A USEFUL FORMULA FROM BOUNDED COHOMOLOGY

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Contents

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1. Introduction

In [25] a theory of continuous bounded cohomology for locally compact groups was developed which proved itself to be rather useful and flexible at the same time. Bounded cohomology was originally defined by Gromov in 1982 and has already been used by several authors. The point of the theory developed in [25] is the introduction in this context of relative homological algebra methods in the continuous setting. Based on this theory, the authors developed a machinery which has already proved itself very fruitful in showing several rigidity results for actions of finitely generated groups or in finding new proofs of known results. We want to list here in very telegraphic style some results in which either bounded cohomology or continuous bounded cohomology play an essential role.

Minimal Volume (Gromov [37]): A geometric application to control the minimal volume of a smooth compact manifold by its simplicial volume, that is the seminorm of the fundamental class in ℓ^1-homology.

Actions on the Circle (Ghys [35]): E. Ghys observed that the Euler class of a group action by homeomorphisms on the circle admits a bounded representative, leading thus to the bounded Euler class of this action, which he showed determines it up to semiconjugacy (see § 2.3).

Maximal Representations in $\mathrm{Homeo}_+(\mathbb{S}^1)$ (Matsumoto, [48], see also [42]): A characterization of representations of surface groups which are semiconjugate to a hyperbolization as those with maximal Euler number (see § 3.2).

Stable Length (Bavard [5]): The stable length of commutators of a finitely generated group Γ vanishes if and only if the comparison map between bounded cohomology and ordinary cohomology

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}) \to \mathrm{H}^2(\Gamma,\mathbb{R})$$

is injective.

Characterization of Gromov Hyperbolic Groups (Mineyev [50, 51]): A finitely generated group is Gromov hyperbolic if and only if for every Banach Γ-module V, the comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma,V) \to \mathrm{H}^2(\Gamma,V)$ is surjective.

Central Extensions and their Geometry (Gersten [34]): If

$$0 \to \mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to 1$$

is a central extension of a finitely generated group Γ given by a bounded two-cocycle, then $\widetilde{\Gamma}$ is quasiisometric to $\Gamma\times\mathbb{Z}$. Applying this to $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$ and to the inverse image $\widetilde{\Gamma}$ of Γ in the universal covering of

$\mathrm{Sp}(2n,\mathbb{R})$, one obtains that $\widetilde{\Gamma}$ is quasiisometric to $\Gamma\times\mathbb{Z}$; since $\widetilde{\Gamma}$ has property (T) for $n\geq 2$, while $\Gamma\times\mathbb{Z}$ does not, this shows that property (T) is not a quasiisometry invariant.

Boundedness of Characteristic Classes (Gromov [37], Bucher-Karlsson [12]): Recently, M. Bucher-Karlsson, strengthening a result of Gromov, showed that characteristic classes of flat bundles admit cocycle representatives taking finitely many values, hence in particular they are bounded.

Orbit Equivalence (Monod–Shalom [54]): If Γ is a finitely generated group, then the nonvanishing of $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma,\ell^2(\Gamma)\big)$ is an invariant of measure equivalence, and this can be applied to show rigidity of certain products under measure equivalence (see § 2.3).

Theory of Amenable Actions (Burger–Monod [25]): For a locally compact group G acting on a standard measure space (S,μ), the amenability of the G-action (in the sense of Zimmer, [65]) is equivalent to the injectivity of the G-module $\mathrm{L}^\infty(S,\mu)$ in a sense appropriate for bounded cohomology (see Definition 2.8).

Rigidity Questions for Group Actions on Hermitian Symmetric Spaces: When G is a connected semisimple Lie group with finite center such that the associated symmetric space $\mathcal{X}$ is Hermitian, then there is a canonical continuous bounded class $\kappa\in\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ constructed using the Kähler form on $\mathcal{X}$. This allows to associate to any homomorphism $\rho:\Gamma\to G$ an invariant $\rho^{(2)}(\kappa)\in\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$, coined the *bounded Kähler class of* ρ, and which contains substantial information about the homomorphism. This is put to use to study various aspects of group actions on $\mathcal{X}$.

The first one concerns the case where Γ is a lattice in $\mathrm{SU}(1,p)$, ρ is the homomorphism $\Gamma\to\mathrm{SU}(1,q)$ obtained from injecting $\mathrm{SU}(1,p)$ in a standard way into $\mathrm{SU}(1,q)$ $(1\leq p\leq q)$, and the question is the local rigidity of ρ in the variety of representations of Γ into $\mathrm{SU}(1,q)$ (see § 5); an important part of this paper will be devoted to developing certain tools in continuous bounded cohomology which are instrumental in answering this question – see § 4 – (Burger–Iozzi [19, 14, 16], Koziarz–Maubon [45]).

The second aspect deals with the case in which G is in general the isometry group of a Hermitian symmetric space, Γ is the fundamental group of a compact oriented surface and the question concerns the geometric understanding of certain components of the representation variety of Γ into G, namely those formed by the set of *maximal representations* – see § 3.3 – (Burger–Iozzi–Wienhard [23, 21, 22], Wienhard [61], Burger–Iozzi–Labourie–Wienhard [13]).

The third aspect is when Γ is an arbitrary, say finitely generated, group: remarkably, if $\rho : \Gamma \to G$ has Zariski dense image and $\mathcal{X}$ is not of tube type, then the bounded Kähler class of ρ determines ρ up to conjugation, (Burger–Iozzi [18], Burger–Iozzi–Wienhard [20]).

The scope of these notes is to give a description of one underlying feature in continuous bounded cohomology common to these last results ([19, 14, 16, 23, 21]) as well as to the proof in [42] of Matsumoto's theorem in [48] and to Gromov's proof of Mostow rigidity theorem in [59]. More specifically, we prove an integral formula which involves specific representatives of bounded cohomology classes. Particular instances of this formula were proven already in [42] and [23], while here we give a treatment which unifies at least the first half of the statement. As this is rather technical, we postpone its statement to § 2 (Proposition 2.44 and also Principle 3.1) and § 4.2 (Proposition 4.9 and Principle 4.11), where the patient reader will be gently guided.

The paper is organized as follows. In § 2 we lay the foundation of continuous bounded cohomology for the noninitiated reader, who will be lead to the statement of an easy version of the main result in § 4. In § 3 we describe the instances in which the results of § 2.7 are used. In § 4 we give a complete proof of a more general version of the Formula in Proposition 2.44 from which the statements in Proposition 2.44 can be easily obtained. Finally, in § 5 we give the application which triggered Proposition 4.9, namely an original proof of a deformation rigidity theorem announced in 2000 in [19] and [42] with a sketch of a proof and also proven by analytic methods in 2004 in [45].

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2. Bounded Cohomology Preliminaries

We refer to [25], [53] and [17] for a complete account of different parts of the theory.

2.1. **Definition via the Bar Resolution.**

Let G be a locally compact group.

Definition 2.1. A *coefficient G-module E* is the dual of a separable Banach space on which G acts continuously and by linear isometries.

EXAMPLES 2.2. Relevant examples of coefficient G-modules in this note are:

(1) $\mathbb{R}$ with the trivial G-action;
(2) Any separable Hilbert space $\mathcal{H}$ with a continuous G-action by unitary operators;
(3) $\mathrm{L}^\infty(G/H)$ with the G-action by translations, where H is a closed subgroup of the second countable group G.
(4) $\mathrm{L}^\infty_{\mathrm{w}*}(S,E)$, that is the space of (equivalence classes of) weak*-measurable maps $f : S \to E$ from a G-space (S,μ) into a coefficient module. We recall that a regular G-space is a standard Borel measure space (S,μ) with a measure class preserving G-action such that the associated isometric representation on $\mathrm{L}^1(S,\mu)$ is continuous.

Now we proceed to define the standard complex whose cohomology is the continuous bounded cohomology with values in a coefficient module E. Let

$$\begin{aligned}\mathrm{C_b}(G^n,E) := \big\{ f : G^n \to E :\ & f \text{ is continuous and} \\ & \|f\|_\infty = \sup_{g_1,\dots,g_n\in G} \|f(g_1,\dots,g_n)\|_E < \infty \big\}\end{aligned}$$

be endowed with the G-module structure given by the G-action

$$(hf)(g_1,\dots,g_n) := hf(h^{-1}g_1,\dots,h^{-1}g_n)\,,$$

and let $\mathrm{C_b}(G^n,E)^G$ be the corresponding submodule of G-invariant vectors. Then the *continuous bounded cohomology* $\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$ of G with coefficients in E is defined as the cohomology of the complex

$$0 \longrightarrow \mathrm{C_b}(G,E)^G \xrightarrow{\ d\ } \mathrm{C_b}(G^2,E)^G \xrightarrow{\ d\ } \mathrm{C_b}(G^3,E)^G \xrightarrow{\ d\ } \dots \tag{2.1}$$

where $d : \mathrm{C_b}(G^n,E) \to \mathrm{C_b}(G^{n+1},E)$ is the usual homogeneous coboundary operator

$$(df)(g_0,\dots,g_n) := \sum_{j=0}^n (-1)^j f(g_0,\dots,\hat{g}_j,\dots,g_n)\,.$$

More precisely,

$$\mathrm{H}^n_{\mathrm{cb}}(G,E) := \mathcal{Z}\mathrm{C_b}(G^{n+1},E)^G / \mathcal{B}\mathrm{C_b}(G^{n+1},E)^G\,,$$

where

$$\mathcal{Z}\mathrm{C_b}(G^{n+1},E)^G := \ker\big\{ d : \mathrm{C_b}(G^{n+1},E)^G \to \mathrm{C_b}(G^{n+2},E)^G \big\}$$

are the homogeneous G-invariant n-cocycles, and

$$\mathcal{B}\mathrm{C_b}(G^{n+1},E)^G = \mathrm{im}\big\{ d : \mathrm{C_b}(G^n,E)^G \to \mathrm{C_b}(G^{n+1},E)^G \big\}$$

are the homogeneous G-invariant n-coboundaries.

Remark 2.3. If $\kappa \in \mathrm{H}^n_{\mathrm{cb}}(G,E)$, then we define

$$\|\kappa\| := \inf\left\{\|c\|_\infty : c \in \mathcal{Z}\mathrm{C}_{\mathrm{b}}(G^{n+1},E),\ [c]=\kappa\right\},$$

so that $\mathrm{H}^n_{\mathrm{cb}}(G,E)$ is a seminormed space with the quotient seminorm.

If we drop the hypothesis of boundedness in (2.1), we obtain the *continuous cohomology* of G, which we denote by $\mathrm{H}^n_{\mathrm{c}}(G,E)$. Thus continuous bounded cohomology appears as the cohomology of a subcomplex of the complex defining continuous cohomology, and thus we have a natural comparison map

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G,E) \to \mathrm{H}^\bullet_{\mathrm{c}}(G,E)\,.$$

Note however that in the case of continuous cohomology, the appropriate coefficients are just topological vector spaces with a continuous G-action (See [38], [9] and [6] for the corresponding homological algebra theory in continuous cohomology.)

If the group G is discrete, the assumption of continuity is of course redundant, and in this case the cohomology and bounded cohomology will be simply denoted by $\mathrm{H}^\bullet(G,E)$ or $\mathrm{H}^\bullet_{\mathrm{b}}(G,E)$ respectively. Observe that in this case a homological algebra approach was already initiated by R. Brooks [11] and later developed by N. Ivanov [43] and G. Noskov [56].

Exercise 2.4. Write down the complex of inhomogeneous cochains and the formula for the coboundary map in this setting; compare with [53, § 7.4] and/or [38].

2.2. **Low Degree.** We indicate briefly here what the bounded cohomology computes in low degrees. Notice however that in order to verify our assertions, one should mostly use the nonhomogeneous definition of continuous bounded cohomology (see Exercise 2.4).

- <u>*Degree $n=0$*</u> Since $\mathcal{B}\mathrm{C}_{\mathrm{b}}(G,E)=0$, then

$$\begin{aligned}\mathrm{H}^0_{\mathrm{cb}}(G,E) &= \mathcal{Z}\mathrm{C}_{\mathrm{b}}(G,E)^G\\ &= \left\{f : G\to E\ : f \text{ is constant and } G\text{-invariant}\right\}\\ &= E^G\,,\end{aligned}$$

that is the space of G-fixed vectors in E and, in fact, there is no difference between continuous cohomology and continuous bounded cohomology in degree 0.

- <u>*Degree $n=1$*</u> If we denote by ρ the (linear) isometric action of G on E, the cohomology group $\mathrm{H}^1_{\mathrm{c}}(G,E)$ classifies the continuous affine actions of G with linear part ρ, while $\mathrm{H}^1_{\mathrm{cb}}(G,E)$ the continuous affine actions of G with linear part ρ and with bounded orbits. In particular, if $E=\mathbb{R}$ is the trivial module, then $\mathrm{H}^1_{\mathrm{cb}}(G,\mathbb{R})=\mathrm{Hom}_{\mathrm{cb}}(G,\mathbb{R})=0$, and the same holds also if $E=\mathcal{H}$ is a separable Hilbert space (exercise) and if E is any reflexive separable Banach module, [53, Proposition 6.2.1].

• *Degree $n=2$* If G is a discrete group and A is an Abelian group (in particular, $A=\mathbb{Z}$ or $\mathbb{R}$) , it is a classical result that $\mathrm{H}^2(G,A)$ classifies the equivalence classes of central extensions $\widetilde{G}$ of G by A, that is the equivalence classes of short exact sequences

$$0\longrightarrow A\longrightarrow \widetilde{G}\longrightarrow G\longrightarrow 0\,,$$

such that the image of A in $\widetilde{G}$ is contained in the center. There is no such a characterization of second continuous bounded cohomology in degree two. However, as alluded to in the introduction, Gersten gave a very useful geometric property of central extensions which admit associated bounded cocycles.

Note that in general, a lot of information in degree two can be obtained from the comparison map

$$\mathrm{H}^2_{\mathrm{cb}}(G,E)\longrightarrow \mathrm{H}^2_{\mathrm{c}}(G,E)$$

as illustrated for instance by Bavard's and Mineyev's theorems in the introduction. It is easy to verify that if $E=\mathbb{R}$, the kernel of the comparison map in degree two (the so called "exact part of the continuous bounded cohomology") is identified with the space of continuous quasimorphisms

$$\begin{aligned}\mathrm{QM}(G,\mathbb{R}):=\big\{f:G\to\mathbb{R}:\ & f \text{ is continuous and}\\ &\sup_{g,h\in G}|f(gh)-f(g)-f(h)|<\infty\big\}\end{aligned}$$

up to homomorphisms and continuous bounded functions – the "trivial quasi-morphisms" – namely

$$\begin{aligned}\mathrm{EH}^2_{\mathrm{cb}}(G,\mathbb{R}) &:= \ker\big\{\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})\to\mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})\big\}\\ &=\mathrm{QM}(G,\mathbb{R})/\big(\mathrm{Hom}_{\mathrm{c}}(G,\mathbb{R})\oplus \mathrm{C}_{\mathrm{b}}(G,\mathbb{R})\big)\,.\end{aligned}$$

2.3. **Examples.** We give here a few examples of cocycles, most of which will be used in the sequel.

• *Bounded Euler class*. Let $G=\mathrm{Homeo}_+\big(\mathbb{S}^1\big)$ (thought of as a discrete group, for simplicity). The *Euler class* $e\in\mathrm{H}^2\big(\mathrm{Homeo}_+(\mathbb{S}^1),\mathbb{R}\big)$ can be represented by a nonhomogeneous cocycle arising from the central extension of $\mathrm{Homeo}_+(\mathbb{S}^1)$ by $\mathbb{Z}$ defined by the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of the real line which commute with integral translations

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "p"] & \mathrm{Homeo}_+(\mathbb{S}^1) \arrow[r] \arrow[l, bend left, "s"] & 0\,.
\end{tikzcd}
$$

Then, if one chooses a section s of the projection p in such a way that $s(f)(0) \in [0,1)$, for $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$, the cocycle associated to the central extension is bounded and hence defines a bounded cohomology class called the *bounded Euler class* $e_{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\big(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{R}\big)$, independent of the section chosen. A homogeneous cocycle whose class is a multiple of the bounded Euler class is the *orientation cocycle* defined by

$$
c(g_0, g_1, g_2) := \begin{cases} 1 & \text{if } (g_0x, g_1x, g_2x) \text{ is positively oriented} \\ -1 & \text{if } (g_0x, g_1x, g_2x) \text{ is negatively oriented} \\ 0 & \text{otherwise} \end{cases}, \tag{2.2}
$$

where $x \in \mathbb{S}^1$ is a fixed basepoint and $g_0, g_1, g_2 \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Then it is an exercise to show that $-2e_{\mathrm{b}} = [c] \in \mathrm{H}^2_{\mathrm{b}}\big(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{R}\big)$.

• Dupont cocycle (first instance). Let $G = \mathrm{PU}(1,1)$ and consider the unit disk $\mathbb{D}^2$ with Poincaré metric $(1-|z|^2)^{-2}|dz|^2$ and associated area form $\omega_{\mathbb{D}^2} = (1-|z|^2)^{-2} dz \wedge d\bar{z}$. If $x \in \mathbb{D}^2$, then

$$
b_{\mathbb{D}^2}(g_0, g_1, g_2) := \int_{\Delta(g_0x, g_1x, g_2x)} \omega_{\mathbb{D}^2}\,, \tag{2.3}
$$

where $\Delta(g_0x, g_1x, g_2x)$ is the geodesic triangle with vertices g_0x, g_1x, g_2x, is a $\mathrm{PU}(1,1)$-invariant cocycle which is bounded since

$$
|b_{\mathbb{D}^2}(g_0, g_1, g_2)| < \pi\,.
$$

Moreover, if the basepoint x is chosen on the boundary $\partial\mathbb{D}^2 \cong \mathbb{S}^1$, then the cocycle $\beta_{\mathbb{D}^2}$ that one can define analogously by integration on ideal triangles is also $\mathrm{PU}(1,1)$-invariant and bounded, and, in fact, if $c|_{\mathrm{PU}(1,1)}$ denotes the restriction to $\mathrm{PU}(1,1) < \mathrm{Homeo}_+(\mathbb{S})^1$ of the orientation cocycle c defined in (2.2), then

$$
\pi c|_{\mathrm{PU}(1,1)} = \beta_{\mathbb{D}^2}\,.
$$

• Cartan invariant. If $\langle\ ,\ \rangle$ is a Hermitian form of signature $(p,1)$ on $\mathbb{C}^{p+1}$, a model of complex hyperbolic p-space $\mathcal{H}^p_{\mathbb{C}}$ is given by the cone of negative lines. In this model the visual boundary $\partial\mathcal{H}^p_{\mathbb{C}}$ is the set of isotropic lines. A basic invariant of three vectors $x, y, z \in \mathbb{C}^{p+1}$ is their Hermitian triple product

$$
[x, y, z] = \langle x, y\rangle\langle y, z\rangle\langle z, x\rangle \in \mathbb{C}
$$

which can be projectivized to give a well defined map $\big(\partial\mathcal{H}^p_{\mathbb{C}}\big) \to \mathbb{R}^\times\backslash\mathbb{C}$ whose composition with $\frac{1}{\pi}\arg$ gives the Cartan invariant (*invariant angulaire*, [26])

$$c_p : (\partial\mathcal{H}^p_{\mathbb{C}})^3 \to [-1,1]\,. \tag{2.4}$$

A chain in $\partial\mathcal{H}^p_{\mathbb{C}}$ is by definition the boundary of a complex geodesic in $\mathcal{H}^p_{\mathbb{C}}$, that is an isometrically and holomorphically embedded copy of $\mathbb{D}^2$; as such, it is a circle equipped with a canonical orientation, and it is uniquely determined by any two points lying on it. When restricted to a chain, the Cartan invariant is nothing but the orientation cocycle (2.2); furthermore, the Cartan invariant is a strict alternating cocycle on $(\partial\mathcal{H}^p_{\mathbb{C}})^3 \cong \big(\mathrm{SU}(1,p)/P\big)^3$, where $P < \mathrm{SU}(1,p)$ is a (minimal) parabolic subgroup.

The area form $\omega_{\mathbb{D}^2}$ can be generalized in different directions: the first uses the fact that the area form on the Poincaré disk is obviously also its volume form and will be illustrated in the next example; the subsequent three examples use instead that the area form on the Poincaré disk is its *Kähler form*, that is a nonvanishing differential two-form which is $\mathrm{PU}(1,1)$-invariant (hence closed). The existence of a Kähler form $\omega_{\mathcal{X}}$ is what distinguishes, among all symmetric spaces, the *Hermitian* ones.

• *Volume cocycle*. Let $G = \mathrm{PO}(1,n)^\circ$ be the connected component of the group of isometries of real hyperbolic space $\mathcal{H}^n_{\mathbb{R}}$. Then the volume of simplices with vertices in $\mathcal{H}^n_{\mathbb{R}}$ is uniformly bounded, hence defines a G-invariant alternating continuous bounded cocycle. Likewise, the volume of *ideal* simplices in $\mathcal{H}^n_{\mathbb{R}}$ (that is simplices with vertices on the sphere at infinity $\partial\mathcal{H}^n_{\mathbb{R}}$ of $\mathcal{H}^n_{\mathbb{R}}$) defines a G-invariant alternating bounded cocycle.

• *Dupont cocycle*. Let G be a connected semisimple group with finite center and $\mathcal{X}$ the associated symmetric space which we assume to be Hermitian. In the sequel we will normalize the Hermitian metric such that the minimum of the holomorphic sectional curvature is -1. Letting now $\omega_{\mathcal{X}}$ be the Kähler form and $x \in \mathcal{X}$ a basepoint, integration over simplices $\Delta(x,y,z)$ with vertices x,y,z and geodesic sides gives rise to a continuous G-invariant cocycle on G defined by

$$b_{\mathcal{X}}(g_0,g_1,g_2) := \int_{\Delta(g_0x,g_1x,g_2x)} \omega_{\mathcal{X}},$$

for $g_0,g_1,g_2 \in G$, which turns out to be bounded (Dupont [29]). In fact, more precisely we have that

$$\|b_{\mathcal{X}}\|_\infty = \pi r_{\mathcal{X}}\,,$$

where $r_{\mathcal{X}}$ is the rank of $\mathcal{X}$, (Domic–Toledo [28] and Clerc–Ørsted [27]).

Notice that, contrary to the constant sectional curvature case, the geodesic triangle $\Delta(g_0x, g_1x, g_2x)$ has uniquely defined geodesic sides, but not uniquely defined interior: however the integral is well defined because the Kähler form is closed.

• *Bergmann cocycle*. To extend the previous picture to the boundary, recall that any Hermitian symmetric space $\mathcal{X}$ has a realization as a bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^n$. Let $\check{S}$ be the *Shilov boundary* of $\mathcal{D}$, that is the only closed G-orbit in the topological boundary $\partial\mathcal{D}$ of $\mathcal{D}$. While in the rank one case any two points in the topological boundary $\partial\mathcal{D}$ of $\mathcal{D}$ can be connected by a geodesic, the same is not true in higher rank. Let $\check{S}^{(3)}$ be the subset of $\check{S}^3$ consisting of triples of points which can be joined pairwise by geodesics. Then for $x_1, x_2, x_3 \in \check{S}^{(3)}$ one can define heuristically

$$\beta_{\mathcal{X}}(x_1, x_2, x_3) := \int_{\Delta(x_1,x_2,x_3)} \omega_{\mathcal{X}} \tag{2.5}$$

which turns out to be, once again, an invariant alternating cocycle, which is also bounded since

$$\|\beta_{\mathcal{X}}\|_\infty = \pi r_{\mathcal{X}} \,.$$

We refer to [27] for a justification of this heuristic formula.

• *Maslov index*. Let V be a real vector space with a symplectic form $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$, $G = \mathrm{Sp}(V) = \big\{g \in \mathrm{GL}(V) : g \text{ preserves } \langle\ ,\ \rangle\big\}$ and $\mathcal{X}$ the associated symmetric space. Then $\mathcal{X}$ is a classical example of Hermitian symmetric space and the Grassmannian $\mathcal{L}(V)$ of Lagrangian subspaces is in a natural way identified with the Shilov boundary of the bounded domain realization of $\mathcal{X}$. The Bergmann cocycle $\beta_{\mathcal{X}}$ defined above is here equal to πi_V, where $i_V(L_1, L_2, L_3)$ is the Maslov index of three Lagrangians $L_1, L_2, L_3 \in \mathcal{L}(V)$ defined, following Kashiwara, as the index of the quadratic form

$$\begin{aligned} L_1 \times L_2 \times L_3 &\longrightarrow \qquad\qquad \mathbb{R} \\ (v_1, v_2, v_3) &\mapsto \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle \,. \end{aligned}$$

In fact, i_V is defined for all triples of Lagrangians and it is an $\mathrm{Sp}(V)$-invariant cocycle taking integer values in the interval $[-n, n]$, where $2n = \dim V$. For a more thorough discussion of all objects involved see [13].

Here is finally an example of a cocycle with nontrivial coefficients.

• *Gromov–Sela–Monod cocycle*. Let $\mathcal{T} := (\mathcal{E}, \mathcal{V})$ be a tree with vertices $\mathcal{V}$, edges $\mathcal{E}$, let $G := \mathrm{Aut}(\mathcal{T})$ be its automorphism group and let d

be the combinatorial distance on $\mathcal{T}$. For any $n \in \mathbb{Z}$, we are going to construct a bounded cocycle with values in $\ell^2\big(\mathcal{E}^{(n)}\big)$, where $\mathcal{E}^{(n)}$ is the set of oriented geodesic paths in $\mathcal{T}$ of length n. For $n \in \mathbb{N}$, define

$$\begin{aligned} \omega^{(n)} : G \times G &\to \ell^2(\mathcal{E}^{(n)}) \\ (g_0, g_1) &\mapsto \omega^{(n)}_{g_0,g_1} \end{aligned}$$

to be

$$\omega^{(n)}_{g_0,g_1}(\gamma) := \begin{cases} 1 & \text{if } d(g_0x, g_1x) \geq n \text{ and } \gamma \subset \gamma_{g_0x,g_1x} \\ -1 & \text{if } d(g_0x, g_1x) \geq n \text{ and } \gamma^{-1} \subset \gamma_{g_0x,g_1x} \\ 0 & \text{otherwise} \end{cases},$$

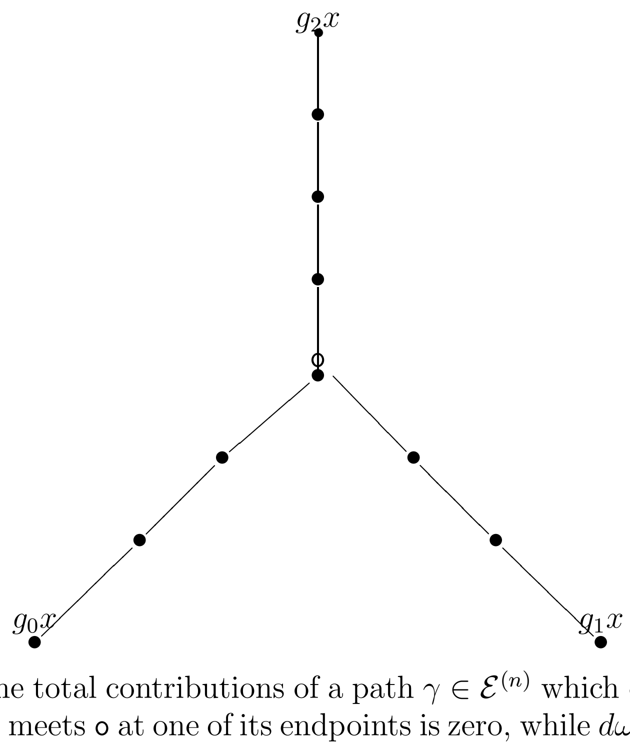
where, if $x \in \mathcal{T}$ is a basepoint and $\gamma_{y,z}$ denotes the oriented geodesic path from y to z. By definition of coboundary operator, and by observing that $\omega^{(n)}$ is alternating, one has

$$d\omega^{(n)}(g_0, g_1, g_2) = \omega^{(n)}(g_0, g_1) + \omega^{(n)}(g_1, g_2) + \omega^{(n)}(g_2, g_0)\,,$$

and it is easy to verify that

$$d\omega^{(n)} = 0 \qquad \text{if and only if} \qquad n = 1\,.$$

In fact, the support of $d\omega^{(n)}(g_0, g_1, g_2)$ is contained in the tripod with vertices g_0x, g_1x, g_2x and center o,



and the total contributions of a path $\gamma \in \mathcal{E}^{(n)}$ which does not meet o or which meets o at one of its endpoints is zero, while $d\omega^{(n)}(g_0, g_1, g_2)(\gamma) \neq$

0 if and only if γ is a path which contains the center o of the tripod in its interior. However, since

$$\sup_{g_0,g_1,g_2} \|d\omega^{(n)}(g_0,g_1,g_2)\|_2 < \infty\,,$$

then

$$c^{(n)} := d\omega^{(n)} : G \times G \times G \to \ell^2\big(\mathcal{E}^{(n)}\big)$$

is a bounded G-invariant cocycle, even though $\omega^{(n)}$ is not a bounded cochain – in fact, $\|\omega^{(n)}_{g_0,g_1}\|_2^2 = 2(d(g_0x, g_1x) - n)$. Let $c^{(n)}_{GSM} := [c^{(n)}] \in \mathrm{H}^2_{\mathrm{b}}\big(\operatorname{Aut}(\mathcal{T}), \ell^2(\mathcal{E}^{(n)})\big)$ be the bounded cohomology class so defined, which is independent of the chosen basepoint x. These classes turn out to be nontrivial in the following rather strong sense:

THEOREM 2.5 (Monod–Shalom [55]). *Let Γ be a finitely generated group and $\rho : \Gamma \to \operatorname{Aut}(\mathcal{T})$ an action of Γ by automorphisms on $\mathcal{T}$. Then the following are equivalent:*

(i) *the action of Γ is not elementary;*
(ii) *the pullback $\rho^{(2)}\big(c^{(2)}_{GSM}\big) \in \mathrm{H}^2_{\mathrm{b}}\big(\Gamma, \ell^2(\mathcal{E}^{(2)})\big)$ is nonzero;*
(iii) *the pullback $\rho^{(2)}\big(c^{(n)}_{GSM}\big) \in \mathrm{H}^2_{\mathrm{b}}\big(\Gamma, \ell^2(\mathcal{E}^{(n)})\big)$ is nonzero for every $n \geq 2$.*

A similar but more elaborate construction for Gromov hyperbolic graphs of bounded valency due to Mineyev, Monod and Shalom gives the following general nonvanishing result:

THEOREM 2.6 (Mineyev–Monod–Shalom [52]). *Let Γ be a countable group admitting a proper nonelementary action on a hyperbolic graph of bounded valency. Then $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma, \ell^2(\Gamma)\big)$ is nontrivial.*

REMARK 2.7. Another illustration of the relevance of bounded cohomology with coefficients is provided by the result of Monod and Shalom already alluded to in the introduction. They proved that if two groups Γ_1 and Γ_2 are finitely generated and measure equivalent, then $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma_1, \ell^2(\Gamma_1)\big) \neq 0$ if and only if $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma_2, \ell^2(\Gamma_2)\big) \neq 0$.

Recall that two groups are measure equivalent if there exists a space X with a σ-finite measure μ, such that the actions of the Γ_i's on (X,μ) are measure preserving, commute, and admit a finite volume fundamental domain. The typical example of measure equivalent groups are lattices Γ_1, Γ_2 in a locally compact second countable group G, where one can take $(X,\mu) = (G, dg)$, where dg is the Haar measure on G.

It is interesting to compare this with a result of Gaboriau [31, 33] asserting the equality of ℓ^2 Betti numbers if Γ_1 and Γ_2 are orbit equivalent. Recall that ℓ^2-Betti numbers of Γ are computed using the ordinary cohomology groups $\mathrm{H}^\bullet\big(\Gamma, \ell^2(\Gamma)\big)$ [32], and that two actions are orbit equivalent if there exists a measure class preserving measurable isomorphism of the underlying spaces which sends almost every Γ_1-orbit to a Γ_2-orbit.

Now that we have some examples at hand, it is clear that, just like in the case of continuous cohomology, there is the need of more flexibility than allowed by the bar resolution as, for instance, some of the cocycles defined above – e. g. the Dupont cocycle in (2.3) – are not continuous.

2.4. **Homological Algebra Approach to Continuous Bounded Cohomology.** Let V be a Banach G-module. As for ordinary continuous cohomology, there is a notion of relatively injective Banach G-module appropriate in this context.

Definition 2.8. A Banach G-module V (that is a Banach space with an action of G by linear isometries) is *relatively injective* if it satisfies an extension property, namely:

- given two continuous Banach G-modules A and B and a G-morphism $i : A \to B$ between them (that is a continuous linear G-map), such that there exists a left inverse $\sigma : B \to A$ with norm $\|\sigma\| \leq 1$ (which is linear but not necessarily a G-map), and
- given any G-morphism $\alpha : A \to V$,

there exists a G-morphism $\beta : B \to V$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, hook, "\imath"'] \arrow[dr, "\alpha"'] & & B \arrow[dl, dotted, "\exists\beta"] \\ & V & \end{tikzcd}$$

commutes, and $\|\beta\| \leq \|\alpha\|$.

We remark that the existence of such σ is a rather severe restriction on $\imath$, as it implies that there exists a splitting of $B = A + C$, where C is a Banach complement of A in B; if however we were to require that σ is a G-morphism, then the splitting would be G-invariant and hence all Banach G-modules would be relatively injective.

Example 2.9. For any coefficient G-module E, let $\mathrm{L}^\infty_{\mathrm{w}*,\mathrm{alt}}(G^{n+1}, E)$ be the subspace of (equivalence classes of) alternating functions in $\mathrm{L}^\infty_{\mathrm{w}*}(G^{n+1}, E)$, that is $f \in \mathrm{L}^\infty_{\mathrm{w}*}(G^{n+1}, E)$ and for any permutation σ on $n+1$ symbols, we have that

$$f\big(\sigma(g_0, \dots, g_n)\big) = \mathrm{sign}(\sigma) f(g_0, \dots, g_n)\,.$$

It is not difficult to show that the Banach G-modules $\mathrm{C_b}(G^{n+1},E)$, $\mathrm{L}^\infty_{\mathrm{w}*}(G^{n+1},E)$ and $\mathrm{L}^\infty_{\mathrm{w}*,\mathrm{alt}}(G^{n+1},E)$ are relatively injective. In fact, if V is any of the above function spaces, $\alpha : A \to V$ is a G-morphism, and $B,\sigma,\imath$ are as in Definition 2.8, then one can define $\beta(b)(g_0,\dots,g_n) := \alpha\big(g_0\sigma(g_0^{-1}b)\big)(g_0,\dots,g_n)$ and verify that it has the desired properties.

Definition 2.10. Let E be a coefficient G-module.

(1) A *resolution* $(E_\bullet,\mathfrak{d}_\bullet)$ of E is an exact complex of Banach G-modules such that $E_0 = E$ and $E_n = 0$ for all $n \leq -1$

$$0 \longrightarrow E \xrightarrow{\mathfrak{d}_0} E_1 \xrightarrow{\mathfrak{d}_1} \cdots \xrightarrow{\mathfrak{d}_{n-2}} E_{n-1} \xrightarrow{\mathfrak{d}_{n-1}} E_n \xrightarrow{\mathfrak{d}_n} \cdots$$

(2) The *continuous submodule* $\mathcal{C}E$ of E is the submodule defined as the subspace of E of vectors on which the action of G is norm-continuous, that is $v \in \mathcal{C}E$ if and only if $\|gv - v\| \to 0$ as $g \to e$. Then a *strong resolution* of E by relatively injective G-modules is a resolution where the E_js are relatively injective G-modules, with a *contracting homotopy* defined on the subcomplex $(\mathcal{C}E_\bullet,\mathfrak{d}_\bullet)$ of continuous vectors, that is a map $h_{n+1} : \mathcal{C}E_{n+1} \to \mathcal{C}E_n$ such that:
   - $\|h_{n+1}\| \leq 1$, and
   - $h_{n+1}\mathfrak{d}_n + \mathfrak{d}_{n-1}h_n = Id_{E_n}$ for all $n \geq 0$.

Given two strong resolutions of a coefficient Banach G-module E by relatively injective G-modules, the extension property in Definition 2.8 allows to extend the identity map of the coefficients to a G-morphism of the resolutions, which in turn results in an isomorphism of the corresponding cohomology groups. In general however, the isomorphism that one thus obtains is only an isomorphism of topological vector spaces, not necessarily isometric. More precisely,

Exercise 2.11. (1) Let $(E_\bullet,\mathfrak{d}_\bullet)$ be a strong resolution of a coefficient module E, and $(F_\bullet,\mathfrak{d}'_\bullet)$ a strong resolution of E by relatively injective modules. Then there exists a G-morphism of complexes $\alpha_\bullet : \mathcal{C}E_\bullet \to \mathcal{C}F_\bullet$ which extends the identity $Id : E \to E$ and such that $\eta_0 = Id$.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\mathfrak{d}_0"] \arrow[d, "Id"] & E_1 \arrow[r, "\mathfrak{d}_1"] \arrow[d, dotted, "\eta_1"] & \cdots \arrow[r, "\mathfrak{d}_{n-2}"] & E_{n-1} \arrow[r, "\mathfrak{d}_{n-1}"] & E_n \arrow[r, "\mathfrak{d}_n"] & \cdots \\
0 \arrow[r] & E \arrow[r, "\mathfrak{d}'_0"] & F_1 \arrow[r, "\mathfrak{d}'_1"] & \cdots \arrow[r, "\mathfrak{d}'_{n-2}"] & F_{n-1} \arrow[r, "\mathfrak{d}'_{n-1}"] & F_n \arrow[r, "\mathfrak{d}'_n"] & \cdots
\end{tikzcd}$$

Note that the existence of $\eta_1 = \mathfrak{d}_0 Id$ is just an application of the definition of relative injectivity while, for $j \geq 2$, in the construction of η_j one has to use the contracting homotopy to deal with the kernel of $\mathfrak{d}_{j-1}$.

(2) If in addition also $(E_\bullet, \mathfrak{d}_\bullet)$ is by relatively injective modules, then there exists a G-homotopy equivalence between G-morphisms of complexes which induces an isomorphism in cohomology.

Recall that if $\alpha_\bullet$ and $\eta_\bullet$ are G-morphisms of complexes $(E_\bullet, \mathfrak{d}_\bullet)$ and $(F_\bullet, \mathfrak{d}'_\bullet)$, a G-homotopy $\sigma_\bullet : \alpha_\bullet \to \eta_{\bullet-1}$ is a sequence of G-morphisms $\sigma_n : E_n \to F_{n-1}$,

$$\begin{tikzcd}
E_{n-1} \arrow[r, "\mathfrak{d}_{n-1}"] \arrow[d, shift right] \arrow[d, shift left] & E_n \arrow[r, "\mathfrak{d}_n"] \arrow[dl, "\sigma_n"'] \arrow[d, shift right, "\alpha_n"'] \arrow[d, shift left, "\eta_n"] & E_{n+1} \arrow[dl, "\sigma_{n+1}"'] \arrow[d, shift right] \arrow[d, shift left] \\
F_{n-1} \arrow[r, "\mathfrak{d}'_{n-1}"] & F_n \arrow[r, "\mathfrak{d}'_n"] & F_{n+1}
\end{tikzcd}$$

such that

$$\sigma_{n+1}\mathfrak{d}_n + \mathfrak{d}'_{n-1}\sigma_n = \alpha_n - \eta_n \,.$$

Corollary 2.12. *The continuous bounded cohomology of a locally compact group G with coefficients in the coefficient module E is isomorphic (as a topological vector space) to the cohomology of the subcomplex of invariants of any strong resolution of E by relatively injective G-modules.*

We want to present a case in which the isomorphism is indeed isometric, together with providing a realization of bounded cohomology which turns out to be very useful from the geometric point of view.

2.5. **Amenable Actions.** The notion of amenable action is a relativized notion of that of an amenable group and we refer to [65, Chapter 4] and [53] for details and proofs (see also [3, Chapter 4] for a groupoid point of view). We start our discussion with the definition most useful for us, although not necessarily the most transparent among those available.

Definition 2.13. A locally compact group G is *amenable* if and only if there exists a (left) G-invariant mean on $\mathrm{L}^\infty(G)$, that is a norm-continuous G-invariant linear functional $m : \mathrm{L}^\infty(G) \to \mathbb{R}$ such that $m(f) \geq 0$ if $f \geq 0$, $m(1) = 1$, and hence has norm $\|m\| = 1$.

Analogously, one has:

Definition 2.14. Let (S, μ) be a G-space with a quasiinvariant measure. *The action of G on (S, μ) is amenable* if and only if there exists a G-equivariant projection $m : \mathrm{L}^\infty(G \times S) \to \mathrm{L}^\infty(S)$ which is $\mathrm{L}^\infty(S)$-linear and such that $m(1_{G\times S}) = 1_S$, $m(f) \geq 0$ if $f \geq 0$ and hence m has norm $\|m\| = 1$.

Examples of amenable groups include Abelian, compact, and solvable groups as well as all extensions of amenable groups by amenable groups and inductive limits of amenable groups. For example, let P be a minimal parabolic subgroup in a Lie group G: since P is a compact extension of a solvable group, then P is amenable. Moreover one can show that, although a noncompact semisimple Lie group G is never amenable, it acts amenably on the homogeneous space G/P [65, Chapter 4].

This is not by chance, in fact:

Proposition 2.15 (Zimmer [65, Proposition 4.3.2]). *Let G be a locally compact group and $H \leq G$ any closed subgroup. The action of G on G/H is amenable (with respect to the quotient class of the Haar measure) if and only if H is an amenable group.*

Corollary 2.16. *A group acts amenably on a point if and only if it is amenable.*

We want to illustrate now a characterization of amenable action (which was actually the original definition [65, Chapter 4]) modeled on the characterization of amenable groups by a fixed point property. Namely a locally compact group G is amenable if and only if there is a fixed point on any G-invariant compact convex subset in the unit ball (in the weak*-topology) of the dual of a separable Banach space. on which G acts continuously by linear isometries. The concept of amenable action once again relativizes that of amenable group. We start illustrating it in terms of bundles in the case of a transitive action .

Let us consider the principal H-bundle $G \to G/H$. For any separable Banach space E and any continuous isometric action $H \to \mathrm{Iso}(E)$, we can consider the associated bundle with fiber the dual E^* of E endowed with the weak*-topology. Since the group G acts by bundle automorphisms on the bundle $G \to G/H$ it preserves the subbundle of the associated bundle with fiber the unit ball $E_1^* \subset E^*$. Let A be a G-invariant weak*-measurable subset of $G\times_H E_1^*$ which is fiberwise weak*-compact and convex. Then we say that G acts amenably on G/H if and only if whenever in the above situation, there exists an A-valued measurable section of the associated bundle which is G-invariant.

The following definition is just the translation of the above picture in the more general case of "virtual group actions" (see [47] and [57] for a description of the philosophy behind it and [66] for the explicit correspondence between principal bundle automorphisms and measurable cocycles – see the next definition).

Definition 2.17 (Zimmer [65]). Let G be a locally compact group acting continuously and by linear isometries on a separable Banach space E, and let (S,ν) be a (right) G-space with a quasiinvariant measure. Let $\alpha : S\times G \to \mathrm{Iso}(E)$ be a measurable cocycle (that is a measurable map such that $\alpha(s,gh)=\alpha(s,g)\alpha(sg,h)$ for almost all $s\in S$ and for all $g,h\in G$) and let $s\mapsto A_s$ a Borel assignment of a compact convex subset of the unit ball E_1^* of the dual such that $\alpha(s,g)^*A_s=A_{sg}$. Let

$$F\big(S,\{A_s\}_{s\in S}\big) := \big\{f : S\to E_1^* : f \text{ is measurable and } f(s)\in A_s \text{ for } a.\,e. s\in S\big\}$$

be endowed with the G-action $(gf)(s):=\alpha(s,g)^*f(sg)$. The above data $\big(S,\{A_s\}_{s\in S}\big)$ is an *affine* action of G over (S,ν).

Proposition 2.18 (Zimmer [63], Adams [1], Adams–Elliott–Giordano [2]). *A locally compact group G acts amenably on S if and only if for every affine action of G over S there is a fixed point, that is a measurable function $f\in F\big(S,\{A_s\}_{s\in S}\big)$ such that $f(s)=\alpha(s,g)^*f(sg)$ for almost every $s\in S$ and $g\in G$.*

Example 2.19. Any action of an amenable group is amenable. We saw already that a (nonamenable) group acts amenably on a homogeneous space with amenable stabilizer. Another important example of an amenable action of a (nonamenable) group is that of a free group $\mathbb{F}_r$ in r-generators ($r\geq 2$) on the boundary $\partial\mathcal{T}_r$ of the associated tree $\mathcal{T}_r$, with respect to the measure $m\big(C(x)\big)=\big(2r(2r-1)^{n-1}\big)^{-1}$, where x is a reduced word of length n and $C(x)$ is the subset of $\partial\mathcal{T}_r$ consisting of all infinite reduced words starting with x.

The relevance (as well as a new characterization) of amenability of an action with respect to bounded cohomology is given by the following:

Theorem 2.20 (Burger–Monod [25]). *Let (B,ν) be a regular G-space with a quasi-invariant measure. Then the following are equivalent:*

(i) *The G-action on B is amenable;*
(ii) $\mathrm{L}^\infty(B)$ *is relatively injective;*
(iii) $\mathrm{L}^\infty_{\mathrm{w}*}(B^n,E)$ *is relatively injective for every coefficient G-module E and every $n\geq 1$.*

We precede the proof with the following observation. Let $\theta : V\to W$ be a G-morphism of Banach G-modules V, W such that there is a left inverse G-morphism $\zeta : W \to V$ with $\|\zeta\| \leq 1$; assume moreover that W is relatively injective. Then V is also relatively injective as one can easily see from the diagram

$$\begin{tikzcd}
A \arrow[r, hook, "\imath"'] \arrow[dr, "\alpha"'] & B \arrow[l, bend right, "\sigma"'] \arrow[d, dotted] \arrow[dr, dotted] & \\
& V \arrow[r, "\theta"] & W \arrow[l, bend left, "\zeta"] .
\end{tikzcd}$$

Proof. We give an idea of the proof of the equivalence of the first two statements. Since $\mathrm{L}^\infty(G \times B) \cong \mathrm{L}^\infty_{\mathrm{w}*}\big(G, \mathrm{L}^\infty(B)\big)$, and because we have already observed that $\mathrm{L}^\infty_{\mathrm{w}*}\big(G, \mathrm{L}^\infty(B)\big)$ is relatively injective (Example 2.9), by the above observation with $V = \mathrm{L}^\infty(B)$ and $W = \mathrm{L}^\infty_{\mathrm{w}*}\big(G, \mathrm{L}^\infty(B)\big)$ it is enough to find a left inverse G-morphism of norm one of the inclusion $\mathrm{L}^\infty(B) \hookrightarrow \mathrm{L}^\infty(G \times B)$; but this is implied by the definition we gave of amenable action.

Conversely, consider the diagram

$$\begin{tikzcd}
\mathrm{L}^\infty(B) \arrow[rr, hook, "\imath"'] \arrow[dr, "Id"'] & & \mathrm{L}^\infty(B \times G) \\
& \mathrm{L}^\infty(B) &
\end{tikzcd}$$

and observe that $\imath$ admits a left inverse of norm one given by

$$\sigma(F)(b) := \int_G F(b, g)\psi(g)dg \,,$$

where $\psi \geq 0$ is some continuous function with compact support and integral one. If $\mathrm{L}^\infty(B)$ is injective, there is a G-map $\beta : \mathrm{L}^\infty(B \times G) \to \mathrm{L}^\infty(B)$ of norm one making the above diagram commutative; in particular $\beta(f \otimes 1_G) = f$ for all $f \in \mathrm{L}^\infty(B)$, which, together with the G-equivariance implies that β is $\mathrm{L}^\infty(B)$-linear. □

This gives us yet another characterization of the amenability of a group.

Corollary 2.21. *Let G be a locally compact group. The following are equivalent:*

(i) *The group G is amenable;*
(ii) *The trivial G-module $\mathbb{R}$ is relatively injective;*
(iii) *Every coefficient G-module is relatively injective.*

Exercise 2.22. Show that if G is amenable, then $\mathrm{H}^n_{\mathrm{cb}}(G, E) = 0$ for any $n \geq 1$ and any coefficient G-module E.

Putting together Theorem 2.20, Corollary 2.12 and some extra work finally one obtains the following

Corollary 2.23 (Burger–Monod [25]). *There is a canonical isometric isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G,E) \cong \mathrm{H}^\bullet\big(\mathrm{L}^\infty_{\mathrm{w*,alt}}(B^\bullet,E)^G\big)\,.$$

The first important application of the above corollary is in degree 2. Let Ξ be a class of coefficient Banach G-modules. In our case, we shall be mostly concerned with the case in which either Ξ consists of all separable Hilbert G-modules $\Xi^{\mathrm{sep}\mathcal{H}}$, or, more simply, of the trivial module $\Xi = \{\mathbb{R}\}$.

Definition 2.24 (Burger–Monod [25]). Let (S,ν) be a G-space with a quasiinvariant measure ν. We say that B is a *doubly Ξ-ergodic* space if for every coefficient G-module $E \in \Xi$, every measurable G-equivariant map $f : B \times B \to E$ is essentially constant.

Note that in the case in which Ξ consists only of the trivial module, a doubly ergodic action is nothing but the classical concept of a "mixing" action.

The following are then fundamental examples of doubly Ξ-ergodic spaces.

Examples 2.25.
1. If G is a semisimple Lie group with finite center and $Q < G$ is any parabolic subgroup, the action of G on G/Q is doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodic;
2. If $\Gamma < G$ is a lattice in a locally compact group G and (B,ν) is a doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodic G-space, then (B,ν) is a doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodic Γ-space;
3. The action of $\mathrm{Aut}(\mathbb{F}_r)$ – and hence, by the previous example, of $\mathbb{F}_r$ – on $\partial\mathcal{T}_r$ is doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodic.

These examples are just a reformulation of the Mautner property [53, Corollary 11.2.3 and Proposition 11.2.2].

While we shall make essential use in practice of the double $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodicity of the G-action on G/P, where P is a minimal parabolic, the double $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodicity in the second example is used in an essential way in the proof of the following result (at least for finitely generated groups, as we shall indicate). Its proof is due to Burger and Monod for compactly generated groups [25] and to Kaimanovich [44] in the general case.

Theorem 2.26 (Burger–Monod [25], Kaimanovich [44]). *Let G be a σ-compact locally compact group. Then there always exists a G-space*

(B,ν) with a quasiinvariant measure such that the action of G is both amenable and doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodic.

Before we start, we recall here Mackey's point realization construction which is used in the proof. Let (X,μ) be a measure space. Associated to any weak*-closed C^*-subalgebra $\mathcal{A}$ of $\mathrm{L}^\infty(X,\mu)$ there is a measure space (Y,ν) and a map $p:(X,\mu)\to(Y,\nu)$ such that $p_*\mu=\nu$ and $\mathcal{A}=p^*\big(\mathrm{L}^\infty(Y,\nu)\big)$. If in addition X is a G-space, the measure μ is quasiinvariant, and the subalgebra $\mathcal{A}$ is G-invariant, then the space Y inherits a G-action and the map p is a G-map [46].

Proof. We sketch here the proof in the case in which the group is discrete and finitely generated. Fix a set of r generators of G and a presentation $\tau:\mathbb{F}_r\to G$ with kernel N. Then the space of N-invariant functions $\mathrm{L}^\infty(\partial\mathcal{T}_r)^N$ is contained in $\mathrm{L}^\infty(\partial\mathcal{T}_r)$ as a weak*-closed subalgebra whose point realization is a measure G-space (B,ν) with a quasiinvariant measure. Hence $\mathrm{L}^\infty(B)\cong\mathrm{L}^\infty(\partial\mathcal{T}_r)^N$ and there is a measure preserving $\mathbb{F}_r$-map $(\partial\mathcal{T}_r,m)\to(B,\nu)$, so that double $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodicity and amenability follow from the corresponding properties of the $\mathbb{F}_r$-action on $\partial\mathcal{T}_r$. $\square$

As an immediate consequence of the above results, we have the following:

Corollary 2.27 (Burger–Monod [25]). *Let G be a σ-finite locally compact group and (B,ν) any G-space on which G acts amenably and doubly $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodically. For any separable Hilbert G-module $\mathcal{H}\in\Xi^{\mathrm{sep}\mathcal{H}}$, we have an isometric isomorphism*

$$\mathrm{H}^2_{\mathrm{cb}}(G,\mathcal{H})\cong\mathcal{Z}\mathrm{L}^\infty_{\mathrm{w*,alt}}(B^3,\mathcal{H})^G\,.$$

Proof. The double $\Xi^{\mathrm{sep}\mathcal{H}}$-ergodicity implies that $\mathrm{L}^\infty(B^2,\mathcal{H})^G=\mathbb{R}$ and hence $\mathrm{L}^\infty_{\mathrm{w*,alt}}(B^2,\mathcal{H})^G=0$, so that the assertion follows from Corollary 2.23. $\square$

In particular,

Corollary 2.28. *For every separable Hilbert G-module $\mathcal{H}\in\Xi^{\mathrm{sep}\mathcal{H}}$, the second continuous bounded cohomology space $\mathrm{H}^2_{\mathrm{cb}}(G,\mathcal{H})$ is a Banach space.*

2.6. **Toolbox of Useful Results.** We briefly recall here without proof some results from [25] and [24], the first two of which will be used in disguise in the sequel, while the others are here for illustration for the reader more inclined toward the cohomological aspects than their applications. For example, using an appropriate resolution one can prove:

THEOREM 2.29. *Let G be a locally compact group and $N \trianglelefteq G$ a closed amenable normal subgroup. Then there is an isometric isomorphism*

$$\mathrm{H}^{\bullet}_{\mathrm{cb}}(G, E) \cong \mathrm{H}^{\bullet}_{\mathrm{cb}}\big(G/N, E^N\big)\,.$$

If we restrict our attention to degree two and trivial coefficients, then one has:

THEOREM 2.30. *Let G_j, $j = 1, \dots, n$, be locally compact groups. Then*

$$\mathrm{H}^2_{\mathrm{cb}}\bigg(\prod_{j=1}^{n} G_j, \mathbb{R}\bigg) \cong \bigoplus_{j=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(G_j, \mathbb{R})\,.$$

THEOREM 2.31. *Let G_1, G_2 be locally compact groups, let $\Gamma < G_1 \times G_2$ be a lattice with dense projection in each factor, and let $\mathcal{H}$ be a Hilbert Γ-module. Then*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathcal{H}) \cong \mathrm{H}^2_{\mathrm{cb}}(G_1, \mathcal{H}_1) \oplus \mathrm{H}^2_{\mathrm{cb}}(G_2, \mathcal{H}_2)\,,$$

where $\mathcal{H}_i$ is the maximal Γ-invariant subspace of $\mathcal{H}$ such that the restricted action extends continuously to $G_1 \times G_2$ factoring via G_j, where $i \neq j$, $1 \leq i, j \leq 2$.

The last results are an analog of a Lyndon–Hochschild–Serre exact sequence.

THEOREM 2.32. *Let $1 \to N \to G \to Q \to 1$ be an exact sequence of locally compact second countable groups, with N compactly generated, and let $\mathcal{H}$ be a coefficient Banach G-module. If $Z_G(N)$ denotes the centralizer of N in G, the sequence*

$$0 \longrightarrow \mathrm{H}^2_{\mathrm{cb}}\big(Q, \mathcal{H}^N\big) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(G, \mathcal{H}) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}\big(N, \mathcal{H}^{Z_G(N)}\big)^Q \longrightarrow$$

$$\longrightarrow \mathrm{H}^3_{\mathrm{cb}}\big(Q, \mathcal{H}^N\big) \longrightarrow \mathrm{H}^3_{\mathrm{cb}}(G, \mathcal{H})$$

is exact.

An analog of the Eckmann–Shapiro lemma is available also in bounded cohomology:

THEOREM 2.33. *Let G be a locally compact second countable group, $H < G$ a closed subgroup and $\mathcal{H}$ a separable Hilbert G-module. Then induction of cocycles induces the following isomorphism in all degrees*

$$\mathrm{H}^n_{\mathrm{cb}}(H, \mathcal{H}) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{cb}}\big(G, \mathrm{L}^\infty(G/H, \mathcal{H})\big)\,.$$

If $H = \Gamma$ is a lattice in G, one can show that the inclusion of coefficient G-modules

$$\mathrm{L}^\infty(G/H, \mathcal{H}) \hookrightarrow \mathrm{L}^2(G/H, \mathcal{H})$$

induces an injection in cohomology in degree two. Together with Theorem 2.33, this implies the following

Corollary 2.34. *With the above hypotheses there is an injection*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathcal{H}) \hookrightarrow \mathrm{H}^2_{\mathrm{cb}}\big(G, \mathrm{L}^2(G/H, \mathcal{H})\big)\,.$$

Moreover, if we restrict our attention to Lie groups and to cohomology with trivial coefficients, we have:

Theorem 2.35. *Let G be a connected Lie group with finite center. Then $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \cong \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R})$.*

Notice that the surjectivity of the comparison map follows from the arguments used in the discussion of the examples at the beginning of this section. The injectivity follows from the interplay between Mautner property and properties of quasimorphisms. If however we consider lattices even in semisimple Lie groups, then the comparison map is definitely not an isomorphism. In fact we have the following:

Theorem 2.36 (Epstein–Fujiwara [30]). *If Γ is a nonelementary Gromov hyperbolic group, then $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ is an infinite dimensional Banach space.*

This applies for instance to the case where Γ is a cocompact lattice in a connected Lie group G of real rank one and with finite center. In contrast, lattices in higher rank Lie groups exhibit, once again, strong rigidity phenomena:

Theorem 2.37 (Burger–Monod [24, 25]). *Let $\Gamma < G$ be an irreducible lattice in a connected semisimple Lie group with finite center and real rank at least two. Then the comparison map*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \to \mathrm{H}^2(\Gamma, \mathbb{R})$$

is injective in degree two and its image coincides with the restriction to Γ of G-invariant classes.

This is somehow a perfect example which illustrates how the cohomology theory for discrete groups and with trivial coefficients does not suffice, as the proof of the above result depends in an essential way on the Corollary 2.34 where the cohomology of Γ with trivial coefficients is related to the cohomology of the ambient (nondiscrete) group G with coefficients in the induced Hilbert G-module $\mathrm{L}^2(G/\Gamma)$.

There is however also a version of Theorem 2.37 with coefficients, namely:

THEOREM 2.38 (Monod–Shalom [55]). *Assume that Γ is a lattice in a connected simple Lie group G with finite center and real rank at least two and E is any separable coefficient Γ-module. Then*

$$\dim \mathrm{H}^2_{\mathrm{b}}(\Gamma, E) = \begin{cases} \dim E^\Gamma & \text{if } \pi_1(G) \text{ is infinite} \\ 0 & \text{otherwise.} \end{cases}$$

2.7. An Easy Version of "The Formula".

2.7.1. *The Pullback.* The use of resolutions consisting of L^∞ functions, although very useful, has its side effects. For example, given a continuous homomorphism $\rho : G \to G'$ of locally compact groups, it is obvious that the induced pullback in bounded cohomology $\rho^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G', E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$ could be implemented simply by pulling back cocycles if we were using the bar resolution: however the pullback, even via a continuous map, of a function in L^∞ (hence an equivalence class of functions) does not necessarily give a well defined equivalence class of functions. We recall here how it is however possible to implement the pullback in a rather natural way in the case of cocycles which arise in geometric situations, once again using homological algebra.

In fact, if X is a measurable G'-space, it is shown in [17, Proposition 2.1] that the complex $\mathcal{B}^\infty(X^\bullet)$ of bounded measurable functions is a strong resolution of $\mathbb{R}$. Not knowing whether the modules are relatively injective, we cannot conclude that the cohomology of the subcomplex of G'-invariants computes the continuous bounded cohomology of G', however we can deduce the existence of a functorially defined map

$$\epsilon^\bullet_X : \mathrm{H}^\bullet\big(\mathcal{B}^\infty(X^\bullet)^{G'}\big) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G', \mathbb{R})$$

such that to any bounded measurable G'-invariant cocycle $c : X^{n+1} \to \mathbb{R}$ corresponds canonically a class $[c] \in \mathrm{H}^n_{\mathrm{cb}}(G', \mathbb{R})$, [17, Corollary 2.2].

Let us now assume that there exists a Γ-equivariant measurable map $\varphi : G/P \to X$, where $P < G$ is a closed amenable subgroup.

EXAMPLE 2.39. An example of such situation occurs when X is the space $\mathcal{M}^1(G'/P')$ of probability measures on the homogeneous space G'/P' with G' a semisimple Lie group and P' a minimal parabolic subgroup, in which case the existence of the map φ follows immediately from the characterization of amenability given in Proposition 2.18, as one can easily see by taking E to be the space of continuous functions on G'/P', $A_s = \mathcal{M}^1(G'/P')$ (hence constant with respect to $s \in S$), and $\alpha(s, h) = \rho(h)$.

The main point of [17] is to show that the map φ can be used to implement the composition

$$\mathrm{H}^\bullet\big(\mathcal{B}^\infty(X^\bullet)^{G'}\big) \xrightarrow{\epsilon_X^\bullet} \mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \xrightarrow{\rho_{\mathrm{b}}^\bullet} \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R})\,. \tag{2.6}$$

More specifically, we recall here the following definition

Definition 2.40. Let X be a measurable G'-space. We say that the measurable map $c : X^{n+1} \to \mathbb{R}$ is a *strict measurable cocycle* if c is defined everywhere and satisfies everywhere the relation $dc = 0$.

Then we have:

Theorem 2.41 (Burger–Iozzi [17]). *Let G, G' be locally compact groups, let $\rho : L \to G'$ be a continuous homomorphism from a closed subgroup $L < G$, let $P < G$ be a closed amenable subgroup and let $\varphi : G/P \to X$ a ρ-equivariant measurable map into a measurable G'-space X. If $\kappa \in \mathrm{H}^n_{\mathrm{cb}}(G',\mathbb{R})$ is a bounded cohomology class representable by a G-invariant bounded strict measurable cocycle $c \in \mathcal{B}^\infty(X^{n+1})^{G'}$, then the image of the pullback $\rho_{\mathrm{b}}^{(n)}(\kappa) \in \mathrm{H}^n_{\mathrm{cb}}(L,\mathbb{R})$ can be represented canonically by the cocycle in $\mathcal{Z}\mathrm{L}^\infty((G/P)^{n+1})^L$ defined by*

$$(x_0,\dots,x_n) \mapsto c\big(\varphi(x_0),\dots,\varphi(x_n)\big)\,. \tag{2.7}$$

Exercise 2.42. Let $\Gamma < \mathrm{PU}(1,1)$ be a (cocompact) surface group and $(\partial\mathbb{D}^2,\lambda)$ the boundary of the hyperbolic disk $\mathbb{D}^2$ with the round measure λ. Then (Corollary 2.23)

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}) \cong \mathcal{Z}\mathrm{L}^\infty_{\mathrm{alt}}\big((\partial\mathbb{D}^2)^3,\mathbb{R}\big)^\Gamma\,.$$

Give an example of a class in $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ which cannot be represented by a strict pointwise Γ-invariant Borel cocycle on $(\partial\mathbb{D}^2)^3$.

This illustrates the fact that given a measurable G-invariance cocycle, while it is easy to make the cocycle either strict (see [65, Appendix B]) or everywhere G-invariant, obtaining both properties at the same time is sometimes not possible.

2.7.2. *The Transfer Map.* We need only one last bounded cohomological ingredient. If $L < G$ is a closed subgroup the injection $L \hookrightarrow G$ induces by contravariance in cohomology the restriction map

$$\mathrm{r}^\bullet_{\mathbb{R}} : \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R})\,.$$

If we assume that $L\backslash G$ has a G-invariant probability measure μ, then the *transfer map*

$$\mathrm{T}^\bullet : \mathrm{C}_{\mathrm{b}}(G^\bullet)^L \to \mathrm{C}_{\mathrm{b}}(G^\bullet)^G\,,$$

defined by integration

$$\mathrm{T}^{(n)} f(g_1, \dots, g_n) := \int_{L\backslash G} f(gg_1, \dots, gg_n) d\mu(g)\,, \tag{2.8}$$

for all $(g_1, \dots, g_n) \in G^n$, induces in cohomology a left inverse of $\mathrm{r}^\bullet_{\mathbb{R}}$ of norm one

$$\mathrm{T}^\bullet_{\mathrm{b}} : \mathrm{H}^\bullet_{\mathrm{cb}}(L, \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R})\,,$$

(see [53, Proposition 8.6.2, pp.106-107]).

Notice that the functorial machinery does not apply directly to the transfer map, as it is not a map of resolutions but it is only defined on the subcomplex of invariant vectors. However, the following result, which will be obtained in greater generality in § 4.1.2, allows us anyway to use the resolution of L^∞ functions on amenable spaces.

Lemma 2.43 (Monod [53]). *Let $P, L < G$ be closed subgroups with P amenable, and let*

$$\mathrm{T}^\bullet_{G/P} : \big(\mathrm{L}^\infty\big((G/P)^\bullet\big)^L, d^\bullet\big) \to \big(\mathrm{L}^\infty\big((G/P)^\bullet\big)^G, d^\bullet\big) \tag{2.9}$$

be defined by

$$\mathrm{T}^{(n)}_{G/P} f(x_1, \dots, x_n) := \int_{L\backslash G} f(gx_1, \dots, gx_n) d\mu(g)\,, \tag{2.10}$$

for $(x_1, \dots, x_n) \in (G/P)^n$, where μ is the G-invariant probability measure on $L\backslash G$. Then the diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) & \xrightarrow{\mathrm{T}^\bullet_{\mathrm{b}}} & \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) & \xrightarrow{\mathrm{T}^\bullet_{G/P}} & \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \end{array} \tag{2.11}$$

commutes, where the vertical arrows are the canonical isomorphisms in bounded cohomology extending the identity $\mathbb{R} \to \mathbb{R}$.

Putting together all of these ingredients, one has a general formula which has several applications to rigidity questions.

Proposition 2.44. *Let G, G' be locally compact second countable groups and let $L < G$ be a closed subgroup such that $L\backslash G$ carries a G-invariant probability measure μ. Let $\rho : L \to G'$ be a continuous homomorphism, X a measurable G'-space and assume that there exists an L-equivariant measurable map $\varphi : G/P \to X$, where $P < G$ is a closed subgroup. Let $\kappa' \in \mathrm{H}^n_{\mathrm{cb}}(G', \mathbb{R})$ and let $\kappa := \mathrm{T}^{(n)}_{\mathrm{b}}\big(\rho^{(n)}_{\mathrm{b}}(\kappa')\big) \in \mathrm{H}^n_{\mathrm{cb}}(G, \mathbb{R})$. Let $c \in \mathrm{L}^\infty\big((G/P)^{n+1}\big)^G$ and $c' \in \mathcal{B}^\infty(X^{n+1})^{G'}$ be alternating cocycles* *representing κ and κ' respectively. If we assume that c' is strict and that P is amenable then we have that*

(2.12)
$$\int_{L\backslash G} c'\big(\varphi(gx_0),\dots,\varphi(gx_n)\big)d\mu(g) = c(x_0,\dots,x_n) + \textit{coboundary},$$

for almost every $(x_0,\dots,x_n)\in(G/P)^{n+1}$.

3. First Applications of "The Formula"

The above proposition is really just a careful reformulation of the implementation of the *bounded Toledo map*, defined as the composition

$$\mathrm{T}_{\mathrm{b}}^{\bullet}(\rho) := \mathrm{T}^{\bullet}\circ\rho_{\mathrm{b}}^{\bullet} : \mathrm{H}_{\mathrm{cb}}^{\bullet}(G',\mathbb{R}) \to \mathrm{H}_{\mathrm{cb}}^{\bullet}(G,\mathbb{R})$$

of the pullback followed by the transfer map in continuous bounded cohomology. Likewise, its generalization (Proposition 4.9) will be a reformulation of the implementation of the pullback followed by the transfer map and by an appropriate change of coefficients. While for these two statements there is a unified treatment, in the applications – which require that both $\mathrm{H}_{\mathrm{c}}^n(G,\mathbb{R})$ and $\mathrm{H}_{\mathrm{cb}}^n(G,\mathbb{R})$ are one dimensional – we have to resort to a case by case study. The situation can be however summarized in the following:

Principle 3.1. Let G, G' be locally compact second countable groups and let $L<G$ be a closed subgroup such that $L\backslash G$ carries a G-invariant probability measure μ. Let $\rho: L\to G'$ be a continuous homomorphism, X a measurable G'-space and assume that there exists an L-equivariant measurable map $\varphi: G/P\to X$, where $P<G$ is a closed subgroup. Let $\kappa'=[c']\in\mathrm{H}_{\mathrm{cb}}^n(G',\mathbb{R})$ and let $\mathrm{H}_{\mathrm{c}}^n(G,\mathbb{R})\cong\mathrm{H}_{\mathrm{cb}}^n(G,\mathbb{R})=\mathbb{R}\kappa=\mathbb{R}[c]$, where $c\in\mathcal{Z}\mathrm{L}^\infty\big((G/P)^{n+1}\big)^G$ and $c'\in\mathcal{B}^\infty(X^{n+1})^{G'}$ are alternating cocycles and c' is strict. If P is amenable, then there exists an <u>explicit</u> constant $\lambda_{\kappa'}\in\mathbb{R}$ such that

(3.1)
$$\int_{L\backslash G} c'\big(\varphi(gx_0),\dots,\varphi(gx_n)\big)d\mu(g) = \lambda_{\kappa'}c(x_0,\dots,x_n) + \textit{coboundary},$$

for almost every $(x_0,\dots,x_n)\in(G/P)^{n+1}$.

Remark 3.2. (1) Notice that if for example the action of G on $(G/P)^n$ is ergodic, then there is no coboundary term, as ergodicity is equivalent to the nonexistence of G-invariant measurable maps $(G/P)^n\to\mathbb{R}$ which are not constant. This is going to be the case in all of our applications.

(2) Clearly the above formula would not be useful as is if we were interested in the values of the measurable function φ on sets of measure zero. It is for this purpose that in the application to deformation rigidity of lattices in complex hyperbolic spaces, where we need to gather information about the "values" of φ on a chain in the boundary of complex hyperbolic space (see § 5), we need to recur to the use of coefficients coupled with the use of fibered products. This will be done in § 4, after that we illustrate in the next section some of the applications of Proposition 2.44 and Principle 3.1.

While there is no general proof of this principle, in each case the identification of the constant $\lambda_{\kappa'}$ will follow from the interplay between the bounded Toledo map and the corresponding map in continuous cohomology, that is from the commutativity of the following diagram

$$\begin{tikzcd} \mathrm{H}^n_{\mathrm{cb}}(G',\mathbb{R}) \arrow[r] \arrow[d, "\rho^\bullet_{\mathrm{b}}"'] & \mathrm{H}^n_{\mathrm{c}}(G',\mathbb{R}) \arrow[d, "\rho^\bullet"] \\ \mathrm{H}^n_{\mathrm{cb}}(L,\mathbb{R}) \arrow[d, "\mathrm{T}^\bullet_{\mathrm{b}}"'] & \mathrm{H}^n_{\mathrm{c}}(L,\mathbb{R}) \arrow[d, "\mathrm{T}^\bullet"] \\ \mathrm{H}^n_{\mathrm{cb}}(G,\mathbb{R}) \arrow[r, "\cong"] & \mathrm{H}^n_{\mathrm{c}}(G,\mathbb{R}) \end{tikzcd} \tag{3.2}$$

where the horizontal arrows are the obvious comparison maps between continuous bounded and continuous cohomology, the map $\rho^\bullet$ is the pullback in ordinary continuous cohomology, and the transfer map $\mathrm{T}^\bullet$: $\mathrm{H}^\bullet_{\mathrm{c}}(L,\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R})$ is defined by integration on $L\backslash G$ if in addition this space is compact. This is the case in the applications we present in this section, and we refer the reader to § 5 for a further discussion on this important point.

We give now a very short list of some situations in which the above formula is of use. (Note that in all our examples, as remarked before. one can conclude that there are no coboundaries.) Not all results are new, and our method does not even provide a new proof in some cases. Nevertheless, we deem appropriate to discuss here possible applicability of this method, as well as its present limitations.

3.1. **Mostow Rigidity Theorem.** The celebrated theorem of Mostow asserts that, in dimension $n \geq 3$, any two compact hyperbolic manifolds M_1 and M_2 which are homotopy equivalent are isometric. In his notes

[59], Thurston provides a new proof of this result, using measure homology (a generalization of ℓ^1-homology) as well as the determination of the maximal ideal simplices in hyperbolic geometry. Since this last result (later obtained by Haagerup and Munkholm [40]) was available at that time only for $n = 3$, Thurston's proof of Mostow Rigidity Theorem is limited to this case. However, the proof contains in disguise exactly our formula (2.12) for all $n \geq 3$, while with our method we succeed only in proving the formula in the case in which $n = 3$, because in the general case we do not have enough information about the comparison map in higher degrees. In fact, while $\mathrm{H}^n_{\mathrm{c}}\big(\mathrm{SO}(1,n)\big) \cong \mathbb{R}$, in general it is not known whether the comparison map $\mathrm{H}^n_{\mathrm{cb}}\big(\mathrm{SO}(1,n)\big) \to \mathrm{H}^n_{\mathrm{c}}\big(\mathrm{SO}(1,n)\big)$ is injective for all n: if $n = 3$ this follows from a result of Bloch, [7, 8].

So let M_1, M_2 be compact hyperbolic 3-manifolds with isomorphic fundamental groups, set $\Gamma := \pi_1(M_1) < \mathrm{SO}(1,3) =: G$ which is a cocompact lattice and let $\Gamma' := \pi_1(M_2) < \mathrm{SO}(1,3) =: G'$. Let $G/P = X = \mathbb{S}^2 = \partial\mathcal{H}^3_{\mathbb{R}}$ and let $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ be the ρ-equivariant boundary homeomorphism, where $\rho : \Gamma \to \Gamma'$ is an isomorphism. Let $c = c'$ be the volume 3-cocycle on ideal 3-simplices defining cohomology classes $\kappa = \kappa' \in \mathrm{H}^3_{\mathrm{cb}}\big(\mathrm{SO}(1,3), \mathbb{R}\big)$. Then one obtains that $\lambda_{\kappa'} = \frac{\mathrm{vol}(M_2)}{\mathrm{vol}(M_1)}$, and hence the formula (2.12) reads

(3.3)
$$\int_{\Gamma\backslash\mathrm{SO}(1,3)} \mathrm{vol}(\varphi(g\xi_0), \dots, \varphi(g\xi_3))d\mu(g) = \frac{\mathrm{vol}(M_2)}{\mathrm{vol}(M_1)}\,\mathrm{vol}(\xi_0, \dots, \xi_3)\,,$$

where $\mathrm{vol}(\xi_0, \dots, \xi_3)$ is the volume of the ideal simplex in $\mathcal{H}^3_{\mathbb{R}}$ with vertices $\xi_0, \dots, \xi_3$, μ is the normalized Haar measure on $\Gamma\backslash\mathrm{SO}(1,3)$ and equality holds almost everywhere. Because the measure μ is a probability measure, it follows that $\mathrm{vol}(M_2) \leq \mathrm{vol}(M_1)$, from which, interchanging the role of M_1 and M_2 one obtains that M_1 and M_2 have the same volume. Now observe that both sides of (3.3) are continuous functions on $(\mathbb{S}^2)^4$ which, coinciding almost everywhere, are therefore equal for all values of $(\xi_0, \dots, \xi_3) \in (\mathbb{S}^2)^4$. Thus, whenever $\mathrm{vol}(\xi_0, \dots, \xi_3)$ is maximal, we deduce from (3.3) taking into account that $\mathrm{vol}(M_1) = \mathrm{vol}(M_2)$ and μ is a probability measure, that $\mathrm{vol}\,\big(\varphi(\xi_0), \dots, \varphi(\xi_3)\big)$ is maximal as well. From this, one deduces like in [59], that the isomorphism between the fundamental groups extends to an isomorphism between the ambient connected groups.

Let us relate this to the ℓ^1-homology approach of Gromov–Thurston. If $f : \mathcal{H}^3_{\mathbb{R}} \to \mathcal{H}^3_{\mathbb{R}}$ denotes a lift of a homotopy equivalence associated to the isomorphism $\rho : \Gamma \to \Gamma'$, then Thurston's smearing technique implies that if $\sigma : \Delta^3 \to \mathcal{H}^3_{\mathbb{R}}$ is any straight simplex, then

$$\int_{\Gamma\backslash \mathrm{SO}(1,3)} \mathrm{vol}\big(f\big(g\sigma(0)\big),\dots,f\big(g\sigma(3)\big)\big)d\mu(g) = \mathrm{vol}(\sigma)\frac{\mathrm{vol}(M_2)}{\mathrm{vol}(M_1)}\,.$$

One can then follow an idea of Pansu, using the fact that f extends continuously to the boundary with extension $f : \mathbb{S}^2 \to \mathbb{S}^2$ and let the vertices of σ tend to $(\xi_0,\dots,\xi_3) \in (\mathbb{S}^2)^4$ to obtain (3.3).

The strength of this argument is that it extends to all real hyperbolic spaces $\mathcal{H}^n_{\mathbb{R}}$. Its limitation however lies in the fact that it requires very strong conditions on ρ in order to have a map extending "nicely" to the boundary. Besides, it cannot be applied for example in Matsumoto's theorem since there is no symmetric space associated to $\mathrm{Homeo}_+(\mathbb{S}^1)$.

3.2. **Matsumoto's Theorem.** Let Γ_g be the fundamental group of a compact oriented surface Σ_g of genus $g \geq 2$ and let $\rho : \Gamma_g \to \mathrm{Homeo}_+(\mathbb{S}^1)$ be an action of Γ_g on the circle by orientation preserving homeomorphisms. Let $e \in \mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1),\mathbb{Z})$ be the Euler class defined by the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 0\,,$$

where $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the group of homeomorphisms of the real line which commute with the integral translations. Then $\rho^{(2)}(e) \in \mathrm{H}^2(\Gamma_g,\mathbb{Z})$ and, since Σ_g is a $K(\Gamma_g,1)$ and hence $\mathrm{H}^2(\Gamma_g,\mathbb{Z}) \cong \mathrm{H}^2(\Sigma_g,\mathbb{Z})$, we can evaluate $\rho^{(2)}(e)$ on the fundamental class $[\Sigma_g] \in \mathrm{H}_2(\Sigma_g,\mathbb{Z})$ of Σ_g. We thus obtain a numerical invariant attached to the representation ρ, called the *Euler number* $eu(\rho)$ of ρ,

$$eu(\rho) := \big\langle \rho^{(2)}(e), [\Sigma_g]\big\rangle\,,$$

which turns out to be uniformly bounded with respect to the representation. In fact, we have the Milnor–Wood inequality [49, 62]

$$|eu(\rho)| \leq |\chi(\Sigma_g)|\,,$$

and we say that a representation is *maximal* if $|eu(\rho)| = |\chi(\Sigma_g)|$. Examples of maximal representations are for instance hyperbolizations (that is, faithful representations into $\mathrm{PU}(1,1)$ such that the image $\rho(\Gamma_g)$ is a lattice in $\mathrm{PU}(1,1)$. Matsumoto's theorem provides some kind of converse to this statement, namely:

Theorem 3.3 (Matsumoto [48]). *If ρ is maximal, then ρ is semiconjugate to a hyperbolization.*

Recall that a semiconjugacy in this context is the map given on $\mathbb{S}^1$ by a monotone increasing map from the real line to itself which commutes with translations by integers.

Our proof in [42] follows once again from the formula in (2.12), where we take $G = \mathrm{PU}(1,1)$, $G' = \mathrm{Homeo}_+(\mathbb{S}^1)$, $\mathrm{PU}(1,1)/P = \mathbb{S}^1$, $X = \mathcal{M}^1\big(\mathrm{PU}(1,1)/P\big)$, c the orientation cocycle and c' its restriction to $\mathrm{PU}(1,1)$. Then one can prove that $\lambda = \frac{eu(\rho)}{\chi(\Sigma_g)}$ and thus one obtains that (2.12) reads

$$\int_{\Gamma\backslash\mathrm{PU}(1,1)} \big(\varphi(gb_0)\varphi(gb_1)\varphi(gb_2)\big)(c)d\mu(g) = \frac{eu(\rho)}{\chi(\Sigma_g)}c(b_0,b_1,b_2)\,,$$

for almost every $b_0, b_1, b_3 \in \mathbb{S}^1$ and where $d\mu$ is the normalized measure on $\Gamma\backslash\mathrm{PU}(1,1)$. Since μ is a probability measure, once again we obtain the Milnor–Wood inequality $|eu(\rho)| \leq \big|\chi(\Sigma_g)\big|$. Moreover, if we have equality, then the above formula implies that the boundary map φ takes values in $\mathbb{S}^1$ itself. It follows that φ is "almost" order preserving, in the sense that it preserves the order of almost all triples of points in $\mathbb{S}^1$. An "inverse" of φ in an appropriate sense provides the explicit semiconjugacy between the representation ρ and an hyperbolization [42].

3.3. **Maximal Representations.** Before Matsumoto, Goldman proved in his thesis the full converse of the above statement for a representation into $\mathrm{PU}(1,1)$, namely he showed that if $\rho : \Gamma_g \to \mathrm{PU}(1,1)$ is maximal, then it is indeed a hyperbolization. The generalization of this result to representations into the (connected component of the) isometry group of a Hermitian symmetric space was the starting point of the results exposed in this section.

So let, as above, Σ_g be a compact oriented surface of genus $g \geq 2$ and fundamental group $\Gamma_g := \pi_1(\Sigma_g)$, and let $\rho : \Gamma_g \to G'$ be a homomorphism into the connected component $G' = \mathrm{Iso}(\mathcal{X}')^\circ$, of the isometry group of a Hermitian symmetric space $\mathcal{X}'$. Associated to ρ we can define an invariant τ_ρ as follows: let $f : \Sigma_g \to \Sigma_g \times_\rho \mathcal{X}'$ be a smooth section of the flat bundle with fiber $\mathcal{X}'$ associated to the principal Γ_g-bundle $\widetilde{\Sigma_g} \to \Sigma_g$, and let $\tilde f : \widetilde{\Sigma_g} \to \mathcal{X}'$ be a smooth Γ_g-equivariant lift of f. The Kähler form $\omega_{\mathcal{X}'}$ on $\mathcal{X}'$ pulls back to a Γ_g-invariant closed two-form $\tilde f^*\omega_{\mathcal{X}'}$ on $\widetilde{\Sigma_g}$, which hence descends to a closed two form on Σ_g. Since the map $\tilde f$ is unique up to Γ_g-equivariant homotopy, the integral

$$\tau_\rho := \int_{\Sigma_g} \tilde f^*\omega_{\mathcal{X}'}$$

depends only on ρ and defines the *Toledo invariant* of ρ. Moreover, while the above definition would not have been possible in the case of a representation of Γ_g into $\mathrm{Homeo}_+(\mathbb{S}^1)$, we could have defined τ_ρ

analogously to § 3.2 as

$$\tau_\rho := \left\langle \rho^{(2)}(\kappa), [\Sigma_g] \right\rangle$$

and in fact it can be proven that the two definitions coincide. For more interpretations of the Toledo invariant see for instance [13]. At any rate, we also have an analogue of the Milnor–Wood inequality, namely

$$|\tau_\rho| \leq \left|\chi(\Sigma_g)\right| r_{\mathcal{X}'}\,, \tag{3.4}$$

where $r_{\mathcal{X}'}$ is the rank of the symmetric space $\mathcal{X}'$ [28, 27], and we say that ρ is maximal if $|\tau_\rho| = \left|\chi(\Sigma_g)\right| r_{\mathcal{X}'}$.

Before we state the next result, recall that an important subclass of Hermitian symmetric spaces consists of those of *tube type*, that is those, like for instance the Poincaré disk, which are biholomorphically equivalent to $\mathbb{R}^n \times iC$, where $C \subset \mathbb{R}^n$ is a convex open cone. There are several characterizations of the Hermitian symmetric spaces of tube type, but the relevant one here lies in the fact that it is only for these Hermitian symmetric spaces that the cocycle $\beta_{\mathcal{X}'}$ in (2.5) takes a finite number of values [20]. Then we have:

THEOREM 3.4 (Burger-Iozzi-Wienhard [23, 21]). *Let $\rho : \Gamma_g \to G'$ be a maximal representation. Then ρ is faithful with discrete image. Moreover the Zariski closure of the image of ρ is reductive and the associated symmetric space is of tube type.*

A thorough study of maximal representations has been carried out in several papers, see [60, 41, 23, 21, 10] for example, and many additional interesting properties have been proven. We have limited ourselves here to present the features which are a direct consequence of Proposition 2.44 and Corollary 3.1. To illustrate the technique, we suppose here that the image of the representation ρ is Zariski dense in G'. In this case we have that $G = \mathrm{SU}(1,1)$, and Γ is the image of the compact surface group via a hyperbolization, $\mathrm{SU}(1,1)/P \cong \mathbb{S}^1$, X is the Shilov boundary of $\mathcal{X}'$ (that is the unique closed G'-orbit in the topological boundary of the bounded domain realization of $\mathcal{X}'$), while $c' = \beta_{\mathcal{X}'}$ and $c = \beta_{\mathbb{D}^2}$ as defined in (2.5). Then one obtains that $\lambda_{\kappa'} = \frac{\tau_\rho}{\left|\chi(\Sigma_g)\right|}$ and hence (2.12) reads

$$\int_{\Gamma\backslash \mathrm{PU}(1,1)} \beta_{\mathcal{X}'}\big(\varphi(gx_0), \varphi(gx_1), \varphi(gx_2)\big) d\mu(g) = \frac{\tau_\rho}{\left|\chi(\Sigma_g)\right|} \beta_{\mathbb{D}^2}(x_0, x_1, x_2)\,.$$

Once again, since μ is a probability measure, we obtain the inequality (3.4), and if ρ is maximal we have that

$$\beta_{\mathcal{X}'}\big(\varphi(x_0), \varphi(x_1), \varphi(x_2)\big) = r_{\mathcal{X}'} \beta_{\mathbb{D}^2}(x_0, x_1, x_2) \tag{3.5}$$

for almost all $(x_0, x_1, x_2) \in (\mathbb{S}^1)^3$.

The equality (3.5) has then far reaching consequences. In fact, $e^{2\pi i \beta_{\mathcal{X}}}$ is on $\check{S}^{(3)}$ a rational function and tube type domains are characterized by the property that this rational function is constant (Burger–Iozzi [18] and Burger–Iozzi–Wienhard [20]); but the equality (3.5) implies, taking into account that $\rho(\Gamma)$ is Zariski dense, that $e^{2\pi i \beta_{\mathcal{X}}}$ is constant on a Zariski dense subset of $\check{S}^{(3)}$, hence constant, which implies that $\mathcal{X}'$ is of tube type. Using then that if $\mathcal{X}'$ is of tube type the level sets of $\beta_{\mathcal{X}'}$ on $\check{S}^{(3)}$ are open, one deduces easily that $\rho(\Gamma)$ is not dense and, being Zariski dense, is therefore discrete. The fact that ρ is injective requires more elaborate arguments in which (3.5) enters essentially [23, 21].

4. Toward "The Formula" with Coefficients

In this section we develop some tools in bounded cohomology for locally compact groups and their closed subgroups which will be applied to our specific situation. In particular we prove a formula in § 4.2 of which Proposition 2.44 is a particular case.

4.1. **With the Use of Fibered Products.**

The invariants we consider in this paper are bounded classes with trivial coefficients; however applying a judicious change of coefficients – from $\mathbb{R}$ to the L^∞ functions on a homogeneous space – we capture information which otherwise would be lost by the use of measurable maps (see the last paragraph of § 2 and Remark 5.8).

In doing so, we first find ourselves to have to deal with a somewhat new situation. More precisely, while the functorial machinery developed in [25], [53] and [17] applies in theory to general strong resolutions, in practice one ends up working mostly with spaces of functions on Cartesian products. In this section we deal with spaces of functions on fibered products (of homogeneous spaces), whose general framework would be that of complexes of functions on appropriate sequences $(\mathcal{S}_n, \nu_n)$ of (amenable) spaces which would be analogues of simplicial sets in the category of measured spaces.

In particular we shall first show in § 4.1.1 that we can compute the continuous bounded cohomology with some L^∞ coefficients as the cohomology of the complex of L^∞ functions on appropriate fibered products, then in § 4.1.2 and § 4.1.3 respectively we shall see how to implement the transfer map and the pullback using this particular resolution.

4.1.1. *Realization on Fibered Products.* The goal of this section is to define the fibered product of homogeneous spaces and prove that the complex of L^∞ functions on fibered products satisfies all properties necessary to be used to compute bounded cohomology. Observe that because of the projection in (4.1), we shall deal here with cohomology with coefficients.

Let G be a locally compact, second countable group and P, H closed subgroups of G such that $P \leq H$. We define the n-fold fibered product $(G/P)^n_f$ of G/P with respect to the canonical projection $p : G/P \to G/H$ to be, for $n \geq 1$, the closed subset of $(G/P)^n$ defined by

$$(G/P)^n_f := \big\{(x_1, \dots, x_n) \in (G/P)^n \ : \ p(x_1) = \cdots = p(x_n)\big\}\,,$$

and we set $(G/P)^n_f = G/H$ if $n = 0$. The invariance of $(G/P)^n_f$ for the diagonal G-action on $(G/P)^n$ induces a G-equivariant projection

$$p_n : (G/P)^n_f \to G/H \tag{4.1}$$

whose typical fiber is homeomorphic to $(H/P)^n$.

A useful description of $(G/P)^n_f$ as a quotient space may be obtained as follows. Considering H/P as a subset of G/P, the map

$$\begin{aligned} q_n : G \times (H/P)^n &\to \quad (G/P)^n_f \\ (g, x_1, \dots, x_n) &\mapsto (gx_1, \dots, gx_n) \end{aligned} \tag{4.2}$$

is well defined, surjective, G-equivariant (with respect to the G-action on the first coordinate on $G \times (H/P)^n$ and the product action on $(G/P)^n_f$) and invariant under the right H-action on $G \times (H/P)^n$ defined by

$$(g, x_1, \dots, x_n)h := (gh, h^{-1}x_1, \dots, h^{-1}x_n)\,. \tag{4.3}$$

It is then easy to see that q_n induces a G-equivariant homeomorphism

$$\big(G \times (H/P)^n\big)/H \to (G/P)^n_f\,,$$

which hence realizes the fibered product $(G/P)^n_f$ as a quotient space.

Let now μ and ν be Borel probability measures respectively on G and H/P, such that μ is in the class of the Haar measure on G and ν is in the H-invariant measure class on H/P. The pushforward $\nu_n = (q_n)_*(\mu \times \nu^n)$ of the probability measure $\mu \times \nu^n$ under q_n is then a Borel probability measure on $(G/P)^n_f$ whose class is G-invariant and thus gives rise to Banach G-modules $\mathrm{L}^\infty((G/P)^n_f)$ and G-equivariant (norm) continuous maps

$$d_n : \mathrm{L}^\infty\big((G/P)^n_f\big) \to \mathrm{L}^\infty\big((G/P)^{n+1}_f\big)\,, \text{for } n \geq 0\,,$$

defined as follows:

– $d_0 f(x) := f(p(x))$, for $f \in \mathrm{L}^\infty(G/H)$, and
– $d_n f(x) = \sum_{i=1}^{n+1}(-1)^{i-1} f\big(p_{n,i}(x)\big)$, for $f \in \mathrm{L}^\infty\big((G/P)^n_f\big)$ and $n \geq 1$,

where

$$p_{n,i} : (G/P)^{n+1}_f \to (G/P)^n_f \tag{4.4}$$

is obtained by leaving out the i-th coordinate. Observe that from the equality $(p_{n,i})_*(\nu_{n+1}) = \nu_n$, it follows that d_n is a well defined linear map between L^∞ spaces.

Then:

Proposition 4.1. *Let $L \leq G$ be a closed subgroup.*

(i) *The complex*

$$0 \longrightarrow \mathrm{L}^\infty(G/H) \longrightarrow \dots \longrightarrow \mathrm{L}^\infty\big((G/P)^n_f\big) \xrightarrow{d_n} \mathrm{L}^\infty\big((G/P)^{n+1}_f\big) \longrightarrow \dots$$

is a strong resolution of the coefficient L-module $\mathrm{L}^\infty(G/H)$ by Banach L-modules.

(ii) *If P is amenable and $n \geq 1$, then the G-action on $(G/P)^n_f$ is amenable and $\mathrm{L}^\infty\big((G/P)^n_f\big)$ is a relatively injective Banach L-module.*

Using [25, Theorem 2] (see also § 2), this implies immediately the following:

Corollary 4.2. *Assume that P is amenable. Then the cohomology of the complex of L-invariants*

$$0 \longrightarrow \mathrm{L}^\infty(G/P)^L \longrightarrow \mathrm{L}^\infty\big((G/P)^2_f\big)^L \longrightarrow \cdots$$

is canonically isomorphic to the bounded continuous cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}\big(L, \mathrm{L}^\infty(G/H)\big)$ of L with coefficients in $\mathrm{L}^\infty(G/H)$.

Remark 4.3. Just like for the usual resolutions of L^∞ functions on the Cartesian product of copies of an amenable space (see § 2 or [25]), it is easy to see that the statements of Proposition 4.1 and of Corollary 4.2 hold verbatim if we consider instead the complex $\big(\mathrm{L}^\infty_{\mathrm{alt}}\big((G/P)^\bullet\big), d^\bullet\big)$, where $\mathrm{L}^\infty_{\mathrm{alt}}\big((G/P)^n_f\big)$ is the subspace consisting of functions in $\mathrm{L}^\infty\big((G/P)^n_f\big)$ which are alternating (observe that the symmetric group in n letters acts on $(G/P)^n_f$).

Proof of Proposition 4.1. The proof of Proposition 4.1(i) consists in the construction of appropriate contracting homotopy operators. Since it is rather long and technical, it will be given in the appendix at the end of this paper.

To prove Proposition 4.1(ii), we start by observing that if $n \geq 1$ we have by definition the inclusion $(G/P)^n_f \subset (G/P)^n$ and hence there is a map of G-spaces

$$\pi : (G/P)^n_f \to G/P,$$

obtained by projection on the first component. Since $\pi_*(\nu_n) = \nu$, π realizes the measure G-space $(G/P)^n_f$ as an extension of the measure G-space G/P. If P is amenable, the latter is an amenable G-space and hence the G-space $(G/P)^n_f$ is also amenable [64]. Since L is a closed subgroup, $(G/P)^n_f$ is also an amenable L-space [65, Theorem 4.3.5] and hence $\mathrm{L}^\infty\big((G/P)^n_f\big)$ is a relatively injective L-module, (Theorem 2.20 or [25]). $\square$

4.1.2. *An Implementation of the Transfer Map.* We recalled in (2.8) the definition of the transfer map, and remarked that the functorial machinery does not apply directly because $\mathrm{T}^\bullet$ is not a map of resolutions but is defined only on the subcomplex of invariant vectors. The point of this subsection is to see how the transfer map can be implemented, in a certain sense, on the resolution by L^∞ functions on the fibered product defined in § 4.1.1.

Let H, P be closed subgroups of G such that $P < H$. We assume that P is amenable so that, by Proposition 4.1, the complex $\big(\mathrm{L}^\infty\big((G/P)^\bullet_f\big), d^\bullet\big)$ is a strong resolution of the coefficient module $\mathrm{L}^\infty(G/H)$ by relatively injective L-modules. For $n \geq 1$, $\phi \in \mathrm{L}^\infty\big((G/P)^n_f\big)^L$, and $(x_1, \dots, x_n) \in (G/P)^n_f$, let

$$(\tau^{(n)}_{G/P}\phi)(x_1, \dots, x_n) := \int_{L\backslash G} \phi(gx_1, \dots, gx_n) d\mu(\dot{g})\,. \tag{4.5}$$

This defines a morphism of complexes

$$\tau^\bullet_{G/P} : \big(\mathrm{L}^\infty\big((G/P)^\bullet_f\big)^L\big) \to \big(\mathrm{L}^\infty\big((G/P)^\bullet_f\big)^G\big)$$

and gives a left inverse to the inclusion

$$\big(\mathrm{L}^\infty\big((G/P)^\bullet_f\big)^G\big) \hookrightarrow \big(\mathrm{L}^\infty\big((G/P)^\bullet_f\big)^L\big)\,.$$

The induced map in cohomology

$$\tau^\bullet_{G/P} : \mathrm{H}^\bullet_{\mathrm{cb}}\big(L, \mathrm{L}^\infty(G/H)\big) \to \mathrm{H}^\bullet_{\mathrm{cb}}\big(G, \mathrm{L}^\infty(G/H)\big)$$

is thus a left inverse of the restriction map $\mathrm{r}^\bullet_{\mathrm{L}^\infty(G/H)}$.

Lemma 4.4. *With the above notations, and for any amenable group P, the diagram*

$$
\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \arrow[r, "\mathrm{T}^\bullet_{\mathrm{b}}"] \arrow[d, "\theta^\bullet_L"'] & \mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R}) \arrow[d, "\theta^\bullet_G"] \\
\mathrm{H}^\bullet_{\mathrm{cb}}\big(L,\mathrm{L}^\infty(G/H)\big) \arrow[r, "\tau^\bullet_{G/P}"] & \mathrm{H}^\bullet_{\mathrm{cb}}\big(G,\mathrm{L}^\infty(G/H)\big)
\end{tikzcd}
\tag{4.6}
$$

commutes, where $\theta^\bullet$ is the canonical map induced in cohomology by the morphism of coefficients $\theta:\mathbb{R}\to\mathrm{L}^\infty(G/H)$.

Observe that if in the above lemma we take $H=G$, then the fibered product $(G/P)^n_f$ becomes the usual Cartesian product $(G/P)^n$, and the cohomology of the complex of L-invariants $\big(\mathrm{L}^\infty\big((G/P)^\bullet\big)^L,d^\bullet\big)$ computes as usual the bounded cohomology of L with trivial coefficients. Hence we obtain once again Lemma 2.43.

Proof of Lemma 4.4. Let G^n_f be the n-fold fibered product with respect to the projection $G\to G/H$. The restriction of continuous functions defined on G^n to the subspace $G^n_f\subset G^n$ induces a morphism of strong L-resolutions by L-injective modules

$$R^\bullet:\mathrm{C}_{\mathrm{b}}(G^\bullet)\to\mathrm{L}^\infty(G^\bullet_f)$$

extending $\theta:\mathbb{R}\to\mathrm{L}^\infty(G/H)$, so that the diagram

$$
\begin{tikzcd}
\mathrm{C}_{\mathrm{b}}(G^n)^L \arrow[r, "\mathrm{T}^{(n)}"] \arrow[d, "R^{(n)}_L"'] & \mathrm{C}_{\mathrm{b}}(G^n)^G \arrow[d, "R^{(n)}_G"] \\
\mathrm{L}^\infty(G^n_f)^L \arrow[r, "\tau^{(n)}_G"] & \mathrm{L}^\infty(G^n_f)^G
\end{tikzcd}
\tag{4.7}
$$

commutes.

Likewise, the projection $\beta_n:G^n_f\to(G/P)^n_f$, for $n\geq1$, gives by precomposition a morphism of strong L-resolutions by L-injective modules

$$\beta^\bullet:\mathrm{L}^\infty((G/P)^\bullet_f)\to\mathrm{L}^\infty(G^\bullet_f)$$

extending the identity $\mathrm{L}^\infty(G/H)\to\mathrm{L}^\infty(G/H)$ and, as before, the diagram

$$
\begin{tikzcd}
\mathrm{L}^\infty(G^n_f)^L \arrow[r, "\tau^{(n)}_G"] & \mathrm{L}^\infty(G^n_f)^G \\
\mathrm{L}^\infty\big((G/P)^n_f\big)^L \arrow[r, "\tau^{(n)}_{G/P}"] \arrow[u, "\beta^{(n)}_L"] & \mathrm{L}^\infty\big((G/P)^n_f\big)^G, \arrow[u, "\beta^{(n)}_G"']
\end{tikzcd}
\tag{4.8}
$$

commutes.

The composition of the map induced in cohomology by $R^\bullet$ with the inverse of the isomorphism induced by $\beta^\bullet$ in cohomology realizes therefore the canonical map

$$\theta_L^\bullet : \mathrm{H}_{\mathrm{cb}}^\bullet(L,\mathbb{R}) \to \mathrm{H}_{\mathrm{cb}}^\bullet\big(L,\mathrm{L}^\infty(G/H)\big) \tag{4.9}$$

induced by the change of coefficient $\theta : \mathbb{R} \to \mathrm{L}^\infty(G/H)$, [53, Proposition 8.1.1]. Hence the commutative diagrams induced in cohomology by (4.7) and (4.8) can be combined to obtain a diagram

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{cb}}^\bullet(L,\mathbb{R}) \arrow[r, "\mathrm{T}_{\mathrm{b}}^\bullet"] \arrow[d, "R_L^\bullet"'] \arrow[dd, bend right=60, "\theta_L^\bullet"'] & \mathrm{H}_{\mathrm{cb}}^\bullet(G,\mathbb{R}) \arrow[d, "R_G^\bullet"] \arrow[dd, bend left=60, "\theta_G^\bullet"] \\
\mathrm{H}_{\mathrm{cb}}^\bullet\big(L,\mathrm{L}^\infty(G/H)\big) \arrow[r, "\tau_G^\bullet"] \arrow[d, "(\beta_L^\bullet)^{-1}"', "\cong"] & \mathrm{H}_{\mathrm{cb}}^\bullet\big(G,\mathrm{L}^\infty(G/H)\big) \arrow[d, "\cong"', "(\beta_G^\bullet)^{-1}"] \\
\mathrm{H}_{\mathrm{cb}}^\bullet\big(L,\mathrm{L}^\infty(G/H)\big) \arrow[r, "\tau_{G/P}^\bullet"] & \mathrm{H}_{\mathrm{cb}}^\bullet\big(G,\mathrm{L}^\infty(G/H)\big)
\end{tikzcd}$$

whose commutativity completes the proof. □

4.1.3. *An Implementation of the Pullback.* In this section we shall use the results of § 2.7.1 (see also [17]) to implement the pullback in bounded cohomology followed by the change of coefficients, by using the resolution by L^∞ functions on the fibered product.

We saw already in § 2.7.1 how to implement the composition (2.6) with the use of a boundary map $\varphi : G/Q \to X$, where $Q < G$ is an amenable subgroup and X is a measurable G'-space. The point of this section is to move one step further and to show how to represent canonically the composition of the above maps with the map $\theta_L^\bullet$ in (4.9).

To this purpose, let P, H, Q be closed subgroups of G such that $P \leq H \cap Q$, and let us consider the map

$$\begin{aligned} G \times H/P &\to G/Q \\ (g, xP) &\mapsto gxQ \end{aligned}$$

which, composed with φ, gives a measurable map $\widetilde{\varphi} : G \times H/P \to X$ which has the properties of being:

- L-equivariant with respect to the action by left translations on the first variable: $\widetilde{\varphi}(\gamma g, \dot{x}) = \rho(\gamma)\widetilde{\varphi}(g, \dot{x})$ for all $\gamma \in L$ and a. e. $(g, \dot{x}) \in G \times H/P$;
- H-invariant with respect to the right action considered in (4.3): $\widetilde{\varphi}(gh^{-1}, h\dot{x}) = \widetilde{\varphi}(g, \dot{x})$ for all $h \in H$ and all $(g, \dot{x}) \in G \times H/P$.

For every $n \geq 1$, the measurable map

$$\begin{aligned} \widetilde{\varphi}_f^n : G \times (H/P)^n &\longrightarrow \quad X^n \\ (g, \dot{x}_1, \dots, \dot{x}_n) &\mapsto \big(\widetilde{\varphi}(g, \dot{x}_1), \dots, \widetilde{\varphi}(g, \dot{x}_n)\big) \end{aligned}$$

gives, in view of (4.2), (i) and (ii), a measurable L-equivariant map $\varphi_f^n : (G/P)_f^n \to X^n$ defined by the composition

$$\varphi_f^n : (G/P)_f^n \xrightarrow{q_n^{-1}} \big(G \times (H/P)^n\big)/H \xrightarrow{\widetilde{\varphi}_f^n} X^n \,, \tag{4.10}$$

such that for every $1 \leq i \leq n+1$ the diagram

$$\begin{tikzcd} (G/P)_f^{n+1} \arrow[r, "\varphi_f^{n+1}"] \arrow[d, "p_{n,i}"'] & X^{n+1} \arrow[d] \\ (G/P)_f^n \arrow[r, "\varphi_f^n"] & X^n \end{tikzcd}$$

commutes, where $p_{n,i}$ was defined in (4.4) and the second vertical arrow is the map obtained by dropping the i-th coordinate. Precomposition by φ_f^n gives thus rise to a morphism of strong L-resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & \mathcal{B}^\infty(X^n) \arrow[r] \arrow[d, "\varphi_f^{(n)}"] & \cdots \\ 0 \arrow[r] & \mathrm{L}^\infty(G/H) \arrow[r] & \cdots \arrow[r] & \mathrm{L}^\infty\big((G/P)_f^n\big) \arrow[r] & \cdots \end{tikzcd}$$

extending the inclusion $\mathbb{R} \hookrightarrow \mathrm{L}^\infty(G/H)$. Let us denote by

$$\varphi_f^\bullet : \mathrm{H}^\bullet\big(\mathcal{B}^\infty(X^\bullet)^{G'}\big) \to \mathrm{H}^\bullet_{\mathrm{cb}}\big(L, \mathrm{L}^\infty(G/H)\big) \tag{4.11}$$

the map obtained in cohomology.

One more technical result which collects many functoriality statements needed in this paper is a small modification of a lemma in [53].

Lemma 4.5. *Let G, G' be locally compact groups, $\rho : G \to G'$ a continuous homomorphism, E a G-coefficient module and F a G'-coefficient module. Let $\alpha : F \to E$ be a morphism of G-coefficient modules, where the G-module structure on F is via ρ. Let $(E_\bullet)$ be a strong G-resolution of E by relatively injective G-modules, and let $(F_\bullet)$ be a strong G'-resolution of F. Then any two extensions of the morphism α to a morphism of G-complexes induce the same map in cohomology*

$$\mathrm{H}^\bullet\big(F_\bullet^{G'}\big) \to \mathrm{H}^\bullet\big(E_\bullet^G\big) \,.$$

Proof. By [53, Lemma 7.2.6] any two extensions of α are G-homotopic and hence induce the same map in cohomology

$$\mathrm{H}^\bullet\big(F_\bullet^{\rho(G)}\big) \to \mathrm{H}^\bullet\big(E_\bullet^G\big) .$$

Moreover, the inclusion of complexes $F_\bullet^{G'} \subset F_\bullet^{\rho(G)}$ induces a unique map in cohomology

$$\mathrm{H}^\bullet\big(F_\bullet^{G'}\big) \to \mathrm{H}^\bullet\big(F_\bullet^{\rho(G)}\big) ,$$

hence proving the lemma. □

PROPOSITION 4.6. *Assume that P is amenable. Then the map $\varphi_f^\bullet$ defined in (4.11) coincides with the composition*

$$\mathrm{H}^\bullet\big(\mathcal{B}^\infty(X^\bullet)^{G'}\big) \xrightarrow{\epsilon_X^\bullet} \mathrm{H}^\bullet_{\mathrm{cb}}(G',\mathbb{R}) \xrightarrow{\rho_{\mathrm{b}}^\bullet} \mathrm{H}^\bullet_{\mathrm{cb}}(L,\mathbb{R}) \xrightarrow{\theta_L^\bullet} \mathrm{H}^\bullet_{\mathrm{cb}}\big(L,\mathrm{L}^\infty(G/H)\big) .$$

Proof. By Proposition 4.1 $\big(\mathrm{L}^\infty\big((G/P)_f^\bullet\big), d^\bullet\big)$ is a strong resolution by relatively injective L-modules, so it is enough to apply Lemma 4.5 with $G = L$, $E = \mathrm{L}^\infty(G/H)$, $F = \mathbb{R}$ the trivial coefficient G'-module, $F_\bullet = \mathcal{B}^\infty(X^\bullet)$, and $E_\bullet = \big(\mathrm{L}^\infty(G/P)_f^\bullet\big)$. □

For further use we record the explicit reformulation of the above proposition:

COROLLARY 4.7. *Let G, G' be locally compact second countable groups, $L, H, P, Q \leq G$ closed subgroups with $P \leq H \cap Q$, and assume that P is amenable. Let $\rho : L \to G'$ be a continuous homomorphism, X a measurable G'-space and assume that there is an L-equivariant measurable map $\varphi : G/Q \to X$. Let $\kappa' \in \mathrm{H}^n_{\mathrm{cb}}(G',\mathbb{R})$ be a bounded cohomology class which admits as representative a bounded strict G'-invariant measurable cocycle $c' : X^{n+1} \to \mathbb{R}$. Then the class*

$$\theta_L^{(n)}\big(\rho_{\mathrm{b}}^{(n)}(\kappa')\big) \in \mathrm{H}^n_{\mathrm{cb}}\big(L, \mathrm{L}^\infty(G/H)\big)$$

is represented by the L-invariant essentially bounded measurable cocycle

$$\tilde{c}' : (G/P)_f^{n+1} \to \mathbb{R}$$

defined by

$$\tilde{c}'(x_0, x_1, \dots, x_n) := c'\big(\varphi_f^n(x_0, x_1, \dots, x_n)\big) , \tag{4.12}$$

where φ_f^n is defined in (4.10).

REMARK 4.8. Consider now that case in which $L = G = G'$, $\rho = Id$ (so that we can take $\varphi = Id$) and $X = G/Q$; if the class $\kappa \in \mathrm{H}^n_{\mathrm{cb}}(G,\mathbb{R})$ admits as representative a bounded strict G-invariant Borel cocycle c :

$(G/Q)^{n+1} \to \mathbb{R}$, then under the change of coefficients $\mathbb{R} \to \mathrm{L}^\infty(G/H)$, the class

$$\theta_G^{(n)}(\kappa) \in \mathrm{H}_{\mathrm{cb}}^{(n)}\big(G, \mathrm{L}^\infty(G/H)\big)$$

is represented by the bounded strict G-invariant Borel cocycle

$$\tilde{c} : (G/P)_f^{n+1} \to \mathbb{R}$$

defined by

$$\tilde{c}(x_1, \dots, x_{n+1}) := c(x_1 Q, \dots, x_{n+1} Q)\,. \tag{4.13}$$

4.2. **"The Formula", Finally.** We apply now all the results obtained so far to prove finally a generalization of the Formula in Proposition 2.44. In this section we have the following standing assumptions:

- G and G' are locally compact second countable groups,
- $L, H, P, Q \leq G$ are closed subgroups with $P \leq H \cap Q$,
- $L\backslash G$ carries a G-invariant probability measure μ,
- X is a measurable G'-space,
- there is a ρ-equivariant measurable map $\varphi : G/Q \to X$, where $\rho : L \to G'$ be a continuous homomorphism, and φ_f^{n+1} is the map defined in (4.10),
- $\kappa' \in \mathrm{H}_{\mathrm{cb}}^n(G', \mathbb{R})$ is represented by an alternating strict cocycle $c' \in \mathcal{B}^\infty(X^{n+1})^{G'}$, and $\tilde{c}' : (G/P)_f^{n+1} \to \mathbb{R}$ is the corresponding alternating cocycle defined in (4.12).
- $\kappa \in \mathrm{H}_{\mathrm{cb}}^n(G, \mathbb{R})$ is represented by an alternating cocycle $c \in \mathcal{Z}\mathrm{L}^\infty\big((G/Q)^{n+1}\big)^G$ and $\tilde{c} : (G/P)_f^{n+1} \to \mathbb{R}$ is the corresponding alternating cocycle defined in (4.13).

Proposition 4.9. *If $\kappa := \mathrm{T}_{\mathrm{b}}^{(n)}\big(\rho_{\mathrm{b}}^{(n)}(\kappa')\big) \in \mathrm{H}_{\mathrm{cb}}^n(G, \mathbb{R})$ and P is amenable, we have*

$$\int_{L\backslash G} \tilde{c}'\big(\varphi_f^{n+1}(gx_0, \dots, gx_{n+1})\big) d\mu(\dot{g}) = \tilde{c}(x_1, \dots, x_{n+1}) + \textit{coboundary}$$

for a. e. $(x_1, \dots, x_{n+1}) \in (G/P)_f^{n+1}$.

Remark 4.10. If H were to be ergodic on $(H/P)^n$, as for instance it is often the case if $n = 2$, then there would be no coboundary. In fact, in this case G would act ergodically on $(G/P)_f^2$ because it acts on the basis of the fibration $(G/P)_f^2 \to G/H$ transitively with stabilizer H, which then by hypothesis acts ergodically on the typical fiber homeomorphic to $(H/P)^2$. Hence $\mathrm{L}^\infty((G/P)_f^2)^G = \mathbb{R}$. Thus any coboundary would be constant and hence zero, being the difference of two alternating functions.

Principle 4.11. If $\mathrm{H}_{\mathrm{c}}^n(G,\mathbb{R}) \cong \mathrm{H}_{\mathrm{cb}}^n(G,\mathbb{R}) = \mathbb{R}\kappa = \mathbb{R}[c]$, and P is amenable, there exists an explicit constant $\lambda_{\kappa'} \in \mathbb{R}$ such that

$$\int_{L\backslash G} \tilde{c}'\big(\varphi_f^{n+1}(gx_0,\dots,gx_{n+1})\big)d\mu(\dot{g}) = \lambda_{\kappa'}\tilde{c}(x_1,\dots,x_{n+1}) + \textit{coboundary}$$

for a. e. $(x_1,\dots,x_{n+1}) \in (G/P)_f^{n+1}$.

Proof of Proposition 4.9. The commutativity of the square in the following diagram (see Lemma 4.4)

$$\begin{tikzcd}
\mathrm{H}^n\big(\mathcal{B}^\infty(X^{n+1})^{G'}\big) \arrow[r, "\omega_X^{(n)}"] & \mathrm{H}_{\mathrm{cb}}^n(G',\mathbb{R}) \arrow[r, "\rho_{\mathrm{b}}^{(n)}"] & \mathrm{H}_{\mathrm{cb}}^n(L,\mathbb{R}) \arrow[r, "\theta_L^{(n)}"] \arrow[d, "\mathrm{T}_{\mathrm{b}}^{(n)}"] & \mathrm{H}_{\mathrm{cb}}^n\big(L,\mathrm{L}^\infty(G/H)\big) \arrow[d, "\tau_{G/P}^{(n)}"] \\
\mathrm{H}^n\big(\mathcal{B}^\infty((G/Q)^{n+1})^{G}\big) \arrow[rr, "\omega_{G/Q}^{(n)}"] & & \mathrm{H}_{\mathrm{cb}}^n(G,\mathbb{R}) \arrow[r, "\theta_G^{(n)}"] & \mathrm{H}_{\mathrm{cb}}^n\big(G,\mathrm{L}^\infty(G/H)\big)
\end{tikzcd}$$

applied to the class $\rho_{\mathrm{b}}^{(n)}(\kappa') \in \mathrm{H}_{\mathrm{cb}}^n(L,\mathbb{R})$ reads

$$\tau_{G/P}^{(n)}\Big(\theta_L^{(n)}\big(\rho_{\mathrm{b}}^{(n)}(\kappa')\big)\Big) = \theta_G^{(n)}\Big(\mathrm{T}_{\mathrm{b}}^{(n)}\big(\rho_{\mathrm{b}}^{(n)}(\kappa')\big)\Big) = \theta_G^{(n)}(\kappa)\,.$$

Hence the representatives for the classes $\theta_G^{(n)}(\kappa)$ and $\theta_L^{(n)}\big(\rho_{\mathrm{b}}^{(n)}(\kappa')\big)$ chosen according to Corollary 4.7 satisfy the relation

$$\tau_{G/P}^{(n)}(\tilde{c}') = \tilde{c} + db\,,$$

where $b \in \mathrm{L}^\infty\big((G/P)_f^n\big)^G$, which, using the definition of $\tau_{G/P}^{(n)}$ in (4.5) implies that

$$\int_{L\backslash G} \tilde{c}'\big(\varphi_f^{n+1}(gx_0,\dots,gx_n)\big)d\mu(\dot{g}) = \tilde{c}(x_0,\dots,x_n) + db$$

for a. e. $(x_0,\dots,x_n) \in (G/P)_f^{n+1}$. $\square$

Notice that if $G = H$ and $Q = P$, we obtain Proposition 2.44.

5. One More Application of "The Formula": Deformation Rigidity of Lattices of Hyperbolic Isometries

As alluded to at the beginning of § 3, the transfer map

$$\mathrm{T}^\bullet : \mathrm{H}_{\mathrm{c}}^\bullet(L,\mathbb{R}) \to \mathrm{H}_{\mathrm{c}}^\bullet(G,\mathbb{R}) \tag{5.1}$$

makes sense only if $L\backslash G$ is compact as the restriction map (of which the transfer map would be a left inverse) is often not injective if L is only of finite covolume, (see [14]). So the diagram (3.2) is not complete, but in some cases, as for instance if G is a connected semisimple Lie group, the missing arrow can be replaced by a more complicated diagram involving the complex of L^2 differential forms on the corresponding symmetric space. For a very thorough discussion of this point we refer the reader to [14] from where we extract what we need in the following discussion.

The Kähler form ω_p on complex hyperbolic spaces $\mathcal{H}^p_{\mathbb{C}}$ is the unique (up to scalars) two-form on $\mathcal{H}^p_{\mathbb{C}}$ which is invariant for the action of $\mathrm{SU}(1,p)$. Let κ_p be the *Kähler class*, that is the continuous cohomology class in $\mathrm{H}^2_{\mathrm{c}}\big(\mathrm{SU}(1,p),\mathbb{R}\big)$ corresponding to ω_p under the Van Est isomorphism $\mathrm{H}^2_{\mathrm{c}}\big(\mathrm{SU}(1,p)\big)\cong\Omega^2(\mathcal{H}^p_{\mathbb{C}})^{\mathrm{SU}(1,p)}$. If $\Gamma<\mathrm{SU}(1,p)$ is a (torsionfree) lattice, let $\mathrm{H}^\bullet_2(M)$ denote the L^2-cohomology of the finite volume hyperbolic manifold $M:=\Gamma\backslash\mathcal{H}^p_{\mathbb{C}}$, that is the cohomology of the complex of smooth differential forms α on M such that α and $d\alpha$ are in L^2. Under the assumption that $p\geq 2$, Zucker proved [67] that $\mathrm{H}^2_2(M)$ injects into the de Rham cohomology $\mathrm{H}^2_{\mathrm{dR}}(M)\cong\mathrm{H}^2(\Gamma,\mathbb{R})$, while if Γ is cocompact (and p is arbitrary) we have by Hodge theory that $\mathrm{H}^2_2(M)=\mathrm{H}^2_{\mathrm{dR}}(M)$. Furthermore, if $\rho^{(2)}_{\mathrm{dR}}(\kappa_p)$ denotes the class in $\mathrm{H}^2_{\mathrm{dR}}(M)$ which corresponds to the pullback $\rho^{(2)}(\kappa_p)\in\mathrm{H}^2(\Gamma,\mathbb{R})$, we have the following:

Proposition 5.1. [14, Corollary 4.2] *Let $\rho:\Gamma\to\mathrm{PU}(1,q)$ be a homomorphism of a lattice $\Gamma<\mathrm{SU}(1,p)$. The pullback $\rho^{(2)}_{\mathrm{dR}}(\kappa_p)$ of the Kähler class is in $\mathrm{H}^2_2(M)\hookrightarrow\mathrm{H}^2_{\mathrm{dR}}(M)$.*

Denoting by $\langle\,,\,\rangle$ the scalar product in $\mathrm{H}^2_2(M)$ and by $\omega_M\in\mathrm{H}^2_2(M)$ the L^2-cohomology class defined by the Kähler form on M induced by ω_p, we define an invariant associated to the homomorphism $\rho:\Gamma\to\mathrm{PU}(1,q)$, by

$$i_\rho:=\frac{\langle\rho^{(2)}_{\mathrm{dR}}(\kappa_q),\omega_M\rangle}{\langle\omega_M,\omega_M\rangle}\,. \tag{5.2}$$

Proposition 5.2. [16] *If either Γ is cocompact or $p\geq 2$, the map $\rho\mapsto i_\rho$ is constant on connected components of the representation variety* $\mathrm{Rep}\big(\Gamma,\mathrm{PU}(1,q)\big)$.

We have then the following global rigidity result:

Theorem 5.3 (Burger–Iozzi [19, 16], Koziarz–Maubon [45]). *Assume that $\Gamma<\mathrm{SU}(1,p)$ is a lattice and $p\geq 2$. Then $|i_\rho|\leq 1$ and equality holds if and only if there is an isometric embedding of the corresponding complex hyperbolic spaces $\mathcal{H}^p_{\mathbb{C}}\to\mathcal{H}^q_{\mathbb{C}}$ which is ρ-equivariant.*

Corollary 5.4 (Burger–Iozzi [19, 16], Koziarz–Maubon [45]). *There are no nontrivial deformations of the restriction to Γ of the standard embedding* $\mathrm{SU}(1,p)\hookrightarrow\mathrm{SU}(1,q)$.

Our proof of the above theorems relies on the techniques developed in this paper (Proposition 4.9 in particular), on [14] and on [16]. An alternative proof using harmonic maps, as well as an overview of the history and context of the topic, can be found in the paper by Koziarz and Maubon [45]. The above corollary in the case in which Γ is cocompact is a result of Goldman and Millson [36]. If on the other hand $p = 1$, Gusevskii and Parker [39] constructed nontrivial quasi-Fuchsian deformations of a noncocompact lattice $\Gamma < \mathrm{SU}(1,1)$ into $\mathrm{PU}(1,2)$; however, it is still possible to conclude the following result which generalizes the case in which Γ is a compact surface group, [60]:

THEOREM 5.5 ([15, 14]). *Let $\Gamma < \mathrm{SU}(1,1)$ be a lattice and $\rho : \Gamma \to \mathrm{PU}(1,q)$ a representation such that $|i_\rho| = 1$. Then $\rho(\Gamma)$ leaves a complex geodesic invariant.*

We turn now to a short description of how Theorem 5.3 follows from Proposition 4.9 (using also results from [14] and [16]).

The ideal boundary $\partial\mathcal{H}^p_{\mathbb{C}}$ of complex hyperbolic p-space $\mathcal{H}^p_{\mathbb{C}}$ is identified with the projectivized cone of null vectors

$$\partial\mathcal{H}^p_{\mathbb{C}} = \mathbb{C}^\times \backslash \left\{ x \in \mathbb{C}^{p+1} : (x,x) = 0 \right\}.$$

and carries a rich geometry whose "lines" are the *chains*, namely boundaries of complex geodesics in $\mathcal{H}^p_{\mathbb{C}}$. The "geometry of chains" was first studied by E. Cartan who showed that, analogously to the Fundamental Theorem of Projective Geometry [4, Theorem 2.26], any automorphism of the incidence graph of the geometry of chains comes, for $p \geq 2$, from an isometry of $\mathcal{H}^p_{\mathbb{C}}$, [26]. Closely connected to this is Cartan's *invariant angulaire* c_p introduced in the same paper [26] and recalled in (2.4) in § 2.3. Observe that $|c_p| = 1$ exactly on triples of points which belong to a chain; moreover it represents the multiple of the bounded Kähler class $\frac{1}{\pi}\kappa^{\mathrm{b}}_p \in \mathrm{H}^2_{\mathrm{cb}}\big(\mathrm{SU}(1,p),\mathbb{R}\big)$ [18], that is of the bounded cohomology class which corresponds to the Kähler class $\kappa_p \in \mathrm{H}^2_{\mathrm{c}}\big(\mathrm{SU}(1,p),\mathbb{R}\big)$ under the isomorphism in Theorem 2.35.

Let us assume now that $L = \Gamma < \mathrm{SU}(1,p)$ is a lattice and move to the main formula, which will be an implementation of Proposition 4.9 in our concrete situation. Let $\mathcal{C}_p$ be the set of all chains in $\partial\mathcal{H}^p_{\mathbb{C}}$ and, for any $k \geq 1$, let

$$\mathcal{C}^{\{k\}}_p := \left\{ (C, \xi_1, \ldots, \xi_k) : C \in \mathcal{C}_p, (\xi_1, \ldots, \xi_k) \in C^k \right\}$$

be the space of configurations of k-tuples of points on a chain. Both $\mathcal{C}_p$ and $\mathcal{C}^{\{1\}}_p$ are homogeneous spaces of $\mathrm{SU}(1,p)$. In fact, the stabilizer H in G of a fixed chain $C_0 \in \mathcal{C}_p$ is also the stabilizer of a plane of signature

$(1,1)$ in $\mathrm{SU}(1,p)$ and hence isomorphic to $\mathrm{S}\big(\mathrm{U}(1,1)\times\mathrm{U}(\mathrm{p}-1)\big)$. Then $\mathrm{SU}(1,p)$ acts transitively on $\mathcal{C}_p$ (for example because it acts transitively on pairs of points in $\partial\mathcal{H}^p_{\mathbb{C}}$ and any two points in $\partial\mathcal{H}^p_{\mathbb{C}}$ determine uniquely a chain) and H acts transitively on C_0, so that, if $P=Q\cap H$, where Q is the stabilizer in $\mathrm{SU}(1,p)$ of a fixed basepoint $\xi_0\in C_0$, there are $\mathrm{SU}(1,p)$-equivariant (hence measure class preserving) diffeomorphisms

$$\begin{aligned} \mathrm{SU}(1,p)/H &\to \mathcal{C}_p\\ gH &\mapsto gC_0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{SU}(1,p)/P &\to \mathcal{C}_p^{\{1\}}\\ gP &\mapsto (gC_0,g\xi_0)\,. \end{aligned}$$

Moreover, the projection $\pi:\mathcal{C}_p^{\{1\}}\to\mathcal{C}_p$ which associates to a point $(C,\xi)\in\mathcal{C}_p^{\{1\}}$ the chain $C\in\mathcal{C}_p$ is a $\mathrm{SU}(1,p)$-equivariant fibration, the space $\mathcal{C}_p^{\{k\}}$ appears then naturally as k-fold fibered product of $\mathcal{C}_p^{\{1\}}$ with respect to π, and for every $k\geq 1$, the map

$$\begin{aligned} \big(\mathrm{SU}(1,p)/P\big)^k_f &\to \mathcal{C}_p^{\{k\}}\\ (x_1P,\dots,x_kP) &\mapsto (gC_0,x_1\xi_0,\dots,x_k\xi_0) \end{aligned} \tag{5.3}$$

where $x_iH=gH$, $1\leq i\leq k$, is a $\mathrm{SU}(1,p)$-equivariant diffeomorphism which preserves the $\mathrm{SU}(1,p)$-invariant Lebesgue measure class. Using Fubini's theorem, one has that for almost every $C\in\mathcal{C}_p$ the restriction

$$\varphi_C:C\to\partial\mathcal{H}^q_{\mathbb{C}}$$

of φ to C is measurable and for every $\gamma\in\Gamma$ and almost every $\xi\in C$

$$\varphi_{\gamma C}(\gamma\xi)=\rho(\gamma)\varphi_C(\xi)\,.$$

This allows us to define

$$\begin{aligned} \varphi^{\{3\}}:\quad \mathcal{C}_p^{\{3\}} &\to (\partial\mathcal{H}^q_{\mathbb{C}})^3\\ (C,\xi_1,\xi_2,\xi_3) &\mapsto \big(\varphi_C(\xi_1),\varphi_C(\xi_2),\varphi_C(\xi_3)\big)\,. \end{aligned}$$

Then Proposition 4.9 can be reinterpreted as follows:

THEOREM 5.6. *Let i_ρ be the invariant defined in (5.2). Then for almost every chain $C\in\mathcal{C}_p$ and almost every $(\xi_1,\xi_2,\xi_3)\in C^3$,*

$$\int_{\Gamma\backslash\mathrm{SU}(1,p)} c_q\big(\varphi^{\{3\}}(gC,g\xi_1,g\xi_2,g\xi_3)\big)d\mu(g)=i_\rho c_p(\xi_1,\xi_2,\xi_3)\,,$$

where c_p is the Cartan invariant and μ is the $\mathrm{SU}(1,p)$-invariant probability measure on $\Gamma\backslash\mathrm{SU}(1,p)$.

Corollary 5.7. *Assume that $i_\rho = 1$. Then for almost every $C \in \mathcal{C}_p$ and almost every $(\xi_1, \xi_2, \xi_3) \in C^3$*

$$c_q\big(\varphi_C(\xi_1), \varphi_C(\xi_2), \varphi_C(\xi_3)\big) = c_p(\xi_1, \xi_2, \xi_3)\,.$$

Proof of Theorem 5.6. Let $H, Q, P < \mathrm{SU}(1,p)$ such as in the above discussion. Since Q is the stabilizer of a basepoint $\xi_0 \in \partial\mathcal{H}^p_{\mathbb{C}}$, it is a minimal parabolic subgroup and hence the closed subgroup P is amenable. Moreover, H acts ergodically on $H/P \times H/P$ since in $H/P \times H/P$ there is an open H-orbit of full measure. We can hence apply Proposition 4.9 with $G = \mathrm{SU}(1,p)$, $G' = \mathrm{PU}(q,1)$ and $\kappa' = \kappa^{\mathrm{b}}_q$. Moreover, by [14, (5.1), (5.4), and Lemma 5.3] we have that $\kappa = i_\rho \kappa^{\mathrm{b}}_p$. Set $G/Q = \partial\mathcal{H}^p_{\mathbb{C}}$, $c' = i_\rho c_p \in \mathcal{B}^\infty\big((\partial\mathcal{H}^p_{\mathbb{C}})^3\big)^{\mathrm{SU}(1,p)}$, $X = \partial\mathcal{H}^p_{\mathbb{C}}$ and $c' = c_q \in \mathcal{B}^\infty\big((\partial\mathcal{H}^p_{\mathbb{C}})^3\big)^{\mathrm{PU}(q,1)}$. Then the conclusion of the theorem is immediate if we observe that the identification in (5.3) transforms the map φ^3_f defined in (4.10) into the map $\varphi^{\{3\}}$ defined above. $\square$

Remark 5.8. It is now clear what is the essential use of the fibered product: the triples of points that lie on a chain form a set of measure zero in $\big(\partial\mathcal{H}^p_{\mathbb{C}}\big)^3$, and hence we would not have gained any information on these configuration of points by the direct use of the more familiar formula as in Principle 3.1.

Corollary 5.7 states that if the invariant takes its maximal value then the boundary map φ maps chains into chains. A modification of a theorem of Cartan [16] allows then to conclude the existence of the embedding in Theorem 5.3.

Appendix A. Proof of Proposition 4.1

For the proof of Proposition 4.1(i) we need to show the existence of norm one contracting homotopy operators from $\mathrm{L}^\infty\big((G/P)^{n+1}_f\big)$ to $\mathrm{L}^\infty\big((G/P)^n_f\big)$ sending L-continuous vectors into L-continuous vectors.

To this purpose we use the map q_n which identifies the complex of Banach G-modules $\big(\mathrm{L}^\infty(G/P)^\bullet_f\big)$ with the subcomplex $\big(\mathrm{L}^\infty(G \times (H/P)^\bullet)^H\big)$ of H-invariant vectors of the complex $\big(\mathrm{L}^\infty(G \times (H/P)^\bullet)\big)$, where now the differential d_n is given by

$$d_n f(g, x_1, \dots, x_n) = \sum_{i=0}^{n} (-1)^i f(g, x_1, \dots, \hat{x}_i, \dots, x_n)\,,$$

and we show more generally that:

LEMMA A.1. *For every $n \geq 0$ there are linear maps*

$$h_n : \mathrm{L}^\infty\big(G \times (H/P)^{n+1}\big) \to \mathrm{L}^\infty\big(G \times (H/P)^n\big)$$

such that:

(i) *h_n is norm-decreasing and H-equivariant;*
(ii) *for any closed subgroup $L < G$, the map h_n sends L-continuous vectors into L-continuous vectors, and*
(iii) *for every $n \geq 1$ we have the identity*

$$h_n d_n + d_{n-1} h_{n-1} = Id\,.$$

The Lemma A.1 and the remarks preceding it imply then Proposition 4.1.

The construction of the homotopy operator in Lemma A.1 requires the following two lemmas, the first of which showing that the measure ν on H/P can be chosen to satisfy certain regularity properties, and the second constructing an appropriate Bruhat function for $H < G$.

Let dh and $d\xi$ be the left invariant Haar measures on H and P.

LEMMA A.2. *There is an everywhere positive continuous function $q : H \to \mathbb{R}^+$ and a Borel probability measure ν on H/P such that*

$$\int_{H/P}\int_P f(x\xi)d\xi\, d\nu(x) = \int_H f(h)q(h)dh\,,$$

for every $f \in \mathrm{C}_{00}(H)$, where $\mathrm{C}_{00}(H)$ denotes the space of continuous functions on H with compact support.

Proof. Let $q_1 : H \to \mathbb{R}^+$ be an everywhere positive continuous function satisfying

$$q_1(x\eta) = q_1(x)\frac{\Delta_P(\eta)}{\Delta_H(\eta)}\,, \forall \eta \in P \text{ and } \forall x \in H\,,$$

where Δ_P, Δ_H are the respective modular functions (see [58]), and let ν_1 be the corresponding positive Radon measure on H/P such that the above formula holds. Then choose $q_2 : H/P \to \mathbb{R}^+$ continuous and everywhere positive, such that $q_2 d\nu_1$ is a probability measure. Then the lemma holds with $q = q_1 q_2$ and $\nu = q_2 d\nu_1$. □

A direct computation shows that

$$\int_{H/P} f(y^{-1}x)d\nu(x) = \int_{H/P} f(x)\lambda_y(x)d\nu(x)\,, \tag{A.1}$$

where $\lambda_y(x) = q(yx)/q(x)$, for all $f \in \mathrm{C}_{00}(H/P)$ and $h \in H$. In particular, the class of ν is H-invariant since λ_y is continuous and everywhere positive on H/P.

LEMMA A.3. *There exists a function $\beta : G \to \mathbb{R}^+$ such that*

(i) *for every compact set $K \subset G$, β coincides on KH with a continuous function with compact support;*
(ii) *$\int_H \beta(gh)dh = 1$ for all $g \in G$, and*
(iii) *$\lim_{g_0 \to e} \sup_{g \in G} \int_H \left|\beta(g_0 gh) - \beta(gh)\right| dh = 0$*

Proof. Let β_0 be any function satisfying (i) and (ii) (see [58]) and let $f \in \mathrm{C}_{00}(G)$ be any nonnegative function normalized so that

$$\int_G f(x) d_r x = 1 \,,$$

where $d_r x$ is a right invariant Haar measure on G. Define

$$\beta(g) = \int_G f(gx^{-1})\beta_0(x) d_r x \,, \qquad g \in G \,.$$

It is easy to verify that also β satisfies (i) and (ii), and, moreover, it satisfies (iii) as well. In fact, we have that for all $g_0, g \in G$, $h \in H$

$$\beta(g_0 gh) - \beta(gh) = \int_G \big(f(g_0 gx^{-1}) - f(gx^{-1})\big)\beta_0(xh) d_r x \,,$$

which implies, taking into account that $\int_G \beta_0(xh)dh = 1$ and the invariance of $d_r x$, that

$$\int_H \left|\beta(g_0 gh) - \beta(gh)\right| dh \leq \int_G \left|f(g_0 x^{-1}) - f(x^{-1})\right| d_r x \,,$$

so that

$$\lim_{g_0 \to e} \sup_{g \in G} \int_H \left|\beta(g_0 gh) - \beta(gh)\right| dh \leq \lim_{g_0 \to e} \int_G \left|f(g_0 x^{-1}) - f(x^{-1})\right| d_r x = 0 \,.$$

$\square$

Proof of Lemma A.1. Let ν be as in Lemma A.2 and β as in Lemma A.3. define a function

$$\psi : G \times H/P \to \mathbb{R}^+$$

by

$$\psi(g, x) := \int_H \beta(gh)\lambda_{h^{-1}}(x) dh \,,$$

where $\lambda_h(x)$ is as in (A.1). The following properties are then direct verifications:

- $\psi(gh^{-1}, hx)\lambda_h(x) = \psi(g, x)$ for all $g \in G$, $h \in H$ and $x \in H/P$;
- $\int_{H/P} \psi(g, x) d\nu(x) = 1$, for all $g \in G$;
- $\psi \geq 0$ and is continuous.

This being, define for $n \geq 0$ and $f \in \mathrm{L}^\infty(G \times (H/P)^{n+1})$:

$$h_n f(g, x_1, \dots, x_n) = \int_{H/P} \psi(g, x) f(g, x_1, \dots, x_n, x) d\nu(x) \,.$$

Then, $h_n f \in \mathrm{L}^\infty\big(G \times (H/P)^n\big)$ and (ii) implies that $\|h_n f\|_\infty \leq \|f\|_\infty$. The fact that h_n is an H-equivariant homotopy operator is a formal consequence of (i) and (ii).

Finally, let $L < G$ be a closed subgroup and $f \in \mathrm{L}^\infty\big(G \times (H/P)^{n+1}\big)$ an L-continuous vector, that is

$$\lim_{l \to e} \|\theta(l) f - f\|_\infty = 0 \,,$$

where

$$\big(\theta(l) f\big)(g, x_1, \dots, x_{n+1}) = f(lg, x_1, \dots, x_n) \,.$$

Then

$$\begin{aligned} &h_n f(lg, x_1, \dots, x_n) - h_n f(g, x_1, \dots, x_n) \\ &= \int_{H/P} \psi(lg, x) \big(f(lg, x_1, \dots, x_n, x) - f(g, x_1, \dots, x_n, x) \big) d\nu(x) \\ &+ \int_{H/P} \big(\psi(lg, x) - \psi(g, x) \big) f(g, x_1, \dots, x_n, x) d\nu(x) \,. \end{aligned}$$

The first term is bounded by $\|\theta(l) f - f\|_\infty$ taking into account (ii), while the second is bounded by $\|f\|_\infty \int_{H/P} \big(\psi(lg, x) - \psi(g, x)\big) d\nu(x)$. Now

$$\psi(lg, x) - \psi(g, x) = \int_H \big(\beta(lgh) - \beta(gh) \big) \lambda_{h^{-1}}(x) dh \,,$$

which, taking into account that $\int_{H/P} \lambda_{h^{-1}}(x) d\nu(x) = 1$, implies that

$$\int_{H/P} \big| \psi(lg, x) - \psi(g, x) \big| d\nu(x) \leq \int_{H/P} \big| \beta(lgh) - \beta(gh) \big| dh \,.$$

Thus

$$\begin{aligned} \|\theta(l) h_n f - h_n f\|_\infty \leq & \|\theta(l) f - f\|_\infty \\ & + \|f\|_\infty \sup_{g \in G} \int_H \big| \beta(lgh) - \beta(gh) \big| dh \end{aligned}$$

which, using Lemma A.3, implies that

$$\lim_{l \to e} \|\theta(l) h_n f - h_n f\|_\infty = 0$$

and shows that $h_n f$ is an L-continuous vector. $\square$

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