]). We say that φ^* is the *inverse* of φ .

The following definition, due to Ghys, is not standard, but it turns out to be the appropriate one in this context.

Definition 5.3 ([10]). Let $\pi_i : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$, for $i = 1, 2$ be homomorphisms. We say that π_1 is *semiconjugate* to π_2 if there exists a degree one monotone map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ such that, for all $\gamma \in \Gamma$, $f \circ \pi_1(\gamma) = \pi_2(\gamma) \circ f$. The map f is then a *semiconjugacy* from π_1 to π_2 .

It is clear now what is the meaning of “inverse” for a degree one monotone map, as defined in Remark 5.2: φ is a semiconjugacy from π_1 to π_2 if and only if φ^* is a semiconjugacy from π_2 to π_1 , that is semiconjugacy is an equivalence relation on the set of homomorphisms of Γ into $\text{Homeo}_+(\mathbf{S}^1)$.

Definition 5.4. • Let $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ be a measurable map:

- φ is *order preserving (reversing)* if for almost every $(x_1, x_2, x_3) \in (\mathbf{S}^1)^3$, the triples (x_1, x_2, x_3) and $(\varphi(x_1), \varphi(x_2), \varphi(x_3))$ have the same (opposite) orientation;
- φ is *weakly order preserving* if for almost every $(x_1, x_2, x_3) \in (\mathbf{S}^1)^3$, either the triples (x_1, x_2, x_3) and $(\varphi(x_1), \varphi(x_2), \varphi(x_3))$ have the same orientation or some of the points in the triple $(\varphi(x_1), \varphi(x_2), \varphi(x_3))$ coincide.
- If $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is a continuous map, we say that φ is *order preserving* or *weakly order preserving* if the above statements hold everywhere.

We shall use the following notation. If $x_1, x_2 \in \mathbf{S}^1$ are distinct points, we denote by (x_1, x_2) the set of points $x \in \mathbf{S}^1$ such that the triple (x_1, x, x_2) is positively oriented, and we refer to it as to the *open interval* in \mathbf{S}^1 with beginning and endpoint x_1 and x_2 respectively.

Proposition 5.5. *Let $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ be a measurable order-preserving map. Then there exists a degree one monotone map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ with the following properties:*

- f is continuous;
- f is surjective;
- the inverse of f (in the sense of Remark 5.2) coincides with φ almost everywhere.

Proof. Let λ be the Lebesgue measure on \mathbf{S}^1 . If $\Phi : \mathbf{S}^1 \rightarrow \mathbf{S}^1 \times \mathbf{S}^1$ is defined by $\Phi(x) = (x, \varphi(x))$, let $\nu = \Phi_*(\lambda)$, that is $\nu(A) = \lambda(\Phi^{-1}(A)) = \lambda\{x \in \mathbf{S}^1 : (x, \varphi(x)) \in A\}$. Define $F = \text{supp}(\nu) \subset \mathbf{S}^1 \times \mathbf{S}^1$, that is the essential graph of φ . Since λ has no atoms, it is immediate that F has no isolated points. Moreover, F has the property that all almost everywhere statements for the function φ , hold in fact everywhere on F . In particular:

Lemma 5.6. *If $(x_i, \xi_i) \in F$, $i = 1, 2, 3$ are points with pairwise distinct first coordinates, then:*

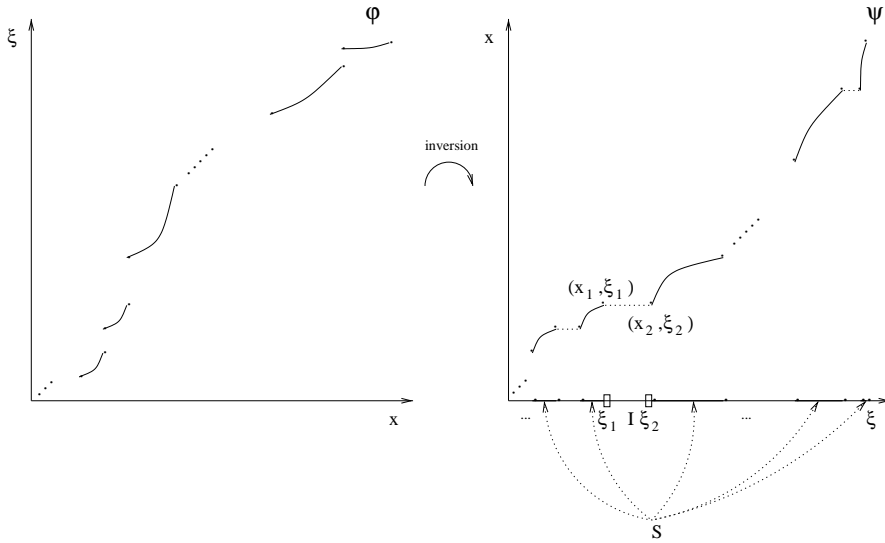
1. the ξ_i 's are pairwise distinct, and
2. the triples (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3) have the same orientation.

This Lemma is an essential stepping stone in the construction of the degree one monotone map f . However its proof is rather technical and will be postponed until later.

Let $p_i : \mathbf{S}^1 \times \mathbf{S}^1 \rightarrow \mathbf{S}^1$ be the projection onto the i -th component, $i = 1, 2$, and let $S = p_2(F)$. Then S is a closed subset of \mathbf{S}^1 and, because of Lemma 5.6(1), F is the graph of a continuous function ψ on S , namely

$$F = \{(\psi(\xi), \xi) : \xi \in S\} .$$

We want to extend now ψ to be defined on the whole of \mathbf{S}^1 . Since S is closed in \mathbf{S}^1 , its complement is a union of open intervals. Let $I = [\xi_1, \xi_2]$ be the closure of one of the open intervals which forms a connected component in $\mathbf{S}^1 \setminus S$. Since $\xi_1, \xi_2 \in S$, there exist $x_1, x_2 \in \mathbf{S}^1$ such that $(x_1, \xi_1) \in F$ and $(x_2, \xi_2) \in F$. We claim that we necessarily have that $x_1 = x_2$. In fact, if this were not the case, we could choose $(t, s) \in F$ such that (x_1, t, x_2) is positively oriented. Since by Lemma 5.6, (ξ_1, s, ξ_2) is also positively oriented, this would imply that $s \in I^\circ$ (the interior of I), contradicting the fact that $I^\circ \subset \mathbf{S}^1 \setminus S$.



With this in hand, we can now extend ψ to \mathbf{S}^1 , namely defining $\hat{\psi} : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ to be

$$\hat{\psi}(\xi) = \begin{cases} \psi(\xi) & \text{if } \xi \in S \\ x_1 & \text{if } \xi \in I \subset \mathbf{S}^1 \setminus S . \end{cases}$$

We shall prove now that the function $\hat{\psi}$ as defined is continuous on \mathbf{S}^1 , by showing that $\hat{F} = \text{graph}(\hat{\psi})$ is closed in $\mathbf{S}^1 \times \mathbf{S}^1$. Observe that either $\mathbf{S}^1 \setminus S$ consists of a finite union of intervals, or of a countable union of open intervals I_n with $\lambda(I_n) \rightarrow 0$ as $n \rightarrow \infty$. In any case we have that

$$\hat{F} = \cup_{n=1}^{\infty} (\{x_n\} \times I_n) \cup F ,$$

where $\hat{\psi}|_{I_n} = x_n$ and with the convention that all but finitely many I_n 's might be empty. Let $\{p_j\} \in \hat{F}$ be a sequence and assume that $p_j \rightarrow p$ as $j \rightarrow \infty$. Since F is closed, we may assume that $\{p_j\} \subset \cup_{n=1}^{\infty} (\{x_n\} \times I_n)$. If there exists $J > 0$ and $\bar{n} > 0$ such that for all $j \geq J$, $p_j \in \{x_{\bar{n}}\} \times I_{\bar{n}}$, then $p \in \{x_{\bar{n}}\} \times I_{\bar{n}} \subset \hat{F}$. Hence, we may assume that each $\{x_n\} \times I_n$ contains only finitely many points of the sequence. Passing to a subsequence, we may finally assume that $p_n = (x_n, \xi_n) \in \{x_n\} \times I_n$. Let $I_n = [\xi_1^n, \xi_2^n]$; since $\lambda(I_n) \rightarrow 0$ as $n \rightarrow \infty$, we have that as $n \rightarrow \infty$ both $d(\xi_n, \xi_1^n) \rightarrow 0$ and $d(\xi_n, \xi_2^n) \rightarrow 0$, that is $p \in F$ and \hat{F} is closed.

Observe now that $\hat{\psi}$ is weakly order preserving. To see this, consider a triple $(\eta_1, \eta_2, \eta_3) \in (\mathbf{S}^1)^3$ and assume that no two points among the η_i 's, $i = 1, 2, 3$ belong to the same interval I_n , so that the points $\hat{\psi}(\eta_1)$, $\hat{\psi}(\eta_2)$ and $\hat{\psi}(\eta_3)$ are all distinct. By construction, we have that if, for $i = 1, 2, 3$, $y_i \in S$ are such that $\hat{\psi}(\eta_i) = y_i$, then $\eta_i = \varphi(y_i)$ and hence the fact that φ is order preserving implies that $\hat{\psi}$ is order preserving on S and weakly order preserving on $\mathbf{S}^1 \setminus S$.

Furthermore, $\hat{\psi} : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is surjective, since, by construction, it takes every value in $p_1(F) = \mathbf{S}^1$.

Let $\tilde{\psi} : \mathbb{R} \rightarrow \mathbb{R}$ be a lift of $\hat{\psi} : \mathbf{S}^1 \rightarrow \mathbf{S}^1$, that is a continuous map which covers $\hat{\psi}$. Since $\hat{\psi}$ is weakly order preserving, it is immediate that $\tilde{\psi}$ is monotone increasing. Since $\tilde{\psi}$ covers $\hat{\psi}$, we have that for some $d \in \mathbb{N}$, $d \geq 1$, $\tilde{\psi}(t+1) = \tilde{\psi}(t) + d$ for all $t \in \mathbb{R}$.

We shall now prove that $d = 1$, hence showing that $\hat{\psi}$ is a degree one monotone map. The idea is to show that if $d > 1$, then we could wrap around $\hat{\psi}$ fast enough to contradict the fact that $\hat{\psi}$ is weakly order preserving.

Suppose, by contradiction, that $d > 1$ and suppose, for simplicity, that $\hat{\psi}(0) = 0$, if necessary composing with a rotation. Choose $t_1 \in (0, 1)$ such that

$$0 < \tilde{\psi}(t_1) = s_1 < 1$$

(this exists since $\tilde{\psi}$ is surjective and continuous). Then choose $s_2 \in (0, s_1)$ and $t_2 \in (t_1, 1)$ such that for some integer d' , with $1 \leq d' \leq d - 1$, we have that

$$\tilde{\psi}(t_2) = s_2 + d' .$$

Note that this is possible since $\tilde{\psi}(t_1) = s_1$, $\tilde{\psi}(1) = d$ and

$$s_1 < \tilde{\psi}(t_2) = s_2 + d' < s_1 + d - 1 < d .$$

Let $p : \mathbb{R} \rightarrow \mathbf{S}^1$ be the natural projection. Since $0 < t_1, t_2 < 1$, the triple $(p(0), p(t_1), p(t_2))$ is positively oriented, while $(p(\tilde{\psi}(0)), p(\tilde{\psi}(t_1)), p(\tilde{\psi}(t_2))) = (p(0), p(s_1), p(s_2))$ is negatively oriented, contrary to the assumption that $\hat{\psi}$ is weakly order preserving. \square

The last missing step is the:

Proof of Lemma 5.6. (1) Suppose by contradiction that there exist $(x_1, \xi) \in F$ and $(x_2, \xi) \in F$ such that $x_1 \neq x_2$. Let us define the intervals $I_j = (x_j - \epsilon, x_j + \epsilon)$ and the corresponding half open intervals $I_j^- = (x_j - \epsilon, x_j]$ and $I_j^+ = [x_j, x_j + \epsilon)$ containing x_j . Let $V = (\xi_1, \xi_2)$ be an open interval containing ξ . We shall use the following notation: if I, V are any intervals, then

$$I(V) = \{x \in I : \varphi(x) \in V\}.$$

Since $(x_j, \xi) \in F$ then, by definition, $\nu(I_j \times V) > 0$ for $j = 1, 2$. However, to prove the assertion, we need a more attentive examination of which parts of the intervals have actually positive measure. To this purpose, define $V^- = (\xi_1, \xi]$ and $V^+ = [\xi, \xi_2)$.

Claim. $\nu(I_j^- \times V^+) = 0$ and $\nu(I_j^+ \times V^-) = 0$.

We show only the first equality, the second being completely analogous. If $\nu(I_j^- \times V^+) > 0$, then we could choose open intervals $V'' \subset V' \subset V^+$ containing ξ such that $\nu(I_j^- \times V'')\nu(I_j^- \times (V' \setminus V''))\nu(I_j^- \times (V \setminus V')) > 0$. But then $I_j^-(V'') \times I_j^-(V' \setminus V'') \times I_j^-(V \setminus V')$ would be a set of positive measure consisting of positively oriented triples, which is sent by φ into the set $V'' \times (V' \setminus V'') \times (V \setminus V')$ consisting of negatively oriented triples. Hence the assert of the Claim.

Observe that, as a consequence of this Claim, there are only two possibilities: for $\delta \in \{-, +\}$, either

- (i) $\nu(I_1^\delta \times V^\delta)\nu(I_2^{-\delta} \times V^{-\delta}) > 0$, or
- (ii) $\nu(I_1^\delta \times V^\delta)\nu(I_2^\delta \times V^{-\delta}) > 0$.

Although the proofs are very similar, we are going to have to deal with the two cases separately. Observe however, that in both cases, because φ is order preserving, it cannot be constant on any interval, in particular on the interval $[x_1, x_2]$. Hence there exists $y \in (x_1, x_2)$ such that $(y, \eta) \in F$ for some $\eta \neq \xi$. Choose open intervals J and W containing y and η respectively and disjoint from I_j and V respectively. By hypothesis $\nu(J \times W) > 0$ and hence $\lambda(J(W)) > 0$.

(i) Suppose $\delta = -$. Then $I_1^-(V^-) \times J(W) \times I_2^+(V^+)$ is a set of positive measure consisting of positively oriented triples which is sent by φ into the set $V^- \times W \times V^+$ consisting of negatively oriented triples.

The case $\delta = +$ is treated analogously, but with the point y chosen in the interval (x_2, x_1) (on which, as observed already, the function φ cannot be constant).

(ii) Suppose $\delta = -$. Choose an open interval $V' \subset V^-$ such that $\nu(I_2^- \times V')\nu(I_1^- \times (V \setminus V')) > 0$. Then the set $I_1^-(V \setminus V') \times J(W) \times I_2^-(V')$ is a set of positive measure consisting of positively oriented triples which is sent by φ into the set $(V \setminus V') \times W \times V'$ consisting of negatively oriented triples.

The case $\delta = +$ is treated completely analogously, but with the open interval V' chosen in V^+ so as to satisfy $\nu(I_1^+ \times V')\nu(I_2^+ \times (V \setminus V')) > 0$.

(2) As before, if this were not true, one could choose pairwise disjoint intervals V_j containing ξ_j , so that $I_1(V_1) \times I_2(V_2) \times I_3(V_3)$ would be a set of positive measure consisting of positive oriented triples sent by φ into the set $V_1 \times V_2 \times V_3$ consisting of negatively oriented triples. \square

Corollary 5.7. *Let $\pi : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ be a homomorphism with maximal Euler number and let $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$ be a Γ -equivariant measurable map. Then φ takes values in \mathbf{S}^1 and is either an order preserving or an order reversing map.*

Proof. Because of Corollary 4.4, the map φ takes values in \mathbf{S}^1 . Since by hypothesis $|eu(\pi)| = |\chi(\Sigma_g)|$, the formula (2) becomes

$$\begin{aligned} |c(x_1, x_2, x_3)| &= \left| \int_{\pi_0(\Gamma) \setminus G} \varphi(gx_1) \otimes \varphi(gx_2) \otimes \varphi(gx_3)(c) d\mu(g) \right| \\ &= \left| \int_{\pi_0(\Gamma) \setminus G} c(\varphi(gx_1), \varphi(gx_2), \varphi(gx_3)) d\mu(g) \right|, \end{aligned}$$

from which the assertion follows immediately. \square

Corollary 5.8 ([17]). *If π_0 and π are representations of the fundamental group of Σ_g such that π_0 is standard and $|eu(\pi)|$ is maximal, then there exists a continuous surjective semiconjugacy from π to π_0 (possibly after conjugating by an orientation reversing homeomorphism of \mathbf{S}^1). In particular, π is injective with discrete image.*

Proof. Since the $\pi_0(\Gamma)$ -action on \mathbf{S}^1 is amenable, by [8] there exists a Γ -invariant measurable map $\varphi : \mathbf{S}^1 \rightarrow \mathcal{M}(\mathbf{S}^1)$. Maximality of the Euler number and Corollary 4.4 imply that $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$. Moreover, because of Corollary 5.7, we have that φ is either order preserving or order reversing. In the former case we can apply Proposition 5.5 directly. In the latter, we can apply Proposition 5.5 to the map $\varphi' = s \circ \varphi$, where $s \in \text{Homeo}(\mathbf{S}^1)$ is an orientation reversing homeomorphism of the circle; then we shall prove that in this case there is a continuous semiconjugacy from π' to π_0 , where $\pi'(\gamma) = s \circ \pi(\gamma) \circ s^{-1}$. In either cases we deduce the existence of a degree one monotone map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ for which we already showed most of the properties required to be a semiconjugacy from π to π_0 . Indeed the only property left to prove is the Γ -equivariance of f . We unfortunately have to refer back

to the Proof of Proposition 5.5, so that we adopt here again the notation used there.

Let $f = \hat{\psi}$. To show that $\hat{\psi}$ is Γ -equivariant, we shall prove that its graph \hat{F} is a Γ -invariant subset of $\mathbf{S}^1 \times \mathbf{S}^1$, where the action on the first and second coordinate is via π_0 and π respectively. We shall prove, in particular, that both F and $\cup(\{x_I\} \times I)$, where the union ranges over the closure of the connected components of $\mathbf{S}^1 \setminus S$, are Γ -invariant sets. Let $(\psi(\xi), \xi) \in F$. Then, if $\psi(\xi) = x$, and using the Γ -invariance of φ , we have

$$\begin{aligned} \gamma(\psi(x), x) &= (\pi_0(\gamma)\psi(\xi), \pi(\gamma)\xi) = (\pi_0(\gamma)x, \pi(\gamma)\varphi(x)) \\ &= (\pi_0(\gamma)x, \varphi(\pi_0(\gamma)x)) \in F . \end{aligned}$$

so that F is Γ -invariant.

Observe that since S is a $\pi(\Gamma)$ -invariant closed subset of \mathbf{S}^1 , its complement is a $\pi(\Gamma)$ -invariant open set the closures of whose connected components, consisting of the intervals I , are permuted by $\pi(\Gamma)$. Since for $\gamma \in \Gamma$ $\gamma \cup_I (\{x_I\} \times I) = \cup_I (\{\pi_0(\gamma)x_I\} \times \pi(\gamma)I)$, to prove the γ -invariance of \hat{F} it will be enough to show that $\pi_0(\gamma)x_I = x_{\pi(\gamma)I}$.

Fix an interval $I = [\xi_1, \xi_2]$ as above. Then, since by construction the set $(\mathbf{S}^1 \times I) \cap F$ consists only of the two points (x_I, ξ_1) and (x_I, ξ_2) , there exists a well defined map pr from the space of the (closure of the) connected components of $\mathbf{S}^1 \setminus S$ to \mathbf{S}^1 , defined by $pr(I) = p_1((\mathbf{S}^1 \times I) \cap F) = x_I$. Then the Γ -invariance of $\cup_I (\{x_I\} \times I)$ follows immediately from the Γ -equivariance of pr , since

$$\begin{aligned} \pi_0(\gamma)x_I &= \pi_0(\gamma)pr(I) = p_1(\gamma((\mathbf{S}^1 \times I) \cap F)) \\ &= p_1((\gamma\mathbf{S}^1 \times \pi(\gamma)I) \cap \gamma F) = p_1((\mathbf{S}^1 \times \pi(\gamma)I) \cap F) \\ &= x_{\pi(\gamma)I} . \end{aligned}$$

We have to show now that π is injective and that $\pi(\Gamma)$ is discrete. Injectivity follows immediately from the fact that if $\pi(\gamma) = Id$, then $f(x) = \pi_0(\gamma)f(x)$ for all $x \in \mathbf{S}^1$ from which surjectivity of f implies that $\gamma = e$.

Endow $\text{Homeo}_+(\mathbf{S}^1)$ with the compact-open topology, that is the topology of uniform convergence. Let $\pi(\gamma_n)$ be a sequence in $\text{Homeo}_+(\mathbf{S}^1)$ such that $\pi(\gamma_n) \rightarrow Id$ uniformly. Then for every $\delta > 0$ there exists $N > 0$ such that if $n \geq N$, then $d(\pi(\gamma_n)x, x) < \delta$ for every $x \in \mathbf{S}^1$. Since f is continuous, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$. Now fix $\epsilon > 0$ and choose $\delta = \delta(\epsilon)$ and $N = N(\delta(\epsilon))$ as above. Then if $n \geq N$ we have that $d(\pi(\gamma_n)x, x) < \delta$ and hence $d(\pi_0(\gamma_n)f(x), f(x)) = d(f(\pi(\gamma_n)x), f(x)) < \epsilon$ for all $x \in \mathbf{S}^1$. Surjectivity of f implies that $d(\pi_0(\gamma_n)y, y) < \epsilon$ for all $y \in \mathbf{S}^1$ and for all $n \geq N$. Since $\pi_0(\Gamma)$ is discrete, there exists $N' \geq N$ such that for $n \geq N'$, $\pi_0(\gamma_n) = e$. Hence for $n \geq N'$, $\pi(\gamma_n) = Id$, that is $\pi(\Gamma)$ is discrete. \square

Proof of Corollary 1.10. Since the Euler number is maximal, it follows from Corollary 5.8 that π is injective and has discrete image. Hence we need only to show that $\pi(\Gamma)$ is cocompact.

First of all observe that since $\pi(\Gamma)$ is discrete, it acts properly discontinuously on $\mathcal{H}_{\mathbb{R}}^2$. Since it is torsion free, it acts without fixed points, so that $\pi(\Gamma)$ is the fundamental group of $\mathcal{H}_{\mathbb{R}}^2/\pi(\Gamma)$. If $\mathcal{H}_{\mathbb{R}}^2/\pi(\Gamma)$ were not compact we would have that $\pi(\Gamma)$ is a free group, which is impossible since Γ is not free and π is injective. Hence $\mathcal{H}_{\mathbb{R}}^2/\pi(\Gamma)$ is a compact hyperbolic manifold so that $\Gamma \simeq \pi(\Gamma)$. \square

It is of interest to understand when a given semiconjugacy has some additional properties. To this end, recall that an action of a discrete group Γ on a topological space is minimal if there are no non-trivial closed Γ -invariant sets or, equivalently, if all Γ -orbits are dense. In particular, the action of $\Gamma = \pi(\Sigma_g)$ as above on \mathbf{S}^1 is minimal. In [10], Ghys observes that if $\pi_i : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$, $i = 1, 2$, are homomorphisms, and if $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is a semiconjugacy from π_1 to π_2 , then φ is injective (surjective, respectively) if the action of Γ via π_1 (π_2 , respectively) is minimal.

Definition 5.9. We say that two homomorphisms $\pi_i : \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ are *topologically conjugate* if there exists $f \in \text{Homeo}_+(\mathbf{S}^1)$ such that $f \circ \pi_1(\gamma) = \pi_2(\gamma) \circ f$.

Remark 5.10. The fact that any two standard representations of Γ are topologically conjugate is well known. However, it follows immediately from the observation before Definition 5.9 that since the action of Γ , and hence of $\pi(\Gamma)$ is minimal, f is injective and hence a topological conjugacy. It is worth noticing that the proof of Proposition 5.5 highlights the content of the above remark. Namely, the set $\overline{\mathbf{S}^1 \setminus S}$ is a closed $\pi(\Gamma)$ -invariant set, and it is precisely the set where the semiconjugacy f fails to be injective.

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