On the cohomology of foliations with amenable groupoid

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Abstract. We illustrate the proof of a vanishing theorem for the tangential de Rham cohomology of a compact foliated space with amenable fundamental groupoid, by using the existence of bounded primitives of closed bounded differential forms in degree above the rank (for an appropriate notion). In the case of foliated bundles we give a proof of a related theorem asserting the vanishing of the tangential singular cohomology, by using methods in homological algebra.

1. A discussion of the main result

Given a differentiable manifold M, it is a classical problem to study the relation between the topology and the geometry of M, in particular which restriction the fundamental group of M imposes on the possible Riemannian geometries of M. A fundamental result in this direction is the following:

Theorem 1.1 ([15], [8]). *Let M be a compact Riemannian manifold with non-positive sectional curvature $\kappa \leq 0$ and solvable fundamental group $\pi_1(M)$. Then $\kappa = 0$ (and, in fact, $\pi_1(M)$ is virtually abelian).*

More generally,

Theorem 1.2 ([16]). *Let M be a compact Riemannian manifold such that $\kappa \leq 0$ and $\pi_1(M)$ is amenable. Then $\kappa = 0$.*

We refer the reader to § 2.3 for a discussion of amenability and related topics, and we limit ourselves to point out here that solvable groups are amenable.

The purpose of this paper is to illustrate some results whose motivation stems from a proof of Theorem 1.2 for negatively curved manifolds which is due to Gromov and Thurston and can be summarized in two steps:

1. The bounded cohomology $\mathrm{H}^*_{\mathrm{b}}(X)$ of any topological space X is defined as the singular cohomology of X, where we restrict our attention only to bounded cochains, that is cochains c such that

$$\|c\|_\infty = \sup\{|c(\sigma)| : \sigma \text{ is a singular simplex in } M\} < \infty\,.$$

Then we have the following striking result (for a complete proof see [10]):

Theorem 1.3 ([9],[3]). *For any countable* CW *complex* X,

$$\mathrm{H}_{\mathrm{b}}^*(X) \simeq \mathrm{H}_{\mathrm{b}}^*(\pi_1(X)) .$$

Since the bounded cohomology of an amenable group vanishes (see Remark 2.9 and Corollary 4.2 with $T = \{\mathrm{pt}\}$), it follows that if X has amenable fundamental group, then $\mathrm{H}_{\mathrm{b}}^*(X) = 0$.

2. The second part of the proof follows from the following result:

   Theorem 1.4 ([14]). *Let M be a compact manifold with strictly negative sectional curvature. Then there is a surjection*

   $$\mathrm{H}_{\mathrm{b}}^j(M) \twoheadrightarrow \mathrm{H}^j(M) ,$$

   in degree $j > 1$.

Hence for a compact manifold, since $\mathrm{H}^{\dim(M)}(M) \neq 0$, Theorems 1.3 and 1.4 imply the incompatibility between the amenability of the fundamental group and strictly negative sectional curvature. However, if one extends the realm of generality of the above results, one can obtain the following vanishing theorem:

Theorem 1.5 ([6]). *Let $(X, \mathcal{F})$ be a compact foliated space whose leaves are uniformly of rank at most r. If the fundamental groupoid of the foliation is amenable, then the tangential de Rham cohomology $\mathrm{H}_{\mathrm{dR}}^j(X, \mathcal{F})$ vanishes for all $j > r$.*

We refer the reader to § 2 for all the relevant definitions. However we mention here that a prominent example of such situation is a compact space foliated by locally symmetric spaces of $\mathbb{R}$-rank r. Moreover tangential de Rham cohomology has been considered by several authors with different degrees of regularity in the the direction transverse to the leaves (see [13], for example, for an extensive list of references), thus obtaining different theories (see § 2.1 for an example).

The initial approach to the proof of Theorem 1.5 was along the lines of Gromov's proof of Theorem 1.2, in the special case of foliated bundles whose leaves have strictly negative curvature. The proof that eventually appeared in print in [6], and whose outline is presented in § 3, does not make any use of bounded cohomology, but uses rather a direct approach via an analogue of a Poincaré lemma with estimates (Lemma 3.1). In that original approach Gromov's definition of bounded cohomology was used. Here we want to present instead a proof of a related vanishing theorem in the special case of foliated bundles (see Example 2.1 for the definition), in which the functorial approach to the bounded cohomology of locally compact groups developed by Burger and Monod in [4] is exploited. Although the definitions are *ad hoc*, it indicates a possible use of a systematic development of the theory of the bounded cohomology of groupoids applied to general foliations.

Theorem 1.6. *Let Y be a compact locally* CAT(-1) *space with fundamental group Γ and universal covering $\widetilde{Y}$. If (T, μ) is a standard measure space with a measure*

class preserving amenable Γ*-action and* $X = (Y \times T)/\Gamma$*, then the tangential singular cohomology* $\mathrm{H}^j_{\mathrm{s}}(X, \mathcal{F})$ *vanishes for all* $j > 1$.

2. Definitions and Examples

We collect here the definitions needed in the sequel. We shall often prefer to give illustrative examples rather than technical definitions.

2.1. Foliations

Let $(X, \mathcal{F})$ be a topological space X with a foliation $\mathcal{F}$ whose leaves are smooth Riemannian manifolds and such that the Riemannian structure is smooth along the leaves and globally continuous. Assume that there is a measure λ which is obtained by combining a transverse measure, whose class is invariant under holonomy, with the Lebesgue measure along the leaves.

Example 2.1.

- Any locally free smooth action of a connected Lie group on a manifold determines a foliation.
- The space $X = \mathbb{R}^p \times \mathbb{R}^{n-p}$ is a foliation, and, in fact, any foliation looks locally like a product $U \times \Sigma$, where $U \subset \mathbb{R}^{n-p}$ is an open set and Σ is a topological space. More generally, if Y is a Riemannian manifold and Σ is a topological space, then $X = Y \times \Sigma$ is a topological space with a foliation whose leaves are $Y \times \{\sigma\}$, $\sigma \in \Sigma$.
- If Y and Σ are as above, if Γ acts properly discontinuously on Y and with no fixed points and, moreover, if Γ acts on Σ, then $X = (Y \times \Sigma)/\Gamma$ is a topological space with a foliation with leaves $(Y \times \{\sigma\})/\Gamma_\sigma$, where Γ_σ is the stabilizer of $\sigma \in \Sigma$. The foliated space X is often referred to as a foliated bundle.

Since the leaves of the foliation are Riemannian manifolds, they admit tangent spaces which can then be assembled together to form the foliated tangent space $T\mathcal{F}$. Let $T^*\mathcal{F}$ be the foliated cotangent bundle and $\Lambda^j T^*\mathcal{F}$ be its j-th exterior power.

Definition 2.2 ([6]). If $(X, \mathcal{F})$ is a foliated space, its tangential de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X, \mathcal{F})$ is the cohomology of the complex

$$\Omega^j(X, \mathcal{F}) = \{\omega : X \to \Lambda^j T^*\mathcal{F} \,:\, \omega, d\omega \in L^\infty(X, \Lambda^j T^*\mathcal{F}), \text{ and } \omega, d\omega \text{ are } C^\infty \text{ along the leaves}\},$$

where the differential is taken in the direction of the leaves and where

$$\begin{aligned}\|\omega\|_\infty &= \operatorname*{esssup}_{x\in X}\|\omega_x\| \\ &= \operatorname*{esssup}_{x\in X}\sup\{|\omega_x(v_1\wedge\cdots\wedge v_j)|\,:\, v_1,\dots,v_j\in T_x\mathcal{F} \text{ are orthonormal}\}\,.\end{aligned}$$

As mentioned in § 1, one can choose to require different degrees of regularity in the direction transversal to the leaves. For instance if one takes differential forms which are just measurable on the total space without any assumption of boundedness, then it was observed by Zimmer that the tangential de Rham cohomology thus defined vanishes in degree above one, provided that almost every leaf is contractible (see [6]).

2.2. Fundamental groupoid

Definition 2.3. A groupoid $\mathcal{G}$ is a small category in which each morphism is an isomorphism.

Hence the information which characterizes a groupoid is encoded by the set of units $\mathrm{Obj}(\mathcal{G})$ and the set of morphisms $\mathrm{Mor}(\mathcal{G})$. We have moreover source and target maps, $s,t : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Obj}(\mathcal{G})$ which determine when two morphisms m_1, m_2 are composable, namely if and only if $s(m_2) = t(m_1)$, in which case the multiplication is $(m_1, m_2) \mapsto m_1 \circ m_2$. A few examples will serve the purpose of clarifying this concept:

Example 2.4.

- Let G be a group acting on a space X. Then the groupoid $\mathcal{G}$ associated to the action is such that $\mathrm{Obj}(\mathcal{G}) = X$ and $\mathrm{Mor}(\mathcal{G}) = \big\{(x,g) \in X \times G\big\}$; moreover $s : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Obj}(\mathcal{G})$ and $t : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Obj}(\mathcal{G})$ are respectively defined by $s(x,g) := x$ and $t(x,g) := xg$, and two morphisms (x,g) and (x',g') are composable if and only if $xg = x'$, in which case $(x,g)\circ(x',g') = (x,gg')$.

- Let $\mathcal{R} \subset X \times X$ be an equivalence relation on X. Then the groupoid $\mathcal{G}_\mathcal{R}$ associated to $\mathcal{R}$ is such that $\mathrm{Obj}(\mathcal{G}_\mathcal{R}) = X$ and $\mathrm{Mor}(\mathcal{G}_\mathcal{R}) = \big\{(x,y) \in \mathcal{R}\big\}$; Here $s(x,y) = x$ and $t(x,y) = y$, so that two morphisms (x,y), $(z,w) \in \mathcal{R}$ are composable if and only if $y = z$, in which case $(x,y)\circ(y,w) = (x,w)$.

- If X is any topological space, its fundamental groupoid $\mathcal{G}_X$ is such that $\mathrm{Obj}(\mathcal{G}_X) = X$ and $\mathrm{Mor}(\mathcal{G}_X)$ is the set of homotopy classes (with fixed endpoints) of paths. Evidently two morphisms are composable if and only if the endpoint of a path (or, more precisely, of an equivalence class of paths) coincides with the beginning point of the other path.

- As a generalization of the previous example, we finally have the definition of the fundamental groupoid of a foliated topological space:

Definition 2.5. If $(X, \mathcal{F})$ is a foliated topological space, the fundamental groupoid of the foliation $\mathcal{G}_{(X,\mathcal{F})}$ is the groupoid whose set of units $\mathrm{Obj}(\mathcal{G}_{(X,\mathcal{F})})$ is X and whose set of morphisms $\mathrm{Mor}(\mathcal{G}_{(X,\mathcal{F})})$ is the set of homotopy classes (endpoints fixing) of paths contained in a leaf.

2.3. Amenability

One of the many classical equivalent definitions of amenability of a topological group G requires that for every compact metric space X on which G acts continuously, there exists on X a G-invariant Borel probability measure μ. Note that the space $C(X)$ of continuous functions on X with the supremum norm is a separable Banach space with an isometric G-action, and the space of Borel probability measures $\mathcal{M}(X)$ is a compact convex G-invariant subset of the unit ball of the dual $C^*(X)_1$ (in the weak-$*$ topology). Then an invariant measure $\mu \in \mathcal{M}(X)$ is nothing but a fixed point for the G-action on $\mathcal{M}(X)$, and one is hence lead to the following definition:

Definition 2.6. A group G is amenable if and only if there exists a fixed point in any affine G-space, that is in any compact convex G-invariant subset $A \subset E_1^*$ in the unit ball of the dual of a separable Banach space on which G acts isometrically and continuously.

We mentioned already that cyclic groups and, more generally, solvable groups are amenable (see for instance [17, Ch. 4, § 1]). Moreover, we shall use in what follows that, among the parabolic subgroups of Lie groups, the only ones which are amenable are the minimal parabolics.

In order to extend the definition of amenability of a group to a groupoid, we first need to define the notion of action of a groupoid. Let E be a separable Banach space, $V \to X$ an isometric Banach bundle with fiber E (that is a fiber bundle with fiber E such that there is a covering of X and a corresponding trivialization of V with transition functions in $\mathrm{Iso}(E)$), and let $V^* \to X$ be its dual Banach bundle. If V_x is the fiber of $V \to X$ above the point $x \in X$, let $\mathrm{Iso}(V)$ be the groupoid with $\mathrm{Obj}(\mathrm{Iso}(V)) = X$ and morphisms $\mathrm{Mor}(\mathrm{Iso}(V)) = \left\{ \mathrm{Iso}(V_x, V_y) : x, y \in X \right\}$, that is the linear isomorphisms between fibers.

Definition 2.7. An action of a groupoid $\mathcal{G}$ on V is a functor from $\mathcal{G}$ to $\mathrm{Iso}(V)$ which is the identity on objects, that is a map

$$\begin{array}{rccc} \rho : & \mathrm{Mor}(\mathcal{G}) & \to & \mathrm{Mor}(\mathrm{Iso}(V)) \\ & (g : x \to y) & \mapsto & (\rho(g) : V_x \to V_y) \end{array}$$

such that $\rho(gh) = \rho(g)\rho(h)$ whenever g and h are composable.

Once we have an action of $\mathcal{G}$ on V, a field of compact convex subsets of V^* parametrized by X is a subset $\mathcal{A} \subset V^*$ such that each subset $A_x \subset V_x^*$ is a

compact convex subset of V_x^*. We say that $\mathcal{A}$ is ρ-invariant if for any morphism $g : x \to y$ in $\mathrm{Mor}(\mathcal{G})$ and almost every $x \in X$, we have that $\rho(g^{-1})^* A_x \subset A_y$, where $\rho(g^{-1})^* : V_x^* \to V_y^*$. We finally have:

Definition 2.8. A groupoid $\mathcal{G}$ is amenable is for every Borel representation of $\mathcal{G}$ on an isometric Banach bundle $V \to X$ with separable fiber and any ρ-invariant Borel field $\mathcal{A}$ of compact convex subsets of V^*, there exists a ρ-invariant section of $\mathcal{A}$, that is a Borel map $s : X \to V^*$ with $s(x) \in A_x$ and such that $\rho(g^{-1})^*(s(x)) = s(y)$ for almost every $x \in X$ and all morphisms $g : x \to y$.

Remark 2.9.

- If $\mathcal{G}$ is the groupoid of an action, then $\mathcal{G}$ is amenable if and only if the action is amenable [17, Definition 4.3.1].
- Recall that a transitive action is amenable if and only if the stabilizer of a point is amenable. More generally, an action is amenable if and only if the equivalence relation of the action is amenable and the stabilizers are amenable ([1] or [2]). Analogously, the fundamental groupoid of a foliation is amenable if and only if the foliation is amenable (that is the equivalence relation induced on any transversal is amenable) and the fundamental groups of the leaves are amenable (see for example [2]).

We can now give examples of foliations with amenable fundamental groupoid.

Example 2.10.

- Let M be a compact Riemannian manifold with negative sectional curvature and universal cover $\widetilde{M}$, and let $\Sigma = \widetilde{M}(\infty)$ be the set of equivalence classes of asymptotic geodesic rays. If $\Gamma = \pi_1(M)$, then $X = (\widetilde{M} \times \Sigma)/\Gamma$ is a foliated space with amenable fundamental groupoid, since the equivalence relation of the transversal Σ is amenable and the fundamental group Γ_σ of the leaf $\mathcal{L}\sigma = (\widetilde{M} \times \{\sigma\})/\Gamma_\sigma$ is amenable since cyclic.
- Let Y be a symmetric space of non-compact type, $G = \mathrm{Iso}(Y)$ be its isometry group (hence a semisimple group), $\Gamma < G$ a cocompact torsion-free lattice and Q a parabolic subgroup. Then $X = (Y \times (G/Q))/\Gamma$ is a space foliated by leaves $(Y \times [x])/\Gamma_{[x]}$ and the fundamental groupoid of the foliation is amenable if and only if Q is the minimal parabolic. Note that in this case the non-amenability of Q is reflected in the non-amenability of the foliation, although the fundamental groups of the leaves might still be amenable. This is the case, for instance, if $G = \mathrm{SL}((p-1)/2, \mathbb{C})$ for p a prime congruent to 3 modulo 4 and Q is the parabolic subgroup which stabilizes the vector $(1, 0, \ldots, 0) \in \mathbb{C}^{(p-1)/2}$, in which case one might choose Γ so that for each $[x] = gQ$ the leaf $\mathcal{L}_{[x]}$ has abelian fundamental group $\Gamma_{[x]} = g^{-1}\Gamma g \cap Q$, [6, Corollary 4.5].

2.4. Rank of a manifold

The notion of rank that is needed in Theorem 1.5 is somewhat different from any of the standard definitions. We say that a manifold M of non-positive curvature

has rank r at a point m and with respect to a tangent vector $v \in TM_m$ if r is the largest dimension of a subspace $W \subset TM_m$ containing v such that every plane in W containing v has sectional curvature zero. The uniform notion of rank that is needed is then the following:

Definition 2.11. Let M be a complete simply connected Riemannian manifold with non-positive sectional curvature. We say that M is uniformly of rank at most r if there is a positive constant C such that, for every subspace of dimension $r+1$ of every tangent space to M and every non-zero vector v in the subspace, there is a plane with sectional curvature at most $-C$ containing v.

Notice that if M is a symmetric space this notion of rank coincides with the usual one in terms of maximal dimension of flats.

2.5. Remarks

We give here some indication of examples which show that the hypotheses of Theorem 1.5 are sharp. For instance one cannot expect to have vanishing of the tangential cohomology in degree smaller or equal than the rank of the manifold, since, already for the one leaf foliation consisting of a flat torus, the de Rham cohomology does not vanish in top degree.

Moreover, also the full strength of the amenability of the fundamental groupoid is necessary. In fact, on the one hand one can consider once again the foliation consisting of just one leaf which is a compact quotient of a symmetric space of non-compact type. In this case the equivalence relation on a transversal is amenable (being the trivial one), but the fundamental group of the leaf is typically not amenable. In many of such examples one has non-vanishing of the de Rham cohomology in degree above the rank, as one can see for instance by taking any compact quotient of any symmetric space of non-compact type, in which case the volume form gives a non-vanishing class in top degree.

On the other hand, one can construct examples of foliated bundles with non-amenable equivalence relation but such that the leaves have abelian fundamental groups and for which the tangential de Rham cohomology groups do not vanish in some degree above the rank. In fact:

Proposition 2.12 ([6]). *For $n \neq 3$, let $G = \mathrm{SL}(n,\mathbb{C})$, $\Gamma < \mathrm{SL}(n,\mathbb{C})$ a cocompact lattice, $Q < \mathrm{SL}(n,\mathbb{C})$ the parabolic subgroup which stabilizes the vector $(1,0,\ldots,0) \in \mathbb{C}^n$ and $Y = \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$. Then for all j odd, with $3 \leq j \leq 2n-3$, $\mathrm{H}^j_{\mathrm{dR}}((Y \times (G/Q))/\Gamma) \neq 0$.*

Collecting the information from the above proposition and from Example 2.10, if $p \geq 7$ is a prime such that $p \equiv 3 \pmod 4$, for all $(p-1)/2 \leq j \leq p-4$ and j odd, $\mathrm{H}^j_{\mathrm{dR}}((Y \times (G/Q))/\Gamma) \neq 0$ despite the fact that $\mathbb{R}$-rank$(\mathrm{SL}((p-1)/2,\mathbb{C})) = (p-3)/2$.

We want to conclude this section by mentioning a possible relation between our theorem and the main theorem in [16]. There Zimmer considered the case of a

measure space X with a Riemannian measurable foliation $\mathcal{F}$ of finite total volume, such that almost every leaf is a complete simply-connected manifold of non-positive sectional curvature. He proved that if the foliation is amenable and if there exists a transversally invariant measure, then almost every leaf is flat. Although this theorem is much more general in that, for example, there is no rank assumption on the leaves, Theorem 1.5 should imply this result in the case in which both can be applied. In fact, in view of the simple-connectivity of the leaves, amenability of the foliation coincides with amenability of the fundamental groupoid. Now suppose that the leaves satisfy the uniform rank condition in Definition 2.11, for instance are locally symmetric spaces of dimension n. Then, if one were to prove an analogue of a theorem of Ruelle and Sullivan (see [13, Corollary 4.25], for example), the existence of an absolutely continuous transversally invariant measure would imply the existence of a non-zero class in $\mathrm{H}^n_{\mathrm{dR}}(X, \mathcal{F})$. Hence, by Theorem 1.5, we must have that $n \leq \mathbb{R}\text{-rank}(G)$, that is the leaves are flat.

3. A sketch of the proof of Theorem 1.5

The idea is simple. For each leaf $\mathcal{L}$ of the foliation and each leafwise closed differential form α of degree at least equal to the rank, there exists a canonical convex set of bounded primitives of α, once α is restricted to the leaf $\mathcal{L}$ and lifted to its universal cover $\widetilde{\alpha}$. Then, by using the amenability of the fundamental groupoid, it is possible to choose primitives from there convex sets coherently for all leaves. More specifically:

Lemma 3.1. *Let M be a complete simply-connected Riemannian manifold with non-positive sectional curvature which is uniformly of rank at most r and let $\alpha \in \Omega^j(M)$ be a bounded smooth closed differential j-form, $r < j \leq \dim M$. If $M(\infty)$ is the boundary consisting of equivalence classes of asymptotic geodesic rays, then there exists a Borel map $\beta : M(\infty) \to \Omega^{j-1}(M)$, $\beta(\xi) := \beta_\xi$, such that $d\beta_\xi = \alpha$ and $\|\beta\| = \sup_\xi \|\beta_\xi\| < \infty$. Moreover β is equivariant with respect to isometries.*

The proof of the Lemma is basically the same as the proof of Poincaré lemma with estimates. Let $\varphi_\xi(t)$ be the gradient flow associated to the gradient vector field of the Busemann function $b_\xi : M \to \mathbb{R}$. Define a map $\Phi_\xi : M \times [0,1] \to M$, by $\Phi_\xi(m,t) = \varphi_\xi(t)(m)$ to use as a homotopy in the classical Poincaré lemma. Namely, if $\Phi_\xi^*(\alpha) = \omega_0(t) + \omega_1(t) \wedge dt$, define $\beta_\xi = \int_0^\infty \omega_1(t)dt$. Note that the existence of the map β uses the fact that $\varphi_\xi(t)$ is a contraction on j tangent vector, $j \geq r+1$, that is that $\|\varphi_\xi(t)(X_1 \wedge \cdots \wedge X_k)\|$ decays exponentially.

We observe now the first consequence of the amenability of the fundamental groupoid $\mathcal{G}_{(X,\mathcal{F})}$, for which we need to define an appropriate action. Let $\mathcal{L}_{x_0}$ be the leaf through $x_0 \in X$, and if $x \in \mathcal{L}_{x_0}$, let

$$\widetilde{\mathcal{L}_x} = \{([d], z) : z \in \mathcal{L}_{x_0}, [d] \text{ is a homotopy class of paths from } x \text{ to } z\}$$

be its universal covering based at x. If y is another point in $\mathcal{L}_{x_0}$, any homotopy class $[c]$ from x to y defines an isometry $\rho([c]) : \widetilde{\mathcal{L}_x} \to \widetilde{\mathcal{L}_y}$, by $\rho([c])([d], z) = ([c^{-1} \circ d], z)$, which extends to a homeomorphism of the associated ideal boundaries $\rho([c]) : \widetilde{\mathcal{L}_x}(\infty) \to \widetilde{\mathcal{L}_y}(\infty)$. Since $\widetilde{\mathcal{L}_x}(\infty)$ is a compact and metrizable, for every $x \in X$ the space of continuous functions $C(\widetilde{\mathcal{L}_x}(\infty))$ is a separable Banach space, so that we can consider the isometric Banach bundle $V \to X$ with fiber $C(\widetilde{\mathcal{L}_x}(\infty))$, on which $\mathcal{G}_{(X,\mathcal{F})}$ acts via $\rho : \mathrm{Mor}(\mathcal{G}_{(X,\mathcal{F})}) \to \mathrm{Mor}(\mathrm{Iso}(V))$.

Hence we have a field of compact convex subsets of V^* parametrized by X, $x \to \mathcal{M}(\widetilde{\mathcal{L}_x}) \subset C(\widetilde{\mathcal{L}_x})_1^*$ consisting of probability measures on $\widetilde{\mathcal{L}_x}$, which can be easily seen to be $\mathcal{G}_{(X,\mathcal{F})}$-invariant. The amenability of $\mathcal{G}_{(X,\mathcal{F})}$ implies the existence of a $\mathcal{G}_{(X,\mathcal{F})}$-invariant Borel section $s : X \to \mathcal{M}(\widetilde{\mathcal{L}_x})$.

To conclude, let now $\alpha \in \Omega^j(X, \mathcal{F})$ be a closed form. If $p_x : \widetilde{\mathcal{L}_x} \to \mathcal{L}_x$ is the projection, let us consider $p_x^*(\alpha|_{\mathcal{L}_x}) \in \Omega^j(\widetilde{\mathcal{L}_x})$, where $\alpha|_{\mathcal{L}_x}$ is the restriction of α to $\mathcal{L}_x$. By Lemma 3.1, there exists a Borel map $\beta : \widetilde{\mathcal{L}_x}(\infty) \to \Omega^{j-1}(\widetilde{\mathcal{L}_x})$, such that $d\beta_\xi = p_x^*(\alpha|_{\mathcal{L}_x})$ for every $\xi \in \widetilde{\mathcal{L}_x}$ and such that β is bounded uniformly in ξ. Define now

$$\beta_x = \int_{\widetilde{\mathcal{L}_x}(\infty)} \beta_\xi \, ds_x(\xi) \in \Omega^{j-1}(\widetilde{\mathcal{L}_x}) \,,$$

which has still the property that $d\beta_x = p_x^*(\alpha|_{\mathcal{L}_x})$. Now we use twice the invariance of the section s. Firstly, since s is invariant for morphisms $x \mapsto x$ (that is for homotopy paths in $\pi_1(\mathcal{L}_x)$), we obtain that there exists $\omega_x \in \Omega^j(\mathcal{L}_x)$ such that $\beta_x = p_x^*(\omega_x)$; secondly, since s is $\mathcal{G}_{(X,\mathcal{F})}$-invariant (that is invariant with respect to all morphisms $x \mapsto y$), we deduce that the differential form ω_x is independent of the choice of the basepoint, namely that $\omega_x = \omega_y$ if $\mathcal{L}_x = \mathcal{L}_y$. We have hence defined a tangential form $\omega \in \Omega^{j-1}(X, \mathcal{F})$ which inherits its Borel measurability from s. □

4. Proof of Theorem 1.6

Given a discrete group Γ, if $\mathrm{C_b}(\Gamma^j)$ denotes the space of bounded functions on the j-fold cartesian product Γ^j, the bounded cohomology of Γ can be defined as the cohomology of the complex

$$0 \longrightarrow \mathrm{C_b}(\Gamma)^\Gamma \longrightarrow \mathrm{C_b}(\Gamma^2)^\Gamma \longrightarrow \mathrm{C_b}(\Gamma^3)^\Gamma \longrightarrow \dots$$

with the usual homogeneous coboundary operator. However, just like in the case of ordinary group cohomology, one can use instead an homological algebraic approach which has the advantage of being more flexible in that one can use resolutions which are more appropriate to specific situations, as long as they satisfy certain properties. In other words, it can be proven that the cohomology of any admissible resolution by relatively injective Γ-modules, is isomorphic to the bounded

cohomology of Γ. As in the case of ordinary group cohomology, admissibility of a resolution involves the existence of homotopy operators which in this case should be bounded in norm. Moreover, amenability of a Γ-action is intimately related to certain functions spaces being relatively injective Γ-modules, which makes this theory particularly fitting in this case and, more generally, whenever there is a suitable boundary. All of this is very vague and it is just to give some of the flavor of what follows: we refer the reader to [4], [12] and [5], where this theory was developed (in much greater generality) for the background and the precise definitions.

Let Y be a countable cellular space, $\pi_1(Y) = \Gamma$ its fundamental group, $\widetilde{Y}$ its universal covering and (T, μ) a standard measure space with a measure class preserving Γ-action. If $S_j(\widetilde{Y})$ is the space of singular simplices in $\widetilde{Y}$, let $L^\infty_{\mathrm{w}*}(T, \ell^\infty(S_j(\widetilde{Y})))$ denote the space of (equivalence classes of) maps $\alpha : T \to \ell^\infty(S_j(\widetilde{Y}))$ which are measurable when $\ell^\infty(S_j(\widetilde{Y}))$ is endowed of the weak-$*$ topology as the dual of $\ell^1(S_j(\widetilde{Y}))$, and which are essentially bounded. We then define the singular tangential bounded cohomology $\mathrm{H}^*_{\mathrm{s,b}}(X, \mathcal{F})$ of the foliated bundle $X = (\widetilde{Y} \times T)/\Gamma$ as the cohomology of the complex

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(T, \ell^\infty(S_0(\widetilde{Y})))^\Gamma \xrightarrow{d} L^\infty_{\mathrm{w}*}(T, \ell^\infty(S_1(\widetilde{Y})))^\Gamma \xrightarrow{d} \dots \tag{4.1}$$

with boundary operator

$$d\alpha(t)(s) = \alpha(t)(ds)\,,$$

where $\alpha \in L^\infty_{\mathrm{w}*}(T, \ell^\infty(S_j(\widetilde{Y})))$ and $s \in S_{j+1}(\widetilde{Y})$.

The first application of the homological algebra approach to bounded cohomology is the following:

Proposition 4.1. $\mathrm{H}^*_{\mathrm{s,b}}(X, \mathcal{F}) \simeq \mathrm{H}^*_{\mathrm{b}}(\Gamma, L^\infty(T))$.

Proof. First of all observe that we have the identification

$$L^\infty_{\mathrm{w}*}(T, \ell^\infty(S_j(\widetilde{Y}))) \simeq L^\infty(T \times S_j(\widetilde{Y})) \simeq \ell^\infty(S_j(\widetilde{Y}), L^\infty(T))\,,$$

so that the complex in (4.1) can be rewritten as the complex

$$0 \longrightarrow \ell^\infty(S_0(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \ell^\infty(S_1(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \dots\,.$$

Being this the non-augmented subcomplex of invariant vectors of the complex

$$0 \longrightarrow L^\infty(T)) \longrightarrow \ell^\infty(S_0(\widetilde{Y}), L^\infty(T)) \longrightarrow \ell^\infty(S_1(\widetilde{Y}), L^\infty(T)) \longrightarrow \dots\,, \tag{4.2}$$

to prove the proposition it will be enough to show that (4.2) is an admissible resolution by relatively injective Γ-modules (see [4] or [12]).

We start by observing that the properness of the action of Γ on $\widetilde{Y}$ implies that, for all $j \geq 0$, the spaces $\ell^\infty(S_j(\widetilde{Y}), L^\infty(T))$ are relatively injective objects in the category of isometric Γ-Banach spaces, [12, Definition 4.1.2 and Theorem 4.5.2].

We need to define now appropriate homotopy operators. By using the usual coning procedure (since $\widetilde{Y}$ is contractible) there are homotopy operators

$$\ell^\infty(S_{j-1}(\widetilde{Y})) \underset{h_j}{\overset{d}{\rightleftarrows}} \ell^\infty(S_j(\widetilde{Y}))$$

which are norm continuous, and such that $\|h_j\| \leq 1$, ([9], [10]). We can now define contracting homotopy operators ([12, § 7.1])

$$\ell^\infty(S_{j-1}(\widetilde{Y}), L^\infty(T)) \xleftarrow{H_j} \ell^\infty(S_j(\widetilde{Y}), L^\infty(T))$$

as follows: let $\alpha : S_j(\widetilde{Y}) \to L^\infty(T)$ be a cochain, and for $f \in L^1(T)$, define $\alpha_f : S_j(\widetilde{Y}) \to \mathbb{R}$ by $\alpha_f(s_{(j)}) := \langle \alpha(s_{(j)}), f \rangle$, for $s_{(j)} \in S_j(\widetilde{Y})$. Then $f \mapsto h_j(\alpha_f)(s_{(j-1)})$ is a continuous linear form on $L^1(T)$, giving thus an element in $L^\infty(T)$ denoted $H_j(\alpha)(s_{(j-1)})$. This defines a norm continuous H_j and hence the cohomology of the complex

$$0 \longrightarrow \ell^\infty(S_0(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \ell^\infty(S_1(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \dots$$

is isomorphic to $\mathrm{H}^*_{\mathrm{b}}(\Gamma, L^\infty(T))$, [12, Proposition 8.1.1]. □

This is the point where the amenability of the Γ-action on T plays an essential role.

Corollary 4.2. *If Γ acts amenably on T, then $\mathrm{H}^*_{\mathrm{s,b}}(X, \mathcal{F}) = 0$.*

Proof. The amenability of the Γ-action implies that $L^\infty(T)$ is a relatively injective Γ-module, which in turns implies easily that $\mathrm{H}^*_{\mathrm{b}}(\Gamma, L^\infty(T)) = 0$, [12, Proposition 7.4.1]. □

Now we need to relate the ordinary group cohomology of Γ to the singular cohomology of the foliated bundle. The idea is to use spaces very similar to those used in the case of singular bounded cohomology but with no requirement on the boundedness in the direction of the leaves. To this purpose, If Y is a compact locally CAT(-1) space (that is a generalization, in the singular context, of a $\mathbb{R}$-rank one symmetric space), let $\sigma_j(\widetilde{Y})$ denote the set of j-simplices lifted to $\widetilde{Y}$ of any finite simplicial decomposition of Y. Observe that $\sigma_j(\widetilde{Y})$ is countable. Let $L_\infty(T, \mathrm{Maps}(\sigma_j(\widetilde{Y}), \mathbb{R}))$ be the space of all maps $\alpha : T \to \mathrm{Maps}(\sigma_j(\widetilde{Y}), \mathbb{R})$ such that for every $s_{(j)} \in \sigma_j(\widetilde{Y})$, the function $t \mapsto \alpha(t)(s_{(j)})$ is in $L^\infty(T)$ and define the singular tangential cohomology $\mathrm{H}^*_{\mathrm{s}}(X, \mathcal{F})$ of $\mathcal{F}$, as the cohomology of the complex

$$0 \longrightarrow L_\infty(T, \mathrm{Maps}(\sigma_0(\widetilde{Y}), \mathbb{R}))^\Gamma \longrightarrow L_\infty(T, \mathrm{Maps}(\sigma_1(\widetilde{Y}), \mathbb{R}))^\Gamma \longrightarrow \dots .$$

Observe that $L_\infty(T, \mathrm{Maps}(\sigma_j(\widetilde{Y}), \mathbb{R})) \simeq \mathrm{Maps}(\sigma_j(\widetilde{Y}), L^\infty(T))$; then a classical argument in ordinary group cohomology analogous to the one in the proof of Proposition 4.1 shows that the resolution

$$0 \longrightarrow \mathrm{Maps}(\sigma_0(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \mathrm{Maps}(\sigma_1(\widetilde{Y}), L^\infty(T))^\Gamma \longrightarrow \dots$$

is an admissible resolution by relatively injective modules (where all the concepts have to be interpreted now in ordinary group cohomology) and hence its cohomology computes $\mathrm{H}^*(\Gamma, L^\infty(T))$.

Now that all cohomology spaces have been defined, finally the punchline. Since Y is a compact locally CAT(-1) space then its fundamental group Γ is a Gromov-hyperbolic group, [7]. The essential step now is a result of Mineyev [11], which states that the map

$$\mathrm{H}^j_{\mathrm{b}}(\Gamma, V) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^j(\Gamma, V)$$

is surjective for all $j \geq 2$ and all isometric Banach Γ-modules V. In particular the map

$$\mathrm{H}^j_{\mathrm{b}}(\Gamma, L^\infty(T)) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^j(\Gamma, L^\infty(T)) \tag{4.3}$$

is surjective for $j \geq 2$.

Collecting the isomorphisms $\mathrm{H}^*_{\mathrm{s,b}}(X, \mathcal{F}) \simeq \mathrm{H}^*_{\mathrm{b}}(\Gamma, L^\infty(T))$ and $\mathrm{H}^*_{\mathrm{s}}(X, \mathcal{F}) \simeq \mathrm{H}^*(\Gamma, L^\infty(T))$, and using (4.3), we have:

Corollary 4.3. *The map*

$$\mathrm{H}^j_{\mathrm{s,b}}(X, \mathcal{F}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^j_{\mathrm{s}}(X, \mathcal{F})$$

is surjective for every $j \geq 2$.

Then Corollaries 4.2 and 4.3 immediately imply Theorem 1.6 if the Γ-action on T is amenable.

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