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ALGEBRAIC HULLS AND THE FØLNER PROPERTY

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1. Introduction

Let G be a connected semisimple Lie group with finite center and no non-trivial compact factors, and let Γ be an irreducible lattice in G. The Borel density theorem then asserts that Γ is Zariski dense in G ([B1], [Fu2]). This result has proven valuable in a variety of different contexts and several authors have extended it to more general situations (see [D1,2], [Msk], [MosMsk], [S1,2], [W], for example). An immediate consequence of the Borel density theorem is the following: if π is any rational homomorphism of G and Γ is an irreducible lattice in G, then the Zariski closure of $\pi(\Gamma)$ coincides with $\pi(G)$. A natural problem that arises here is to extend this result to other 'large' subgroups of G, not necessarily of finite co-volume.

In the study of automorphism groups of geometric structures an important notion is that of a *cocycle* of a group action on a manifold (see §2 for the definition), which can be viewed as a generalization of the familiar notion of a group homomorphism. There is a corresponding generalization of the notion of Zariski closure. This point of view, proposed by G. Mackey [M2], was systematically developed by R.J. Zimmer and lead to his superrigidity theorem for cocycles [Z3], an extension of the superrigidity theorem due to G.A. Margulis [Ma]. The Borel density theorem plays a fundamental role in both proofs.

To explain how cocycles appear as generalizations of homomorphisms, note that a cocycle for an action on a point is a homomorphism and, a less obvious case, that to (the cohomology class of) a cocycle for a transitive action on G/G_0 corresponds uniquely (the conjugacy class of) a homomorphism of G_0.

It is natural, in view of the previous remarks, to try to find extensions of the Borel density theorem that will apply to cocycles. The notion corresponding to the Zariski closure of the image of the homomorphism is the

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algebraic hull of a cocycle [Z4] (see §2): given a cocycle from an ergodic action into an algebraic group, the algebraic hull of the cocycle is the (unique conjugacy class of the) smallest algebraic subgroup in which a cohomologous cocycle takes values. In the case of a transitive action for example, the algebraic hull of a cocycle on G/G_0 coincides with the Zariski closure of the image of G_0 under the corresponding homomorphism.

The first result in this direction appeared in [I1]. There it is proven that if Γ is an irreducible lattice in a real semisimple group G without non-trivial compact factors acting ergodically on a space S with quasi-invariant measure, then the algebraic hull of the G-action and of the Γ-action are the same, provided that S is not isomorphic to G/G_0, with G_0 compact. This result has numerous applications both to ergodic theory and to differential geometry (see [I1,2 and 3]).

We will extend this conclusion (under some additional assumptions) to subgroups $\Lambda \subset G$, which will be shown to be Zariski dense in G, but may have infinite co-volume. The subgroups are termed *co-Følner* subgroups of G, and are defined as follows. If G is a locally compact group, we say that the action of G on a space S with a G-invariant measure ν has the Følner property if S admits an asymptotically G-invariant sequence of sets of positive finite measure. If $S = G/\Lambda$ has a G-invariant measure and G acts on G/Λ with the Følner property, we say that Λ is co-Følner in G (see §3).

Before proceeding with the statements of our results, we introduce one more definition. Let k denote the field $\mathbb{R}$, $\mathbb{C}$ or any local field of characteristic zero. We will consider cocycles taking values in an algebraic group H_k, consisting of the k-points of an algebraically connected semi-simple group defined over k.

DEFINITION 1.1. (a) *A cocycle $\alpha : S \times G \to H_k$ is k-**unbounded** if, given any proper normal k-subgroup N_k of H_k, the cocycle $\rho \circ \alpha : S \times G \to H'_k = H_k/N_k$ is not equivalent to a cocycle taking values in a compact subgroup of H'_k.*

(b) *In particular, if S is a point, a homomorphism $\pi : G \to H_k$ is k-**unbounded** if, given any proper normal k-subgroup N_k of H_k, the image of $\pi(G)$ in $H'_k = H_k/N_k$ is not contained in a compact subgroup.*

If α is a cocycle for a G-action on S, let us denote by α_Λ the restriction of α to the Λ-action on S, $\alpha_\Lambda = \alpha|_{S\times\Lambda}$. We prove the following:

Theorem 5.4. *Let G be a locally compact second countable group, S a G-space with a quasi-invariant measure and Λ a co-Følner subgroup of G acting ergodically on S. Let H_k denote the k-points of an algebraically connected semisimple algebraic group defined over k. If $\alpha : S \times G \to H_k$*

is a k-unbounded measurable cocycle which has algebraic hull H_k, then $\alpha_\Lambda : S \times \Lambda \to H_k$ has algebraic hull H_k as well.

The simplest example of a co-Følner subgroup is a lattice in G. Hence this theorem implies the result mentioned above on algebraic hulls [I1] when the algebraic hull of the cocycle is semisimple and k-unbounded.

In view of the condition of semi-simplicity imposed on H_k, we should note that the conclusion of Theorem 5.4 fails when the algebraic hull H_k of α has a non-trivial solvable radical, even if H_k is reductive with (non-trivial) compact center (see the example in §6). Hence the assumption of semi-simplicity of the algebraic hull of α is necessary. It is quite a natural assumption in this context, and appears for example in the cocycle super-rigidity theorem due to R.J. Zimmer. The assumption of k-unboundedness implies that H_k has no non-trivial compact factors, which is a necessary assumption in the Borel density theorem.

We now consider a simple corollary of Theorem 5.4, in the case where there exists a finite G-invariant measure on S. Before formulating the next result we make the following

Definition 1.2. *A locally compact second countable group G is said to have the* **Howe-Moore property** *if for every unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$ which does not contain G-invariant vectors, the functions $g \mapsto \langle \pi(g)v, w\rangle$ $(v, w \in \mathcal{H}_\pi)$ converge to 0 as g leaves compact sets in G.*

For example, simple non-compact algebraic groups over local fields are Howe-Moore groups. For other examples of Howe-Moore groups we refer to [LMoz]. We can now state

Corollary 5.6. *Let G be a locally compact second countable group with the Howe-Moore property, Λ a closed non-compact co-Følner subgroup, and S be an ergodic G-space with quasi-invariant measure. Let H_k denote k-points of an algebraically connected semisimple algebraic group defined over k. Let $\alpha : S \times G \to H_k$ denote a k-unbounded measurable cocycle with algebraic hull H_k. Then the algebraic hull of α_Λ is H_k as well.*

Proof. If G is a Howe-Moore group, and Λ is non-compact, then Λ is ergodic (in fact, mixing) on any ergodic G-space with finite invariant measure. Hence the result follows from Theorem 5.4. □

Remark: The same conclusions apply to co-Følner subgroups of semi-simple Lie groups with finite center and no non-trivial compact factors (or more generally S-arithmetic groups without non-trivial compact factors [Z4]), provided we assume that the action is irreducible, i.e. that every nontrivial normal subgroup of G acts ergodically on S. Clearly, in such an action, every non-compact subgroup acts ergodically, and if a co-Følner subgroup of G is compact, then G must be amenable (see Corollary 3.6 below).

For the case $k = \mathbb{R}$, we prove

Theorem 5.7. *Let G be a locally compact second countable group, S a G-space with a quasi-invariant measure and Λ a co-Følner subgroup of G acting ergodically on S. Let $H_{\mathbb{R}}$ denote the $\mathbb{R}$-points of an algebraically connected semisimple algebraic group defined over $\mathbb{R}$, with no non-trivial compact factors. If $\alpha : S \times G \to H_{\mathbb{R}}$ is a measurable cocycle which has algebraic hull $H_{\mathbb{R}}$, then $\alpha_\Lambda : S \times \Lambda \to H_{\mathbb{R}}$ has algebraic hull $H_{\mathbb{R}}$ as well.*

We remark that the assumption of the absence of non-trivial compact factors in Theorem 5.7 is clearly necessary. Indeed, if $\Lambda \subset G$ is a co-Følner subgroup, then of course it is also a co-Følner sugroup of $G \times K$, for any compact group K. If G and K are real algebraic groups, then clearly $\Lambda \subset G$ is not Zariski dense in $G \times K$. Hence the conclusion fails when S is a one point space, $H = G \times K$, Λ is a lattice of G and α is the identity homomorphism.

As shown in §6 the semi-simplicity of $H_{\mathbb{R}}$ is also necessary. Theorem 5.7 shows that restricting a real valued cocycle to a co-Følner subgroup will result in no loss of information regarding the simple non-compact factors of the algebraic hull of the cocycle. By the foregoing remarks, this is the best possible conclusion that can be obtained.

Theorems 5.4, 5.7 and Corollary 5.6 have more general formulations that apply to pull-backs of cocycles, rather than their restrictions. These generalization will be formulated and proved in §5.

The results above rely upon a lemma due to Furstenberg [Fu1] (see also [Z4] and [D1]), which is the main tool in Furstenberg's proof of the Borel density theorem. It is here that the assumption that the cocycle α is k-unbounded becomes crucial, and is used to compensate for the fact that Λ is not necessarily a lattice. In this regard, see the remark after Proposition 5.5 in §5.

In §3 we give examples of co-Følner subgroups. In particular, subgroups $\Lambda \subset G$ such that G/Λ has subexponential growth (as a Riemannian manifold, see §3) are co-Følner subgroups. It is shown in [S2] that such subgroups are Zariski dense; in fact the same proof shows that if Λ is a co-Følner subgroup and G is semisimple with finite center and no compact factors, then Λ is Zariski dense. It should be pointed out that if G is a semi-simple group possessing a non-compact factor group with property T, and Λ is co-Følner subgroup which projects densely to each factor group of G of lower dimension, then Λ must necessarily have co-finite volume in G. Hence examples of co-Følner subgroups interesting in our context arise only from simple groups of $\mathbb{R}$-rank equal to 1 or semisimple groups all of whose factors have $\mathbb{R}$-rank one (and do not have property T).

As an application of the results above, we note the following homomorphism theorems, generalizing the Borel density Theorem:

Theorem 6.1. *Let G be any locally compact second countable group, Λ a closed co-Følner subgroup and let $\pi : G \to H_k$ be any representation into the k-points of an algebraically connected semisimple algebraic group defined over k. If $\pi(G)$ is Zariski dense and π is k-unbounded, then $\pi(\Lambda)$ is Zariski dense as well.*

Theorem 6.2. *Let G be a locally compact second countable group. Let Λ denote a closed non-compact co-Følner subgroup, and $\pi : G \to H_{\mathbb{R}}$ a measurable homomorphism of G into the $\mathbb{R}$-points $H_{\mathbb{R}}$ of an algebraically connected semi-simple algebraic group defined over $\mathbb{R}$, with no non-trivial compact factors. Then, if $\pi(G)$ is Zariski dense, the same holds for $\pi(\Lambda)$.*

Another application is the identification of algebraic hulls, as follows:

Theorem 6.3. *Let $L \subset G$ be a closed subgroup of a locally compact second countable group G and let $\pi : G \to H_k$ be any representation into the k-points of an algebraic group defined over k, such that $\pi|_L$ is k-unbounded. Let $\alpha : G/\Lambda \times G \to H_k$ be the cocycle corresponding to the (restriction to Λ of the) representation π. If the Zariski closure of $\pi(L)$ is semisimple and L is ergodic on G/Λ, then the algebraic hull of α_L is the Zariski closure of $\pi(L)$.*

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2. Preliminaries

Given a G-space S with a quasi-invariant Borel measure μ, we say that μ is ergodic if any measurable G-invariant subset $A \subset S$ is either null or conull. Whenever this does not happen, we can decompose S into subsets on which the above holds true for suitable measures. In fact, there exist a measurable metrizable space $(E_S^G, \tilde{\mu})$, that is the space of G-ergodic components of S, and a measurable map $\sigma : S \to E_S^G$ with the properties that, for almost every $e \in E_S^G$, the space $(\sigma^{-1}(e), \mu_e)$ is an ergodic probability G-space, where the measures μ_e arise from the decomposition $\mu(A) = \int_{e \in E_S^G} \mu_e(A \cap \sigma^{-1}(e)) d\tilde{\mu}(e)$, for any measurable $A \subseteq S$, (see [V] for μ a finite G-invariant measure and [E], [R] for the general case).

Let $\alpha : S \times G \to H$ be a measurable cocycle, that is a measurable function satisfying the relation $\alpha(s, g_1 g_2) = \alpha(s, g_1)\alpha(sg_1, g_2)$ for a.e. $s \in S$ and for all $g_1, g_2 \in G$. We say that two cocycles $\alpha, \beta : S \times G \to H$ are equivalent, or cohomologous, and we write $\alpha \sim \beta$, if there exists a Borel map $\theta : S \to H$ such that, for all $g \in G$, $\alpha(s, g) = \theta(s)\beta(s, g)\theta(sg)^{-1}$, for a.e. $s \in S$.

In the case in which the G-action on S is transitive, namely $S \sim G/G_0$, we mentioned already that there is a bijective correspondence between equivalence classes of cocycles from $G/G_0 \times G$ into H and conjugacy classes of homomorphisms of G_0 into H [Z4, Prop. 4.2.13]. If $\gamma : G/G_0 \to G$ is a measurable section satisfying $\gamma([G_0]) = e$ then a cocycle $\alpha_\pi : G/G_0 \times G \to H$ corresponding to the homomorphism $\pi : G_0 \to H$ is defined, for every $x \in G/G_0$ and all $g \in G$, by $\alpha_\pi(x, g) = \pi(\gamma(x) g \gamma(xg)^{-1})$. It is easy to see that the equivalence class of α_π is independent of the choice of γ. Moreover, we have $\alpha_\pi(G/G_0 \times G) = \pi(G_0)$.

Given a locally compact group G_0, let (T, m) be a left G_0-space with a quasi-invariant Borel measure m and let $\alpha : S \times G \to G_0$ be a measurable cocycle. Then we can define the skew product G-space $(S \times_\alpha T, \mu \times m)$ where the right action is given by $(s, t)g = (sg, \alpha(s, g)^{-1} t)$. It is easy to see that equivalent cocycles give isomorphic G-spaces. Moreover when $G = G_0$, $S \times_\alpha T$ is equivalent to $S \times T$ (with the product action of G) if and only if α is equivalent to the identity cocycle $(s, g) \to g$.

A particularly important example of a skew product G-space arises when $S = G/G_0$ and the cocycle α corresponds to the identity homomorphism $G_0 \to G_0$, in which case the space $G/G_0 \times_\alpha T$ is called the G-space induced from the G_0-space T. In this case one can give an alternative description as follows: the action of G_0 on $G \times T$ given by $(g, t)g_0 = (gg_0, g_0^{-1} t)$ and the action of G on $G \times T$ given by $(g, t)h = (h^{-1} g, t)$ commute and it is easy to show that $(G \times T)/G_0$ and $G/G_0 \times_\alpha T$ are isomorphic G-spaces. One more particular case that will be of interest to us is when $T = G_0/G_1$, where $G_1 \subset G_0 \subset G$ are all closed subgroups; it is once again easy to see that as measurable G-spaces G/G_1, $(G \times G_0/G_1)/G_0$ and $G/G_0 \times_\alpha G_0/G_1$ are all isomorphic.

If S is an ergodic G-space and H is an algebraic group, a construction due to R.J. Zimmer assigns to each cocycle α a conjugacy class of algebraic subgroups. The conjugacy class is an invariant of the cohomology class of

α, and it is given by the following:

PROPOSITION 2.1 [Z4, Prop. 9.2.1]. *Let (S, μ) be an ergodic G-space, H be a k-algebraic group with k-points H_k and $\alpha : S \times G \to H_k$ be a measurable cocycle. Then there exists a unique (up to conjugacy) k-algebraic subgroup $L \subseteq H$ such that α is cohomologous to a cocycle taking values in L_k but not equivalent to a cocycle taking values in the k-points of a proper k-algebraic subgroup of L. L_k is called the* algebraic hull *of α.*

For cocycles taking values in compact groups a notion of a compact hull was introduced by R.J. Zimmer in [Z1].

PROPOSITION 2.2 [Z1]. *Given a cocycle $\beta : S \times G \to C$, where C is compact, there exists a unique (up to conjugacy) smallest compact subgroup $C' \subset C$ such that β is equivalent to a cocycle taking values in C', but is not equivalent to a cocycle taking values in a proper closed subgroup. C' is called the* compact hull *of β.*

REMARK: It is well known that compact real groups are real algebraic and that, however, this property does not hold over $\mathbb{C}$ or any local field. For example, $SU(2, \mathbb{C})$ is Zariski dense in $SL(2, \mathbb{C})$, and $SL(2, \mathbb{Z}_p)$ is Zariski dense in $SL(2, \mathbb{Q}_p)$.

Suppose now a cocycle α taking values in a non-compact algebraic group is equivalent to a cocycle taking values in a compact subgroup. As a consequence of the foregoing remarks, when $k = \mathbb{R}$, α cannot be Zariski dense. For $k \neq \mathbb{R}$, α may be both Zariski dense and equivalent to a cocycle taking values in a compact subgroup. This fact implies that the application of Furstenberg's Lemma yields a weaker conclusion in this case, and accounts for the assumption of k-unboundedness of the cocycle that appears in Theorems 5.4, 5.8 and 6.1.

3. Følner Actions and Co-Følner Subgroups

DEFINITION 3.1. *Let G be a locally compact second countable acting on a space X with G-invariant measure ν. We say that the G-action on (X, ν) has the* Følner property *if for every $\epsilon > 0$ and for every compact set $K \subset G$ there exists a measurable set $A \subset X$ of positive finite measure, such that*

$$\frac{\nu(gA \bigtriangleup A)}{\nu(A)} < \epsilon$$

for every $g \in K$.

REMARK: It is easy to see that the above definition is equivalent to the following. The G-action on (X, ν) has the Følner property if there exists a sequence $\{A_n\} \subset X$ of measurable sets of positive finite measure, such that

$$\lim_{n\to\infty} \frac{\nu(gA_n \triangle A_n)}{\nu(A_n)} \to 0$$

uniformly on compact subsets of G. Such a sequence is usually called an asymptotically invariant sequence, or a *Følner sequence.* We should also point out that such actions were first introduced by F. Greenleaf (under the name of amenable actions) in [G], where it is proven that the existence of a Følner sequence is implied by the existence of a topological left invariant mean on $L^\infty(X, \nu)$. For further discussion of such actions we refer to [Ey], [Z6].

The two simplest examples of actions with the Følner property, which motivate the definition above, are the action of an amenable group G on itself, and the action of a group on a space with finite invariant measure. To better illustrate other examples, we recall the following characterization of the Følner property in terms of representation theory.

Recall that if π is a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$, an $n \times n$ submatrix of π is a function $f : G \to M^{n\times n}(\mathbb{C})$ (the space of $n \times n$ complex matrices), defined by $f(g)_{ij} = \langle \pi(g)v_i, v_j\rangle$, where $\{v_1, \dots, v_n\}$ is an orthonormal set in $\mathcal{H}_\pi$. The following definition is due to Fell, [F1,2].

DEFINITION 3.2. *Let π, σ be unitary representations of G on $\mathcal{H}_\pi, \mathcal{H}_\sigma$ respectively. We say that σ is* weakly contained *in π, and we write $\sigma \prec \pi$ if for any compact set $K \subset G$, any $\epsilon > 0$ and any $n \times n$ submatrix f_σ of σ, there exists an $n \times n$ submatrix f_π of π, which approximates f_σ on K, namely such that $||f_\pi(g) - f_\sigma(g)|| < \epsilon$ for any $g \in K$.*

The following theorem is well known, and generalizes the fact that if the regular representation of G weakly contains the identity representation then the action of G on itself by translations has the Følner property, or, in other words, G is amenable (see [Ey] or [L], for instance). Since this result is a main source of examples of actions with the Følner property (as will be seen below), we reproduce its proof here for completeness.

Let $\pi : G \to \mathcal{U}(L^2(X, \nu))$ denote the unitary representation which, for any $f \in L^2(X, \nu)$ is defined by $(\pi(g)f)(x) = f(g^{-1}x)$.

Theorem 3.3 ([G],[Ey]). *The action of any locally compact second countable G on (X, ν) has the Følner property if and only if the representation $\pi : G \to \mathcal{U}(L^2(X, \nu))$ weakly contains the one-dimensional trivial representation.*

Proof. It is straightforward to check that π weakly contains the one-dimensional identity representation if and only if for every $\epsilon > 0$ and any compact set $K \subset G$, there exists $f \in L^2(X, \nu)$ with $||f||_2 = 1$, such that $||\pi(g)f - f||_2 < \epsilon$, for every $g \in K$.

With this in mind, assume now that the G-action on X has the Følner property; then, if $\epsilon > 0$ and $K \subset G$ is a compact set, let $N_0 > 0$ such that, if $n > N_0$ and $g \in K$, then $\frac{\nu(gA_n \triangle A_n)}{\nu(A_n)} < \epsilon$, where $\{A_n\} \subset X$ is a Følner sequence. Then, if we define $f = \frac{\chi_{A_n}}{\nu(A_n)^{1/2}}$, it is straightforward to check that $||\pi(g)f - f||_2^2 = \frac{\nu(gA_n \triangle A_n)}{\nu(A_n)} < \epsilon$, for $g \in K$.

To prove the converse, let λ denote the left Haar measure on the group G. Consider the following:

Claim. For any compact set $K \subset G$ of positive measure, any $\epsilon > 0$ and any $\delta > 0$, there exists a measurable set $A \subset X$ of positive finite measure such that $\frac{\nu(gA \triangle A)}{\nu(A)} < \epsilon$ for every $g \in N$, where $N \subset K$ satisfies $\lambda(K \backslash N) < \delta$.

Proving the claim will suffice, and to see why, fix $\epsilon > 0$ and a compact set $K \subset G$, of positive measure. Define another compact set K' by $K' = K \cup KK$. Then, by the above claim, for any $\delta > 0$, there exist measurable sets $A \subset X$ and $N \subset K'$ with $\lambda(K' \setminus N) < \delta$ such that $\frac{\nu(gA \triangle A)}{\nu(A)} < \frac{\epsilon}{2}$, for any $g \in N$. Moreover, if $g_1, g_2 \in N$, it is easy to see that

$$\nu(g_1 g_2^{-1} A \triangle A) \leq \nu(g_1 A \triangle A) + \nu(g_2 A \triangle A) < \epsilon \nu(A)$$

and we shall have proven our theorem (assuming the claim) if we can show that $K \subseteq NN^{-1}$. To this purpose, let us choose $\delta = \frac{1}{4}\lambda(K)$. Then, using the left invariance of the Haar measure, we have that for any $k \in K$

$$4\delta = \lambda(K) = \lambda(kK) \leq \lambda(kK' \cap K') \leq \lambda(kN \cap N) + 2\delta \ ,$$

that is $\lambda(kN \cap N) > 0$; this implies that $K \subset NN^{-1}$, as stated.

To prove the claim, note that by hypothesis, given a compact set $K \subset G$ of positive measure, any $\delta > 0$ and $\epsilon > 0$, if we set $\epsilon_0 = \frac{\epsilon^2 \delta^2}{4\lambda(K)^2}$, there exists $f \in L^2(X, \nu)$ such that $||f||_2 = 1$ and $||\pi(g)f - f||_2 < \epsilon_0$, for $g \in K$. We can assume (passing to the real or imaginary part of f if necessary) that f is real, and define $F = f^2$, so that $||F||_1 = 1$. By an easy computation using the Cauchy-Schwartz inequality, $||\pi(g)F - F||_1 < 2\sqrt{\epsilon_0}$, for $g \in K$. For any $t \in \mathbb{R}$, $t > 0$, let $A_t = \{x \in X : F(x) \geq t\}$; then clearly $1 = ||F||_1 =$

$\int_0^\infty \nu(A_t)dt$ and $||\pi(g)F - F||_1 = \int_0^\infty \nu(gA_t \triangle A_t)dt$. But then

$$\int_0^\infty \int_K \nu(gA_t \triangle A_t)d\lambda\, dt = \int_K \int_0^\infty \nu(gA_t \triangle A_t)dt\, d\lambda = \int_K ||\pi(g)F - F||_1 d\lambda <$$

$$< 2\sqrt{\epsilon_0}\lambda(K) = \epsilon\delta = \epsilon\delta \int_0^\infty \nu(A_t)dt = \int_0^\infty \epsilon\delta\nu(A_t)dt \ ,$$

which implies that for some $t \in \mathbb{R}$

$$\int_K \nu(gA_t \triangle A_t)d\lambda < \epsilon\delta\nu(A_t) \ .$$

Hence the λ-measure of the $k \in K$ such that $\frac{\nu(gA_t \triangle A_t)}{\nu(A_t)} \geq \epsilon$ has to be less than δ, completing the proof. □

Definition 3.4. *Let G be a locally compact group and Λ be a closed subgroup. We say that Λ is a* co-Følner *subgroup of G if the action of G on G/Λ has a G-invariant measure and satisfies the Følner property.*

Proposition 3.5. *Let $\Lambda \subset \Gamma \subset G$ be locally compact groups such that Λ is co-Følner in Γ and Γ is co-Følner in G. Then Λ is co-Følner in G.*

Proof. Because of Theorem 3.3, the trivial one-dimensional representation of Γ is weakly contained in the representation of Γ on $L^2(\Gamma/\Lambda)$ by translations. Inducing both representations up to G and recalling that weak containment is preserved by this operation [F2], we obtain that the representation of G on $L^2(G/\Gamma)$ is weakly contained in the representation induced to G from the representation of Γ on $L^2(\Gamma/\Lambda)$. But the former weakly contains the trivial representation of G, and latter is equivalent to the representation of G on $L^2(G/\Lambda)$ by induction in stages [M1]. By transitivity of weak containment and Theorem 3.3 again, we conclude that Λ is a co-Følner of G. □

Corollary 3.6. *If G is a non-amenable locally compact second countable group and Λ is a co-Følner subgroup, then Λ is not amenable (so, in particular, Λ is not compact).*

Definition 3.7. *Let G be a locally compact, compactly generated, topological group, Λ a closed subgroup of G and $p : G \to G/\Lambda$ the canonical projection. We say that G/Λ has* subexponential growth *if G/Λ has a G-invariant measure ν and for every relatively compact neighborhood of the identity $e \in V \subset G$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\big(p(V^n)\big) = 0 \ .$$

A first class of examples of co-Følner subgroups is given by the following:

PROPOSITION 3.8. *Let Λ be a closed subgroup of a locally compact topological group G. If G/Λ has subexponential growth, then Λ is a co-Følner subgroup of G.*

Proof. If G/Λ has subexponential growth there exists a sequence of integers $n_j \to \infty$ such that

$$\lim_{j\to\infty} \frac{\nu(A_{n_j+1}) - \nu(A_{n_j})}{\nu(A_{n_j})} = 0$$

where $A_{n_j} = p(V^{n_j}) \subset G/\Lambda$. We claim that $\{A_{n_j}\}$ is a Følner sequence. Let $K \subset G$ be any compact set and let N be large enough that $K \subset V^N$. Fix $\epsilon > 0$ and let j_0 be such that if $j \geq j_0$

$$\frac{\nu(A_{n_j+N} \bigtriangleup A_{n_j})}{\nu(A_{n_j})} = \frac{\nu(A_{n_j+N}) - \nu(A_{n_j})}{\nu(A_{n_j})} < \frac{\epsilon}{2}$$

where the equality holds since $A_{n_j} \subseteq A_{n_j+N}$. Moreover, it is easy to see that, since $K \subset V^N$ and because of the G-invariance of ν, we have that $2\nu(A_{n_j+N} \bigtriangleup A_{n_j}) \geq \nu(gA_{n_j} \bigtriangleup A_{n_j})$. Hence

$$\frac{\nu(gA_{n_j} \bigtriangleup A_{n_j})}{\nu(A_{n_j})} < \epsilon$$

if $j \geq j_0$ and $g \in K$, so that G acts with the Følner property on G/Λ. □

EXAMPLE 3.9: Let Γ be a cocompact torsion-free lattice in $PSL(2,\mathbb{R}) = G$ and K a maximal compact subgroup, so that $M = K\backslash G/\Gamma$ is a closed orientable surface of genus $g \geq 2$. Let $0 \to \Lambda \to \Gamma \to \mathbb{Z} \to 0$ be an exact sequence. Then G/Λ has linear growth, being the unit tangent bundle of an infinite cyclic cover of M. Using Proposition 3.8, it follows that the action of G on G/Λ has the Følner property. Similar examples exist for lattices $\Gamma \subset SO(1,n)_{\mathbb{R}}$ or $\Gamma \subset SU(1,n)$, such that $H^1(\Gamma,\mathbb{R}) = \mathrm{Hom}(\Gamma,\mathbb{R})$ is not trivial (see for example [S2]). By the normal subgroup theorem due to G.A. Margulis (see [Z4, Ch 8]), the first cohomology group is always trivial for irreducible lattices in higher rank semi-simple groups.

EXAMPLE 3.10: A large class of examples which do not arise from the situation in which G/Λ has subexponential growth are given as follows. Consider the free group on two generators $\mathbb{F}_2 = \Gamma$ embedded as a lattice in $SL(2,\mathbb{R}) = G$. Fix a surjective homomorphism $\varphi : \Gamma \to S$ onto a solvable (or more generally, amenable) group S with exponential growth (see e.g. [Ro]) and let $\Lambda = \operatorname{Ker}\varphi$. Since S is amenable, the regular representation of S weakly contains the one-dimensional identity representation of S; it follows that the one-dimensional identity representation of Γ is weakly contained in the representation of Γ on $L^2(\Gamma/\Lambda)$ by translations. Proposition 3.5 shows that Λ is co-Følner in G.

EXAMPLE 3.11: When a co-Følner subgroup Λ is a normal subgroup of a lattice in a simple non-compact algebraic group G (as are the examples above), it is obviously Zariski dense, since the lattice, and hence G, normalize its Zariski closure. Note however, that if S is any finitely generated amenable group and $S_0 \subset S$ any subgroup, then S_0 is co-Følner in S, since there is an invariant mean on $L^\infty(S/S_0)$, and therefore also a Følner sequence [G]. Hence, if $\varphi : \mathbb{F}_r \to S$ is a presentation, $\varphi^{-1}(S_0)$ is a co-Følner subgroup of $\mathbb{F}_r$, which is normal if and only if S_0 is normal in S. In fact, it is possible to construct a subgroup $\Lambda \subset \mathbb{F}_r$ which does not contain any non-trivial normal subgroup of $\mathbb{F}_r$ and such that the Cayley graph of $\mathbb{F}_r/\Lambda$ has polynomial growth. We thank S. Mozes for bringing this fact to our attention.

4. Invariant Maps and Limits of Measures

Recall the following definitions:

(1) If (S,μ) is a G-space, T is an H-space, and $\alpha : S \times G \to H$ is a measurable cocycle, we say that a map φ is α-invariant if $\varphi(sg) = \alpha(s,g)\varphi(s)$ for every $g \in G$ and almost every $s \in S$.

(2) If $p : S' \to S$ is a G-equivariant map between two G-spaces (S',μ') and (S,μ) satisfying $p_*\mu' = \mu$, and $\alpha : S \times G \to H$ is a cocycle, the pull-back of α is the cocycle $\alpha^* : S' \times G \to H$ given by $\alpha^*(s',g) = \alpha(p(s'),g)$.

The proof of the theorems in the next section will make essential use of the following result, a particular case of which, namely S consisting of one point, was proven in [S2]. In that case, it asserts that if a G-equivariant map exists from a Følner G-space to a compact G-space, then the compact space admits a G-invariant measure.

PROPOSITION 4.1. *Let G, Γ and H be locally compact second countable groups, (S,μ) a G-space with a quasi-invariant measure, Y a compact metric H-space, and $\alpha : S \times G \to H$ a cocycle. Let (X,ν) be a Følner action of Γ, $\beta : S \times G \to \Gamma$ a cocycle and $S \times_\beta X$ the skew product action of G. If there is an α^*-invariant map $\varphi : S \times_\beta X \to Y$, where α^* denotes the pull-back of the cocycle α from S to $S \times_\beta X$, then there exists an α-invariant map*

$\Phi : S \to M(Y)$, where $M(Y)$ denotes the space of probability measures on Y.

Proof. According to the remark after Definition 3.1, let $\{A_n\}$ be a Følner sequence for the Γ-action on X. Define a sequence of probability measures ν_n on X by $\nu_n(h) = \frac{1}{\nu(A_n)} \int_{A_n} h d\nu$, where $h \in L^1(X,\nu)$. The map $\varphi : S \times_\beta X \to Y$ gives rise to a family of maps $\varphi_s : X \to Y$, with the property that, for a.e. $(s,x) \in S \times_\beta X$ and every $g \in G$, $\varphi_{sg}(\beta(s,g)^{-1}x) = \alpha^*(s,x,g)\varphi_s(x) = \alpha(s,g)\varphi_s(x)$, where $\varphi_s(x) = \varphi(s,x)$. We use this family of maps φ_s to define, for almost every $s \in S$, a sequence of measures $\nu_n^s \in M(Y)$, by

$$\nu_n^s(f) = \int_X f d(\varphi_s)_* \nu_n = \frac{1}{\nu(A_n)} \int_{A_n} (f \circ \varphi_s) d\nu$$

where $f \in C(Y)$.

Let us first observe the following invariance property of the sequence ν_n^s, which follows from the Følner property of the Γ-action on X. For almost every $s \in S$, and any continuous function $f \in C(Y)$

$$\begin{aligned}
&\lim_{n\to\infty} \left|\nu_n^{sg}(f) - \nu_n^s(\alpha(s,g)\cdot f)\right| \\
&= \lim_{n\to\infty} \frac{1}{\nu(A_n)} \left| \int\limits_{A_n} (f \circ \varphi_{sg})(x) d\nu(x) - \int\limits_{A_n} \big((\alpha(s,g)\cdot f) \circ \varphi_s\big)(x) d\nu(x) \right| \\
&= \lim_{n\to\infty} \frac{1}{\nu(A_n)} \left| \int\limits_{A_n} f\big(\varphi_{sg}(x)\big) d\nu(x) - \int\limits_{A_n} f\big(\alpha(s,g)\varphi_s(x)\big) d\nu(x) \right| \\
&= \lim_{n\to\infty} \frac{1}{\nu(A_n)} \left| \int\limits_{A_n} f\big(\varphi_{sg}(x)\big) d\nu(x) - \int\limits_{A_n} f\big(\varphi_{sg}(\beta(s,g)^{-1}x)\big) d\nu(x) \right| \\
&= \lim_{n\to\infty} \frac{1}{\nu(A_n)} \left| \int\limits_{A_n \setminus \beta(s,g)A_n} f\big(\varphi_{sg}(x)\big) d\nu(x) - \int\limits_{\beta(s,g)A_n \setminus A_n} f\big(\varphi_{sg}(x)\big) d\nu(x) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{\nu(A_n)} ||f||_\infty \, \nu\big(\beta(s,g)A_n \triangle A_n\big) = 0
\end{aligned}$$

Suppose we could find a subsequence n_k, with the property that for almost all $s \in S$, the measures $\lambda_n^s = \frac{1}{n+1}\sum_{k=0}^n \nu_{n_k}^s$ converge in the w^*-topology in $M(Y)$ to a measure ν^s. This would certainly imply the desired result since, in that case, the measurable map $\Phi : S \to M(Y)$ defined by $\Phi(s) = \nu^s$ would be α-invariant, namely the measures ν^s would satisfy the relation $\nu^{sg} = \alpha(s,g)\nu^s$. Indeed $|\nu^{sg}(f) - \nu^s(\alpha(s,g)\cdot f)| = \lim_{n\to\infty} |\lambda_n^{sg}(f) - \lambda_n^s(\alpha(s,g)\cdot f)| = 0$. Hence we need only establish the following:

PROPOSITION 4.2. *Let (S,μ) be a standard Lebesgue probability space and let Y be a compact metric space. Assume $s \to \nu_n^s \in M(Y)$ is a sequence of measurable functions from S to $M(Y)$, where measurability in $M(Y)$ is determined by the Borel σ-algebra generated by the w^*-topology. Then there exist a subsequence n_k, independent of s, and measures $\nu^s \in M(Y)$ such that for almost every $s \in S$ the uniform averages $\lambda_n^s = \frac{1}{n+1}\sum_{k=0}^n \nu_{n_k}^s$ converge to ν^s in the w^*-topology.*

Proposition 4.2 is based on the following result due to J. Komlós [K]:

Theorem 4.3 [K]. *Let S be a standard Lebesgue probability space, and $u_n(s)$ a sequence of L^1-functions satisfying $||u_n|| \leq C$, $\forall n \geq 0$. Then there exist a subsequence v_n of u_n and an L^1-function u, with the property that for every subsequence w_n of v_n, the uniform averages $\frac{1}{n+1}\sum_{k=0}^n w_k(s)$ converge to $u(s)$ for almost every $s \in S$.*

Proof of Proposition 4.2. Let $\{f_i\} \in C(Y)$ be a sequence of functions which is dense in $C(Y)$ in the uniform norm. Let us define a sequence $u_m^1(s) = \nu_m^s(f_1)$ which is certainly bounded and in L^1 so that we can apply Komlós' theorem to obtain a subsequence $u^1_{m_k^{(1)}}$ of u_m^1 and a function $u_1 \in L^1(S)$ such that uniform averages of any subsequence of $u^1_{m_k^{(1)}}$ converge almost everywhere to u_1. Clearly u_1 is bounded (in fact by the same constant) and we can now iterate the process. After the $(j-1)$-th step, having selected the $(j-1)$-th subsequence $m_k^{(j-1)}$, apply again Komlós' theorem to the sequence of functions given by $u^j_{m_k^{(j-1)}}(s) = \nu^s_{m_k^{(j-1)}}(f_j)$, so as to obtain a subsequence $u^j_{m_k^{(j)}}$ of $u^j_{m_k^{(j-1)}}$ and $u_j \in L^1(S)$, once again with the property that uniform averages of any subsequence of $u^j_{m_k^{(j)}}$ converge almost everywhere to u_j.

Now let us consider the diagonal sequence of measures given by $\nu^s_{m_j^{(j)}}$. By construction, the sequence $u^i_{m_j^{(j)}} = \nu^s_{m_j^{(j)}}(f_i)$ is a subsequence of $u^i_{m_k^{(i)}}$ as soon as $j > i$. Hence it has the property that, if λ_n^s is the probability measure defined by $\lambda_n^s = \frac{1}{n+1}\sum_{j=0}^n \nu^s_{m_j^{(j)}}$, then

$$\lambda_n^s(f_i) = \frac{1}{n+1}\sum_{j=0}^n \nu^s_{m_j^{(j)}}(f_i) = \frac{1}{n+1}\sum_{j=0}^n u^i_{m_j^{(j)}}(s) \to u_i(s)$$

almost everywhere. Hence, for almost every $s \in S$, we have a functional ν^s, defined so far only on the dense sequence $\{f_i\} \subset C(Y)$ by $\nu^s(f_i) = u_i(s) = \lim_{n\to\infty} \lambda_n^s(f_i)$. Now, given $f \in C(Y)$, choose a subsequence of the

f_i's converging to f. Fix $\epsilon > 0$, and assume that $||f_i - f_j||_\infty < \epsilon$ when $i, j \geq N(\epsilon)$. Then estimate as follows:

$$\left|\nu^s(f_i)-\nu^s(f_j)\right| \leq \left|\nu^s(f_i)-\lambda^s_n(f_i)\right|+\left|\lambda^s_n(f_i)-\lambda^s_n(f_j)\right|+\left|\lambda^s_n(f_j)-\nu^s(f_j)\right| < \epsilon$$

if n large enough. Therefore $\nu^s(f_i)$ is a Cauchy sequence, and we can define $\nu^s(f) = \lim_{i\to\infty} \nu^s(f_i)$. It is straightforward to check that $\nu^s \in M(Y)$ and this completes the proof. □

REMARK: We note that the family of probability measures ν^s_n can be viewed as a norm-bounded sequence of elements belonging to the Banach space $L^\infty(S, M(Y)) = L^1(S, C(Y))^*$. By Alaoglu's theorem, the sequence has a subsequence which converges (to an element F, say) in the w^*-topology. The limit function F does indeed satisfy that $F(s)$ is a probability measure for almost all $s \in S$, provided $F_n(s)$ are probability measures for every n and almost every $s \in S$. This argument can be used in the proof of Proposition 4.1, instead of Proposition 4.2. The point of Proposition 4.2 is to show the stronger statement that, for such a sequence F_n, it is possible to choose a subsequence whose uniform averages converge *pointwise almost everywhere for* $s \in S$, in the w^*-topology on $M(Y)$.

5. Algebraic Hulls

Let k be non-discrete, locally compact topological fields of characteristic zero. Given a G-space S and a group H, let us denote by $H^1(S \times G, H)$ the set of equivalence classes of measurable cocycles from $S \times G$ into H. We will use the following :

PROPOSITION 5.1 ([Z2], [I1]). *If L is a closed subgroup of a locally compact second countable group G and S is any L-space with an ergodic quasi-invariant measure, then there is a bijection*

$$\Psi : H^1(S \times L; H) \longrightarrow H^1(S \times (G/L) \times G; H) ,$$

which has the following properties:

(1) *If a cocycle $\alpha \in H^1(S \times L; H)$ is the restriction to L of a cocycle α_1 of a G-action on S, then the cocycle $\alpha^* \in H^1(S \times (G/L) \times G; H)$, defined by $\alpha^*(s, x, g) = \alpha_1(s, g)$, is cohomologous to $\Psi(\alpha)$. Moreover, the cocycle α is cohomologous to a cocycle taking values in a closed subgroup $H_1 \subset H$ if and only if there is an α^*-invariant function $\Phi : S \times (G/L) \to H/H_1$.*

(2) *If $H = H_k$ is an algebraic group, the correspondence preserves the algebraic hulls.*

Recall now that if Y is a topological space, Y is defined to be *tame* if it is T_0 and has a countable base for its topology; if Y carries an action of a locally compact second countable topological group L we define the action to be *tame* if both Y and Y/L are tame.

Examples of tame actions which will be of interest to us arise in the following way. Given any connected k-algebraic group H and any k-subgroup $M \subseteq H$, it was proven by C. Chevalley that there exist a k-rational representation $\pi : H_k \to GL(d+1, k)$ and a point $x \in \mathbb{P}^d(k)$, such that M_k is the stabilizer of x in H_k (see [B2] for a proof). This allows us to identify H_k/M_k with an H_k-orbit in the projective space, via the injection $i : H_k/M_k \to \mathbb{P}^d(k)$. If $M(\mathbb{P}^d(k))$ is the space of probability measures on $\mathbb{P}^d(k)$ and $M(H_k/M_k)$ the space of probability measures on H_k/M_k, we have an injection $i_* : M(H_k/M_k) \to M(\mathbb{P}^d(k))$, given by $i_*(\mu)(f) = \mu(f \circ i)$, where $f \in C(\mathbb{P}^d(k))$. It should be noted that the image of $M(H_k/M_k)$ in $M(\mathbb{P}^d(k))$ will not necessarily be closed. If we let H_k act on $\mathbb{P}^d(k)$ via the representation π, then the action of H_k on $M(\mathbb{P}^d(k))$ is tame by [Z4].

Furthermore, if $\pi : H \to SL(d+1, \mathbb{K})$ is a k-rational representation and the action of H_k on k^{d+1} is irreducible and faithful, then the stabilizer of a measure in $M(\mathbb{P}^d(k))$ will either be compact or will be contained in the k-points of a k-algebraic subgroup of strictly smaller dimension, ([Fu1], [Z4]).

The following result will be used in the proofs of both Theorem 5.4 and Theorem 5.7.

PROPOSITION 5.2. *Let $\alpha : S \times G \to H_k$ be a cocycle with algebraic hull H_k, $N \subset H$ a normal k-subgroup and $\rho : H \to H/N$ the canonical projection. Then the algebraic hull of $\rho \circ \alpha$ is the Zariski closure $\rho(H)_k$ of $\rho(H_k)$, in which $\rho(H_k)$ is of finite index.*

Proof. General facts about algebraic groups (see [Z4, §3.1]) reduce the proposition to the following standard result :

LEMMA 5.3. *Let $\alpha : S \times G \to H$ be a measurable cocycle and, if N is a normal subgroup of H, let $\alpha' = \rho \circ \alpha : S \times G \to H/N = H'$, where $\rho : H \to H/N = H'$ is the canonical projection. If $\rho \circ \alpha$ is equivalent to a cocycle into a subgroup $H'_1 \subset H'$ then α is equivalent to a cocycle into $\rho^{-1}(H'_1)$.*

Proof. By hypothesis there exist a measurable map $\varphi : S \to H'$ and a cocycle $\beta' : S \times G \to H'$ such that $\beta'(S \times G) \subseteq H'_1$ and $\beta'(s,g) = \varphi(s)\alpha'(s,g)\varphi(sg)^{-1}$, for $g \in G$ and a.e. $s \in S$. Now let $\theta : H' \to H$ be a measurable section of the projection ρ. Then the cocycle $\beta : S \times G \to H$ defined by $\beta(s,g) = (\theta \circ \varphi)(s)\alpha(s,g)(\theta \circ \varphi)(sg)^{-1}$ is equivalent to α and takes values in $\rho^{-1}(H_1') \subset H$. □

We are now ready to prove the first of the results on algebraic hulls stated in §1.

Theorem 5.4. *Let G be a locally compact second countable group, S a G-space with a quasi-invariant measure and Λ a co-Følner subgroup of G acting ergodically on S. Let H_k denote the k-points of an algebraically connected semisimple algebraic group defined over k. If $\alpha : S \times G \to H_k$ is a cocycle which is k-unbounded and has H_k as its algebraic hull, then $\alpha_\Lambda : S \times \Lambda \to H_k$ has H_k as its algebraic hull as well.*

Proof. By assumption Λ, and hence G, act ergodically on S. Let us denote the algebraic hull of α_Λ by $(H_1)_k \subseteq H_k$. We assume that $(H_1)_k \neq H_k$, and argue for a contradiction.

Choose a proper maximal algebraic subgroup M_k of H_k containing $(H_1)_k$. Let $\pi_0 : H_k \to SL(n,k)$ be a Chevalley representation such that M_k is the stability group in H_k of a line L_0. We can assume, without loss of generality, that π_0 is irreducible. Indeed, since H_k is semisimple, we can decompose the representation space into irreducible components. Then the projection of the line L_0 to each of the components has a stability group containing M_k, and since M_k is maximal, we can choose some irreducible component which contains a line L whose stability group is exactly M_k. We restrict our attention to this representation π of H_k and note that H_k acts irreducibly in the corresponding projective space $\mathbb{P}^d(k)$. In order to pass to a faithful representation, let $N_k \subset H_k$ be the kernel of the irreducible representation $\pi : H_k \to SL(d+1,k)$. Since by definition of M_k we have $N_k \subset M_k$, and since N_k is normal in H_k, if we define $H_k' = H_k/N_k$ and $M_k' = M_k/N_k$, we see that $\pi' : H_k' \to SL(d+1,k)$ is a faithful representation of H_k' and gives us an embedding $i : H_k/M_k \simeq H_k'/M_k' \to \mathbb{P}^d(k)$.

Now consider $\alpha^* : S \times G/\Lambda \times G \to H_k$ given by $\alpha^*(s,h\Lambda,g) = \alpha(s,g)$. By Proposition 5.1, $(H_1)_k$ is also the algebraic hull of α^* and there exists an α^*-invariant measurable map $\varphi : S \times G/\Lambda \to H_k/M_k$ which, because of the above embedding, can be thought of as an $\pi' \circ \alpha^*$-invariant map $\varphi : S \times G/\Lambda \to \mathbb{P}^d(k)$. Since $G/\Lambda = X$ is a Følner action of G, we can apply Proposition 4.1, where we take $\Gamma = G$, the identity cocycle given by $\beta(s,g) = g$, and the corresponding product action of G on $S \times_\beta X = S \times X$. We thus obtain an α-invariant map $\Phi : S \to M(\mathbb{P}^d(k))$. Because the action of G on S is ergodic and the action of H_k' on $M(\mathbb{P}^d(k))$ is tame, we conclude by the cocycle reduction lemma [Z4, Lemma 5.2.11] that there exists a measure $\omega \in M(\mathbb{P}^d(k))$ such that the map Φ takes values in the H_k'-orbit of ω. In other words, if α' is the cocycle $\alpha' : S \times G \to H_k \to H'_k$, Φ can be viewed as an α'-invariant map $\Phi : S \to H'_k/(H'_k)_\omega$ or equivalently the cocycle α' is cohomologous to a cocycle into $(H'_k)_\omega$. Note that, by Proposition 5.2, α' is Zariski dense in H'_k, and hence $(H'_k)_\omega$ is Zariski dense, and in particular, non-trivial. Since, as we observed before, H'_k acts irreducibly and faithfully, by Furstenberg's lemma ([Fu1] or [Z4, 3.2.15]) either $(H'_k)_\omega$ is compact, or it is contained in the k-points of a k-algebraic group strictly contained in H'_k. The first possibility is ruled out by assumption, and the second since $(H'_k)_\omega$ is Zariski dense in H'_k. We have arrived at a contradiction and the proof is complete. □

Let us note that the assumption that S has a finite G-invariant measure allows us to determine easily the compact hull of the restriction of a cocycle, when the cocycle takes values in a compact group. Indeed, in that case, if G is a non-compact group with the Howe-Moore property, then (see e.g. [Mo]) the restriction to Λ of an ergodic G-action on a space with finite G-invariant measure is still ergodic, provided Λ is any closed non-compact subgroup. We therefore have the following:

Proposition 5.5 [I1]. *Let G be a locally compact second countable group with the Howe-Moore property, $\Lambda \subset G$ a closed non-compact subgroup and (S,μ) a G-space with finite G-invariant measure. Let $\delta : S \times G \to K$ be a cocycle taking values in a compact group K. Then the compact hulls of δ and δ_Λ in K are the same.*

Proof. We sketch here the proof for the sake of completeness; the reader is referred to [I1, §4] for further details.

We may as well assume that the compact hull of δ is K and this is equivalent to the ergodicity of the action of G on $S \times_\delta K$, which is a G-space with finite invariant measure. The non-compactness of Λ and the Howe-Moore property imply that the restriction to Λ of the action on $S \times_\delta K$ is still ergodic and this implies that the compact hull of δ_Λ is K. □

Remark: It is shown in [I1] that Proposition 5.5 holds also in the case of a quasi-invariant measure if Λ is a lattice in a semisimple group G. This is not true anymore for a general co-Følner subgroup. In fact, even assuming that Λ acts ergodically on S (a condition that does not follow now from the ergodicity of G), the ergodicity of the G-action on $S \times_\delta K$ does not imply the ergodicity of Λ on $S \times_\delta K$, when G/Λ does not carry a finite G-invariant measure.

We now consider the case $k = \mathbb{R}$, and prove:

Theorem 5.7. *Let G be a locally compact second countable group, S a G-space with a quasi-invariant measure and Λ a co-Følner subgroup of G*

acting ergodically on S. Let $H_{\mathbb{R}}$ denote the $\mathbb{R}$-points of an algebraically connected semisimple algebraic group defined over $\mathbb{R}$, with no non-trivial compact factors. If $\alpha : S \times G \to H_{\mathbb{R}}$ is a measurable cocycle which has algebraic hull $H_{\mathbb{R}}$, then $\alpha_\Lambda : S \times \Lambda \to H_{\mathbb{R}}$ has algebraic hull $H_{\mathbb{R}}$ as well.

Proof. We follow the method and notation used in the proof of Theorem 5.4, taking $k = \mathbb{R}$. Applying the arguments used in the proof of Theorem 5.4, we conclude as before that α' is equivalent to a cocycle taking values in the stability group of a measure, denoted $(H'_{\mathbb{R}})_\omega$. Again Lemma 5.2 implies that $(H'_{\mathbb{R}})_\omega$ is Zariski dense in $H'_{\mathbb{R}}$, and by [Z4, 3.2.15] $(H'_{\mathbb{R}})_\omega$ must also be compact. Since compact real groups are real algebraic, we conclude that $(H'_{\mathbb{R}})_\omega = H'_{\mathbb{R}}$. Hence $H_{\mathbb{R}}$ has a non-trivial compact factor, contrary to our assumption. This contradiction completes the proof. □

We now consider the pullback of a cocycle rather than its restriction.

Theorem 5.8. *Let G be a locally compact second countable group and (S, μ) be an ergodic G-space, with quasi-invariant measure. Let $\alpha : S \times G \to H_k$ be a k-unbounded cocycle into the k-points H_k of an algebraically connected semisimple algebraic group defined over k, which has H_k as its algebraic hull. Let Γ be a locally compact second countable group, $\beta : S \times G \to \Gamma$ a cocycle and let X be a Følner action of Γ. Let $\alpha^* : S \times_\beta X \times G \to H_k$ be the pull-back of the cocycle α to the skew product action $S \times_\beta X$. If the skew product action is ergodic under G, then the cocycle α^* has H_k as its algebraic hull as well.*

Proof. If the algebraic hull of α^* is strictly contained in H_k, we assume it is contained in a maximal proper algebraic subgroup M_k. Using again the same argument as in the proof of Theorem 5.4, we pass to a quotient group H'_k of H_k that acts irreducibly and faithfully on $\mathbb{P}^d(k)$, via a representation $\pi' : H'_k \to SL(d+1, k)$, in which M'_k is the stability group of a line. As before, we embed H'_k/M'_k in $\mathbb{P}^d(k)$ and deduce that there exists a $\pi' \circ \alpha^*$-invariant map $\varphi : S \times_\beta X \to \mathbb{P}^d(k)$. By Proposition 4.1, there is an α'-invariant map $\Phi : S \to M(\mathbb{P}^d(k))$, where $\alpha' : S \times G \to H_k \to H'_k$. By the cocycle reduction lemma and Furstenberg's lemma (as in the proof of Theorem 5.4), either α' is equivalent to a cocycle taking values in a Zariski dense (see Lemma 5.2) compact subgroup of H'_k, or it is equivalent to a cocycle taking values in a proper algebraic subgroup. Both cases are ruled out by the assumptions, and so it follows that the algebraic hull of α^* is equal to H_k. □

REMARK: (1) The assumption of the ergodicity of the skew product action in Theorem 5.8 is required to insure the existence of the algebraic hull of α^*.

(2) We can obtain Theorem 5.4 from Theorem 5.8 setting $\Gamma = G$, $X = G/\Lambda$ and $\beta(s,g) = g$ and using Proposition 5.1.

Theorem 5.9. *Let G be a locally compact second countable group, and (S,μ) an ergodic G-action with quasi-invariant measure. Let $\alpha : S \times G \to H_{\mathbb{R}}$ be a cocycle into the $\mathbb{R}$-points $H_{\mathbb{R}}$ of an algebraically connected semisimple algebraic group defined over $\mathbb{R}$, with no non-trivial compact factors. Assume the algebraic hull of α equals $H_{\mathbb{R}}$. Let Γ be a locally compact second countable group, $\beta : S \times G \to \Gamma$ a cocycle and let X be a Følner action of Γ. Let $\alpha^* : S \times_\beta X \times G \to H_{\mathbb{R}}$ be the pull-back of the cocycle α to the skew product action $S \times_\beta X$. If the skew product action is ergodic under G, then the cocycle α^* has algebraic hull $H_{\mathbb{R}}$ as well.*

Proof. Combine the arguments in the proof of Theorem 5.8 with those of Theorem 5.7. □

6. Applications

The results in this section will be generalizations of the Borel density theorem, and are simple consequences of those in §5. We prove:

Theorem 6.1. *Let G be any locally compact second countable group, Λ a closed co-Følner subgroup and let $\pi : G \to H_k$ be any representation into the k-points of an algebraically connected semisimple algebraic group defined over k. If $\pi(G)$ is Zariski dense, and the homomorphism π is k-unbounded, then $\pi(\Lambda)$ is Zariski dense as well.*

Proof. Apply Theorem 5.4, taking S to be a one point space. The homomorphism π is a cocycle of the action, which is k-unbounded, by assumption. Hence the restriction of π to Λ has H_k as its algebraic hull, and $\pi(\Lambda)$ is Zariski dense. □

Theorem 6.2. *Let G be a locally compact second countable group, Λ a closed co-Følner subgroup. Let $H_{\mathbb{R}}$ the $\mathbb{R}$ points of an algebraically connected semi-simple algebraic group defined over $\mathbb{R}$, without non-trivial compact factors. Let $\pi : G \to H_{\mathbb{R}}$ be a representation of G, such that $\pi(G)$ is Zariski dense. Then $\pi(\Lambda)$ is Zariski dense as well.*

Proof. As in the proof of Theorem 6.2, take S to be a one point space, and then apply Theorem 5.7. □

EXAMPLE: It is easy to give examples of cocycles with non-semisimple algebraic hull, which do not satisfy the above Theorems.

First, let $\Gamma = \mathbb{F}_2$ and let $h : \Gamma \to \mathbb{R}$ be a homomorphism with dense image. Define $\Lambda = \mathrm{Ker}(h)$, so that Λ is a co-Følner subgroup of Γ. In order to have a target group which is not just unipotent, embed $i : \Gamma = \mathbb{F}_2 \to SL(2,\mathbb{R})$ as a lattice and define the homomorphism $\pi : \Gamma \to SL(2,\mathbb{R}) \times \mathbb{R}$, $\pi(\gamma) = (i(\gamma), h(\gamma))$. Then $\overline{\pi(\Gamma)} = SL(2,\mathbb{R}) \times \mathbb{R}$, but $\overline{\pi(\Lambda)} = SL(2,\mathbb{R})$. The same construction can be repeated with a homomorphism $h' : \Gamma \to \mathbb{R}/\mathbb{Z}$ or any other compact group.

Now regard $\Gamma = \mathbb{F}_2$ as a lattice in $G = SL(2,\mathbb{R})$, and consider the cocycles α that the homomorphisms h, h' and π determine in $H^1(G/\Gamma \times G, H)$, where H is one of the target groups above. The algebraic hull of the cocycles is in each case the Zariski closure of the image of Γ, which equals H, but the algebraic hull of α_Λ is a proper algebraic subgroup.

Such examples show similarly that the assumption of semisimple algebraic hull cannot be deleted in Theorems 5.4, 5.7, 5.8 as well.

Remark: Taking $G = H_k$ in Theorems 6.1, Λ to be a subgroup with subexponential co-growth in H_k, and π to be the identity, we recover several of the results of [S2], pertaining to Zariski density of such subgroups.

Theorem 6.3. *Let $L \subset G$ be a closed subgroup of a locally compact second countable group G and let $\pi : G \to H_k$ be any representation into the k-points of an algebraic group defined over k, such that $\pi|_L$ is k-unbounded. Let $\alpha : G/\Lambda \times G \to H_k$ be the cocycle coming from the (restriction to Λ of the) representation π. If the Zariski closure of $\pi(L)$ is semisimple and L is ergodic on G/Λ, then the algebraic hull of α_L is the Zariski closure of $\pi(L)$.*

Proof. We sketch here the proof, which uses the same techique as in [I3, Theorem 3.1], where we now use our results for co-Følner subgroups. By Proposition 5.1 there are bijections $H^1(G/\Lambda \times L; H_k) \leftrightarrow H^1(G/\Lambda \times G/L \times G; H_k)$ and $H^1(G/L \times \Lambda; H_k) \leftrightarrow H^1(G/L \times G/\Lambda \times G; H_k)$, which, together with the identification $H^1(G/\Lambda \times G/L \times G; H_k) \sim H^1(G/L \times G/\Lambda \times G; H_k)$, imply that there is a bijection in cohomology between cocycles for the action of L on G/Λ and of Λ on G/L, such that if a cocycle $\alpha : G/\Lambda \times L \to H_k$ comes from the restriction to Λ of a homomorphism $\pi : G \to H_k$, then the same is true for the corresponding cocycle for the action of Λ on G/L (see [I3, Lemma 3.3]). Hence the algebraic hull of α_L coincides with the algebraic hull of the cocycle $\beta_\Lambda : G/L \times \Lambda \to H_k$ which is the restriction to Λ of the cocycle over a transitive space coming from the restriction of π to Λ. By Theorem 5.4 the algebraic hull of β_Λ is the same as the algebraic hull of $\beta : G/L \times G \to H_k$, which is in turn the Zariski closure of $\pi(L)$. $\square$

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