Higher Teichmüller Spaces: from $\mathbf{SL}(2,\mathbb{R})$ to other Lie groups

*M. Burger and A. Iozzi** *and A. Wienhard***

Department Mathematik
ETH Zentrum
Rämistrasse 101, CH-8092 Zürich, Switzerland
email: burger@math.ethz.ch

Department Mathematik
ETH Zentrum
Rämistrasse 101, CH-8092 Zürich, Switzerland
email: iozzi@math.ethz.ch

Department of Mathematics
Princeton University
Fine Hall, Washington Road, Princeton, NJ 08540, USA
email: wienhard@math.princeton.edu

Contents

*Work partially supported by the Swiss National Science Foundation project 2000021-127016/2

**Work partially supported by NSF Grant No.DMS-0803216 and NSF CAREER Grant No. DMS-0846408

1 Introduction

Let S be a connected surface of finite topological type. The Teichmüller space $\mathcal{T}(S)$ is the moduli space of marked complex structures on S. It is isomorphic to the moduli space of marked complete hyperbolic structures on S, sometimes called the Fricke space $\mathcal{F}(S)$. Associating to a hyperbolic structure its holonomy representation naturally embeds the Fricke space $\mathcal{F}(S)$ into the variety of representations $\mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)/\,\mathrm{PSU}(1,1)$.

This realization of the classical Teichmüller space as a subset of the representations variety is the starting point of this article. We begin in § 2.1 by defining the space $\mathrm{Hyp}(S)$ of hyperbolic structures on S and constructing in some details the map

$$\delta : \mathrm{Hyp}(S) \to \mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)\,,$$

as well as the embedding

$$\delta' : \mathcal{F}(S) = \mathrm{Diff}_0^+(S)\backslash\mathrm{Hyp}(S) \to \mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)/\,\mathrm{PSU}(1,1)\,,$$

where $\mathrm{Diff}_0^+(S)$ is the group of orientation preserving diffeomorphisms which are homotopic to the identity. Then §§ 2.2 and 3 are devoted to various descriptions of the subset $\delta\big(\mathrm{Hyp}(S)\big) \subset \mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)$. When S is a compact surface, $\delta(\mathrm{Hyp}(S))$ is:

(1) the set of injective orientation preserving homomorphisms with discrete image (see Theorem 2.5 and Corollary 2.13);

(2) identified with one connected component of $\mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)$ (see § 2.2);

(3) the (maximal value) level set of numerical invariants described in § 3 (see Theorem 2.2);

(4) the solution set of a commutator equation (see (4.1));

(5) characterized in terms of bounded cohomology classes (see Corollary 4.5).

When S is a noncompact surface of finite type, the description of $\delta\big(\mathrm{Hyp}(S)\big)$ is more involved, and the characterizations (1) and (2) do not hold in this case. In § 4 we define suitable (bounded cohomological) analogues of the numerical invariants described in § 3, which allow us to give in § 4.5 characterizations of $\delta\big(\mathrm{Hyp}(S)\big)$ for noncompact surfaces S, generalizing (3), (4) and (5) above.

In the second part we ask how much of this "$\mathrm{PSU}(1,1)$ picture" generalizes to an arbitrary Lie group G. We discuss two classes of (semi)simple Lie groups for which one can make this question precise by defining, in very different ways, components (or specific subsets when S is not compact) of $\mathrm{Hom}\big(\pi_1(S), G\big)$ which play the role of Teichmüller space.

The terminology Higher Teichmüller spaces, coined by Fock and Goncharov, has now come to mean subsets of $\mathrm{Hom}\big(\pi_1(S), G\big)$, where G is a simple

Lie group, which share essential geometric and algebraic properties with classical Teichmüller space considered as a subset of $\mathrm{Hom}\big(\pi_1(S), \mathrm{PSU}(1,1)\big)$. Up to now higher Teichmüller spaces are defined for two classes of Lie groups, namely for split real simple Lie groups, e.g. $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n+1)$ or $\mathrm{SO}(n,n)$ and for Lie groups of Hermitian type, e.g. $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(2,n)$, $\mathrm{SU}(p,q)$ or $\mathrm{SO}^*(2n)$.

The invariants defined in § 3 and § 4 can be defined for homomorphisms from $\pi_1(S)$ with values in any Lie group G but, when G is a Lie group of Hermitian type these invariants are particularly meaningful. We describe how the basic objects available in the case of $\mathrm{PSU}(1,1)$ can be generalized to higher rank in § 5. Considering the maximal value level set of the numerical invariant thus constructed leads us to consider the space of maximal representations

$$\mathrm{Hom}_{max}\big(\pi_1(S), G\big) \subset \mathrm{Hom}\big(\pi_1(S), G\big)\,,$$

some of whose properties are discussed in § 5. In particular we state a result ("structure theorem") which describes the Zariski closure in G of the image of a maximal representation; a major part of §§ 5.4 5.5 then offers a guided tour showing the various aspects of the proof of the structure theorem.

The space of maximal representations is an example of a higher Teichmüller space. Hitchin components and spaces of positive representations are other examples of a higher Teichmüller space, defined when G is a split real Lie group. We review the definition of these spaces shortly (§ 6), and then discuss important structures underlying both families of higher Teichmüller spaces (§ 7). In the case of compact surfaces the quest for common structures leads us to consider the concept of Anosov structures (§ 8). This more general notion provides an important framework within which one can start to understand the geometric significance of higher Teichmüller spaces, their quotients by the mapping class group (§ 8.2), the geometric structures parametrized by higher Teichmüller spaces (§ 8.3), as well as further topological invariants (§ 8.4). In § 9 we conclude by mentioning further directions of study.

There are many aspects of higher Teichmüller spaces which we do not touch upon for lack of space and expertise. One of them is the huge body of work studying maximal representations from the point of view of Higgs bundles; such techniques give in particular precise information about the number of connected components of maximal representations, as well as information on the homotopy type of those components (see for example [**?**, **?**, **?**, **?**, **?**, **?**, **?**]).

As a guide to the reader, we mention that the first part of the paper and the discussion of maximal representations is very descriptive and should be read linearly, while starting from the definition of Hitchin representations the paper is a pure survey.

Acknowledgments: We thank A. Papadopoulos for undertaking this project. We thank also O. Guichard and T. Hartnick for carefully reading a preliminary version of this paper providing many helpful comments, W. Goldman for many bibliographical comments and F. Labourie for positive feedback. Our thanks go also to D. Toledo and N. A'Campo for useful discussions on this general topic over the years and to T. Delzant for helpful comments concerning central extensions of surface groups. Finally, we thank the referee for helpful comments.

Part I
Teichmüller Space and Hyperbolic Structures

2 Hyperbolic structures and representations

2.1 Hyperbolic structures

In this section we review briefly how one associates to a hyperbolic structure on a surface a homomorphism of its fundamental group into the group of orientation preserving isometries of the Poincaré disk, and how an appropriate quotient of the set of hyperbolic structures injects into the representation variety.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk endowed with the Poincaré metric $\frac{4|dz|^2}{\left(1-|z|^2\right)^2}$, and let $G := \mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm\mathrm{Id}\}$ denote the quotient of $\mathrm{SU}(1,1)$ by its center. The group G acts on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by linear fractional transformations preserving $\mathbb{D}$ and hence can be identified with the group of orientation preserving isometries of $\mathbb{D}$.

Given a surface S, that is a two-dimensional smooth manifold[1] a hyperbolic metric on S is a Riemannian metric with sectional curvature -1. A $(G, \mathbb{D})$-structure on S is an atlas on S consisting of charts taking values in $\mathbb{D}$, whose change of charts are locally restrictions of elements of G, [**?**]. Assuming from now on that S is orientable, we observe that by the local version of Cartan's theorem, an orientation together with a hyperbolic metric on S determines a $(G, \mathbb{D})$-structure on S (the converse is also true and straightforward). Also, the hyperbolic metric is complete if and only if the same is true for the corresponding $(G, \mathbb{D})$-structure, i.e. if the developing map $\widetilde{S} \to \mathbb{D}$ is a diffeomorphism.

[1] Note that all manifolds here are without boundary. In particular a compact surface is necessarily closed.

The group $\mathrm{Diff}(S)$, and hence its subgroup $\mathrm{Diff}^+(S)$ consisting of orientation preserving diffeomorphisms of S, act on the set $\mathrm{Hyp}(S)$ of complete hyperbolic metrics on S in a contravariant way. In the sequel let $\widetilde{S} = D$ be a smooth oriented disk with a basepoint $* \in D$ and let us fix once and for all a base tangent vector $v \in T_*D$, $v \neq 0$. By the correspondence between hyperbolic metrics and $(G, \mathbb{D})$-structures, let us also consider, for every $h \in \mathrm{Hyp}(D)$, the unique orientation preserving isometry

$$f_h : (D, *) \to (\mathbb{D}, 0)$$

with $df_h(v) \in \mathbb{R}^+ e$, where $e = 1 \in \mathbb{C}$. If $\varphi \in \mathrm{Diff}^+(D)$, then for any $h \in \mathrm{Hyp}(D)$, φ is by definition an orientation preserving isometry between the hyperbolic metrics $\varphi^*(h)$ and h. Therefore

$$c(\varphi, h) := f_h \circ \varphi \circ f^{-1}_{\varphi^*(h)}$$

is an element of G. In this way we obtain a map

$$c : \mathrm{Diff}^+(D) \times \mathrm{Hyp}(D) \to G$$

which verifies the cocycle relation

$$c(\varphi_1 \varphi_2, h) = c(\varphi_1, h) c\big(\varphi_2, \varphi_1^*(h)\big) \,.$$

Let now $(S, *)$ be a connected oriented surface with base point $*$ and assume that $\mathrm{Hyp}(S) \neq \emptyset$. Let $(\widetilde{S}, *) = (D, *)$, let $p : D \to S$ be the canonical projection and

$$\Gamma = \{T_\gamma : \gamma \in \pi_1(S, *)\} < \mathrm{Diff}^+(D)$$

the group of covering transformations. Then the pullback via p gives a bijection between $\mathrm{Hyp}(S)$ and the set $\mathrm{Hyp}(D)^\Gamma$ of Γ-invariant elements in $\mathrm{Hyp}(D)$. Furthermore, it follows from the cocycle identity that, for every $h \in \mathrm{Hyp}(D)^\Gamma$, the map

$$\begin{aligned} \rho_h : \pi_1(S, *) &\longrightarrow \ G \\ \gamma \quad &\mapsto c(T_\gamma, h) \end{aligned}$$

is a homomorphism with respect to which the isometry f_h is equivariant. Thus we obtain the map δ, assigning to a hyperbolic structure its holonomy homomorphism

$$\begin{aligned} \delta :\ \mathrm{Hyp}(S) &\to \mathrm{Hom}\big(\pi_1(S, *), G\big) \\ h \quad &\longmapsto \qquad \rho_{p^*(h)} \,, \end{aligned}$$

which has certain important equivariance properties which we now explain.

To this end, let $\mathcal{N}^+$ be the normalizer of Γ in $\mathrm{Diff}^+(D)$. Then we have the diagram with exact line

$$\begin{tikzcd}
\{e\} \arrow[r] & \Gamma \arrow[r] & \mathcal{N}^+ \arrow[r, "\pi"] \arrow[d, "a"] & \mathrm{Diff}^+(S) \arrow[r] & \{e\} \\
& & \mathrm{Aut}\big(\pi_1(S,*)\big)\,, & &
\end{tikzcd}$$

where π associates to every $\varphi \in \mathcal{N}^+$ the corresponding diffeomorphism of S obtained by observing that φ permutes the fibers of p; the fact that π is surjective follows from covering theory. The homomorphism a is the one associating to φ the automorphism a_φ of Γ, or rather of $\pi_1(S,*)$, obtained by conjugation. With these definitions, a computation gives

$$\rho_{\varphi^*(h)}(\gamma) = c(\varphi,h)^{-1}\rho_h\big(a_\varphi(\gamma)\big)c(\varphi,h)\,, \tag{2.1}$$

for every $\varphi \in \mathcal{N}^+$, $h \in \mathrm{Hyp}(D)^\Gamma$ and $\gamma \in \Gamma$.

In view of (2.1), it is important to determine those $\varphi \in \mathcal{N}^+$ such that a_φ is an inner automorphism of Γ. Let $\mathcal{N}^+_{\mathrm{i}}$ be the subgroup consisting of all such diffeomorphisms. Then we have:

Lemma 2.1. *The map π induces an isomorphism*

$$\Gamma\backslash\mathcal{N}^+_{\mathrm{i}} \cong \mathrm{Diff}^+_0(S)\,,$$

where $\mathrm{Diff}^+_0(S)$ is the subgroup of $\mathrm{Diff}^+(S)$ consisting of those diffeomorphisms which are homotopic to the identity.

Proof. If $f : S \to S$ is homotopic to the identity, then by covering theory the conjugation of Γ by any lift $\tilde{f}$ of f gives an inner automorphism of Γ.

Conversely, if $\varphi T_\gamma \varphi^{-1} = T_{\eta\gamma\eta^{-1}}$ for some η and all γ, then the diffeomorphism $T_\eta^{-1}\varphi : D \to D$ commutes with the Γ-action on D; if we fix $h \in \mathrm{Hyp}(D)^\Gamma$ then the geodesic homotopy from $T_\eta^{-1}\varphi$ to Id_D is Γ-equivariant and hence descends to a homotopy between $\pi(T_\eta^{-1}\varphi) = \pi(\varphi)$ and Id_S. □

Thus combining the inverse of π with a we obtain an injective homomorphism

$$\mathrm{Map}(S) := \mathrm{Diff}^+_0(S)\backslash \mathrm{Diff}^+(S) \xrightarrow{\alpha} \mathrm{Out}\big(\pi_1(S,*)\big)$$

of the *mapping class group* $\mathrm{Map}(S)$ of S into the group of outer automorphisms of $\pi_1(S)$. It follows then from (2.1) that the map which to $h \in \mathrm{Hyp}(S)$ associates the class of the homomorphism $[\rho_{p^*(h)}]$ and which takes values in the quotient $\mathrm{Hom}\big(\pi_1(S,*),G\big)/G$ by the G-conjugation action on the target, is invariant under the $\mathrm{Diff}^+_0(S)$-action so that finally we obtain a map from the *Fricke space* $\mathcal{F}(S) = \mathrm{Diff}_0^+(S)\backslash\mathrm{Hyp}(S)$ to the representation variety

$$\begin{aligned} \delta' : \mathrm{Diff}_0^+(S)\backslash\mathrm{Hyp}(S) &\to \mathrm{Hom}\big(\pi_1(S,*),G\big)/G \\ [h] \qquad &\longmapsto \qquad [\rho_{p^*(h)}] \end{aligned}$$

which is α-equivariant.

Proposition 2.2. *If S is a connected, oriented surface admitting a complete hyperbolic structure, then δ' is injective.*

Proof. If $h_1, h_2 \in \mathrm{Hyp}(D)^\Gamma$ are such that ρ_{h_1} and ρ_{h_2} are conjugated by $g \in G$, then it follows from the definitions that $f_{h_2}^{-1} g f_{h_1}$ is an orientation preserving diffeomorphism sending h_1 to h_2, which furthermore is Γ-equivariant; by the argument used in Lemma 2.1, we get that $\pi(f_{h_2}^{-1} g f_{h_1}) \in \mathrm{Diff}_0^+(S)$. ☐

We now describe the image of the homomorphism α and of the map δ' in the case in which S is a compact oriented surface of genus $g \geq 2$. This latter condition guarantees that the classifying map

$$S \to B\pi_1(S,*) \tag{2.2}$$

is a homotopy equivalence; we use this fact to equip $\mathrm{H}_2\big(\pi_1(S,*),\mathbb{Z}\big)$ with the canonical generator, image of the fundamental class $[S]$ via the isomorphism $\mathrm{H}_2(S,\mathbb{Z}) \to \mathrm{H}_2\big(\pi_1(S,*),\mathbb{Z}\big)$ induced by (2.2). An isomorphism between the fundamental groups of two compact oriented surfaces S_1 and S_2 is said to be orientation preserving if the generator of $\mathrm{H}_2\big(\pi_1(S_1,*),\mathbb{Z}\big)$ is mapped to the generator of $\mathrm{H}_2\big(\pi_1(S_2,*),\mathbb{Z}\big)$.

Theorem 2.3. *Let S_1 and S_2 be compact oriented surfaces of genus $g \geq 1$. Then any orientation preserving isomorphism $\pi_1(S_1,*) \to \pi_1(S_2,*)$ is induced by an orientation preserving diffeomorphism $S_1 \to S_2$.*

Let us denote by $\mathrm{Aut}^+\big(\pi_1(S,*)\big)$ the group of the orientation preserving automorphisms of a compact orientable surface S of genus $g \geq 1$, and by $\mathrm{Out}^+\big(\pi_1(S,*)\big)$ its quotient by the group of inner automorphisms. From Theorem 2.3 we conclude:

Corollary 2.4 (Dehn-Nielsen–Baer Theorem, see [**?**] for a proof). *If S is a compact orientable surface of genus $g \geq 2$, the map*

$$\alpha : \mathrm{Map}(S) = \mathrm{Diff}^+(S)/\,\mathrm{Diff}_0^+(S) \to \mathrm{Out}^+\big(\pi_1(S,*)\big)$$

is an isomorphism.

Let now $\mathrm{Hom}_{d,i}$ denote the subset of Hom consisting of all injective homomorphisms with discrete image. The following classical identification of the image of δ uses the Nielsen realization:

Theorem 2.5. *If S is a compact orientable surface of genus $g \geq 1$, then the image of δ consists precisely of*

$$\big\{\rho \in \mathrm{Hom}_{d,i}\big(\pi_1(S,*),G\big) : \mathrm{im}\rho\backslash\mathbb{D} \text{ is compact and } \rho \text{ is orientation preserving}\big\}\,.$$

Remark 2.6. Since $\pi_1(S)$ is the fundamental group of a compact surface and ρ is a discrete embedding, the quotient $\mathrm{im}\rho\backslash\mathbb{D}$ is automatically compact. Here, we include this property explicitly in order to stress the similarity with the definition of $\mathrm{Hom}_0(\Gamma,G)$ in § 2.2.

As above, the representation ρ is orientation preserving if the induced map ρ_* maps the generator of $\mathrm{H}_2\big(\pi_1(S,*),\mathbb{Z}\big)$ to the generator of $\mathrm{H}_2\big(\pi_1(\mathrm{im}\rho\backslash\mathbb{D},*),\mathbb{Z}\big)$.

Proof. Given $h \in \mathrm{Hyp}(S)$, it is immediate, by using f_h, that ρ_h belongs to the above set.

Conversely, apply Nielsen realization to the orientation preserving isomorphism ρ to get an orientation preserving diffeomorphism $f : S \to \mathrm{im}\rho\backslash\mathbb{D}$; if $h = f^*(h_P)$, where h_P is the Poincaré metric on $\mathrm{im}\rho\backslash\mathbb{D}$, then one verifies that $[\rho] = [\rho_h]$. □

Remark 2.7. Contrary to what happens in the compact case, if S is a non-compact orientable surface of negative Euler characteristic, the inclusion

$$\delta\big(\mathrm{Hyp}(S)\big) \cup \delta\big(\mathrm{Hyp}(\overline{S})\big) \subset \mathrm{Hom}_{d,i}\big(\pi_1(S,*),\mathrm{PSU}(1,1)\big)\,,$$

where $\overline{S}$ denotes the surface S with the opposite orientation is always proper. In fact, if $\rho : \pi_1(S,*) \to G$ is just discrete and injective, the surfaces $\mathrm{im}\rho\backslash\mathbb{D}$ and S need not be diffeomorphic, although they have the "same" fundamental group. For example, the once punctured torus and the thrice punctured sphere have isomorphic fundamental groups $\mathbb{F}_2$ and admit complete hyperbolic structures. We will see in § 4 one way to remedy this problem.

2.2 Representation varieties

In this section we review some basic properties of the set of discrete and faithful representations in $\mathrm{Hom}(\Gamma,G)$ in the general context of a finitely generated group Γ and a connected reductive Lie group G.

One way to approach the problem of determining the image of $\mathrm{Hyp}(S)$ under the map δ is to equip $\mathrm{Hom}(\Gamma,G)$ with a topology. Quite generally if Γ is a discrete group and G is a topological group, $\mathrm{Hom}(\Gamma,G)$ inherits the topology of the product space G^Γ. In case Γ is finitely generated with finite generating set $F \subset \Gamma$, let $p : \mathbb{F}_{|F|} \to \Gamma$ be the corresponding presentation and R a set of generators of the relators $\ker p$. Since every $r \in R$ is a word in $\mathbb{F}_{|F|}$, it determines a product map $m_r : G^F \to G$ by evaluation on G. The map

$$\begin{aligned} \mathrm{Hom}(\Gamma, G) &\longrightarrow G^F \\ \pi &\mapsto \big(\pi(s)\big)_{s\in F} \end{aligned}$$

identifies the topological space $\mathrm{Hom}(\Gamma, G)$ with the closed subset $\cap_{r\in R} m_r^{-1}(e) \subset G^F$. In particular, $\mathrm{Hom}(\Gamma, G)$ is locally compact if G is so, and a real algebraic set if G is a real algebraic group. We record the following

Proposition 2.8 ([**?**, **?**]). *If Γ is finitely generated and G is a real algebraic group, then $\mathrm{Hom}(\Gamma, G)$ has finitely many connected components and each of them is a real semialgebraic set, that is, it is defined by a finite number of polynomial equations and inequalities.*

Remark 2.9. Proposition 2.8 fails if G is not algebraic. An example for this, given in [**?**], is the quotient of the three dimensional Heisenberg group by a cyclic central subgroup, where a simple obstruction class detects infinitely many connected components in the representation variety.

In order to proceed further we assume that Γ is finitely generated, G is a Lie group and introduce (see [**?**, **?**]) the following subset of $\mathrm{Hom}(\Gamma, G)$

$$\mathrm{Hom}_0(\Gamma, G) = \{\rho \in \mathrm{Hom}_{d,i}(\Gamma, G) \text{ such that } \rho(\Gamma)\backslash G \text{ is compact}\},$$

where, as in § 2.1, $\mathrm{Hom}_{d,i}$ refers to the set of injective homomorphisms with discrete image, so that $\mathrm{Hom}_0(\Gamma, G) \subset \mathrm{Hom}_{d,i}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$.

The first result on the topology of $\mathrm{Hom}_{d,i}(\Gamma, G)$ requires a hypothesis on Γ:

Definition 2.10. We say that Γ has property (H) if every normal nilpotent subgroup of Γ is finite.

Observe that this condition is fulfilled by every nonabelian free group and every fundamental group of a compact surface of genus $g \geq 2$. With this we can now state the following

Theorem 2.11 ([**?**]). *Let Γ be a finitely generated group with property* (H) *and G a connected Lie group. Then $\mathrm{Hom}_{d,i}(\Gamma, G)$ is closed in $\mathrm{Hom}(\Gamma, G)$.*

Proof. The essential ingredient is the theorem of Kazhdan–Margulis–Zassenhaus [**?**, Theorem 8.16] saying that there exists an open neighborhood $\mathcal{U} \subset G$ of e such that whenever $\Lambda < G$ is a discrete subgroup, then $\mathcal{U} \cap \Lambda$ is contained in a connected nilpotent group. We fix now such an open neighborhood and assume in addition that [it does not contain any nontrivial subgroup of G; let also ℓ be an upper bound on the degree of nilpotency of connected nilpotent Lie subgroups of G.

Let now $\{\rho_n\}_{n\geq 1}$ be a sequence in $\mathrm{Hom}_{d,i}(\Gamma, G)$ with limit ρ. We show that ρ is injective. For every finite set $E \subset \ker\rho$, we have that $\rho_n(E) \subset \mathcal{U}$ for n large, which, by [**?**, Theorem 8.16], implies that for all $k \geq \ell$, the k-th iterated commutator of $\rho_n(E)$ is trivial, and the same holds therefore for E since ρ_n is injective.

As a result, $\ker\rho$ is nilpotent and hence, by property (H), finite; thus $\rho_n(\ker\rho) \subset \mathcal{U}$ for large n which, by the choice of $\mathcal{U}$ implies that $\rho_n(\ker\rho) = e$ and hence that ρ is injective.

We prove now that ρ is discrete. To this end, let $L = \overline{\rho(\Gamma)}$ be the closure of $\rho(\Gamma)$; then L is a Lie subgroup of G and L^0 is open in L. Let V be an open neighborhood of the identity on L^0 with $V \subset \mathcal{U}$; then $\rho(\Gamma) \cap V$ is dense in V and V generates L^0, from which we conclude that $\rho(\Gamma) \cap V$ generates a dense subgroup of L^0. For every finite set $F' \subset \Gamma$ with $\rho(S) \subset V \subset \mathcal{U}$ we have that $\rho_n(S) \subset \mathcal{U}$ for n large, which implies as before that for all $k \geq l$ the k-th iterated commutator of F', and hence of $\rho(F')$ is trivial; thus L^0 is nilpotent and so is $\rho^{-1}(L^0)$ since ρ is injective. But then $\rho^{-1}(L^0)$ is finite and hence $L^0 = \{e\}$, which shows that $\rho(\Gamma)$ is discrete and concludes the proof. □

Next we turn to the set $\mathrm{Hom}_0(\Gamma, G)$ of faithful, discrete and cocompact realizations of Γ in G; this set was considered by A. Weil as a tool in his celebrated local rigidity theorem in which the following general result played an important role.

Theorem 2.12 ([**?**]). *Assume that Γ is finitely generated and that G is a connected Lie group. Then $\mathrm{Hom}_0(\Gamma, G)$ is an open subset of G.*

There are by now several approaches available: we refer to the paper by Bergeron and Gelander [**?**], where the geometric approach due essentially to Ehresmann and Thurston [**?**] is explained (see also [**?**, **?**, **?**]); this approach, based on a reformulation of the problem in terms of variations of (G, X)-structures leads to a more general stability result also valid for manifolds with boundary.

We content ourselves with noticing the following consequence:

Corollary 2.13. *Assume that Γ is finitely generated, torsion-free and has property* (H). *Assume that G is a connected reductive Lie group and that there exists a discrete, injective and cocompact realization of Γ in G. Then*

$$\mathrm{Hom}_0(\Gamma, G) = \mathrm{Hom}_{d,i}(\Gamma, G)$$

and both sets are therefore open and closed, in particular a union of connected components of $\mathrm{Hom}(\Gamma, G)$.

Proof. Let $\rho_0 \in \mathrm{Hom}_0(\Gamma, G)$ and let $K < G$ be a maximal compact subgroup; since by the Iwasawa decomposition $X := G/K$ is contractible and since $\rho_0(\Gamma)$

acts on X as a group of covering transformations with compact quotient, we have that, for $n = \dim X$, $\mathrm{H}^n(\Gamma, \mathbb{R}) = \mathrm{H}^n\big(\rho_0(\Gamma), \mathbb{R}\big) \neq 0$. Therefore, if $\rho : \Gamma \to G$ is any discrete injective embedding, we have that $\mathrm{H}^n\big(\rho(\Gamma)\backslash X, \mathbb{R}\big)$ does not vanish and hence $\rho(\Gamma)\backslash X$ is compact, thus implying that $\rho \in \mathrm{Hom}_0(\Gamma, G)$. $\square$

Applying the preceding discussion to our compact surface S of genus $g \geq 2$, we conclude using Theorem 2.5 and Corollary 2.13 that

$$\delta\big(\mathrm{Hyp}(S)\big) \subset \mathrm{Hom}\big(\pi_1(S, *), \mathrm{PSU}(1,1)\big)$$

is a union of components of the representation variety $\mathrm{Hom}\big(\pi_1(S, *), \mathrm{PSU}(1,1)\big)$.

3 Invariants, Milnor's inequality and Goldman's theorem

In this section we will discuss various aspects of a fundamental invariant attached to a representation $\rho : \pi_1(S, *) \to G$, where $G = \mathrm{PSU}(1,1)$, namely the *Euler number* of ρ. This leads to a quite different way of characterizing the image of

$$\delta : \mathrm{Hyp}(S) \to \mathrm{Hom}\big(\pi_1(S, *), G\big)$$

in the case in which S is compact. This invariant can also be defined for targets belonging to a large class of Lie groups G, and this leads to natural generalizations of Teichmüller space (see the discussion in § 5 and § 7).

In the sequel S denotes a compact surface of genus $g \geq 2$ and fixed orientation. We drop moreover the basepoint in the notation $\pi_1(S, *)$ and we set $D = \widetilde{S}$.

3.1 Flat G-bundles

Given a connected Lie group G and a homomorphism $\rho : \pi_1(S) \to G$, we obtain, in the notation of § 2.1, a proper action without fixed points on $D \times G$ by

$$\gamma_*(x, g) = \big(T_\gamma x, \rho(\gamma) g\big)$$

whose quotient $\pi_1(S)\backslash(D \times G)$ is the total space $G(\rho)$ of a flat principal (right) G-bundle over S, where the projection map comes from the projection $D \times G \to D$ on the first factor.

Given a G-bundle $\mathcal{E}$ over S the first obstruction to find a continuous section of $\mathcal{E} \to S$ lies in $\mathrm{H}^2\big(S, \pi_1(G)\big)$. Namely, let K be a triangulation of S; choose preimages in $\mathcal{E}$ for the vertices of K and extend this section over the 0-skeleton of K to the 1-skeleton by using that G is connected; for each 2-simplex σ we have thus a section over its boundary $\partial\sigma$. Using the flat connection, this section of $\mathcal{E}$ over $\partial\sigma$ can be deformed into a loop lying in a single fixed fiber; identifying this fiber with G we get for every σ a free homotopy class of loops in G and hence a well defined element $c(\sigma) \in \pi_1(G)$, since the latter is Abelian. The map c is a simplicial 2-cocycle on K with values in $\pi_1(G)$ and hence defines an element in $\mathrm{H}^2\big(S, \pi_1(G)\big)$ which depends only on ρ. In this way we obtain a map

$$o_2 : \mathrm{Hom}\big(\pi_1(S), G\big) \to \mathrm{H}^2\big(S, \pi_1(G)\big)$$

which assigns to ρ the obstruction $o_2(\rho) \in \mathrm{H}^2\big(S, \pi_1(G)\big)$ of the flat G-bundle $G(\rho)$. An important observation is that if ρ_1 and ρ_2 lie in the same component of $\mathrm{Hom}\big(\pi_1(S), G\big)$, then the associated G-bundles $G(\rho_1)$ and $G(\rho_2)$ are isomorphic. As a result, the invariant o_2 is constant on connected components of $\mathrm{Hom}\big(\pi_1(S), G\big)$. (See also [**?**] for a discussion of characteristic classed and representations.)

3.2 Central extensions

An invariant closely related to the one defined above is obtained by considering the central extension of G given by the universal covering

$$\{e\} \longrightarrow \pi_1(G) \longrightarrow \widetilde{G} \xrightarrow{\ p\ } G \longrightarrow \{e\}\,,$$

where the neutral element is taken as basepoint.

A homomorphism $\rho : \pi_1(S) \to G$ then gives a central extension Γ_ρ of $\pi_1(S)$ by $\pi_1(G)$ in the familiar way

$$\Gamma_\rho = \big\{(\gamma, g) \in \pi_1(S) \times \widetilde{G} :\ \rho(\gamma) = p(g)\big\}\,.$$

Observing now that the isomorphism classes of central extensions of $\pi_1(S)$ by $\pi_1(G)$ are classified by $\mathrm{H}^2\big(\pi_1(S), \pi_1(G)\big)$, we get a map

$$c_2 : \mathrm{Hom}\big(\pi_1(S), G\big) \to \mathrm{H}^2\big(\pi_1(S), \pi_1(G)\big)\,.$$

So far the discussion in §§ 3.1 and 3.2 applies to any connected Lie group G. In case $G = \mathrm{PSU}(1,1)$, we get a canonical generator of $\pi_1(G)$ from the orientation of $\mathbb{D} \subset \mathbb{C}$; by considering the loop

$$\begin{aligned} [0,1] \to &\quad \mathrm{PSU}(1,1) \\ s \ \mapsto &\begin{pmatrix} e^{i\pi s} & 0 \\ 0 & e^{-i\pi s} \end{pmatrix}, \end{aligned}$$

we identify $\pi_1\big(\mathrm{PSU}(1,1)\big)$ with $\mathbb{Z}$; we will denote by $t \in \pi_1\big(\mathrm{PSU}(1,1)\big)$ the image of $1 \in \mathbb{Z}$.

3.3 Description of $\mathrm{H}^2\big(\pi_1(S),\mathbb{Z}\big)$, a digression

Let $g \geq 1$ be the genus of S. Then $\pi_1(S)$ admits as presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g}[a_i, b_i] = e \right\rangle .$$

The orientation of S is built in, in that, when drawing the lifts to $\widetilde{S}$ of the loop $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \ldots a_g^{-1} b_g^{-1}$, one gets a $4g$-gone whose boundary is traveled through in the positive sense.

Now define

$$\overline{\Gamma}_g := \left\langle A_1, B_1, \ldots, A_g, B_g, z : \prod_{i=1}^{g}[A_i, B_i] = z \text{ and } [z, A_i] = [z, B_i] = e \right\rangle .$$

This group $\overline{\Gamma}_g$ surjects onto $\pi_1(S)$ with kernel the cyclic subgroup generated by z, which incidentally is central. In order to see that z has infinite order, observe first that $\overline{\Gamma}_1$ is isomorphic to the integer Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

by

$$A_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad z \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then conclude by considering the surjection

$$\overline{\Gamma}_g \to \overline{\Gamma}_1$$

obtained by sending A_i and B_i to e for $i \geq 2$. Thus $\overline{\Gamma}_g$ gives a central extension of $\pi_1(S)$ by $\mathbb{Z}$; denoting by $\big[\overline{\Gamma}_g\big]$ its image in $\mathrm{H}^2\big(\pi_1(S_g),\mathbb{Z}\big)$, we have the following

Proposition 3.1. $\mathrm{H}^2\big(\pi_1(S_g),\mathbb{Z}\big) = \mathbb{Z}\big[\overline{\Gamma}_g\big]$.

In fact if

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;i\;} \Lambda \longrightarrow \pi_1(S_g) \longrightarrow \{e\}$$

is any central extension by $\mathbb{Z}$, take lifts $\alpha_j, \beta_j \in \Lambda$ of a_j, b_j: then $\prod_{j=1}^{g}[\alpha_j, \beta_j]$ is independent of all choices and the image under i of a well defined $n \in \mathbb{Z}$. Using the Baer product of extensions [**?**] one shows, by recurrence on n, that

$$[\Lambda] = n\,[\overline{\Gamma}_g] .$$

Now we come back to the invariant associated in § 3.2 to $\rho \in \mathrm{Hom}\big(\pi_1(S), G\big)$ and use the identification

$$\begin{aligned} \pi_1(G) &\to \mathbb{Z} \\ t \quad &\mapsto 1 \end{aligned}$$

to get $c_2(\rho) \in \mathrm{H}^2\big(\pi_1(S), \mathbb{Z}\big)$. In terms of central extension we have then that

$$c_2(\rho) = z_2(\rho)[\overline{\Gamma}_g]\,,$$

where $z_2(\rho) \in \mathbb{Z}$ is defined by the formula

$$\prod_{i=1}^{g}[\alpha_i, \beta_i] = t^{z_2(\rho)}\,,$$

where $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g \in \widetilde{G}$ are lifts of $\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)$.

3.4 The Euler class

The following discussion is specific to the case where $G = \mathrm{PSU}(1,1)$; it takes as point of departure the observation that the action of G by homographies on $\mathbb{D}$ gives an action on the circle $\partial\mathbb{D}$ bounding $\mathbb{D}$, by orientation preserving homeomorphisms. It will become apparent that considering homomorphisms with values in G as homomorphisms with target the group $\mathrm{Homeo}_+(S^1)$ of orientation preserving homeomorphisms of the circle gives additional flexibility.

Let us take the quotient $\mathbb{Z}\backslash\mathbb{R}$ of $\mathbb{R}$ by the group generated by the integer translations $T(x) = x + 1$ of the real line, and consider, as model of S^1,

$$\mathcal{H}^+_{\mathbb{Z}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \text{increasing homeomorphisms commuting with } T\}\,.$$

We obtain then the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R}) \xrightarrow{\ p\ } \mathrm{Homeo}_+(S^1) \longrightarrow 0$$

which realizes $\mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$ as universal covering of the group $\mathrm{Homeo}_+(S^1)$, the latter being endowed with the compact open topology. One obtains a section of p by associating to every $f \in \mathrm{Homeo}_+(S^1)$ the unique lift $\overline{f} : \mathbb{R} \to \mathbb{R}$ with $0 \leq \overline{f}(0) < 1$. The extent to which $f \mapsto \overline{f}$ is not a homomorphism is measured by an integral 2-cocycle ϵ given by

$$\overline{f} \circ \overline{g} = \overline{f \circ g} \circ T^{\epsilon(f,g)}\,,$$

where T is the image in $\mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$ of the generator $1 \in \mathbb{Z}$. The Euler class is then the cohomology class $e \in \mathrm{H}^2\big(\mathrm{Homeo}_+(S^1), \mathbb{Z}\big)$ defined by ϵ.

Definition 3.2. The *Euler number* $e(\rho)$ of a representation

$$\rho : \pi_1(S) \to \mathrm{Homeo}_+(S^1)$$

is the integer $\langle \rho^*(e), [S] \rangle$ obtained by evaluation of the pullback $\rho^*(e) \in \mathrm{H}^2\big(\pi_1(S), \mathbb{Z}\big)$ of e on the fundamental class $[S]$, or rather on its image under the isomorphism

$$\mathrm{H}_2(S, \mathbb{Z}) \to \mathrm{H}_2\big(\pi_1(S), \mathbb{Z}\big)$$

considered in (2.2).

3.5 Kähler form and Toledo number

Contrary to § 3.4 the viewpoint we present here emphasizes the fact that the Poincaré disk $\mathbb{D}$ is an instance of a *Hermitian symmetric space* with G-invariant Kähler form

$$\omega_{\mathbb{D}} := \frac{dz \wedge d\overline{z}}{(1 - |z|^2)^2} \,,$$

where $G = \mathrm{PSU}(1,1)$. Given a homomorphism $\rho : \pi_1(S) \to G$, consider then the bundle with total space the quotient $\mathbb{D}(\rho) := \pi_1(S)\backslash(D \times \mathbb{D})$ of $D \times \mathbb{D}$ by the properly discontinuous and fixed point free action $\gamma(x, z) := \big(T_\gamma x, \rho(\gamma) z\big)$, and with basis $S = \pi_1(S)\backslash D$. Since the typical fiber $\mathbb{D}$ is contractible, one can construct, adapting the procedure described in § 3.1, a continuous and even a a smooth section. Equivalently there is a smooth equivariant map $F : D \to \mathbb{D}$. As a result, the pullback $F^*(\omega_{\mathbb{D}})$ is a $\pi_1(S)$-invariant 2-form on D which gives a 2-form on S denoted again, with a slight abuse of notation, by $F^*(\omega_{\mathbb{D}})$. The *Toledo number* $\mathrm{T}(\rho)$ of the representation ρ is then

$$\mathrm{T}(\rho) := \frac{1}{2\pi} \int_S F^*(\omega_{\mathbb{D}}) \,.$$

Recall that we have fixed an orientation on S once and for all.

Remark 3.3. One verifies, using again geodesic homotopy, that any two ρ-equivariant smooth maps $D \to \mathbb{D}$ are homotopic and hence, by Stokes' theorem, one concludes that the de Rham cohomology class $\big[F^*(\omega_{\mathbb{D}})\big] \in \mathrm{H}^2_{\mathrm{dR}}(S, \mathbb{R})$ is independent of F. This shows that $\mathrm{T}(\rho)$ is independent of the choice of F.

3.6 Toledo number and first Chern classes

Let $L \to \mathbb{D}$ be a Hermitian complex line bundle over the Poincaré disk $\mathbb{D}$ and G' a finite covering of $\mathrm{PSU}(1,1)$ acting by bundle isomorphisms on L; then the curvature form Ω_L is a G'-invariant 2-form on $\mathbb{D}$. Given a representation $\rho : \pi_1(S) \to G'$ and a smooth equivariant map $F : \mathbb{D} \to \mathbb{D}$, $\pi_1(S)$ acts by bundle automorphisms on $F^*L \to \mathbb{D}$ and, by passing to the quotient, we get a complex line bundle $L(\rho)$ over S. Then $\frac{1}{2\pi\imath} F^*\Omega_L$ descends to a 2-form $\omega_{L(\rho)}$

on S which, by Chern-Weil theory, represents the first Chern class of $L(\rho)$, i. e.

$$c_1\big(L(\rho)\big) = \int_S \omega_{L(\rho)} \in \mathbb{Z}\,. \tag{3.1}$$

Applying this to specific line bundles we obtain integrality properties for the Toledo number. Namely, let $\theta \to \mathbb{D} \subset \mathbb{CP}^1$ be the restriction of the tautological bundle over $\mathbb{CP}^1$ and θ^2 be its square. Then $\Omega_\theta = \frac{1}{2i}\omega_{\mathbb{D}}$ and $\Omega_{\theta^2} = \frac{1}{i}\omega_{\mathbb{D}}$. The group $\mathrm{PSU}(1,1)$ acts by isomorphisms on θ^2, and (3.1) implies

$$T(\rho) = \frac{1}{2\pi}\int_S F^*\omega_{\mathbb{D}} = -\int_S \omega_{\theta^2_\rho} = -c_1(\theta^2_\rho)\, \in \mathbb{Z}\,.$$

The group $\mathrm{SU}(1,1)$ acts naturally on θ, so for representations $\rho : \pi_1(S) \to \mathrm{SU}(1,1)$ the relation in (3.1) gives

$$T(\rho) = \frac{1}{2\pi}\int_S F^*\omega_{\mathbb{D}} = -2\int_S \omega_{\theta_\rho} = -2c_1(\theta_\rho)\, \in 2\cdot\mathbb{Z}\,.$$

In particular, a representation $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$ lifts to $\mathrm{SU}(1,1)$ if and only if its Toledo number is divisible by 2.

3.7 Relations between the various invariants

For $G = \mathrm{PSU}(1,1)$ we identify in the sequel $\pi_1(G)$ with $\mathbb{Z}$ as described in § 3.2 and obtain for a representation $\rho \in \mathrm{Hom}\big(\pi_1(S), G\big)$ the obstruction class $o_2(\rho) \in \mathrm{H}^2(S,\mathbb{Z})$ and the class $c_2(\rho) \in \mathrm{H}^2\big(\pi_1(S),\mathbb{Z}\big)$; using the specific description of the latter in terms of central extensions as $\mathbb{Z}[\overline{\Gamma}_g]$, we get the invariant $z_2(\rho) \in \mathbb{Z}$ by setting $c_2(\rho) = z_2(\rho)[\overline{\Gamma}_g]$. Then

$$\langle o_2(\rho), [S]\rangle = -z_2(\rho)\,, \tag{3.2}$$

(see [**?**, Lemma 2] and [**?**]). Turning to the Euler class, we observe that the injection $\mathrm{PSU}(1,1) \hookrightarrow \mathrm{Homeo}^+\big(S^1\big)$ is a homotopy equivalence as both groups retract on the (common) group of rotations. Therefore the restriction $e|_{\mathrm{PSU}(1,1)} \in \mathrm{H}^2\big(\,\mathrm{PSU}(1,1),\mathbb{Z}\big)$ classifies the universal covering of $\mathrm{PSU}(1,1)$ and hence for $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$ we have

$$\rho^*(e) = z_2(\rho)[\overline{\Gamma}_g]\,,$$

which implies that

$$e(\rho) = \langle \rho^*(e), [S]\rangle = -z_2(\rho)\,. \tag{3.3}$$

To relate the previous invariant to the Toledo number we will recall the very general principle that invariant forms on a symmetric space form a complex, with 0 as derivative, which equals the continuous cohomology of the connected group of isometries. In our special case of the Poincaré disk, this takes the form

$$\Omega^2(\mathbb{D})^G \cong \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R}),$$

where, given $\omega_{\mathbb{D}}$, we get a continuous cocycle

$$c(g_1,g_2) := \frac{1}{2\pi}\int_{\Delta\big(0,g_1(0),g_1g_2(0)\big)} \omega_{\mathbb{D}}\,,$$

where $\Delta\big(0,g_1(0),g_1g_2(0)\big)$ denotes the oriented geodesic triangle having vertices at the points $0,g_1(0),g_1g_2(0)\in\mathbb{D}$. We call the resulting class κ_G the *Kähler class*. In fact, it is not difficult to show that under the change of coefficients

$$\mathrm{H}^2_{\mathrm{c}}(G,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})$$

the Euler class e goes to the Kähler class κ_G. If $\rho\in\mathrm{Hom}\big(\pi_1(S),G\big)$, this implies that

$$e(\rho) = \langle \rho^*(e),[S]\rangle = \frac{1}{2\pi}\int_S F^*(\omega_{\mathbb{D}}) = \mathrm{T}(\rho)\,. \tag{3.4}$$

3.8 Milnor's inequality and Goldman's theorem

In his seminal paper [?], J. Milnor treated the problem of characterizing those classes in $\mathrm{H}^2(S,\mathbb{Z})$ which are Euler classes of flat principal GL_2^+-bundles. The fact that, in general there are restrictions on the characteristic classes of flat principal G-bundles and in particular on $o_2(\rho)\in\mathrm{H}^2\big(S,\pi_1(G)\big)$ comes from the following observation: o_2 is constant on connected components of $\mathrm{Hom}\big(\pi_1(S),G\big)$ and the latter is a real algebraic set when G is a real algebraic group, thus possesses only finitely many connected components (see Proposition 2.8). To get explicit restrictions, however, is not a trivial matter. In the case of $G=\mathrm{PSU}(1,1)$-bundles this restriction, known as the *Milnor–Wood inequality*, is the following:

Theorem 3.4 ([?, ?]). *Let $\rho\in\mathrm{Hom}\big(\pi_1(S),G\big)$ and let g be the genus of S. Then*

$$\big|\big\langle o_2(\rho),[S]\big\rangle\big| \le 2g-2\,.$$

In light of subsequent generalizations of this inequality it is instructive to give an outline of the original arguments. Consider the retraction

$$r:\mathrm{PSU}(1,1)\to K = \left\{\pm\begin{pmatrix} e^{\imath s\pi} & 0\\ 0 & e^{-\imath s\pi}\end{pmatrix} : s\in\mathbb{R}/\mathbb{Z}\right\}$$

given by decomposing $g = r(g)h(g)$ as a product of a rotation $r(g)$ with a Hermitian matrix $h(g)$. Now lift r to the universal covering

$$\tilde{r} : \widetilde{\mathrm{PSU}(1,1)} \to \mathbb{R}$$

in such a way that $\tilde{r}(e) = 0$. Then:

(1) $\tilde{r}(t^n g) = n + \tilde{r}(g)$, where t is the generator of $\pi_1\big(\mathrm{PSU}(1,1)\big)$;

(2) $\tilde{r}(g^{-1}) = -\tilde{r}(g)$;

(3) $\big|\tilde{r}(ab) - \tilde{r}(a) - \tilde{r}(b)\big| < \frac{1}{2}$ for all $a, b \in \mathrm{PSU}(1,1)$.

This construction as well as the proof of these properties are given in [**?**] in the case of $\mathrm{GL}_2^+(\mathbb{R})$ (see also [**?**]).

Given $\rho \in \mathrm{Hom}\big(\pi_1(S), G\big)$ let now α_i, β_i be lifts to $\widetilde{\mathrm{PSU}(1,1)}$ of $\rho(a_i)$ and $\rho(b_i)$ (see § 3.3). Then

$$t^{z_2(\rho)} = \prod_{i=1}^{g} [\alpha_i, \beta_i]\,.$$

On applying the above properties several times we obtain

$$\big|z_2(\rho)\big| = |\tilde{r}(t^{z_2(\rho)})| = |\tilde{r}(\prod_{i=1}^{g}[\alpha_i, \beta_i])| \leq (4g-1)\frac{1}{2} = 2g - \frac{1}{2}$$

which, since $z_2(\rho)$ is an integer, implies that

$$\big|z_2(\rho)\big| \leq 2g - 1\,.$$

This is not quite the announced result. An additional argument is needed and can be found in [**?**]; we present instead an argument in the spirit of Gromov's trick to compute the simplicial area of a surface. Namely, let $p : S' \to S$ be a covering of degree $n \geq 1$ and let $p_* : \pi_1(S') \to \pi_1(S)$ be the resulting morphism. The inequality above, applied to $\rho \circ p_*$ gives

$$\big|\langle o_2(\rho \circ p_*), [S']\rangle\big| \leq 2g' - \frac{1}{2}\,,$$

where g' is the genus of S'. Since o_2 is a characteristic class, we have

$$o_2(\rho \circ p_*) = p^*\big(o_2(\rho)\big)\,,$$

where $p^* : \mathrm{H}^2(S, \mathbb{Z}) \to \mathrm{H}^2(S', \mathbb{Z})$ and thus

$$\big\langle o_2(\rho \circ p_*), [S]\big\rangle = \big\langle p^*\big(o_2(\rho)\big), [S']\big\rangle = \big\langle o_2(\rho), p_*[S']\big\rangle = n\big\langle o_2(\rho), [S]\big\rangle\,,$$

since p is of degree n. Using the relation $g' - 1 = n(g-1)$, we obtain

$$\big|\langle o_2(\rho), [S]\rangle\big| < \frac{4n(g-1)+3}{2n}\,,$$

which gives the desired inequality as soon as $n \geq 2$, since the left hand side is an integer.

In Milnor's paper [**?**] the construction and the property (3) of $\tilde{r}$ come as a complete surprise. With hindsight, it is an instance of a *quasimorphism* and it is in the context of bounded cohomology that its relation to the Euler class and the specific constant $\frac{1}{2}$ in (3) are explained.

Concerning the optimality of the inequality, it is shown also in [**?**] that every integer between $-(2g-2)$ and $2g-2$ is attained. In particular the inequality is optimal and one way to see this is to compute the Toledo invariant of a homomorphism $\rho_h : \pi_1(S) \to G$ corresponding to a hyperbolic structure h on S. For this we have at our disposal the orientation preserving isometry $f_h : D \to \mathbb{D}$ and hence the form $f_h^*(\omega_{\mathbb{D}})$ on S coincides with the area 2-form ω_h given by the hyperbolic structure. Thus

$$\mathrm{T}(\rho_h) = \frac{1}{2\pi}\int_S \omega_h = \big|\chi(S)\big| = 2g-2\,.$$

It should be observed that the value of the area of S can be obtained directly from the formula of the area of a geodesic triangle in $\mathbb{D}$ applied to the triangulation of a "standard" fundamental polygon, taking into account that the sum of the internal angles if 2π. In light of this computation it is a very natural question what is the nature of the homomorphisms ρ for which $\mathrm{T}(\rho) = 2g-2$. The answer is given by Goldman in his thesis [**?**]:

Theorem 3.5 ([**?**]). *A representation $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$ corresponds to a hyperbolic structure on S if and only if $\mathrm{T}(\rho) = 2g-2$.*

A reformulation of Theorem 3.5 is given in (4.1). In particular, the image of $\mathrm{Hyp}(S)$ in $\mathrm{Hom}\big(\pi_1(S), G\big)$ being the preimage of $2g-2$ under T is hence a union of components. In fact a little later Goldman proved that the preimages $\mathrm{T}^{-1}(n)$, for $n \in \mathbb{Z} \cap \big[-(2g-2), 2g-2\big]$ are exactly the components of $\mathrm{Hom}\big(\pi_1(S), G\big)$ [**?**]. The component where $\mathrm{T} = 2-2g$ corresponds to hyperbolic structures on S with the reversed orientation.

3.9 An application to Kneser's theorem

This theorem takes its motivation in the question of what are the possible degrees of continuous maps from a compact oriented surface S to itself. If S is either the sphere or the torus, then maps of arbitrarily high degree exist. This is not the case anymore if the genus of S is at least two, and more generally we have the following:

Theorem 3.6 ([**?**]). *Let $f : S_1 \to S_2$ be a continuous map between compact oriented surfaces S_i of genus at least 2. Then*

$$|\deg f| \leq \frac{|\chi(S_1)|}{|\chi(S_2)|}\,,$$

with equality if and only if f is homotopic to a covering map, necessarily of degree $\frac{\chi(S_1)}{\chi(S_2)}$.

Proof. Let $f_* : \pi_1(S_1) \to \pi_1(S_2)$ be the homomorphism induced on the level of fundamental groups, and pick $\rho \in \mathrm{Hom}\big(\pi_1(S_2), G\big)$. Then

$$\begin{aligned} \big\langle o_2(\rho \circ f_*), [S_1]\big\rangle &= \big\langle f^*\big(o_2(\rho)\big), [S_1]\big\rangle = \big\langle o_2(\rho), f_*\big([S_1]\big)\big\rangle \\ &= \deg f \big\langle o_2(\rho), [S_2]\big\rangle \,. \end{aligned} \tag{3.5}$$

Specializing now to $\rho = \rho_h$, for $h \in \mathrm{Hyp}(S_2)$, we get

$$\big\langle o_2(\rho_h), [S_2]\big\rangle = \big|\chi(S_2)\big|$$

while the Milnor-Wood inequality gives

$$\big|\langle o_2(\rho_h \circ f_*), [S_1]\rangle\big| \le \big|\chi(S_1)\big|$$

which, together with (3.5), gives the inequality on $|\deg f|$.

Assume now that we have equality and, without loss of generality, that

$$\mathrm{T}(\rho_h \circ f_*) = \big|\chi(S_1)\big| \,.$$

Then Goldman's theorem implies that $\rho_h \circ f_*$ corresponds to a hyperbolic structure on S_1 and, in particular, f_* is injective. Letting $p : T \to S_2$ denote the covering of S_2 corresponding to the image of f_*, we have that

$$f_* : \pi_1(S_1) \to \pi_1(T)$$

is an isomorphism, which implies that T is compact and (by Nielsen's theorem) that f_* is induced by a homeomorphism $F : S_1 \to T$.

We have then that the homomorphisms f_* and $(p \circ F)_*$ coincide; let now $\tilde f : \tilde S_1 \to \tilde S_2$ and $\widetilde{p \circ F} : \tilde S_1 \to \tilde S_2$ be lifts of f. These are continuous maps which are equivariant with respect to the same homomorphism $\pi_1(S_1) \to \pi_1(S_2)$. Upon choosing a hyperbolic metric on S_2, we conclude by using a geodesic homotopy that $\widetilde{p \circ F}$ and $\tilde f$ are equivariantly homotopic and hence $p \circ F$ and f are homotopic. $\square$

4 Surfaces of finite type and the Euler number

4.1 Hyperbolic structures on surfaces of finite type and semiconjugations

Let S be a compact (connected, oriented) surface. Then the image of $\mathrm{Hyp}(S)$ in $\mathrm{Hom}\big(\pi_1(S), G\big)$, $G = \mathrm{PSU}(1,1)$, can be described by one equation in the image of the generators; namely, letting t be the generator of $\pi_1(G)$, $a_1, b_1, \dots, a_g, b_g$

the generators of $\pi_1(S)$ defined in § 3.3 and introducing the smooth map

$$\begin{aligned} G \times G &\longrightarrow \widetilde{G} \\ (g,h) &\mapsto [g,h]^{\sim} \end{aligned}$$

where $[g,h]^{\sim}$ is the commutator of any two lifts of g and h, Goldman's theorem (Theorem 3.5) can be restated as

$$\delta\big(\mathrm{Hyp}(S)\big) = \left\{ \rho \in \mathrm{Hom}\big(\pi_1(S), G\big) : \prod_{i=1}^{g} \big[\rho(a_i), \rho(b_i)\big]^{\sim} = t^{2-2g} \right\}. \qquad (4.1)$$

The aim of this section is to present a circle of ideas, rooted in the theory of bounded cohomology, which will, among other things, lead to an analogous explicit description of $\delta\big(\mathrm{Hyp}(S)\big)$ in the case in which S is not compact. We will however always assume that $\pi_1(S)$ is finitely generated; equivalently S is diffeomorphic to the interior of a compact surface with boundary. The genus g of this surface and the number n of boundary components together determine S up to diffeomorphism. We say that S is of *finite topological type*.

The first observation is that the invariants introduced in § 3 are of no use when S is not compact. In fact, for a connected surface S the following are equivalent:

(1) $\mathrm{H}^2(S,\mathbb{Z}) = \mathrm{H}^2\big(\pi_1(S),\mathbb{Z}\big) = 0$;

(2) $\pi_1(S)$ is a free group;

(3) S is not compact.

Elaborating a little on (2), if r is the rank of $\pi_1(S)$ as a free group, we have clearly that $\mathrm{Hom}\big(\pi_1(S), G\big) \cong G^r$ and, as a result, this space of homomorphisms is always connected. Thus $\delta\big(\mathrm{Hyp}(S)\big)$ will not be a connected component.

The second observation, and this will lead us in the right direction, is to consider more closely the inclusions

$$\delta\big(\mathrm{Hyp}(S)\big) \subset \mathrm{Hom}\big(\pi_1(S), G\big) \subset \mathrm{Hom}\big(\pi_1(S), \mathrm{Homeo}^+\big(S^1\big)\big)$$

in the case in which S is compact. For $h_1, h_2 \in \mathrm{Hyp}(S)$, the diffeomorphism $f_{h_1} \circ f_{h_2}^{-1} : \mathbb{D} \to \mathbb{D}$ clearly conjugates ρ_{h_2} to ρ_{h_1} within $\mathrm{Diff}^+(\mathbb{D})$. Since S is compact, $f_{h_1} \circ f_{h_2}^{-1}$ is a quasi isometry; it is then a fundamental fact in hyperbolic geometry that $f_{h_1} \circ f_{h_2}^{-1}$ extends to an (orientation preserving) homeomorphism of $S^1 = \partial\mathbb{D}$. Thus any two representations in $\delta\big(\mathrm{Hyp}(S)\big)$ are conjugate in $\mathrm{Homeo}^+\big(S^1\big)$ and it is an easy exercise to see that any $\rho \in \mathrm{Hom}\big(\pi_1(S), G\big)$ that is conjugate to an element in $\delta\big(\mathrm{Hyp}(S)\big)$ in $\mathrm{Homeo}^+\big(S^1\big)$ in fact belongs to $\delta\big(\mathrm{Hyp}(S)\big)$; indeed such a representation ρ is injective with discrete image. Thus a full invariant of conjugacy on $\mathrm{Hom}\big(\pi_1(S), \mathrm{Homeo}^+\big(S^1\big)\big)$ would lead to a characterization of $\delta\big(\mathrm{Hyp}(S)\big)$ within $\mathrm{Hom}\big(\pi_1(S), G\big)$!

We will now develop this line of thought in the case in which S is of finite topological type. We assume that S has a fixed orientation and let Σ denote a compact surface of genus g with n boundary components such that $S = \mathrm{int}(\Sigma)$. Then $\pi_1(S)$ admits a presentation

$$\left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n : \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = e \right\rangle . \tag{4.2}$$

Here each c_i is freely homotopic to the i-th component of $\partial\Sigma$ with orientation compatible with the chosen orientation on Σ. Let now h be a complete hyperbolic metric on S. We have then two possibilities for $\rho_h(c_i)$:

(1) $\rho_h(c_i)$ is parabolic: it has a unique fixed point $\xi_i \in \partial\mathbb{D}$ and for the interior C_i of an appropriate horocycle based at ξ_i, the quotient $\langle \rho_h(c_i) \rangle \backslash C_i$ is of finite area and embeds isometrically into $\rho_h\big(\pi_1(S)\big)\backslash\mathbb{D}$. It is a neighborhood of the i-th end of S.

(2) $\rho_h(c_i)$ is hyperbolic: it has an invariant axis $a_i \subset \mathbb{D}$ which determines a half plane $H_i \subset \mathbb{D}$ such that ∂H_i and a_i have opposite orientation. The quotient $\langle \rho_h(c_i) \rangle \backslash H_i$ embeds isometrically into $\rho_h\big(\pi_1(S)\big)\backslash\mathbb{D}$. It is of infinite area and a neighborhood of the i-th end.

Let $\Lambda_h \subset \partial\mathbb{D}$ be the limit set of $\rho_h\big(\pi_1(S)\big)$. Then either $\Lambda_h = \partial\mathbb{D}$, equivalently (S, h) is of finite area, or $\Lambda_h \neq \partial\mathbb{D}$, in which cases it is a Cantor set; the connected components of $\partial\mathbb{D} \setminus \Lambda_h$ are then in bijective correspondence with the set of elements in

$$\big\{\gamma \in \pi_1(S) : \gamma \text{ is conjugate to a boundary loop } c_i \text{ s.t. } \rho_h(c_i) \text{ is hyperbolic}\big\}$$

Thus, if $h_1, h_2 \in \mathrm{Hyp}(S)$ are such that h_1 has finite area while h_2 has infinite area, then ρ_{h_1} gives a minimal action of $\pi_1(S)$ on $\partial\mathbb{D}$, while ρ_{h_2} gives an action on $\partial\mathbb{D}$ which admits Λ_{h_2} as minimal set. In particular ρ_{h_1}, ρ_{h_2} cannot be conjugated in $\mathrm{Homeo}^+\big(S^1\big)$. Let us however consider the diffeomorphism $F := f_{h_1} \circ f_{h_2}^{-1} : \mathbb{D} \to \mathbb{D}$.

Proposition 4.1. *Assume that h_1 is of finite area. Then F extends to a continuous map $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ which is weakly monotone.*

This proposition is a slight generalization of a classical result stating that if h_1, h_2 have finite area, then since the isomorphism $\rho_{h_1} \circ \rho_{h_2}^{-1}$ is "type preserving", $f_{h_1} \circ f_{h_2}^{-1}$ extends to a homeomorphism of $\partial\mathbb{D}$.

To explain the statements of the proposition, recall that the circle $S^1 \cong \partial\mathbb{D}$ is equipped with its canonical positive orientation and this gives a natural notion for triples of points to be positively oriented. We have the following

Definition 4.2. An (arbitrary) map $\varphi : S^1 \to S^1$ is *weakly monotone* if whenever $x, y, z \in S^1$ are such that $\varphi(x), \varphi(y), \varphi(z)$ are distinct, then (x, y, z) and $\big(\varphi(x), \varphi(y), \varphi(z)\big)$ have the same orientation.

Typically the map in Proposition 4.1 is collapsing a connected component in $\partial\mathbb{D} \setminus \Lambda_{h_2}$ corresponding to $\gamma \in \pi_1(S)$ to the corresponding fixed point in $\partial\mathbb{D}$ of the parabolic element $\rho_{h_1}(\gamma)$. We have for every $x \in \partial\mathbb{D}$

$$\varphi \circ \rho_{h_2}(\gamma)(x) = \rho_{h_1}(\gamma) \circ \varphi(x)$$

and we say that φ *semiconjugates* ρ_{h_2} to ρ_{h_1}. It is in order to reverse this process that in the definition of weakly monotone map one allows discontinuous maps. For instance $\varphi^{-1}(x)$ is always an interval and if we set $\psi(x)$ equal to the left endpoint of $\varphi^{-1}(x)$, then ψ is weakly monotone and

$$\rho_{h_2}(\gamma) \circ \psi(x) = \psi \circ \rho_{h_1}(\gamma)(x)\,.$$

Thus given now arbitrary homomorphisms $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}^+\big(S^1\big)$ defined on a group Γ, we say that ρ_1 and ρ_2 are semiconjugate if there exists $\varphi : S^1 \to S^1$ weakly monotone such that for every $\gamma \in \Gamma$

$$\varphi \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \varphi\,.$$

It is now clear that semiconjugation is an equivalence relation. In our specific situation we have then

Corollary 4.3. (1) *For any $h_1, h_2 \in \mathrm{Hyp}(S)$, ρ_{h_1} and ρ_{h_2} are semiconjugate.*

(2) *If $h \in \mathrm{Hyp}(S)$ and $\rho \in \mathrm{Hom}\big(\pi_1(S), G\big)$ is semiconjugate to ρ_h, then $\rho \in \delta\big(\mathrm{Hyp}(S)\big)$.*

Proof. The first assertion follows from the discussion above. We will now indicate the main points entering in the proof of the second one.

Let $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ be weakly monotone with

$$\varphi \circ \rho(\gamma) = \rho_h(\gamma) \circ \varphi \tag{4.3}$$

and assume, in virtue of the first part, that h has finite area. Since $\rho_h\big(\pi_1(S)\big)$ acts minimally on $\partial\mathbb{D}$, we must have that $\overline{\mathrm{im}\varphi} = \partial\mathbb{D}$. It is easy to see that for a weakly monotone map this implies that φ is continuous. Then we deduce from (4.3) that ρ is injective and with discrete image. Thus $\Gamma := \rho\big(\pi_1(S)\big)$ is a finitely generated discrete subgroup of $\mathrm{PSU}(1,1)$ and hence $\Gamma\backslash\mathbb{D}$ is topologically of finite type.

One uses then φ to check that the isomorphism $\rho : \pi_1(S) \to \Gamma$ sends $\langle c_i \rangle$, for each i, isomorphically into the fundamental group of a boundary component of $\Gamma\backslash\mathbb{D}$ and that each boundary component is so obtained.

Then an appropriate version of the Nielsen realization implies that ρ is implemented by a diffeomorphism $S \to \Gamma\backslash\mathbb{D}$, by means of which we produce the hyperbolic structure h' for which $\rho = \rho_{h'}$. $\square$

4.2 The bounded Euler class

The discussion of the preceding section shows that semiconjugation is a natural notion of equivalence for group actions by homeomorphisms of the circle, at least in the framework of the questions regarding hyperbolic structures. A different context is provided by a paraphrase of a famous theorem of Poincaré concerning rotation numbers of homeomorphisms, namely two orientation preserving homeomorphisms of the circle are semiconjugate if and only if they have the same rotation number.

Remarkably, there is an invariant generalizing the rotation number of a single homeomorphism to arbitrary group actions and which is a complete invariant of semiconjugacy: it is the bounded Euler class, introduced by Ghys in [**?**] and whose main features we now describe briefly.

For this let us recall that *bounded cohomology* can be defined by restricting to bounded cochains in the (inhomogeneous) bar resolution. Let $A = \mathbb{R}$ or $\mathbb{Z}$, and G be any group. Denote by $C^n(G, A)$ the space of function from G^n to A and by $C^n_b(G, A) := \big\{f \in C^n(G, A) \,:\, \sup_{\underline{g}=(g_1,\cdots,g_n)\in G^n} |f(\underline{g})| < \infty\big\}$ the subspace of bounded functions. Defining the boundary map

$$d_n : C^n(G, A) \to C^{n+1}(G, A)$$

by

$$\begin{aligned} d_n f(g_1, \cdots, g_{n+1}) = {} & f(g_2, \cdots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \cdots, g_n)\,, \end{aligned}$$

we obtain the complex $\big(C^\bullet(G, A), d_\bullet\big)$, whose cohomology is the group cohomology $\mathrm{H}^\bullet(G, A)$, and the sub-complex $\big(C^\bullet_b(G, A), d_\bullet\big)$, whose cohomology is the bounded cohomology $\mathrm{H}^\bullet_{\mathrm{b}}(G, A)$ of G.

Bounded cohomology behaves very differently from usual cohomology, for example, the second bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_r, \mathbb{R})$ of a nonabelian free group is infinite dimensional. This different behavior will allow us to define bounded analogues of the invariants introduced in § 3, which are meaningful when S is noncompact, and give finer information even in the case when S is compact (see e.g. Corollary 4.5).

Recall that, in the notation of § 3.4, a representative cocycle ϵ for the Euler class $e \in \mathrm{H}^2\big(\operatorname{Homeo}^+\big(S^1\big)\,, \mathbb{Z}\big)$ was given by

$$\overline{f} \circ \overline{g} = \overline{f \circ g} \circ T^{\epsilon(f,g)}\,,$$

where $\overline{f}$ and $\overline{g}$ are the unique lifts to $\mathbb{R}$ of $f, g \in \operatorname{Homeo}^+\big(S^1\big)$ such that

$$0 \leq \overline{f}(0), \overline{g}(0) < 1\,,$$

and $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) := x + 1$.

Since $\overline{f}$ is increasing and commutes with T, we have

$$\overline{f}\big(\overline{g}(0)\big) \in \big[\overline{f}(0), \overline{f}(0) + 1\big)$$

and since $\overline{fg}(0) \in [0, 1)$, we obtain that $\epsilon(f, g) \in \{0, 1\}$ and hence in particular ϵ is a bounded cocycle. The class

$$e^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\big(\,\mathrm{Homeo}^+\big(S^1\big)\,, \mathbb{Z}\big)$$

so obtained is called the *bounded Euler class* and given any homomorphism $\rho : \Gamma \to \mathrm{Homeo}^+\big(S^1\big)$, $\rho^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z})$ is called the *bounded Euler class of the action given by* ρ. We have then the following

Theorem 4.4 ([**?**]). *The bounded Euler class of a homomorphism $\rho : \Gamma \to \mathrm{Homeo}^+\big(S^1\big)$ is a full invariant of semiconjugation.*

The relation with the classical *rotation number* is then the following. Recall that the translation number $\tau(\varphi) \in \mathbb{R}$ of a homeomorphism $\varphi \in \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$ is given by

$$\tau(\varphi) := \lim_{n\to\infty} \frac{\varphi^n(0)}{n}\,.$$

Then τ has the following remarkable properties (compare to the properties of $\tilde{r}$ in § 3.8):

(1) τ is continuous;

(2) $\tau(\varphi \circ T^m) = \tau(\varphi) + m$, for $m \in \mathbb{Z}$;

(3) $\tau(\varphi^k) = k\tau(\varphi)$;

(4) $\big|\tau(\varphi\psi) - \tau(\varphi) - \tau(\psi)\big| \leq 1$, for all $\varphi, \psi \in \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$.

In the language of bounded cohomology, this says that τ is a continuous homogeneous quasimorphism. Then for $f \in \mathrm{Homeo}^+\big(S^1\big)$ the rotation number of f is

$$\mathsf{rot}(f) := \tau(\overline{f}) \mod \mathbb{Z}$$

which is well defined in view of (2).

Given now $f \in \mathrm{Homeo}^+\big(S^1\big)$, consider

$$\begin{aligned} h_f : \mathbb{Z} &\to \mathrm{Homeo}^+\big(S^1\big) \\ n &\longmapsto \quad f^n \end{aligned}$$

to obtain an invariant $h_f^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\mathbb{Z}, \mathbb{Z})$. Writing the long exact sequence in bounded cohomology [**?**, Proposition 1.1] associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

we get

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^2_{\mathrm{b}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

and then

$$\big(\delta^{-1} h_f^*(e^{\mathrm{b}})\big)(1) = \mathsf{rot}(f)\,. \tag{4.4}$$

It should be noticed in passing that the definition of $\mathsf{rot}(f)$ involves taking a limit, while the left hand side of (4.4) only involves purely algebraic constructions.[2]

The proof of the not straightforward implication of Ghys' theorem goes as follows. Let $\rho_1,\ \rho_2 : \Gamma \to \mathrm{Homeo}_+(S^1)$ be homomorphisms, with $\rho_1^*(e^{\mathrm{b}}) = \rho_2^*(e^{\mathrm{b}})$. Hence $\rho_1^*(e) = \rho_2^*(e)$ and, by replacing Γ with a suitable central extension by $\mathbb{Z}$ we may assume that ρ_1 and ρ_2 lift to homomorphisms $\widetilde{\rho_1}$, $\widetilde{\rho_2} : \Gamma \to \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$. But then

$$\widetilde{\rho_i}(\gamma) = \overline{\rho_i(\gamma)} T^{c_i(\gamma)}$$

and the hypothesis that ρ_1, ρ_2 have the same bounded Euler class is equivalent to saying that we may choose the lifts $\widetilde{\rho_1}$ and $\widetilde{\rho_2}$ such that

$$c_2 - c_1 : \Gamma \to \mathbb{Z}$$

is bounded. Thus:

$$\widetilde{\varphi}(x) := \sup_{\gamma \in \Gamma} \big\{ \widetilde{\rho_1}(\gamma)^{-1} \widetilde{\rho_2}(\gamma)(x) :\, \gamma \in \Gamma \big\} < +\infty$$

is well defined for every x and gives a monotone map $\mathbb{R} \to \mathbb{R}$ commuting with T and satisfying

$$\widetilde{\varphi}\widetilde{\rho_2}(\eta) = \widetilde{\rho_1}(\eta)\widetilde{\varphi}\,,$$

for all $\eta \in \Gamma$. This shows that ρ_1 and ρ_2 are semiconjugate.

In order to complete one of the descriptions of the image of $\mathrm{Hyp}(S)$ under the map δ in $\mathrm{Hom}\big(\pi_1(S), G\big)$ let now S be again an oriented surface of finite topological type, where we do not exclude the case in which S is compact. We have seen that for any two $h_1, h_2 \in \mathrm{Hyp}(S)$, the homomorphisms ρ_{h_1} and ρ_{h_2} are semiconjugate in $\mathrm{Homeo}^+\big(S^1\big)$ and hence, by the easy direction of Ghys' theorem, we have that

$$\rho_{h_1}^*(e^{\mathrm{b}}) = \rho_{h_2}^*(e^{\mathrm{b}})\,.$$

Let $\kappa_S^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{Z}\big)$ denote the class so obtained. Then

[2] In fact we have used $\mathrm{H}^1_{\mathrm{b}}(\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{H}^2_{\mathrm{b}}(\mathbb{Z}, \mathbb{R}) = 0$. The first equality follows from the fact that there are no (nonzero) bounded homomorphisms into $\mathbb{R}$ while the second follows from the elementary fact that if $\psi : \mathbb{Z} \to \mathbb{R}$ is a quasimorphism and $\alpha := \lim_{n\to\infty} \frac{\psi(n)}{n}$, then ψ is at bounded distance from the homomorphism $n \mapsto n\alpha$.

Corollary 4.5.

$$\delta\big(\mathrm{Hyp}(S)\big) = \big\{\rho \in \mathrm{Hom}\big(\pi_1(S), G\big) : \rho^*(e^{\mathrm{b}}) = \kappa_S^{\mathrm{b}}\big\}\,.$$

Proof. The inclusion $\subset$ has already been discussed.

If now $\rho^*(e^{\mathrm{b}}) = \kappa_S^{\mathrm{b}}$, then Ghys' theorem implies that ρ is semiconjugate to an element in $\delta\big(\mathrm{Hyp}(S)\big)$ and the assertion follows from Corollary 4.3. □

4.3 Bounded Euler number and bounded Toledo number

In this section we describe two ways in which one can associate a (real) number to the bounded Euler class; this will give the two invariants mentioned in the title. The fact that they coincide is then an essential result containing a lot of information.

Recall that S is a surface of finite topological type and hence we may consider it as the interior of a compact surface Σ with boundary $\partial\Sigma$. Let now $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}^+\big(S^1\big)$ be a homomorphism and

$$\rho^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{Z}\big)$$

its bounded Euler class. We proceed now to define the *bounded Euler number* of ρ. First we use that the classifying map $\Sigma \to B\pi_1(\Sigma)$ is a homotopy equivalence in order to obtain a natural isomorphism $\mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{Z}\big) \to \mathrm{H}^2_{\mathrm{b}}\big(\Sigma, \mathbb{Z}\big)$ by means of which we consider, keeping the same notation, the class $\rho^*(e^{\mathrm{b}})$ as a bounded singular class on Σ. (See [?] for the definition of singular bounded cohomology.) The inclusion $\partial\Sigma \hookrightarrow \Sigma$ gives in a straightforward way a long exact sequence in bounded cohomology with coefficients in $A = \mathbb{Z}, \mathbb{R}$, whose relevant part for us reads

$$\mathrm{H}^1_{\mathrm{b}}(\partial\Sigma, A) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, A) \xrightarrow{f_A} \mathrm{H}^2_{\mathrm{b}}(\Sigma, A) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\partial\Sigma, A)$$

which gives for $A = \mathbb{Z}$

$$0 \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{Z}) \xrightarrow{f_{\mathbb{Z}}} \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\partial\Sigma, \mathbb{Z})$$

and for $A = \mathbb{R}$

$$0 \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \xrightarrow{f_{\mathbb{R}}} \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R}) \longrightarrow 0$$

where we have used the following facts (see [?] or see footnote in equality (4.4)):

(1) $\mathrm{H}^1_{\mathrm{b}}(\partial\Sigma, A) = 0$ for $A = \mathbb{R}, \mathbb{Z}$;

(2) $\mathrm{H}^2_{\mathrm{b}}(\partial\Sigma, \mathbb{R}) = 0$.

As a result we have that if we consider $\rho^*(e^{\mathrm{b}})$ as a real bounded class on Σ, it corresponds to a unique relative class

$$f_{\mathbb{R}}^{-1}\big(\rho^*(e^{\mathrm{b}})\big) \in \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R})\,.$$

The latter can then be seen as an ordinary singular relative class and hence can be evaluated on the relative fundamental class, thus leading to the *bounded Euler number*:

$$e^{\mathrm{b}}(\rho) := \big\langle f_{\mathbb{R}}^{-1}\big(\rho^*(e^{\mathrm{b}})\big), [\Sigma, \partial\Sigma]\big\rangle\,. \tag{4.5}$$

Two important remarks are in order here. First, the definition of this invariant not only involves $\pi_1(S)$ but also the surface S itself; this is essential if this invariant is to detect hyperbolic structures on S (see Remark 2.7). Second, let us denote by $\rho^*(e^{\mathrm{b}}_{\mathbb{R}}) \in \mathrm{H}^2_{\mathrm{b}}\big(\mathrm{Homeo}^+\big(S^1\big), \mathbb{R}\big)$ the real bounded class obtained by considering the cocycle ϵ as taking values in $\mathbb{R}$. Then $e^{\mathrm{b}}(\rho)$ depends in fact only on the real class $\rho^*(e^{\mathrm{b}}_{\mathbb{R}})$; the extent to which this (real) class determines ρ (up to semiconjugation), is completely understood (see [**?**]).

The *bounded Euler number* $e^{\mathrm{b}}(\rho)$ is in general not an integer. Remarkably, one can give an explicit formula for the "fractional part" of $e^{\mathrm{b}}(\rho)$; indeed, combining the long exact sequence associated to $\partial\Sigma \to \Sigma$ together with the one associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

leads to the following congruence relation

$$e^{\mathrm{b}}(\rho) = -\sum_{i=1}^{n} \mathsf{rot}\rho(c_i) \mod \mathbb{Z}\,.$$

In fact, using this, one can establish a general formula for $e^{\mathrm{b}}(\rho)$:

Theorem 4.6 ([**?**]). *Let S be an oriented surface of finite topological type with presentation of its fundamental group*

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n : \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = e \right\rangle$$

as defined in (4.2). *Let $\rho : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$ be a homomorphism and let τ denote the translation quasimorphism. Then*

(1) *If S is compact (that is $n = 0$), then*

$$e(\rho) = e^{\mathrm{b}}(\rho) = \tau\left(\prod_{i=1}^{g} \big[\rho(a_i), \rho(b_i)\big]^{\sim}\right)\,.$$

(2) *if S is noncompact (that is $n \geq 1$), then*

$$e^{\mathrm{b}}(\rho) = -\sum_{i=1}^{n} \tau\big(\tilde{\rho}(c_i)\big)\,,$$

where $\tilde{\rho} : \pi_1(S) \to \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$ denotes a homomorphism lifting ρ.

Now we will turn to the description of the bounded Toledo number. Its definition is based on the use of a very general operation in bounded cohomology called "transfer", together with a description of the second bounded cohomology of $G = \mathrm{PSU}(1,1)$.

Let $\Gamma < G$ be a lattice in G. One has the isomorphism

$$\mathrm{H}^{\bullet}_{\mathrm{b}}(\Gamma,\mathbb{R}) \cong \mathrm{H}^{\bullet}_{\mathrm{cb}}\big(G, L^{\infty}(\Gamma\backslash G)\big) \tag{4.6}$$

analogous to the Eckmann–Shapiro isomorphism in ordinary cohomology. Here $\mathrm{H}^{\bullet}_{\mathrm{cb}}$ denotes the bounded continuous cohomology for whose definition the reader is referred to [**?**] or also [**?**, § 2.3]. Thus the bounded cohomology of the discrete group Γ can be computed via the bounded continuous cohomology of the ambient Lie group G, but at the expense of replacing the trivial Γ-module $\mathbb{R}$ by the quite intractable G-module $L^{\infty}(\Gamma\backslash G)$. This principle is very general and does not require Γ to be a lattice, but this hypothesis will now allow us to "simplify" the coefficients: indeed, let μ be the G-invariant probability measure on $\Gamma\backslash G$. Then

$$\begin{aligned} L^{\infty}(\Gamma\backslash G) &\longrightarrow \quad \mathbb{R} \\ f \quad &\longmapsto \int_{\Gamma\backslash G} f(x)\mathrm{d}\mu(x) \end{aligned} \tag{4.7}$$

is a morphism of G-modules, where $\mathbb{R}$ is then the trivial G-module. Composing the induction isomorphism (4.6) with the morphism in cohomology induced by the morphism of coefficients (4.7) and specializing to degree 2 leads to a map, called the *transfer map*

$$\mathrm{T}_{\mathrm{b}} : \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$$

which is linear and norm decreasing. The interest of this construction lies in the fact that, while $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ is infinite dimensional, say when G is a real rank one group, the space $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ is finite dimensional if G is a connected Lie group and in fact one dimensional for $G = \mathrm{PSU}(1,1)$. Considering the cocycle in § 3.7 defining the Kähler class, we see that c is bounded by $\frac{1}{2}$, as the area of geodesic triangles in $\mathbb{D}$ is bounded by π, and therefore we can use c to define a bounded continuous class $\kappa^{\mathrm{b}}_G \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ called the bounded Kähler class.

We have then:

Proposition 4.7. *Let $G = \mathrm{PSU}(1,1) \hookrightarrow \mathrm{Homeo}^+(\partial\mathbb{D})$ be the natural inclusion. Then*

(1) $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}) = \mathbb{R}\kappa^{\mathrm{b}}_G$;

(2) *The restriction* $\mathrm{e}^{\mathrm{b}}_{\mathbb{R}}|_G$ *to* G *of the real bounded Euler class equals the bounded Kähler class* κ^{b}_G *in* $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$.

The first assertion is in fact a very special case of a more general result and we will treat this later in its proper context; suffices it to say here that we already know that the comparison map

$$\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R}) = \mathbb{R}\kappa_G$$

is surjective as κ^{b}_G is sent to κ_G; the kernel of this map is then described by the space of continuous quasimorphisms on $\mathrm{PSU}(1,1)$ and it is easy to see that they must be bounded. Hence the comparison map is injective.

For the second statement one needs an explicit relation between the cocycle ϵ used to define the Euler class and the orientation cocycle on S^1. Recall that the *orientation cocycle*

$$\mathrm{or} : S^1 \times S^1 \times S^1 \to \mathbb{Z}$$

is defined by

$$\mathrm{or}(x,y,z) := \begin{cases} 1 & \text{if } x,y,x \text{ are cyclically positively oriented} \\ 0 & \text{if at least two coordinates coincide} \\ -1 & \text{if } x,y,x \text{ are cyclically negatively oriented.} \end{cases}$$

A formula relating the orientation cocycle directly to hyperbolic geometry is given by

$$\mathrm{or}(x,y,z) = \frac{1}{\pi}\int_{\Delta(x,y,z)} \omega_{\mathbb{D}}$$

where now $x,y,z \in \partial\mathbb{D}$ and $\Delta(x,y,z)$ denotes the oriented geodesic ideal triangle with vertices x,y,z. Here we have taken $\partial\mathbb{D}$ as a model of S^1 with the identification

$$\begin{aligned} \mathbb{Z}\backslash\mathbb{R} &\to \partial\mathbb{D} \\ t &\mapsto e^{2\pi\imath t} \end{aligned}$$

and we denote again by ϵ the corresponding cocycle on $\mathrm{Homeo}^+(\partial\mathbb{D})$.

Lemma 4.8 ([**?**]). *For* $f,g \in \mathrm{Homeo}^+(\partial\mathbb{D})$

$$\epsilon(f,g) = -\frac{1}{2}\mathrm{or}\big(1, f(1), fg(1)\big) + d\beta(f,g)\,,$$

where

$$\beta(f) := \begin{cases} 0 & \textit{if } f(1) = 1 \\ -\frac{1}{2} & \textit{if } f(1) \neq 1\,. \end{cases}$$

The proof of Proposition 4.7(2) is now straightforward: using Stokes' theorem one shows that

$$(g_1, g_2) \mapsto c(g_1, g_2) = \frac{1}{2\pi} \int_{\Delta(0, g_1 0, g_1 g_2 0)} \omega_{\mathbb{D}}$$

and

$$(g_1, g_2) \mapsto \frac{1}{2\pi} \int_{\Delta(1, g_1 1, g_1 g_2 1)} \omega_{\mathbb{D}}$$

are cohomologous in the complex of bounded (Borel) cochains; since the second cocycle is then essentially $\frac{1}{2}\mathrm{or}\big(1, g_1(1), g_1 g_2(1)\big)$, Lemma 4.8 allows to conclude.

Now we are in the position to define the *bounded Toledo number.* Define, using Proposition 4.7(1), the linear form $\mathrm{t_b} : \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \to \mathbb{R}$,

$$\mathrm{T_b}(\alpha) = \mathrm{t_b}(\alpha)\kappa^{\mathrm{b}}_G \,.$$

Then given a surface S of finite topological type as before, fix a hyperbolization, i.e. a homomorphism corresponding to a complete hyperbolic structure on S, $h : \pi_1(S) \to G$ with image a lattice $\Gamma = h\big(\pi_1(S)\big)$ in G. Given now any homomorphism $\rho : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$, we define the bounded Toledo number of ρ as

$$\mathrm{T_b}(\rho, h) := \mathrm{t_b}\big((\rho \circ h^{-1})^*(e^{\mathrm{b}}_{\mathbb{R}})\big) \,.$$

Observe that the hyperbolization h is involved in the definition, but we will see that $\mathrm{T_b}(\rho, h)$ is independent of h as a consequence of the relation between the bounded Toledo and the bounded Euler numbers.

Concerning this relation consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{R}\big) \arrow[r, "\cong"] & \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R}) & \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \arrow[l, "f_{\mathbb{R}}"'] \arrow[d, "\langle\,\cdot\,{,}[\Sigma{,}\partial\Sigma]\rangle"] \\
\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \arrow[u, "h^*"] \arrow[d, "\mathrm{t_b}"] & & \mathbb{R} \\
\mathbb{R} & &
\end{tikzcd}$$

where, as before, $h : \pi_1(S) \to \Gamma$ is a hyperbolization with finite area.

Theorem 4.9 ([**?**, Theorem 3.3]). *For every $\alpha \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{R}\big) \cong \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$, we have that*

$$\mathrm{t_b}\big((h^*)^{-1}(\alpha)\big)\big|\chi(S)\big| = \big\langle f_{\mathbb{R}}^{-1}(\alpha), [\Sigma, \partial\Sigma]\big\rangle \,.$$

Specializing to $\alpha = \rho^*(e^{\mathrm{b}})$ this provides the desired equality between the bounded Toledo number and the bounded Euler number.

4.4 Computations in bounded cohomology

In computing bounded cohomology one faces *a priori* the same difficulties as for the usual cohomology, namely that the bar resolution contains many coboundaries; the ideal situation then would be if one had a complex giving bounded cohomology and where all differentials are zero. While this can be achieved for various ordinary cohomology theories, we do not know of an analogue of either Hodge theory or Van Est isomorphism for bounded cohomology. What can be achieved for the moment is a good model in degree two. This follows from the theory developed in [**?**, **?**, **?**] and of which we recall a few consequences in our case at hand.

Proposition 4.10. *Let $G = \mathrm{PSU}(1,1)$ and $L < G$ a closed subgroup whose action on $\partial\mathbb{D} \times \partial\mathbb{D}$ is ergodic. Then there is a canonical isomorphism*

$$\mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R}) \cong \mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^L .$$

Here

$$\begin{aligned}\mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^L := \big\{ f : (\partial\mathbb{D})^3 \to \mathbb{R} :\ & f \text{ is measurable, essentially bounded,} \\ & \text{alternating, } L\text{-invariant and} \\ f(x_2,x_3,x_4) - f(x_1,x_3,x_4) + f(x_1,&x_2,x_4) - f(x_1,x_2,x_3) = 0 \\ & \text{for a.e. } (x_1,x_2,x_3,x_4) \in (\partial\mathbb{D})^4 \big\} .\end{aligned}$$

In particular for $L = G = \mathrm{PSU}(1,1)$ it is plain that $\mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^L$ is one-dimensional, generated by the orientation cocycle. In addition one can verify that under the isomorphism in Proposition 4.10, κ^{b}_G is sent to $\frac{1}{2}\mathrm{or}$. This implies immediately the following

Corollary 4.11. $\|\kappa^{\mathrm{b}}_G\| = \frac{1}{2}$.

This, in turn, together with the fact that $\mathrm{T_b}$ is norm decreasing, implies:

Corollary 4.12. *For every $\alpha \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$,*

$$\big|\mathrm{t_b}(\alpha)\big| \leq 2\|\alpha\| .$$

Another feature of this model for bounded cohomology is that the transfer $\mathrm{T_b}$ takes a particularly simple and useful form:

Proposition 4.13 ([**?**]). *Let Γ be a lattice and μ the G-invariant probability measure on $\Gamma\backslash G$. Then*

$$\mathrm{T_b} : \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}) \to \mathrm{H}^2_{\mathrm{b}}(G,\mathbb{R})$$

is given by the map

$$\begin{array}{ccc} \mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^\Gamma & \to & \mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^G \\ \alpha & \longmapsto & \mathrm{T_b}(\alpha)\,, \end{array}$$

where

$$\mathrm{T_b}(\alpha)(x,y,z) = \int_{\Gamma\backslash G} \alpha(gx,gy,gz)\,d\mu(g)\,.$$

In particular

$$\int_{\Gamma\backslash G} \alpha(gx,gy,gz)\,d\mu(g) = \frac{\mathrm{t_b}(\alpha)}{2}\mathrm{or}(x,y,z)$$

for almost every $(x,y,z) \in (\partial\mathbb{D})^3$.

With this at hand we can now deduce a characterization of the bounded Kähler class which lies at the heart of our approach:

Theorem 4.14. *Let $\Gamma < G$ be a lattice. For every $\alpha \in \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$*

$$\big|\mathrm{t_b}(\alpha)\big| \leq 2\|\alpha\|$$

with equality if and only if α is proportional to the restriction $\kappa^{\mathrm{b}}_G|_\Gamma$ to Γ of the bounded Kähler class.

Proof. Taking up the formula in Proposition 4.13 in terms of measurable cocycles, $\big|\mathrm{t_b}(\alpha)\big| = 2\|\alpha\|$ reads

$$\int_{\Gamma\backslash G} \alpha(gx,gy,gz)\,d\mu(x) = \|\alpha\|_\infty \mathrm{or}(x,y,z)\,,$$

or

$$\int_{\Gamma\backslash G} \big(\|\alpha\|_\infty \mathrm{or}(gx,gy,gz) - \alpha(gx,gy,gz)\big)\,d\mu(g) = 0\,.$$

For positively oriented triples (x,y,z) this implies that for almost every g

$$\|\alpha\|_\infty \mathrm{or}(gx,gy,gz) = \alpha(gx,gy,gz)$$

and hence $\alpha = \|\alpha\|_\infty \mathrm{or}$ in $\mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3\big)^G$. ☐

Let $\kappa^{\mathrm{b}}_{S,\mathbb{R}} \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S),\mathbb{R}\big)$ denote the class obtained by considering $\kappa^{\mathrm{b}}_S \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S),\mathbb{Z}\big)$ (see the end of § 4.2) as a real class. Using the results of the previous section, we obtain the following important characterization of $\kappa^{\mathrm{b}}_{S,\mathbb{R}}$.

Corollary 4.15. *Let S be of finite topological type realized as the interior of Σ. Then for every $\alpha \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{R}\big) = \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$*

$$\big|\big\langle f_{\mathbb{R}}^{-1}(\alpha), [\Sigma, \partial\Sigma]\big\rangle\big| \leq 2\|\alpha\| \, \big|\chi(S)\big| \,,$$

with equality if and only if α is a multiple of $\kappa^{\mathrm{b}}_{S,\mathbb{R}}$.

4.5 Hyperbolic structures and representations: the noncompact case

In this section we fulfill the promise to give explicit equations for the image in $\mathrm{Hom}\big(\pi_1(S), G\big)$ of $\mathrm{Hyp}(S)$ under the map δ, in the case where S is a surface of finite topological type.

Let thus $\rho : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$ be a homomorphism; then we have the following result which is a first characterization of the maximality of the bounded Euler number of ρ.

Corollary 4.16. *We have*

$$\big|e^{\mathrm{b}}(\rho)\big| \leq \big|\chi(S)\big| \,,$$

where equality holds if and only if $\rho^*(e^{\mathrm{b}}_{\mathbb{R}}) = \pm\kappa^{\mathrm{b}}_{S,\mathbb{R}}$.

Proof. Combine Theorem 4.9 with Corollary 4.15. ☐

In fact, if S has n punctures and is of genus g, then we have that $e^{\mathrm{b}}(\rho) = 2g - 2 + n$ if and only if $\rho^*(e^{\mathrm{b}}_{\mathbb{R}}) = \kappa^{\mathrm{b}}_{S,\mathbb{R}}$; observe that for every $h \in \delta\big(\mathrm{Hyp}(S)\big)$, we have $\kappa^{\mathrm{b}}_{S,\mathbb{R}} = h^*(e^{\mathrm{b}}_{\mathbb{R}})$, so that ρ and h have the same real bounded Euler class. Keeping in mind that $\rho^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{Z}\big)$ determines ρ up to semiconjugacy, this is a rather strong conclusion and in fact we have the following

Theorem 4.17. *Let $\rho : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$ be a homomorphism with*

$$\rho^*(e^{\mathrm{b}}_{\mathbb{R}}) = \kappa^{\mathrm{b}}_{S,\mathbb{R}} \,.$$

Then ρ is semiconjugate to an element in $\delta\big(\mathrm{Hyp}(S)\big)$. In particular

$$\delta\big(\mathrm{Hyp}(S)\big) = \big\{\rho \in \mathrm{Hom}\big(\pi_1(S), \mathrm{Homeo}_+(S^1)\big) : \, \rho^*(e^{\mathrm{b}}_{\mathbb{R}}) = \kappa^{\mathrm{b}}_{S,\mathbb{R}}\big\} \,.$$

Note that Theorem 4.17 combined with Corollary 4.16 gives a generalization of Matsumoto's theorem proved in [**?**] for compact surfaces to surfaces of finite topological type (see also [**?**] for a different proof in the case when S is a compact surface).

Theorem 4.18. *Let S be a surface of finite type and let $\rho_i : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$, $i = 1, 2$, be homomorphisms with*

$$\big|e^{\mathrm{b}}(\rho_i)\big| = \big|\chi(S)\big| .$$

Then ρ_1 and ρ_2 are semiconjugate.

In particular, every $\rho : \pi_1(S) \to \mathrm{Homeo}^+\big(S^1\big)$ with $\big|e^{\mathrm{b}}(\rho)\big| = \big|\chi(S)\big|$is injective with discrete image.

Below we will give the proof of this theorem in the case in which ρ takes values in $G = \mathrm{PSU}(1, 1)$. In general this theorem follows immediately from a recent result in [**?**] stating that the real bounded Euler class of a group homomorphism is a complete invariant of semiconjugacy provided its Gromov norm equals $\frac{1}{2}$.

Proof. Let $h, \rho : \pi_1(S) \to G$ be homomorphisms, and suppose that h is a hyperbolization of finite area and ρ satisfies the hypotheses of the theorem. Then

$$\rho^*(e^{\mathrm{b}}_{\mathbb{R}}) = h^*(e^{\mathrm{b}}_{\mathbb{R}}) .$$

Consider now the exact sequence

$$\mathrm{Hom}\big(\pi_1(S), \mathbb{R}/\mathbb{Z}\big) \xrightarrow{b} \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{Z}\big) \longrightarrow \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{R}\big)$$

in order to conclude that there is a homomorphism $\chi : \pi_1(S) \to \mathbb{R}/\mathbb{Z}$ with

$$\rho(e^{\mathrm{b}}) - h^*(e^{\mathrm{b}}) = b(\chi) . \tag{4.8}$$

We now proceed to show that χ is trivial. Since $\chi|_{[\Gamma,\Gamma]} = 0$, we deduce that

$$\big(\rho|_{[\Gamma,\Gamma]}\big)^*(e^{\mathrm{b}}) = \big(h|_{[\Gamma,\Gamma]}\big)^*(e^{\mathrm{b}})$$

and hence, by Ghys' theorem, there exists a weakly monotone map $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ with

$$\varphi\big(\rho(\eta)x\big) = h(\eta)\big(\varphi(x)\big) ,$$

for all $\eta \in \big[\pi_1(S), \pi_1(S)\big]$ and all $x \in \partial\mathbb{D}$.

If now φ had a point of discontinuity, there would be a nonempty open interval in the complement of $\mathrm{im}\varphi$ and in particular $\overline{\mathrm{im}\varphi} \neq \partial\mathbb{D}$; but $h\big(\pi_1(S)\big)$ and hence $h\big([\pi_1(S), \pi_1(S)]\big)$ act minimally on $\partial\mathbb{D}$ and $\overline{\mathrm{im}\varphi}$ is invariant under the latter subgroup, which is a contradiction. Therefore φ is continuous and surjective. This implies in a straightforward way that $\rho\big([\pi_1(S), \pi_1(S)]\big)$ is a discrete subgroup of G; since the limit set of $\rho\big([\pi_1(S), \pi_1(S)]\big)$ is either a Cantor set or $\partial\mathbb{D}$, we deduce that $\rho\big([\pi_1(S), \pi_1(S)]\big)$, and hence $\rho\big(\pi_1(S)\big)$, is Zariski dense. Thus $\rho\big(\pi_1(S)\big)$ is either dense or discrete in G but since it normalizes a nontrivial discrete subgroup and since G is simple, $\rho\big(\pi_1(S)\big)$ must be discrete.

Restricting the equality $\rho(e^{\mathrm{b}}) - h^*(e^{\mathrm{b}}) = b(\chi)$ to a cyclic subgroup we deduce that

$$\mathsf{rot}\rho(\gamma) - \mathsf{rot}h(\gamma) = \chi(\gamma)$$

for all $\gamma \in \pi_1(S)$. But since $h(\gamma)$ has at least one fixed point in $\partial\mathbb{D}$, we have that $\mathsf{rot}h(\gamma) = 0$ for all $\gamma \in \pi_1(S)$, and hence

$$\mathsf{rot}\rho(\gamma) = \chi(\gamma)$$

for all $\gamma \in \pi_1(S)$. In particular, $\ker\rho \subset \ker\chi$ and hence $\rho|_{\ker\rho}$ is semiconjugate to $h|_{\ker\rho}$, which implies that $h(\ker\rho)$ has a fixed point in $\partial\mathbb{D}$. But since h is a hyperbolization and $\ker\rho$ is normal in $\pi_1(S)$, we deduce that $h(\ker\rho)$ is trivial and hence $\ker\rho$ is trivial, thus showing that ρ is injective.

Now we show that χ is trivial. Let $\gamma \in \pi_1(S)$. We distinguish then three cases:

(1) $\rho(\gamma)$ *is hyperbolic or parabolic.* Hence $\rho(\gamma)$ has a fixed point in $\partial\mathbb{D}$ and hence $\chi(\gamma) = \mathsf{rot}\rho(\gamma) = 0$;

(2) $\rho(\gamma)$ *is elliptic and* $\mathsf{rot}\rho(\gamma) = \chi(\gamma) \notin \mathbb{Q}/\mathbb{Z}$. Then $\rho(\gamma)$ is conjugate in G to an irrational rotation contradicting the fact that $\rho\big(\pi_1(S)\big)$ is discrete;

(3) $\rho(\gamma)$ *is elliptic and* $\mathsf{rot}\rho(\gamma) = \chi(\gamma) \in \mathbb{Q}/\mathbb{Z}$. Let $n \in \mathbb{N}$ be such that $n\chi(\gamma) = 0$. Hence $\mathsf{rot}\rho(\gamma^n) = \chi(\gamma^n) = 0$ and, since $\rho(\gamma^n)$ is elliptic, it is hence the identity. Thus $\gamma^n \in \ker\rho = e$ and since $\pi_1(S)$ has no torsion, $\gamma = e$.

Thus we conclude that χ is trivial, $\rho^*(e^{\mathrm{b}}) = h^*(e^{\mathrm{b}})$ and hence ρ is semiconjugate to h. ☐

Now we will put to use the explicit formula for $e^{\mathrm{b}}(\rho)$ together with the above result in order to restore, in a sense, the setting of the case of surfaces without boundary and interpret $\delta\big(\mathrm{Hyp}(S)\big)$ as a union of connected components. Namely let us introduce

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S), G\big)\\
&\qquad = \big\{\rho \in \mathrm{Hom}\big(\pi_1(S), G\big) : \rho(c_i) \text{ has at least one fixed point in } \partial\mathbb{D}\big\}\\
&\qquad\qquad = \big\{\rho : \pi_1(S) \to G : \big(\mathrm{tr}\rho(c_i)\big)^2 \geq 4, \text{ for } 1 \leq i \leq n\big\}
\end{aligned}$$

which is a real semialgebraic subset of $\mathrm{Hom}\big(\pi_1(S), G\big)$. Clearly for $\rho \in \mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S), G\big)$ we have that $\mathsf{rot}\rho(c_i) = 0$, and hence taking into account that $\rho \mapsto e^{\mathrm{b}}(\rho)$ is continuous, we have the following

Corollary 4.19. *If* $\rho \in \mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S), G\big)$*, then*

$$e^{\mathrm{b}}(\rho) = -\sum_{i=1}^{n} \tau\big(\tilde\rho(c_i)\big)$$

takes integer values and is constant on connected components.

This is in contrast with the fact that on $\mathrm{Hom}\big(\pi_1(S), G\big)$ and when S is not compact, the image of $\rho \mapsto e^{\mathrm{b}}(\rho)$ is the whole interval $\big[-|\chi(S)|, |\chi(S)|\big]$. In any case we obtain finally:

Theorem 4.20. *In the notation of Theorem 4.6, we have*

$$\delta\big(\mathrm{Hyp}(S)\big) = \left\{\rho : \pi_1(S) \to G : \sum_{i=1}^{n} \tau\big(\widetilde{\rho}(c_i)\big) = 2g - 2 + n\right\}.$$

Thus $\delta\big(\mathrm{Hyp}(S)\big)$ is a union of connected components of $\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S), G\big)$ and, in particular, a semialgebraic set.

4.6 Relation with quasimorphisms

In connection with Corollary 4.15, we would like to present a different viewpoint, coming from Ch. Bavard [?] and developed by D. Calegari [?], which relates the second bounded cohomology of $\pi_1(S)$ to the stable commutator length (scl) via quasimorphisms. In order to simplify the discussion, we assume in this section that Γ is a group with

$$\mathrm{H}^2(\Gamma, \mathbb{R}) = 0\,. \tag{4.9}$$

This applies in particular to $\Gamma = \pi_1(S)$, where S is a non-compact surface. In the notation of § 4.2, every class in $\mathrm{H}^2_{\mathrm{cb}}(\Gamma, \mathbb{R})$ admits then a representative of the form $d^1 f$, with $f \in \mathrm{C}^1(\Gamma, \mathbb{R})$. This leads us to make the following definition

Definition 4.21. A quasimorphism on Γ is a function $f : \Gamma \to \mathbb{R}$ such that

$$D(f) := \sup_{a,b \in \Gamma} \big|f(ab) - f(a) - f(b)\big| < +\infty\,,$$

and $D(f)$ is called the *defect* of f.

The vector space $Q(\Gamma, \mathbb{R})$ of all quasimorphisms on Γ contains always the subspace $\ell^\infty(\Gamma, \mathbb{R})$ of bounded functions, as well as the subspace of all homomorphisms $\mathrm{Hom}(\Gamma, \mathbb{R})$. It is clear that d^1 induces an isomorphism of vector spaces

$$Q(\Gamma, \mathbb{R})/\ell^\infty(\Gamma, \mathbb{R}) \oplus \mathrm{Hom}(\Gamma, \mathbb{R}) \xrightarrow{\cong} \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

and the Gromov norm $\|\alpha\|$ of a class $\alpha \in \mathrm{H}^2_{\mathrm{cb}}(\Gamma, \mathbb{R})$ is given by

$$\|\alpha\| = \inf\big\{D(f) : [d^1 f] = \alpha,\ f \in Q(\Gamma, \mathbb{R})\big\}\,.$$

There are various ways of choosing a "special" quasimorphism representing a given class. One is to fix a finite symmetric generating set of Γ and establish the existence of a harmonic representative F for each class $\alpha \in \mathrm{H}^2_{\mathrm{cb}}(\Gamma, \mathbb{R})$: this quasimorphism then minimizes the defect, that is

$$\|\alpha\| = D(f)\,,$$

([**?**]; see also [**?**] for a recent application).

Another way to get a representative, this time canonical, is to consider homogeneous quasimorphisms, that is quasimorphisms satisfying the condition

$$f(x^n) = nf(x) \qquad \text{for all } n \in \mathbb{Z},\ , x \in \Gamma\,.$$

A simple argument (see [**?**]), shows that for $f \in Q(\Gamma, \mathbb{R})$, the limit

$$F(x) := \lim_{n\to\infty} \frac{f(x^n)}{n}$$

exists for all x and gives a homogeneous quasimorphism F with the property that $f - F \in \ell^\infty(\Gamma, \mathbb{R})$. Denoting by $Q_{\mathrm{h}}(\Gamma, \mathbb{R})$ the subspace of homogeneous quasimorphisms, it is thus clear that d^1 induces an isomorphism of vector spaces

$$Q_{\mathrm{h}}(\Gamma, \mathbb{R})/\mathrm{Hom}(\Gamma, \mathbb{R}) \xrightarrow{\cong} \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})\,.$$

Obviously the defect D gives a norm on the left hand side, while the right hand side is endowed with the (canonical) Gromov norm. There is then the following non-trivial relation between these two norms:

Theorem 4.22 ([**?**]). *For every homogeneous quasimorphism f on Γ,*

$$\frac{1}{2}D(f) \le \left\|d^1 f\right\| \le D(f)\,.$$

This inequality is based on the following relation between defect of a homogeneous quasimorphism f and commutators

$$D(f) = \sup_{a,b\in\Gamma} \left|f\big([a,b]\big)\right|\,.$$

Example 4.23. (1) The function $\tilde{r} : \widetilde{\mathrm{PU}(1,1)} \to \mathbb{R}$ in § 3.8 leading to Milnor's inequality is a quasimorphism.

(2) The translation number $\tau : \mathcal{H}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ defined in § 4.2 is a continuous quasimorphism. Identifying $\widetilde{\mathrm{PU}(1,1)}$ with a subgroup of $\mathcal{H}^+_{\mathbb{Z}}(\mathbb{R})$, we have

$$D(\tau) = D\big(\tau|_{\widetilde{\mathrm{PU}(1,1)}}\big) = 1\,.$$

We let as usual $[\Gamma, \Gamma]$ denote the subgroup of Γ generated by the set $\big\{[x,y] : x, y \in \Gamma\big\}$ of all commutators and let $\mathrm{cl}(\gamma)$ denote the word length with respect to this generating set. The stable commutator length is defined by

$$\mathrm{scl}(\gamma) := \lim_{n\to\infty} \frac{\mathrm{cl}(\gamma^n)}{n},$$

where the existence of the limit follows from the fact that the map $n \mapsto \|\gamma^n\|$ is subadditive.

The following result then puts the geometry of the commutator subgroup $[\Gamma, \Gamma]$ in direct relation with quasimorphisms, in fact, homogeneous ones:

Theorem 4.24 ([**?**]). *For every $\gamma \in [\Gamma, \Gamma]$, we have*

$$\mathrm{scl}(\gamma) = \sup\left\{ \frac{|\varphi(\gamma)|}{2D(\varphi)} : \varphi \in Q_{\mathrm{h}}(\Gamma, \mathbb{R}) \right\}.$$

Now every element $\gamma \in [\Gamma, \Gamma]$, seen as a 1-chain, is an element in the vector space $\mathrm{B}_1(\Gamma, \mathbb{R})$ of 1-boundaries in the bar resolution defining group homology. In [**?**] the author extends scl to a seminorm on the vector space $\mathrm{B}_1(\Gamma, \mathbb{R})$ and obtains an extension of the Bavard duality theorem. We refer to the monograph [**?**] for the details and interesting developments, and proceed directly to state a corollary of the main result in [**?**]. If now $\Gamma = \pi_1(S)$, where S is a non-compact surface and

$$\pi_1(S) = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n : \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = e \right\rangle,$$

then $\sum_{i=1}^n c_i$ is clearly in $\mathrm{B}_1(\Gamma, \mathbb{R})$ and

Proposition 4.25 ([**?**]). *With the above assumptions and notation*

$$\mathrm{scl}\left(\sum_{i=1}^{n} c_i \right) = \frac{-\chi(S)}{2}.$$

If now $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{PU}(1,1)}$ is the lift of a fixed hyperbolization ρ of S, we can pullback the translation quasimorphism and obtain $\mathsf{rot}_\rho := \tau \circ \rho$, which defines on Γ a homogeneous quasimorphism taking values in $\mathbb{Z}$; in fact, rot_ρ changes by an element of $\mathrm{Hom}(\Gamma, \mathbb{Z})$ if one takes a different lift of ρ. A corollary to the main result in [**?**] is then the following

Theorem 4.26 ([**?**]). *For any homogeneous quasimorphism f on Γ, we have the inequality*

$$\left| \sum_{i=1}^{n} f(c_i) \right| \leq D(f)\, |\chi(S)|,$$

with equality if and only if f differs from rot_ρ by an element of $\mathrm{Hom}(\Gamma, \mathbb{R})$.

The relation with Corollary 4.15 is the following. Given a class $\alpha \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(S), \mathbb{R}\big) = \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$, let $d^1 F$ be a representative of α, where $F : \Gamma \to \mathbb{R}$ is a homogeneous quasimorphism. Then:

Lemma 4.27. *If $f_{\mathbb{R}} : \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$ is the isomorphism in § 4.3, then*

$$\big\langle f_{\mathbb{R}}^{-1}(\alpha), [\Sigma, \partial\Sigma] \big\rangle = -\sum_{i=1}^{n} F(c_i)\,.$$

Taking into account the inequality $D(F) \leq 2\|\alpha\|$ in Theorem 4.22, one sees that Theorem 4.26 implies Corollary 4.15. For our purposes however, both results contain the same information as far as the characterization of equality is concerned. An intriguing question in this context is whether if Γ is a finitely generated group, $F : \Gamma \to \mathbb{R}$ a homogeneous quasimorphism and $[d^1 F] \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ the bounded class it defines, then the equality

$$\big\|[d^1 F]\big\| = \frac{1}{2} D(F)$$

holds.

It would suffice to show this for nonabelian free groups; in this case all known examples of quasimorphisms satisfy the above equality ([**?**, **?**]).

Part II
Higher Teichmüller Spaces

Our considerations started with the study of how the set of hyperbolic structures on a surface S, of finite topological type, is related to the set of representations of $\pi_1(S)$ into $G = \mathrm{PSU}(1,1)$; the problem of characterizing the image of the map

$$\delta : \mathrm{Hyp}(S) \to \mathrm{Hom}\big(\pi_1(S), G\big)$$

constructed in § 2.1 led us to introduce an invariant (see § 3)

$$\mathrm{T} : \mathrm{Hom}\big(\pi_1(S), G\big) \to \mathbb{R}$$

with various incarnations and whose maximal fiber $\mathrm{T}^{-1}\big(-\chi(S)\big)$ coincides with the image of δ. As indicated in § 3 this invariant can be defined for homomorphisms from $\pi_1(S)$ with values in any Lie group G. This leads to the vague question of how much of the "PSU(1,1) picture" generalizes to an arbitrary Lie group G. Interestingly enough, there are two classes of (semi)simple Lie groups for which one can make this question precise in defining, in very different ways, components (or specific subsets when S is not compact) of

$\mathrm{Hom}\big(\pi_1(S), G\big)$ which should play the role of Teichmüller space: those two classes are on the one hand the *split real groups* (Definition 6.1) and, on the other hand, the *Lie groups of Hermitian type* (Definition 5.1).

Because of the various properties these connected components share with Teichmüller space, we will call them *higher Teichmüller spaces.*

We will now describe in some detail the class given by Lie groups of Hermitian type, and the corresponding subset, namely the *space of maximal representations*

$$\mathrm{Hom}_{max}\big(\pi_1(S), G\big) \subset \mathrm{Hom}\big(\pi_1(S), G\big)\,.$$

We will discuss the other class in § 6.

5 Maximal representations into Lie groups of Hermitian type

Definition 5.1. A Lie group G is of *Hermitian type* if it is connected semisimple with finite center without compact factors and the associated symmetric space X has a G-invariant complex structure.

In this setting we will be able to define a *Toledo invariant* (and a *bounded Toledo invariant* when S is not compact) satisfying a Milnor–Wood type inequality and this will lead us to consider the set $\mathrm{Hom}_{\max}\big(\pi_1(S), G\big)$ of maximal representations into G.

In this section we will describe a certain number of fundamental geometric properties of maximal representations and we will also have something to say about the structure of the set of such representations, all this in the context where S is of finite topological type.

5.1 The cohomological framework

Let G be of Hermitian type, X the associated symmetric space,

$$\langle\,\cdot\,,\,\cdot\,\rangle : X \to \mathrm{Sym}(TX)$$

the Riemannian metric and

$$J : X \to \mathrm{End}(TX)$$

the complex structure. Then $\langle J\cdot\,,\,\cdot\,\rangle$ defines a G-invariant Hermitian metric whose imaginary part ω_X is a real G-invariant 2-form on X. By a general lemma of E. Cartan, the complex of G-invariant forms on any symmetric space

X consists of closed forms. Thus ω_X is closed and the above Hermitian metric is Kähler.

Given S compact and $\rho : \pi_1(S) \to G$ a homomorphism, we can then proceed as in § 3.5 and with the help of a smooth equivariant map

$$f : \widetilde{S} = D \to X$$

define the *Toledo invariant* of ρ,

$$\mathrm{T}(\rho) = \frac{1}{2\pi} \int_S f^*(\omega_X)\,.$$

For the cohomological interpretation we can proceed as for $\mathrm{PSU}(1,1)$, namely, since G acts properly on X, we have the Van Est isomorphism

$$\Omega^2(X)^G \cong \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R})\,.$$

To the Kähler form ω_X we associate the continuous (inhomogeneous) 2-cocycle

$$c(g_1, g_2) = \frac{1}{2\pi} \int_{\Delta(x_0, g_1 x_0, g_1 g_2 x_0)} \omega_X$$

where $x_0 \in X$ is a fixed base point and $\Delta(x, y, z)$ denotes a smooth simplex with geodesic sides connecting the vertices x, y, z; of course such a simplex is not unique, but any two such simplices with fixed vertices have the same boundary and hence, since ω_X is closed, by Stokes' theorem the integral does not depend on it. We denote by $\kappa_G \in \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R})$ the class defined by c and call it the Kähler class. Then, given $\rho : \pi_1(S) \to G$, we have the equality

$$\mathrm{T}(\rho) = \big\langle \rho^*(\kappa_G), [S] \big\rangle\,. \tag{5.1}$$

In fact, the cocycle c defining κ_G turns out to be bounded; this is a consequence of a precise study of the Kähler area of triangles with geodesic sides, due to Domic and Toledo [**?**], and Clerc and Orsted [**?**], and which we will describe later in more details. Here we deduce that c defines a bounded class $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$, called the *bounded Kähler class* and we have the following

Theorem 5.2 (Clerc–Orsted, [**?**]). *Assume that the metric on X is normalized so that its minimal holomorphic sectional curvature is -1. Then the value of the Gromov norm of the bounded Kähler classKähler class!bounded is*

$$\|\kappa_G^{\mathrm{b}}\| = \frac{1}{2} \mathrm{rank}_X\,,$$

where rank_X *is the rank of the symmetric space X.*

When X is irreducible, $\Omega^2(X)^G = \mathbb{R}\omega_X$ and hence the comparison map

$$\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R}) \tag{5.2}$$

is surjective; in general, $\Omega^2(X)^G$ is spanned by the pullbacks under the projections of the Kähler forms of the irreducible factors of X and thus the comparison map (5.2) is surjective as well; since G has finite center (see Definition 5.1), it is injective in all cases.

Let now S be of finite topological type realized as the interior of an oriented compact surface Σ with boundary and $\rho : \pi_1(S) \to G$ a homomorphism. Then we can proceed as in the definition of the bounded Euler number and, in the notation of § 4.3, set

$$\mathrm{T}(\rho) := \big\langle f_{\mathbb{R}}^{-1}\big(\rho^*(\kappa_G^{\mathrm{b}})\big), [\Sigma, \partial\Sigma]\big\rangle\,,$$

where we recall that

$$f_{\mathbb{R}} : \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$$

is the isomorphism given by the natural inclusion.

For $G = \mathrm{PSU}(1,1)$ we saw in § 4 that the Kähler class and the bounded Kähler class come from a (bounded) integral class, namely the (bounded) Euler class, and this turned out to be essential in order to obtain explicit formulas for the Toledo invariant needed in particular when S is noncompact. In the case of $\mathrm{PSU}(1,1)$ we have at our disposal the relation with rotation numbers and the translation quasimorphism; these structures were given to us for free from the fact that $\mathrm{PSU}(1,1)$ acts by orientation preserving homeomorphisms of the circle. For G of Hermitian type one can construct (more sophisticated) analogues of each of these objects; in particular the integral structure on (bounded) cohomology and the analogues of rotation number can be described quite explicitly and this is what we turn to now.

We denote by $\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$ and $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z})$ the (bounded) Borel cohomology, which is defined by considering the complex of (bounded) Borel functions from G^n to $\mathbb{Z}$. We refer the reader to [**?**, §§2.3 and 7.2] for more details.

We have the following

Lemma 5.3. *The comparison map*

$$\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$$

is an isomorphism.

Proof. The long exact sequences in cohomology associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

read as

$$
\begin{tikzcd}
0 = \mathrm{Hom}_{\mathrm{c}}(G, \mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[d, "="] & \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \arrow[r] \arrow[d] & \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}/\mathbb{Z}) \arrow[d, "="] \\
0 = \mathrm{Hom}_{\mathrm{c}}(G, \mathbb{R}/\mathbb{Z}) \arrow[r] & \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z}) \arrow[r] & \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R}) \arrow[r] & \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R}/\mathbb{Z})\,.
\end{tikzcd}
$$

The fact that the third vertical arrow is an isomorphism and the 5-term lemma allow to conclude. □

Actually we will turn to an explicit implementation of the isomorphism in Lemma 5.3. This will also give an alternative treatment of some material in [**?**, § 7]. We start with the observation that the isomorphism

$$\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z}) \cong \mathrm{Hom}\big(\pi_1(G), \mathbb{Z}\big) \tag{5.3}$$

is valid for any connected Lie group G; this follows easily from the fact that $\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$ classifies equivalence classes of topological central extensions of G by $\mathbb{Z}$, together with some covering theory; moreover this isomorphism is natural. If now $K < G$ is a maximal compact subgroup, then by the Iwasawa decomposition $K \hookrightarrow G$ is a homotopy equivalence and hence $\pi_1(K) = \pi_1(G)$; this, together with (5.3), implies that the restriction map

$$\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(K, \mathbb{Z})$$

is an isomorphism. Taking into account that continuous cohomology of compact groups with real coefficients is trivial, we obtain, considering the long exact sequence associated to the coefficient sequence, that

$$\mathrm{Hom}_{\mathrm{c}}(K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\epsilon} \mathrm{H}^2_{\mathrm{c}}(K, \mathbb{Z})$$

is an isomorphism. As a result we obtain an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{c}}(K, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z}) \cong \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z}) \\ \chi & \longmapsto & \kappa \end{array}$$

and we say that κ corresponds to χ and viceversa.

Now we will assume that G is real algebraic and semisimple. In this case we have at our disposal the *refined Jordan decomposition* namely every $g \in G$ is a product

$$g = g_e g_h g_u$$

of pairwise commuting elements, where g_e is contained in a compact subgroup, g_h is in the connected component of the identity of a maximal real split torus and g_u is unipotent. Given then

$$\chi : K \to \mathbb{R}/\mathbb{Z}$$

a continuous homomorphism and denoting by $C(h)$ the conjugacy class of an element $h \in G$, define for $g \in G$

$$\chi_{\text{ext}}(g) := \chi\big(C(g_e) \cap K\big)$$

Then, according to [**?**], χ_{ext} is indeed a well defined *continuous* class function on G extending χ; moreover it satisfies

$$\chi_{\text{ext}}(g^n) = n\chi_{\text{ext}}(g)$$

for all $n \in \mathbb{Z}$ and $g \in G$. Let $\widetilde{\chi_{\text{ext}}} : \widetilde{G} \to \mathbb{R}$ denote the unique continuous lift to the universal covering $\widetilde{G}$ of G, vanishing at e; finally we denote by

$$\chi_* : \pi_1(G) = \pi_1(K) \to \mathbb{Z}$$

the morphism on the level of fundamental groups induced by χ. The following result then gives a precise description of the isomorphism in Lemma 5.3.

Theorem 5.4 ([**?**]). (1) *The function $\widetilde{\chi_{\text{ext}}} : \widetilde{G} \to \mathbb{R}$ is a homogeneous quasimorphism and the map*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{c}}(K, \mathbb{R}/\mathbb{Z}) &\to \mathrm{QH}^{\mathrm{h}}_{\mathrm{c}}(\widetilde{G}, \mathbb{R})_{\mathbb{Z}} \\ \chi \quad &\longmapsto \quad \widetilde{\chi_{\text{ext}}} \end{aligned}$$

establishes an isomorphism with the space of continuous homogeneous quasimorphisms sending $\pi_1(G)$ to $\mathbb{Z}$; in fact

$$\chi_* = \widetilde{\chi_{\text{ext}}}|_{\pi_1(G)}\,.$$

(2) *If $\kappa \in \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$ is the class corresponding to χ, and denote by $[\cdot]$ the integer part, then*

$$(g, h) \to \big[\widetilde{\chi_{\text{ext}}}(gh)\big] - \big[\widetilde{\chi_{\text{ext}}}(g)\big] - \big[\widetilde{\chi_{\text{ext}}}(h)\big]$$

descends to a well defined $\mathbb{Z}$-valued bounded cocycle on $G\times G$ representing the class κ.

Remark 5.5. The only not obvious statement in Theorem 5.4 is the assertion that $\widetilde{\chi_{\text{ext}}}$ is a quasimorphism; in fact, this follows easily from the fact (proved in [**?**]) that for every $k \in \mathbb{N}$, $\widetilde{\chi_{\text{ext}}}$ is bounded on the elements in $\widetilde{G}$ which are products of k commutators. We mention in addition that Theorem 5.4 is also consequence of a different and more general approach taken in [**?**].

In our context, *rotation numbers* arise in the following way. Let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z})$ and, for every $g \in G$ consider, as in § 4, the homomorphism

$$\begin{aligned} h_g : \mathbb{Z} &\to G \\ n &\mapsto g^n \end{aligned}$$

by means of which we obtain a bounded integral class $h_g^*(\kappa) \in \mathrm{H}^2_{\mathrm{b}}(\mathbb{Z},\mathbb{Z})$ and finally by means of the canonical isomorphism

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^2_{\mathrm{b}}(\mathbb{Z},\mathbb{Z}) \longrightarrow 0$$

and element in $\mathbb{R}/\mathbb{Z}$,

$$\mathsf{rot}_\kappa(g) := \delta^{-1}\big(h_g^*(\kappa)\big)(1)\,.$$

From standard homological considerations using the naturality of all constructions involved, one can deduce that if $\chi \in \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z})$ corresponds to the class $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{Z})$ then

$$\mathsf{rot}_\kappa(g) = \widetilde{\chi_{\mathrm{ext}}}(\tilde g) \mod \mathbb{Z}$$

where $\tilde g \in \widetilde G$ is any lift of $g \in G$. Now we fix $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and let as before S be a surface of finite topological type realized as the interior of Σ. Given a homomorphism $\rho : \pi_1(S) \to G$, we define then

$$\mathrm{T}_\kappa(\Sigma,\rho) = \big\langle f_{\mathbb{R}}^{-1}\big(\rho^*(\kappa)\big), [\Sigma,\partial\Sigma]\big\rangle\,.$$

Of course if $\partial\Sigma = \emptyset$, that is if S is compact, the definition takes the simplified form

$$\mathrm{T}_\kappa(S,\rho) = \big\langle \rho^*(\kappa), [S]\big\rangle\,.$$

Then we have the following:

Theorem 5.6 ([?]). *Let S be of finite topological type with presentation*

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n : \prod_{i=1}^g [a_i,b_i] \prod_{j=1}^n c_j = e \right\rangle$$

and $\rho : \pi_1(S) \to G$ a homomorphism. Let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and $\chi \in \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z})$ the corresponding homomorphism.

(1) *Assume that S is compact. Then*

$$\mathrm{T}_\kappa(S,\rho) = -\chi_*\left(\prod_{i=1}^g [\rho(a_i),\rho(b_i)]^{\tilde{}}\right),$$

where $\chi_* : \pi_1(G) = \pi_1(K) \to \mathbb{Z}$ is the morphism induced by χ and $[\cdot,\cdot]^{\tilde{}}$ is the commutator map introduced in § 4.1 .

(2) *Assume that S is not compact. If $\widetilde\rho : \pi_1(S) \to \widetilde G$ is a lift of ρ to $\widetilde G$, then*

$$\mathrm{T}_\kappa(\Sigma,\rho) = -\sum_{j=1}^n \widetilde{\chi_{\mathrm{ext}}}\big(\widetilde\rho(c_j)\big)\,.$$

5.2 Maximal representations and basic geometric properties

Let S be of finite topological type and $\rho : \pi_1(S) \to G$ a homomorphism. Based on our considerations in the case of $\mathrm{PSU}(1,1)$ in § 4 and the computation of the Gromov norm of the bounded Kähler class $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ by Clerc and Orsted, we can conclude

Corollary 5.7. *The Toledo invariant* $\mathrm{T}(\rho)$ *defined in* (5.1) *satisfies the inequality*

$$\big|\,\mathrm{T}(\rho)\big| \leq \mathrm{rank}_X \big|\chi(S)\big|$$

with equality if and only if

$$\rho^*(\kappa_G^{\mathrm{b}}) = \pm \mathrm{rank}_X \kappa_{S,\mathbb{R}}^{\mathrm{b}}\,,$$

where $\kappa_{S,\mathbb{R}}^{\mathrm{b}}$ *is the bounded real class defined in the context of Corollary 4.15.*

Proof. Corollary 4.15 with $\alpha = \rho^*(\kappa_G^{\mathrm{b}})$ implies that

$$\big|\,\mathrm{T}(\rho)\big| \leq 2\big\|\rho^*(\kappa_G^{\mathrm{b}})\big\|\,\big|\chi(S)\big|\,.$$

Using the fact that ρ^* is norm decreasing and the value of $\|\kappa_G^{\mathrm{b}}\|$ (see Theorem 5.2) we obtain that

$$\big|\,\mathrm{T}(\rho)\big| \leq \mathrm{rank}_X \big|\chi(S)\big|\,.$$

Equality implies that

$$\big|\,\mathrm{T}(\rho)\big| = 2\big\|\rho^*(\kappa_G^{\mathrm{b}})\big\|\,\big|\chi(S)\big| = \mathrm{rank}_X \big|\chi(S)\big|\,.$$

It follows from the first equality that

$$\rho^*(\kappa_G^{\mathrm{b}}) = \lambda\, \kappa_{S,\mathbb{R}}^{\mathrm{b}}$$

for some $\lambda \in \mathbb{R}$ (Corollary 4.15), and from the second equality that $|\lambda| = \mathrm{rank}_X$. □

Thus we now introduce the following

Definition 5.8. A representation $\rho : \pi_1(S) \to G$ is called *maximal* if

$$\mathrm{T}(\rho) = \mathrm{rank}_X \big|\chi(S)\big|\,.$$

The basic example of a family of maximal representations is obtained via a geometric fact of fundamental importance called the polydisk theorem. Recall that in a symmetric space of noncompact type there are maximal flat subspaces, they are all G-conjugate and their common dimension is the rank $r = \mathrm{rank}_X$ of X. When X is Hermitian symmetric, a geometric version of a fundamental result of Harish-Chandra says that the complexification of a maximal flat is a maximal polydisk, or, in other words, that the image under the exponential map of the complexified tangent space of a maximal flat is a totally geodesic holomorphic copy of $\mathbb{D}^r$. The fact that the normalized metric on X is taken to be of minimal holomorphic sectional curvature -1 is equivalent to the property that any maximal polydisk embedding

$$\varphi : \mathbb{D}^r \to X$$

is isometric. To such a map corresponds a homomorphism

$$\Phi : \mathrm{SU}(1,1)^r \to G$$

with respect to which φ is equivariant. Given then r hyperbolizations $\rho_1, \dots, \rho_r : \pi_1(S) \to \mathrm{SU}(1,1)$, the homomorphism

$$\rho(\gamma) = \Phi\big(\rho_1(\gamma), \dots, \rho_r(\gamma)\big)$$

is then maximal. The main point is the fact that

$$\Phi^*(\kappa_G^{\mathrm{b}}) = \kappa_{\mathrm{SU}(1,1)^r}^{\mathrm{b}} \,.$$

This follows from a Lie algebra computation which gives

$$\Phi^*(\omega_X) = \omega_{\mathbb{D}^r}$$

and the naturality of the isomorphisms

$$\Omega^2(X)^G \cong \mathrm{H}_{\mathrm{c}}^2(G, \mathbb{R}) \cong \mathrm{H}_{\mathrm{cb}}^2(G, \mathbb{R}) \,.$$

Observe that we could have taken an antiholomorphic embedding $\overline{\varphi} : \mathbb{D}^r \to X$, in which case

$$\overline{\rho}(\gamma) = \overline{\Phi}\big(\rho_1(\gamma), \dots, \rho_r(\gamma)\big)$$

has then

$$\mathrm{T}(\overline{\rho}) = -\mathrm{rank}_X \, \big|\chi(S)\big|$$

as Toledo invariant.

Now, the class κ_G^{b} is not always integral, but there is a specific natural number n_X depending on the root system of G such that $\kappa = n_X \kappa_G^{\mathrm{b}}$ is an integral class. From this and Theorem 5.6 we deduce:

Corollary 5.9 ([**?**]). *The map $\rho \mapsto \mathrm{T}(\rho)$ on $\mathrm{Hom}\big(\pi_1(S), G\big)$ is continuous.*

(1) *If S is compact, it takes values in $\frac{1}{n_X}\mathbb{Z}$ and is constant on connected components.*

(2) *If S is not compact, its range is*

$$\big[-\mathrm{rank}_X \big|\chi(S)\big|, \mathrm{rank}_X \big|\chi(S)\big|\big] \,.$$

Proof. The continuity follows from the formulas in Theorem 5.6. Then (1) is clear and (2) follows from the fact that, since $\pi_1(S)$ is free, then $\mathrm{Hom}\big(\pi_1(S), G\big) \cong G^{2g+n-1}$ is connected and therefore the intermediate value theorem implies the statement. □

In the study of representations of $\pi_1(S)$, where S is of finite topological type, we were several times led to consider the corresponding "completed" compact surface Σ with boundary, for which of course we have $\pi_1(S) = \pi_1(\Sigma)$. In the light of the Fenchel–Nielsen approach to Teichmüller theory, it is natural to ask what happens to Toledo invariants when one glues together two surfaces along a component of their boundary. The answer is given by the following

Proposition 5.10 ([**?**]). *Let Σ be a compact oriented surface with boundary and $\rho : \pi_1(\Sigma) \to G$ a homomorphism.*

(1) *If $\Sigma = \Sigma_1 \cup_C \Sigma_2$ is the connected sum of two subsurfaces Σ_1 and Σ_2 along a separating loop C, then*
$$\mathrm{T}(\Sigma, \rho) = \mathrm{T}(\Sigma_1, \rho_1) + \mathrm{T}(\Sigma_2, \rho_2)\,,$$
where ρ_i is the restriction of ρ to $\pi_1(\Sigma_i)$.

(2) *If Σ' is the surface obtained by cutting Σ along a non-separating loop C and $i : \Sigma' \to \Sigma$ is the canonical map, then*
$$\mathrm{T}(\Sigma', \rho\, i_*) = \mathrm{T}(\Sigma, \rho)\,.$$

Remark 5.11. This result holds also for the invariants $\mathrm{T}_\kappa(\Sigma, \rho)$ for $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z})$ introduced in §5.1.

In the situation of Proposition 5.10, if we take into account that the Euler characteristic is additive under connected sum, we obtain that $\rho \in \mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ is maximal if and only if $\rho_i \in \mathrm{Hom}\big(\pi_1(\Sigma_i), G\big)$ are maximal for $i = 1, 2$.

Concerning the geometric properties of maximal representations, the first fundamental result is:

Theorem 5.12 ([**?**]). *Maximal representations are injective and with discrete image.*

Remark 5.13. As soon as the Lie group G is not locally isomorphic to $\mathrm{PSU}(1,1)$, there are injective representations with discrete image that are not maximal.

The proof of this result relies on the fact (see Corollary 5.7) that if $\rho : \pi_1(S) \to G$ is maximal and rank_X is the real rank of G, then
$$\rho^*(\kappa_G^{\mathrm{b}}) = \mathrm{rank}_X\, \kappa_{S,\mathbb{R}}^{\mathrm{b}}\,.$$

The proof proceeds then by an appropriate reinterpretations of this equality in terms of homogeneous quasimorphisms; this approach works for a larger class of representations, namely the class of *causal* representations which satisfy that $\rho^*(\kappa_G^{\mathrm{b}}) = \lambda\,\kappa_{S,\mathbb{R}}^{\mathrm{b}}$ for some $\lambda \neq 0$; details will appear in [**?**].

While Theorem 5.12 holds for all surfaces of finite type, one has a substantially stronger result when S is compact, namely:

Theorem 5.14 ([**?**]). *Let S be a compact surface, $\rho : \pi_1(S) \to G$ a maximal representation and X the symmetric space associated to G. Then there are constants $A > 0$ and $B \geq 0$ such that*

$$A^{-1}\|\gamma\| - B \leq d_X\big(\rho(\gamma)x_0, x_0\big) \leq A\|\gamma\| + B\,,$$

where $x_0 \in X$ is a basepoint and $\|\cdot\|$ is a word metric on $\pi_1(S)$.

This result is a consequence of the fact that maximal representations are *Anosov* (see § 8); the proof of this fact uses the structure theorem Theorem 5.15 presented in the next section.

5.3 The structure theorem and tube type domains

Almost from its beginning in the 80's, research on maximal representations was driven by "irreducibility" questions. For instance D. Toledo, using tools from the Gromov–Thurston proof of Mostow rigidity for real hyperbolic manifolds, showed in [**?**] that a maximal representation from a compact surface group into $\mathrm{SU}(n,1)$ leaves invariant a complex geodesic, or equivalently its image is contained in a conjugate of $\mathrm{S}\big(\mathrm{U}(n-1)\times\mathrm{U}(1,1)\big)$. Then L. Hernández showed in [**?**] that if $\mathrm{SU}(n,2)$ (for $n \geq 2$) is the target group, the image must be contained in a conjugate of $\mathrm{S}\big(\mathrm{U}(n-2)\times\mathrm{U}(2,2)\big)$. In [**?**] S. Bradlow, O. García-Prada and P. Gothen then showed that a reductive maximal representation with target $\mathrm{SU}(p,q)$, with $p \leq q$, is contained in a conjugate of $\mathrm{S}\big(\mathrm{U}(p,p)\times\mathrm{U}(q-p)\big)$ using methods from the theory of Higgs bundles.

In its most general form the problem presents itself naturally in the following way: given a maximal representations $\rho : \pi_1(S) \to G$ where $G := \mathbf{G}(\mathbb{R})^\circ$ consists of the real points of the connected component of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$, determine the Zariski closure $\mathbf{L} := \overline{\rho\big(\pi_1(S)\big)}^{\mathrm{Z}}$ of the image of ρ. In [**?**] we gave a complete answer to this question and most of this section is devoted to the description of the result and the ingredients of the proof.

Recall that every Hermitian symmetric space X (of noncompact type) is biholomorphic to a bounded domain $\mathcal{D} \subset \mathbb{C}^n$. While this is the natural generalization of the Poincaré disk, the question of the generalization of the upper half plane leads to the notion of tube type domain. We say that X (or $\mathcal{D}$)

is of *tube type* if it is biholomorphic to a domain of the form $V + \imath\Omega$, where V is a real vector space and $\Omega \subset V$ is an open convex proper cone in V. The groups corresponding to irreducible Hermitian symmetric spaces of tube type are $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SU}(p,p)$, $\mathrm{SO}^*(2n)$ (for n even), $\mathrm{SO}(2,n)$ and one of the two exceptional ones. There are many known characterizations of tube type domains, mainly in terms of special geometric structures, or the topology of their Shilov boundary, and we will add a new one in Theorem 5.28.

With the notion of tube type at hand, the structure of the Zariski closure of the image of a maximal representation is described by the following

Theorem 5.15 ([**?**]). *Let $G := \mathbf{G}(\mathbb{R})^\circ$ be a Lie group of Hermitian type with associate symmetric space X. Let $\rho : \pi_1(S) \to G$ be a maximal representation and $\mathbf{L} := \overline{\rho\big(\pi_1(S)\big)}^{\mathrm{Z}}$ the Zariski closure of its image. Then:*

1. *the Lie group $L := \mathbf{L}(\mathbb{R})^\circ$ is reductive with compact centralizer in G;*
2. *the semisimple part of L is of Hermitian type;*
3. *the Hermitian symmetric space $\mathcal{Y}$ associated to L is of tube type and the totally geodesic embedding $\mathcal{Y} \hookrightarrow X$ is tight.*

In statement (3) the embedding $\mathcal{Y} \hookrightarrow X$ is not necessarily holomorphic but it is tight, a notion involving the area of geodesic triangles in $\mathcal{Y}$ and X with respect to ω_X. We will elaborate on this notion in § 5.4.

In order to relate this result to the "irreducibility question" described above, we recall that in every Hermitian symmetric space X, maximal tube type subdomains exists, they are all conjugate and of rank equal to the rank of X. We have then:

Corollary 5.16 ([**?**]). *Let $\rho : \pi_1(S) \to G$ be a maximal representation. Then there is a maximal tube type subdomain which is $\rho\big(\pi_1(S)\big)$-invariant.*

A special case of Theorem 5.15 is when ρ has Zariski dense image in $\mathbf{G}$, in which case $\mathcal{Y} = X$ and hence X is of tube type. This result is optimal, in the sense that every tube type domain admits a maximal representation with Zariski dense image. In order to be more specific, we recall that a diagonal disk in X is a holomorphic totally geodesic embedding

$$d : \mathbb{D} \to X$$

obtained as the composition of a diagonal embedding $\mathbb{D} \to \mathbb{D}^r$ (where $r = \mathrm{rank}_X$) and a maximal polydisk embedding $\mathbb{D}^r \to X$. Let

$$\Delta : \mathrm{SU}(1,1) \to G$$

denote the homomorphism corresponding to d. Let now $\rho : \pi_1(S) \to \mathrm{SU}(1,1)$ be a hyperbolization. Then we have the following

Theorem 5.17 ([**?**]). *Assume that X is of tube type. Then there exists a path of homomorphisms*

$$\rho_t : \pi_1(S) \to G\,,$$

for $t \geq 0$, such that

(1) *ρ_t is maximal for all $t \geq 0$ and $\rho_0 = \Delta \circ \rho$;*

(2) *ρ_t has Zariski dense image for $t > 0$.*

Remark 5.18. Using the structure theory developed in [**?**] which is described in the following section, Kim and Pansu [**?**] recently showed that for fundamental groups of compact surfaces the global rigidity result for maximal representation into non-tube type Hermitian Lie groups given by Theorem 5.15 arises only in this context. For a precise statement of their result see [**?**, Corollary 2].

In the next section we describe the various ingredients entering the proof of the structure theorem (Theorem 5.15).

5.4 Tight homomorphisms, triangles and the Hermitian triple product

Maximal representations are a special case of more general type of homomorphism, namely *tight homomorphisms*; they are defined on any locally compact group L and take values in a Lie group of Hermitian type G.

Definition 5.19. A continuous homomorphism $\rho : L \to G$ is *tight* if

$$\left\|\rho^*(\kappa_G^{\mathrm{b}})\right\| = \|\kappa_G^{\mathrm{b}}\|\,.$$

By inspecting the proof of the Milnor–Wood type inequality in Corollary 5.7, one verifies easily that maximal representations are tight.

In the case in which also L is a Lie group of Hermitian type and $\mathcal{Y}$ is the associated symmetric space, then a continuous homomorphism $h : L \to G$ gives rise to a totally geodesic map $f : \mathcal{Y} \to X$. The geometric condition on f for ρ to be tight is then

$$\sup_{\Delta \subset \mathcal{Y}} \int_\Delta f^*\omega_X = \sup_{\Delta \subset X} \int_\Delta \omega_X \tag{5.4}$$

and we call f tight if it satisfies (5.4). A useful observation is that if $\rho : \pi_1(S) \to L$ is a homomorphism such that $h \circ \rho$ is maximal, then h is tight. For the converse, we need to introduce an additional notion. Namely, recall that the space

$$\mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R}) \cong \mathrm{H}^2_{\mathrm{c}}(L, \mathbb{R}) \cong \Omega^2(\mathcal{Y})^L$$

is generated as a vector space by the pullback to $\mathcal{Y}$ of the Kähler form of the irreducible factors of $\mathcal{Y}$. The open cone generated by the linear combination with strictly positive coefficients of these forms is called the *cone of positive Kähler classes* and denoted by $\mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R})_{>0}$.

Definition 5.20. A continuous homomorphism $h : L \to G$ is positive if

$$h^*(\kappa_G^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R})_{>0}\,.$$

With these definitions we have then:

Proposition 5.21 ([?]). *If $\rho : \pi_1(S) \to L$ is maximal and $h : L \to G$ is tight and positive, then $h \circ \rho : \pi_1(S) \to G$ is maximal.*

This is particularly useful in combination with the following geometric examples.

Proposition 5.22 ([?]). *Let $\mathcal{Y}$ and X be Hermitian symmetric spaces with normalized metrics and let $f : \mathcal{Y} \to X$ be a holomorphic and isometric map. Then f is tight if and only if* $\mathrm{rank}_X = \mathrm{rank}_{\mathcal{Y}}$*, in which case it is also positive. In particular:*

1. *a maximal polydisk $t : \mathbb{D}^r \to X$ is tight and positive;*
2. *the inclusion $T \hookrightarrow X$ of a maximal tube type subdomain is tight and positive;*
3. *a diagonal disk $d : \mathbb{D} \to X$ is tight and positive.*

We stress the fact that all Hermitian spaces involved carry the normalized metric, that is the one with minimal holomorphic sectional curvature -1.

There are many interesting tight embeddings which are not holomorphic, as the following result shows.

Proposition 5.23. *The $2n$-dimensional irreducible representation*

$$\rho_{2n} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$$

is tight and corresponds to a holomorphic map only when $n = 1$.

The main structure theorem concerning tight homomorphisms is then the following:

Theorem 5.24 ([?]). *Let L be a locally compact second countable group, $G = \mathbf{G}(\mathbb{R})^\circ$ a Lie group of Hermitian type and let $\rho : L \to G$ be a continuous tight homomorphism. Then:*

1. *the Zariski closure $\mathbf{H} := \overline{\rho(L)}^{\mathrm{Z}}$ is reductive;*

(2) *the centralizer of* $H := \mathbf{H}(\mathbb{R})^\circ$ *in* G *is compact;*

(3) *the semisimple part of* H *is of Hermitian type and the associated symmetric space* $\mathcal{Y}$ *admits a unique* H*-invariant complex structure such that the inclusion* $H \hookrightarrow G$ *is tight and positive.*

Setting $L = \pi_1(S)$ and assuming ρ to be maximal, the above result accounts then for most of the statements in the structure Theorem 5.15 except the one, essential, that $\mathcal{Y}$ is of tube type. This is specific to the hypothesis that $L = \pi_1(S)$ is a surface group.

An important ingredient of the structure theorem for tight homomorphisms is the work of Clerc and Orsted on the characterization of "ideal triangles with maximal symplectic area" [**?**]. To describe some important features, we will assume for simplicity that G is simple (of Hermitian type), that is the associated symmetric space X is irreducible. Let $\mathcal{D} \subset \mathbb{C}^n$ be the bounded domain realization of X; there is an explicit realization of $\mathcal{D}$, called *Harish-Chandra realization*, and in which the Bergmann kernel

$$K_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}^*$$

of $\mathcal{D}$ can be computed rather explicitly. We let G act on $\mathcal{D}$ via the isomorphism $X \to \mathcal{D}$; the Hermitian metric defined by $K_{\mathcal{D}}$ is of course G-invariant, and it has the interesting feature that its Kähler form comes from an integral class in $\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$. However, this metric is not normalized. One introduces then the normalized Bergmann kernel

$$k_{\mathcal{D}} := K_{\mathcal{D}}^{1/n_{\mathcal{D}}}$$

where $n_{\mathcal{D}} = n_X$ is a specific integer (see Corollary 5.9 and the discussion preceding it). This leads to the normalized metric whose Kähler form is

$$\omega_{\mathcal{D}} = \imath \partial \overline{\partial} \log k_{\mathcal{D}}(z, z)\,.$$

In fact, on the specific formulas for $k_{\mathcal{D}}$ one sees that it is defined and not vanishing on $\mathcal{D}^2$ of course and even on a certain open dense subset $\overline{\mathcal{D}}^{(2)} \subset \overline{\mathcal{D}}^2$ of pairs of points satisfying a certain transversality condition. The domain $\overline{\mathcal{D}}^{(2)}$ being star shaped, we let $\arg k_{\mathcal{D}}$ denote the unique continuous determination of the argument of $k_{\mathcal{D}}$ on $\overline{\mathcal{D}}^{(2)}$ vanishing on the diagonal of $\mathcal{D} \times \mathcal{D}$. Then we have the following formula for the area of a triangle with geodesic sides:

Lemma 5.25 ([**?**, **?**]). *For* $x, y, z \in \mathcal{D}$

$$\int_{\Delta(x,y,z)} \omega_{\mathcal{D}} = -\big(\arg k_{\mathcal{D}}(x, y) + \arg k_{\mathcal{D}}(y, z) + \arg k_{\mathcal{D}}(z, x)\big)\,.$$

Guided by this, we introduce a fundamental object on the set $\overline{\mathcal{D}}^{(3)}$ of triples of pairwise transverse points, namely the *Bergmann cocycle*

$$\beta_{\mathcal{D}}(x,y,z) = -\frac{1}{2\pi}\big(\arg k_{\mathcal{D}}(x,y) + \arg k_{\mathcal{D}}(y,z) + \arg k_{\mathcal{D}}(z,x)\big)\,.$$

Its role is to extend in a meaningful way the notion of area to "ideal triangles". This function is continuous on $\overline{\mathcal{D}}^{(3)}$, G-invariant and satisfies an obvious cocycle property. The following is then a summary of some work of Clerc and Orsted.

Theorem 5.26 ([?]). *Let $\check{S}$ and $\mathrm{rank}_{\mathcal{D}}$ be respectively the Shilov boundary and the real rank of $\mathcal{D}$.*

(1) *Then*

$$-\frac{1}{2}\mathrm{rank}_{\mathcal{D}} \le \beta_{\mathcal{D}}(x,y,z) \le \frac{1}{2}\mathrm{rank}_{\mathcal{D}}\,,$$

with strict inequality if $(x,y,z)\in\mathcal{D}^3$;

(2) *we have that $\beta_{\mathcal{D}}(x,y,z) = \frac{\mathrm{rank}_{\mathcal{D}}}{2}$ if and only if $x,y,z\in\check{S}$ and there exists a diagonal disk*

$$d:\overline{\mathbb{D}}\to\overline{\mathcal{D}}$$

with $d(1)=x$, $d(\imath)=y$ and $d(-1)=z$.

In the sequel we call the restriction of the Bergmann cocycle to the Shilov boundary *Maslov cocycle* and we call a triple of points (x,y,z) on the Shilov boundary *maximal* if

$$\beta_{\mathcal{D}}(x,y,z) = \frac{\mathrm{rank}_{\mathcal{D}}}{2}\,.$$

Observe that a maximal triple is always contained in the boundary of a maximal tube type subdomain of $\mathcal{D}$.

One of the corollaries of the above result is the computation of the Gromov norm of κ_G^{b} (Theorem 5.2). This is based on the following

Corollary 5.27 ([?, ?]). *Under the canonical map*

$$\mathrm{H}^\bullet\big(L^\infty(\check{S}^\bullet)^G\big)\to\mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R})$$

the class defined by $\beta_{\mathcal{D}}$ goes to κ_G^{b}.

Finally we turn to the ingredient which leads to the conclusion in Theorem 5.15 that $\mathcal{Y}$ is of tube type. For this we construct an invariant of triples of points on $\check{S}$ which we call the Hermitian triple product and whose definition goes as follows; recall that the Bergmann kernel satisfies the relation

$$K_{\mathcal{D}}(gz,gw) = j(g,z)K_{\mathcal{D}}(z,w)\overline{j(g,w)}\,,$$

where $j(g,z)$ is the complex Jacobian of g at the point $z \in \mathcal{D}$. Then we define on the set $\check{S}^{(3)}$ of pairwise transverse points the Hermitian triple product

$$\langle\langle x,y,z\rangle\rangle := K_{\mathcal{D}}(x,y)K_{\mathcal{D}}(y,z)K_{\mathcal{D}}(z,x) \mod \mathbb{R}^{\times}\,.$$

Recall that $\check{S}$ is of the form G/Q, where Q is a maximal parabolic subgroup of G, and is hence in a natural way the set of real points of a complex projective variety.

Theorem 5.28 ([**?**]). *The function*

$$\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : \check{S}^{(3)} \to \mathbb{R}^{\times}\backslash\mathbb{C}^{\times}$$

is a G-invariant multiplicative cocycle and, for an appropriate real structure on $\mathbb{R}^{\times}\backslash\mathbb{C}^{\times}$, it is a real rational function. Moreover the following are equivalent:

(1) *$\mathcal{D}$ is not of tube type;*

(2) *$\check{S}^{(3)}$ is connected;*

(3) *the Hermitian triple product is not constant.*

Sketch of the proof that $\mathcal{Y}$ is of tube type. Let $\rho : \pi_1(S) \to G := \mathbf{G}(\mathbb{R})^\circ$ be a maximal representation. Using the structure theorem for tight homomorphisms we may assume that $\rho\big(\pi_1(S)\big)$ is Zariski dense in $\mathbf{G}$ and hence $\mathcal{Y} = X$. We have to show that X is of tube type. Realize $\pi_1(S)$ as a lattice in $\mathrm{PSU}(1,1)$ via an appropriate hyperbolization. Since ρ has Zariski dense image, the action on the Shilov boundary $\check{S}$ is strongly proximal. This together with the amenability of the action of $\pi_1(S)$ on $\partial\mathbb{D}$ via the chosen hyperbolization implies (according to [**?**]) the existence of an equivariant measurable map (see e.g. [**?**] for a description of the construction)

$$\varphi : \partial\mathbb{D} \to \check{S}$$

into the Shilov boundary. From the maximality assumption we deduce that

$$\rho^*(\kappa_G^{\mathrm{b}}) = \mathrm{rank}_{\mathcal{D}}\kappa_{S,\mathbb{R}}^{\mathrm{b}}$$

and hence that

$$\beta_{\mathcal{D}}\big(\varphi(x),\varphi(y),\varphi(z)\big) = \mathrm{rank}_{\mathcal{D}}\beta_{\mathbb{D}}(x,y,z)$$

for almost every (x,y,z). Thus we obtained that for almost every $x,y,z \in \partial\mathbb{D}$

$$\begin{aligned} K_{\mathcal{D}}\big(\varphi(x),\varphi(y)\big)&K_{\mathcal{D}}\big(\varphi(y),\varphi(z)\big)K_{\mathcal{D}}\big(\varphi(z),\varphi(x)\big) \\ &= e^{2\pi\imath n_{\mathcal{D}}\beta_{\mathcal{D}}\left(\varphi(x),\varphi(y),\varphi(z)\right)} \mod \mathbb{R}^{\times+} \\ &= e^{2\pm\pi\imath n_{\mathcal{D}}\mathrm{rank}_{\mathcal{D}}\frac{1}{2}} \mod \mathbb{R}^{\times+} \end{aligned}$$

and as a result the square $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle^2$ of the Hermitian triple product is equal to 1 on $(\mathrm{Ess}\,\mathrm{Im}\,\varphi)^{(3)} \subset \check{S}^{(3)}$, where $\mathrm{Ess}\,\mathrm{Im}\,\varphi \subset \check{S}$ is the essential image of φ.

But, being invariant under ρ, this set is Zariski dense in $\check{S}$ hence the rational function $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle^2$ on $\check{S}^{(3)}$ is identically equal to 1. If now $\mathcal{D}$ were not of tube type, $\check{S}^{(3)}$ would be connected and hence $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ would be identically equal to 1 on $\check{S}^{(3)}$, which is a contradiction. □

5.5 Boundary maps, rotation numbers and representation varieties

We have seen that if $\rho : \pi_1(S) \to G$ is a maximal representation into a group of Hermitian type and $h : \pi_1(S) \to \mathrm{PSU}(1,1)$ is a hyperbolization of S of finite area, then there exists a measurable $\rho \circ h^{-1}$-equivariant map $\varphi : \partial\mathbb{D} \to \check{S}$ and furthermore

$$\beta_{\mathcal{D}}\big(\varphi(x), \varphi(y), \varphi(z)\big) = \mathrm{r}_{\mathcal{D}}\, \beta_{\mathbb{D}}(x, y, z) \tag{5.5}$$

for almost every $(x, y, z) \in (\partial\mathbb{D})^3$. Here $\beta_{\mathcal{D}}$ is the Maslov cocycle on the Shilov boundary of the bounded symmetric domain $\mathcal{D}$; observe that $\beta_{\mathbb{D}}$ is just $\frac{1}{2}$ of the orientation cocycle.

In the case where $G = \mathrm{PSU}(1,1)$ we have see that h and ρ are semiconjugate by using Ghys' theorem; an alternative approach would be to use the equality (5.5) to show that φ coincides almost everywhere with a weakly monotone map; this has been carried out in [**?**]. In fact, this way of "improving" the regularity of φ works in general and the basic idea is presented in [**?**]. One considers the essential graph of φ

$$\mathrm{Ess}\,\mathrm{Gr}(\varphi) \subset \partial\mathbb{D} \times \check{S}$$

which is by definition the support of the direct image of the Lebesgue measure on $\partial\mathbb{D}$ under the map

$$\begin{aligned} \partial\mathbb{D} &\to \partial\mathbb{D} \times \check{S} \\ x &\mapsto \big(x, \varphi(x)\big)\,. \end{aligned}$$

Then one shows that there are exactly two sections φ_- and φ_+ of the projection of $\mathrm{Ess}\,\mathrm{Gr}(\varphi)$ on $\partial\mathbb{D}$ such that:

(1) φ_- and φ_+ are strictly equivariant;

(2) φ_- is right continuous while φ_+ is left continuous;

(3) $\mathrm{Ess}\,\mathrm{Gr}(\varphi) = \big\{\big(x, \varphi_-(x)\big), \big(x, \varphi_+(x)\big) : x \in \partial\mathbb{D}\big\}$;

(4) for every positive triple $x, y, z \in \partial\mathbb{D}$, both triples $\varphi_+(x), \varphi_+(y), \varphi_+(z)$ and $\varphi_-(x), \varphi_-(y), \varphi_-(z)$ are maximal.

This generalizes exactly the $\mathrm{PSU}(1,1)$ picture and, remarkably, the discontinuities of φ_- and φ_+ are simple.

One can summarize the situation as follows:

Theorem 5.29 ([**?**]). *The representation $\rho : \pi_1(S) \to G$ is maximal if and only if there exists a left continuous map $\varphi : \partial\mathbb{D} \to \check{S}$ such that*

(1) *φ is strictly $\rho \circ h^{-1}$-equivariant;*

(2) *φ maps every positively oriented triple in $\partial\mathbb{D}$ to a maximal triple on $\check{S}$.*

The first obvious consequence is the following result on the existence of fixed points:

Corollary 5.30 ([**?**]). *Let $\rho : \pi_1(S) \to G$ be maximal. Then:*

(1) *if γ is freely homotopic to a boundary component, $\rho(\gamma)$ has a fixed point in $\check{S}$;*

(2) *if γ is not conjugate to a boundary component, then $\rho(\gamma)$ has (at least) two fixed points in $\check{S}$, which are transverse.*

We use again the standard presentation of $\pi_1(S)$ (see (4.2)) and define the following subset of $\mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big)$:

$$\begin{aligned}\mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big) = \big\{\rho \in \mathrm{Hom}\big(\pi_1(S), G\big);\ &\text{for every } 1 \le i \le n,\\ &\rho(c_i) \text{ has at least one fixed point in } \check{S}\big\}\end{aligned}$$

Then $\mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big)$ is a semialgebraic subset of $\mathrm{Hom}\big(\pi_1(S), G\big)$ and we have from Corollary 5.30 that

$$\mathrm{Hom}_{\max}\big(\pi_1(S), G\big) \subset \mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big)\,.$$

Theorem 5.31 ([**?**]). *Assume that $\mathcal{D}$ is of tube type. Then the Toledo invariant $\rho \mapsto \mathrm{T}(\Sigma, \rho)$ is locally constant on $\mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big)$. In particular, the subset of maximal representations $\mathrm{Hom}_{\max}\big(\pi_1(S), G\big)$ is a union of connected components of $\mathrm{Hom}_{\check{S}}\big(\pi_1(S), G\big)$ and therefore semialgebraic.*

This result is essentially a consequence of the formulas in § 4.2 for the invariants $\mathrm{T}_\kappa(\Sigma, \rho)$, $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{Z})$ together with the lemma that if Q is the stabilizer in G of a point in $\check{S}$, and if $\mathcal{D}$ is of tube type, then the restriction map

$$\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(Q, \mathbb{R})$$

is identically zero.

We end by mentioning a result which gives additional invariants for maximal representations. Recall that for $\chi \in \mathrm{Hom_c}(K, \mathbb{R}/\mathbb{Z})$ we have introduced a class function $\chi_{\mathrm{ext}} : G \to \mathbb{R}/\mathbb{Z}$ extending χ. We have:

Theorem 5.32 ([**?**, Theorem 13]). *Let $\chi \in \mathrm{Hom_c}(K, \mathbb{R}/\mathbb{Z})$ and fix $\rho_0 : \pi_1(S) \to G$ maximal.*

(1) *For every maximal* $\rho : \pi_1(S) \to G$*, the map*

$$\begin{aligned} R_\chi(S) : \pi_1(S) &\longrightarrow \qquad \mathbb{R}/\mathbb{Z} \\ \gamma \quad &\mapsto \chi_{\mathrm{ext}}\big(\rho(\gamma)\big) - \chi_{\mathrm{ext}}\big(\rho_0(\gamma)\big) \end{aligned}$$

is a homomorphism.

(2) *If* $\mathcal{D}$ *is of tube type,* $R_\chi(S)$ *takes values in* $e_G^{-1}\mathbb{Z}/\mathbb{Z}$ *and*

$$\mathrm{Hom}_{\max}\big(\pi_1(S), G\big) \to \mathrm{Hom}\big(\pi_1(S), e_G^{-1}\mathbb{Z}/\mathbb{Z}\big)$$

is constant on connected components. (Here e_G *is an explicit constant depending on* G*, not just on the symmetric space associated to* G*, e. g* $e_{\mathrm{SL}(2,\mathbb{R})} = 2$*.)*

6 Hitchin representations and positive representations

Hitchin representations and positive representations are defined when G is a split real Lie group.

Definition 6.1. A real simple Lie groups G is *split* if its real rank equals the complex rank of its complexification $G_{\mathbb{C}}$, i. e. the maximal torus is diagonalizable over $\mathbb{R}$.

6.1 Hitchin representations

Let S be a compact surface and G a split real simple adjoint group, e.g. $G = \mathrm{PSL}(n, \mathbb{R})$, $\mathrm{PSp}(2n, \mathbb{R})$, $\mathrm{PO}(n, n+1)$ or $\mathrm{PO}(n, n)$, Hitchin [**?**] singled out a connected component

$$\mathrm{Hom}_{Hit}\big(\pi_1(S), G\big) \subset \mathrm{Hom}\big(\pi_1(S), G\big)$$

which he called *Teichmüller component*, now it is usually called *Hitchin component.*

In order to define the Hitchin component we recall that the Lie algebra $\mathfrak{g}$ of a split real simple adjoint Lie group G contains a (up to conjugation) unique principal three dimensional simple Lie algebra. That is an embedded subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ which is the real form of a subalgebra $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{g}_{\mathbb{C}}$ given (via the theorem of Jacobson–Morozov) by a regular nilpotent element in $\mathfrak{g}_{\mathbb{C}}$. Here $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$ and a nilpotent element is regular if its centralizer is of dimension equal to the rank of $\mathfrak{g}_{\mathbb{C}}$. For more details on principal three dimensional subalgebra we refer the reader to [**?**] or Kostant's original papers [**?**, **?**, **?**].

The embedding $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ gives rise to an embedding $\pi : \mathrm{SL}(2,\mathbb{R}) \to G$. Precomposition of π with a discrete (orientation preserving) embedding of $\pi_1(S)$ into $\mathrm{SL}(2,\mathbb{R})$ defines a homomorphism $\rho_0 : \pi_1(S) \to G$, which we call a principal Fuchsian representation. The Hitchin component $\mathrm{Hom}_{Hit}\big(\pi_1(S), G\big)$ is defined as the connected component of $\mathrm{Hom}\big(\pi_1(S), G\big)$ containing ρ_0. By construction it contains a copy of Teichmüller space.

Remark 6.2. When $G = \mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2m,\mathbb{R})$ or $\mathrm{PO}(m,m+1)$ the embedding π is given by the n-dimensional irreducible representation $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$, which is contained in $\mathrm{PSp}(2m,\mathbb{R})$ if $n = 2m$ and in $\mathrm{PO}(m,m+1)$ if $n = 2m+1$. For $G = \mathrm{PO}(m,m)$, the embedding π is given by the composition of the $2m-1$-dimensional irreducible representation into $\mathrm{PO}(m,m-1)$ with the embedding $\mathrm{PO}(m,m-1)$ into $\mathrm{PO}(m,m)$.

Remark 6.3. When G is a finite cover of a split real simple adjoint Lie group G^{Ad}, one can define the Hitchin components $\mathrm{Hom}_{Hit}\big(\pi_1(S), G\big)$ by taking lifts of the principal Fuchsian representations $\pi_1(S) \to G^{\mathrm{Ad}}$ and the corresponding connected components. Equivalently one can define a Hitchin representation $\rho : \pi_1(S) \to G$ as a representation whose projection $\rho : \pi_1(S) \to G^{\mathrm{Ad}}$ is a Hitchin representation.

Hitchin studied the Hitchin component following an analytic approach, relying on the correspondence between irreducible representations of $\pi_1(S)$ in $\mathrm{PSL}(n,\mathbb{C})$ and stable Higgs bundles. One direction of this correspondence is due to Corlette [**?**] and Donaldson [**?**], following ideas of Hitchin [**?**], the other direction is due to Simpson [**?**, **?**].

The correspondence requires to fix a complex structure j on S; with this choice, there is an isomorphism (see [**?**])

$$h_j : \mathrm{Hom}_{Hit}\big(\pi_1(S), G\big)/G \to H^0\big(S, \oplus_{k=1}^{r} \Omega^{d_k}_{(S,j)}\big)\,, \tag{6.1}$$

where $H^0\big(S, \Omega^d_{(S,j)}\big)$ is the vector space of holomorphic differentials (with respect to the fixed complex structure j on S) of degree d. The coefficients d_k, $k = 1, \cdots, r = \mathrm{rank}(G)$ are the degrees of a basis of the algebra of invariant polynomials on $\mathfrak{g}$. In particular, this proves

Theorem 6.4 ([**?**, Theorem A]). *Let S be a compact surface and G the adjoint group of a split real Lie group. Then the Hitchin component $\mathrm{Hom}_{Hit}\big(\pi_1(S), G\big)/G$ is homeomorphic to $\mathbb{R}^{|\chi(S)|\dim G}$.*

Hitchin pointed out that the analytic approach via Higgs bundles gives no indication about the geometric significance of the representations belonging to this component. The only example supporting the idea that Hitchin components might parametrize geometric structures on S available at that time was given in work of Goldman [?] and Choi and Goldman [?], who showed that for $G = \mathrm{PSL}(3,\mathbb{R})$ the Hitchin component parametrizes convex real projective structures on S. Now we have a better, but not yet satisfactory understanding of the geometric significance of the Hitchin components beyond $\mathrm{PSL}(3,\mathbb{R})$, which will be described in more detail in § 8.3 below.

A direct consequence of Choi and Goldman's result is that any Hitchin representation $\rho : \pi_1(S) \to \mathrm{PSL}(3,\mathbb{R})$ is a discrete embedding with the additional property that for any $\gamma \in \pi_1(S) - \{1\}$, the element $\rho(\gamma)$ is diagonalizable with distinct real eigenvalues. These properties have been generalized to all Hitchin representations into $\mathrm{PSL}(n,\mathbb{R})$ by Labourie [?].

Theorem 6.5 ([?]). *Let $\rho : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Then ρ is a discrete embedding, and for any $\gamma \in \pi_1(S) - \{1\}$, the element $\rho(\gamma)$ is diagonalizable with distinct real eigenvalues.*

Note that by Remark 6.2 similar results hold for Hitchin representations into $\mathrm{Sp}(2m,\mathbb{R})$ and $\mathrm{SO}(m, m+1)$.

6.2 Positive representations

In the case when S is a noncompact surface (of finite type), the generalization of Hitchin's work, which is based on methods from the theory of Higgs bundles, has only been partially carried out as it presents some additional analytic difficulties (see for example [?]). But when G is an adjoint split real Lie group and S is a noncompact surface, there is a completely different approach to define a special subset of $\mathrm{Hom}\big(\pi_1(S), G\big)$ due to Fock and Goncharov [?] which leads to the set of *positive representations* $\mathrm{Hom}_{pos}\big(\pi_1(S), G\big) \subset \mathrm{Hom}\big(\pi_1(S), G\big)$.

In order to describe the definition let us identify S with a punctured surfaces of the same topological type. As recalled in § 3.1 a homomorphism $\rho : \pi_1(S) \to G$ corresponds to a flat principal G-bundle $G(\rho)$ on S. The definition of the space of positive representations relies on considering the space of framed G-bundles on S. Let $\mathcal{B} = G/B$ be the space of Borel subgroups of G.

A framed G-bundle on S is a pair $\big(G(\rho), \beta\big)$, where $G(\rho)$ is a flat principal G-bundle on S and β is a flat section of the associated bundle $G(\rho) \times_G \mathcal{B}$ restricted to the punctures. There is a natural forgetful map from the space of framed G-bundles to the space of flat principal G-bundles sending $\big(G(\rho), \beta\big) \to G(\rho)$. Since there always exists a flat section of $G(\rho) \times_G \mathcal{B}$ over the punctures, this map is surjective.

Given an ideal triangulation of the surface S, i.e. a triangulation whose vertices lie at the punctures of S, one can use the information provided by the section β to define a coordinate system on the space of framed G-bundles. Fock and Goncharov show that these coordinate systems form a positive atlas.

This means in particular that the coordinate transformations are given by rational functions, involving only positive coefficients. Hence the set of positive framed G-bundles, i.e. the set where for a given triangulation all coordinate functions are positive real numbers, is well defined and independent of the chosen triangulation. The space of positive representations $\mathrm{Hom}_{pos}\big(\pi_1(S), G\big)$ is the image of the space of positive framed G-bundles under the forgetful map.

The construction of the coordinates involves the notion of positivity in Lie groups introduced by Lusztig [**?**, **?**], to which we will come back in § 7.1. When $G = \mathrm{PSL}(n, \mathbb{R})$ one can give an elementary description of the coordinates in terms of projective invariants of triples and quadruples of flags (see [**?**, § 9]). In the case when $G = \mathrm{PSL}(2, \mathbb{R})$ the coordinates correspond to shearing coordinates constructed first by Thurston and Penner [**?**] and similar coordinates constructed by Fock [**?**].

Theorem 6.6 ([**?**, Theorem 1.13, Theorem 1.9 and Theorem 1.10]). *The space* $\mathrm{Hom}_{pos}\big(\pi_1(S), G\big)/G$ *of positive representations and* $\mathbb{R}^{|\chi(S)| \dim G}$ *are homeomorphic. Every positive representation is a discrete embedding, and every nontrivial element* $\gamma \in \pi_1(S)$ *which is not homotopic to a loop around a boundary component of* S*, is sent to a positive hyperbolic element.*

The notion of positive representations can be extended to the situation when S is compact, using the characterization of positive representations in terms of equivariant positive boundary maps, which is described in the next section.

Theorem 6.7 ([**?**, Theorem 1.15]). *When* S *is compact,* $\mathrm{Hom}_{pos}\big(\pi_1(S), G\big) = \mathrm{Hom}_{Hit}\big(\pi_1(S), G\big)$.

Remark 6.8. In the case when $G = \mathrm{PSL}(n, \mathbb{R}), \mathrm{PSp}(2n, \mathbb{R})$ or $\mathrm{PO}(n, n+1)$ this theorem also follows from Labourie's work [**?**].

When S is a noncompact surface, the set of positive framed G-bundles carries many more interesting structures. It is a cluster variety, admits a mapping class group invariant Poisson structure and natural quantizations. We will not discuss any of these interesting structures and refer the reader to [**?**] and [**?**] for further reading.

7 Higher Teichmüller spaces – a comparison

In this section we discuss common structures as well as differences between the higher Teichmüller spaces introduced above – comparing maximal representations on one hand and Hitchin representations on the other hand. We explain that, when S is compact, representations in higher Teichmüller spaces fit into the context of Anosov structures, which is a more general concept (for the definition see § 8). From this further geometric information about higher Teichmüller spaces can be obtained.

7.1 Boundary maps

The higher Teichmüller spaces we are discussing here were defined and studied with very different methods and so far we see no unified approach to it. Nevertheless there is a common theme in all these works which highlights an important underlying structure for all higher Teichmüller spaces: The existence of very special *boundary maps*.

Since $\pi_1(S)$ is a word hyperbolic group, the boundary $\partial\pi_1(S)$ of $\pi_1(S)$ is a well defined compact metrizable space. When S is compact $\partial\pi_1(S)$ identifies naturally with a topological circle S^1 endowed with a canonical Hölder structure. When S is not compact there is no natural identification of $\partial\pi_1(S)$ with a subset of S^1 (see the discussion in § 4).

All higher Teichmüller spaces can be characterized as the set of representations for which there exist special equivariant (semi)continuous maps into a flag variety. The special boundary maps all satisfy some positivity condition, where the notion of positivity depends on the context.

For maximal representations, i.e in the case when G is a Lie group of Hermitian type, we saw in § 5 that the Maslov cocycle on the Shilov boundary $\check{S}$ of the symmetric space associated to G gives rise to the notion of a maximal triple of points in $\check{S}$; and Theorem 5.29 characterizes maximal representations as those which admit an equivariant boundary map $\varphi : \partial\pi_1(S) \to \check{S}$, which sends every positively oriented triple to a maximal triple in $\check{S}$.

For Hitchin representations Labourie constructed special boundary maps in [?]. In this case S is compact and $\partial\pi_1(S) = S^1$. A map $\varphi : S^1 \to \mathbb{RP}^{n-1}$ is said to be convex if for every n-tuple of distinct points $x_1, \cdots, x_n \in S^1$ the images $\varphi(x_1), \cdots, \varphi(x_n)$ are in direct sum - or, equivalently, the map is injective and any hyperplane in $\mathbb{RP}^{n-1}$ intersects $\varphi(S^1)$ in at most $(n-1)$ points.

The characterization of Hitchin representation into $\mathrm{PSL}(n,\mathbb{R})$ in terms of convex maps is due to a combination of the construction by Labourie and a result by Guichard:

Theorem 7.1 ([?, ?]). *A representation $\rho : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ lies in the Hitchin component if and only if there exists a ρ-equivariant continuous convex map $\varphi : \partial\pi_1(S) \to \mathbb{RP}^{n-1}$.*

In the context of positive representations the notion of positivity for the boundary maps relies on Lusztig's notion of positivity. Recall that a matrix in $\mathrm{GL}(n,\mathbb{R})$ is totally positive if all its minors are positive numbers. An upper triangular matrix is positive if all not obviously zero minors are positive. The notion of positivity has been extended to all split real semisimple Lie groups G by Lusztig [**?**, **?**]. This can be used to define a notion of positivity for k-tuples in full flag varieties. Let B^+ be a Borel subgroup of G, B^- an opposite Borel subgroup and U the unipotent radical of B^+. Then the set of Borel groups in $\mathcal{B}$ being opposite to B^+ can be identified with the orbit of B^- under U. The notion of positivity gives us a well defined subset $U(\mathbb{R}_{>0}) \subset U$. A k-tuple of points $(B_1, \cdots, B_k)$ in $\mathcal{B}$ is said to be positive if (up to the action of G) it can be written as $\big(B^+, B^-, u_1B^-, \cdots (u_1\cdots u_{k-2})B^-\big)$, where $u_i \in U(\mathbb{R}_{>0})$ for all $i = 1, \cdots, k-2$.

Definition 7.2. A map $\partial\pi_1(S) \to G/B$ is said to be positive if it sends every positively oriented k-tuple in $\partial\pi_1(S)$ to a positive k-tuple of flags in G/B.

Remark 7.3. A map $\partial\pi_1(S) \to G/B$ is positive if and only if it sends every positively oriented triple in $\partial\pi_1(S)$ to a positive tripe of flags in G/B.

Theorem 7.4 ([**?**, Theorem 1.6]). *Let G be a split real simple Lie group and $B < G$ a Borel subgroup. A representation $\rho : \pi_1(S) \to G$ is positive if and only if there exists a ρ-equivariant positive map $\varphi : \partial\pi_1(S) \to G/B$.*

Remark 7.5. Note that Fock and Goncharov choose a different identification of the boundary of $\pi_1(S)$ with a subset of S^1. For their identification the boundary map is indeed continuous.

Let us emphasize that Fock and Goncharov prove Theorem 7.4 when S is a noncompact surface. In the case when S is a compact surface, they use the characterization of Theorem 7.4 in order to define positive representations $\rho : \pi_1(S) \to G$ by requiring the existence of a ρ-equivariant positive map $\varphi : \partial\pi_1(S) = S^1 \to G/B$. In order to prove the equality $\mathrm{Hom}_{Hit}\big(\pi_1(S), G\big) = \mathrm{Hom}_{pos}\big(\pi_1(S), G\big)$ for compact surfaces S, they observe first that the set of positive representation is an open subset of the Hitchin component. Then they study limits of positive representations in order to prove that it is also closed. Hence as a nonempty open and closed subset it is a connected component and thus coincides with the Hitchin component.

For $G = \mathrm{PSL}(n,\mathbb{R})$ there is an intimate relation between positive maps of $\partial\pi_1(S)$ into the full flag variety G/B and convex maps into the the partial flag variety $\mathbb{RP}^{n-1}$. In particular, the projection of a positive map into the flag variety to $\mathbb{RP}^{n-1}$ is a convex map, and convex maps (with some regularity) naturally lift to positive maps, see [**?**, Theorem 1.3] and [**?**, Chapter 5], [**?**, Appendix B] for details.

7.2 The symplectic group

The only simple groups which are both of Hermitian type as well as split real are the real symplectic groups, $G = \mathrm{PSp}(2n, \mathbb{R})$. When $n \geq 2$, Hitchin representations or positive representation, and maximal representation provide different generalizations of Teichmüller space in this situation. It is indeed not difficult to see that the Hitchin component and the space of positive representations are properly contained in the space of maximal representations.

Moreover, for the symplectic group the properties of the boundary maps required in Theorem 7.4 and in Theorem 5.29 are related in the following way. In this situation $\mathcal{F} = G/B$ is the flag variety consisting of full isotropic flags and $\check{S} = G/Q$ is the partial flag variety consisting only of Lagrangian (i.e. maximal isotropic) subspaces. Positive triples of flags in $\mathcal{F}$ in the sense of Definition 7.2 are mapped to maximal triples in the Shilov boundary in the sense of Theorem 5.29 under the natural projection

$$\begin{aligned} \mathcal{F} &\to \check{S} \\ (F_1, \cdots, F_n) &\mapsto F_n \, . \end{aligned}$$

8 Anosov structures

The notion of Anosov structure is a dynamical analogue of the concept of locally homogeneous (G, X)-structures in the sense of Ehresmann, introduced by Labourie in [**?**] to study Hitchin representations into $\mathrm{PSL}(n, \mathbb{R})$. Holonomy representations of Anosov structure are called *Anosov representations*.

The class of Anosov representations is much bigger than the higher Teichmüller spaces discussed above. Anosov representations of fundamental groups of surfaces exist into any semisimple Lie group, and they can be defined more generally also for fundamental groups of arbitrary closed negatively curved manifolds. Nevertheless, when S is a compact surface, representations in higher Teichmüller spaces are examples of Anosov representations and recent results about Anosov representations provide important geometric information about higher Teichmüller spaces.

8.1 Definition, properties and examples

From now on let S be a compact connected oriented surface with a fixed hyperbolic metric. Denote by T^1S its unit tangent bundle and by φ_t the geodesic flow on T^1S. The group $\pi_1(S)$ acts as group of deck transformations on $T^1\widetilde{S}$, commuting with φ_t.

Let G be a semisimple Lie group, given a representation $\rho : \pi_1(S) \to G$ we obtain a proper action of $\pi_1(S)$ on $T^1\widetilde{S} \times G$ by

$$\gamma_*(x, g) = \big(T_\gamma x, \rho(\gamma)g\big)$$

whose quotient $\pi_1(S)\backslash(T^1\widetilde{S} \times G)$ is the total space $G(\rho)$ of a (flat) principal G-bundle over T^1S. (Note that the bundle $G(\rho)$ defined here is the pullback of the bundle $G(\rho)$ over S, defined in § 3.1, under the canonical projection $T^1S \to S$.) The geodesic flow lifts to a flow on $G(\rho)$ defined (with a slight abuse of notation) by $\varphi_t(x, g) = (\varphi_t(x), g)$ on $T^1\widetilde{S} \times G$.

Let $P_+, P_- < G$ be a pair of opposite parabolic subgroup of G. The unique open G-orbit $\mathcal{O} \subset G/P_+ \times G/P_-$ inherits two foliations, whose corresponding distributions we denote by $E^\pm$, i.e. $(E^\pm)_{(z_+,z_-)} \cong T_{z_\pm} G/P_\pm$.

Definition 8.1. [**?**] Let $\mathcal{O}(\rho)$ be the associated $\mathcal{O}$-bundle of $G(\rho)$. An Anosov structure on $\mathcal{O}(\rho)$ is a continuous section σ such that

(1) σ commutes with the flow, and
(2) the action of the flow φ_t on σ^*E^+ (resp. σ^*E^-) is contracting (resp. dilating), *i.e.* there exist constants $A, a > 0$ such that
   - for any e in $\sigma^*(E^+)_m$ and for any $t > 0$ one has
   $$\|\varphi_t e\|_{\varphi_t m} \leq A \exp(-at)\|e\|_m,$$
   - for any e in $\sigma^*(E^-)_m$ and for any $t > 0$ one has
   $$\|\varphi_{-t} e\|_{\varphi_{-t} m} \leq A \exp(-at)\|e\|_m,$$

   where $\|\cdot\|$ is any continuous norm on $\mathcal{O}(\rho)$.

Remark 8.2. The definition of Anosov structure does not depend on the choice of the hyperbolic metric on S.

Definition 8.3. A representation $\rho : \pi_1(S) \to G$ is said to be a (P_+, P_-)-Anosov representation if $\mathcal{O}(\rho)$ carries an Anosov structure.

The conditions on σ are equivalent to requiring that $\sigma(T^1S)$ is a hyberbolic set for the flow φ_t. Stability of hyperbolic sets implies stability for (P_+, P_-)-Anosov representations:

Proposition 8.4 ([**?**]). *The set of (P_+, P_-)-Anosov representations is open in* $\mathrm{Hom}\big(\pi_1(S), G\big)$.

Since we fixed a hyperbolic structure on S we can equivariantly identify the boundary $\partial\pi_1(S)$ with the boundary $S^1 = \partial\mathbb{D}$ of the Poincaré disk as we did in § 4.1.

Proposition 8.5 ([?]). *To every Anosov representation $\rho : \pi_1(S) \to G$ there are associated continuous ρ-equivariant boundary maps*

$$\xi_\pm : S^1 \to G/P_\pm ,$$

with the property that for all $t, t' \in S^1$ distinct, we have $\big(\xi_+(t), \xi_-(t')\big) \in \mathcal{O}$. Moreover, for every element $\gamma \in \pi_1(S) - \{1\}$ with fixed points $\gamma^\pm \in S^1$ the point $\xi_\pm(\gamma^+)$ is the unique attracting fixed point of $\rho(\gamma)$ in $G/P_\pm$ and $\xi_\pm(\gamma^-)$ is the unique repelling fixed point of $\rho(\gamma)$ in $G/P_\pm$.

Sketch of proof. Since the existence of these boundary maps play an important role in some results discussed above, let us sketch how these maps are obtained. Recall that $T^1\widetilde{S}$ is naturally identified with the space of positively oriented triples in S^1, via the map

$$\begin{aligned} T^1\widetilde{S} &\to \quad (S^1)^{(3+)} \\ v \quad &\mapsto (v_+, v_0, v_-) , \end{aligned}$$

where $v_\pm$ are the endpoints at $\pm\infty$ of the unique geodesic g_v determined by v and v_0 is the unique point which is mapped to the basepoint of v under the orthogonal projection to the geodesic g_v and such that (v_+, v_0, v_-) is positively oriented.

The existence of a continuous section σ of $\mathcal{O}(\rho)$ is equivalent to the existence of a ρ-equivariant continuous map $F : T^1\widetilde{S} \to \mathcal{O}$. The Anosov condition (1) on σ is equivalent to F being φ_t-invariant. In particular, the map F only depends of $(v_+, v_-) \in (S^1 \times S^1) - \mathrm{diag} =: (S^1)^{(2)}$. Thus we have a map

$$F = (\xi_+, \xi_-) : (S^1)^{(2)} \to G/P_+ \times G/P_- .$$

It is not difficult to see that due to the contraction properties of the geodesic flow (see Anosov condition (2)) the map $\xi_+(v_+, v_-)$ only depends on v_+ and $\xi_-(v_+, v_-)$ only depends on v_-, and that $\xi_\pm$ satisfy the above properties. $\square$

The property of a representation $\rho : \pi_1(S) \to G$ being (P_+, P_-)-Anosov is indeed (almost) equivalent to the existence of such continuous boundary maps.

Proposition 8.6 ([?]). *Let $\rho : \pi_1(S) \to G$ be a Zariski dense representation and assume that there exists ρ-equivariant continuous boundary maps $\xi_\pm : S^1 \to G/P_\pm$ such that*

1. *for all $t, t' \in S^1$ distinct, we have $\big(\xi_+(t), \xi_-(t')\big) \in \mathcal{O}$, and*
2. *for all $t \in S^1$, the two parabolic subgroups stabilizing $\xi_+(t)$ and $\xi_-(t)$ contain a common Borel subgroup.*

Then ρ is a (P_+, P_-)-Anosov representation.

The Anosov section σ can be easily reconstructed from the boundary map using the identification $T^1\widetilde{S} \cong (S^1)^{(3+)}$.

Let us list several consequences of the existence of such boundary maps, for proofs see [**?**] and [**?**].

(1) The representation ρ is faithful with discrete image.

(2) For every $\gamma \in \pi_1(S) - \{1\}$ the holonomy $\rho(\gamma)$ is conjugate to a (contracting) element in $H = P_+ \cap P_-$.

(3) The orbit map $\pi_1(S) \to X$, $\gamma \mapsto \rho(\gamma)x_0$ for some $x_0 \in X$ is a quasi-isometric embedding with respect to the word metric on $\pi_1(S)$ and any G-invariant metric on the symmetric space $X = G/K$.

(4) The representation ρ is well-displacing.

The concepts of welldisplacing representations and quasiisometric embeddings are discussed in more detail in § 8.2.1.

We already mentioned that representations in higher Teichmüller spaces are Anosov representations. We now describe this in a little more detail using the existence of special boundary map discussed in § 7.1. We want to emphasize that this is not the order in which results are proved. In many cases the proof of the existence of a continuous boundary map is intertwined with the proof of the Anosov property. For example one constructs first a not necessarily continuous boundary map, establishes the contraction properties for a (not continuous) section constructed out of this boundary map. Then using the contraction property one can conclude that the map is indeed continuous, hence defining a genuine Anosov section. (For an illustration of this strategy for maximal representations into the symplectic group we refer the reader to [**?**].) The special "positivity conditions" for the map usually are derived from more specific properties of the representations.

(1) Hitchin representations into $\mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PO}(n,n+1)$ are Anosov representations with $P_\pm$ being minimal parabolic subgroups [**?**]. Using the characterization of Hitchin representations via the existence of convex curves, we are given $\varphi : S^1 \to \mathbb{RP}^{n-1}$ and we can take $\varphi^* : S^1 \to (\mathbb{RP}^{n-1})^*$ to be the dual curve, i.e. $\varphi^*(t)$ is the unique osculating hyperplane of the curve φ containing $\varphi(t)$. We set $\xi^+ = \varphi$, $\xi^- = \varphi^*$, then $(\xi^+, \xi^-) : S^1 \to \mathbb{RP}^{n-1} \times (\mathbb{RP}^{n-1})^*$ satisfies the hypothesis of Proposition 8.6, thus ρ is Anosov with respect to the parabolic subgroup stabilizing a line in $\mathbb{R}^n$. In order to see that Hitchin representations are actually Anosov with respect to the minimal parabolic subgroup, note that for any point on the convex curve we can consider the osculating flag and obtain maps $\xi_\pm = \xi : S^1 \to G/P_{min}$. The convexity of φ implies the transversality condition on $\xi_\pm$ (see [**?**, Chapter 5])

(2) Maximal representations are Anosov representation with $P_\pm$ being stabilizers of points in the Shilov boundary $\check{S}$ of the Hermitian symmetric space [**?**, **?**]. This means in particular that in this case the boundary maps $\xi_\pm = \varphi : S^1 \to \check{S}$ (Theorem 5.29) sending positively oriented triples to

maximal triples are continuous. Since (ξ_+, ξ_-) satisfies the transversality conditions required in Proposition 8.6, maximal representations are (P_+, P_-)-Anosov.

8.2 Quotients of Higher Teichmüller spaces

8.2.1 Action of the mapping class group The automorphism groups of $\pi_1(S)$ and of G act naturally on $\mathrm{Hom}\big(\pi_1(S), G\big)$

$$\begin{array}{ccc} \mathrm{Aut}\,\big(\pi_1(S)\big) \times \mathrm{Aut}(G) \times \mathrm{Hom}\big(\pi_1(S), G\big) & \to & \mathrm{Hom}\big(\pi_1(S), G\big) \\ (\psi, \alpha, \rho) & \longmapsto & \alpha \circ \rho \circ \psi^{-1} \,. \end{array}$$

When we consider the quotient of the representation variety $\mathrm{Hom}\big(\pi_1(S), G\big)/G$ this action descends to an action of the outer automorphism group $\mathrm{Out}\big(\pi_1(S)\big) = \mathrm{Aut}\,\big(\pi_1(S)\big)/\mathrm{Inn}\big(\pi_1(S)\big)$ on $\mathrm{Hom}\big(\pi_1(S), G\big)/G$. As we discussed in § 2.1, if S is closed, the group of orientation preserving outer automorphisms of $\pi_1(S)$ is isomorphic to the mapping class group $\mathrm{Map}(S)$, and we will refer to this action as the action of the mapping class group.

The components of $\mathrm{Hom}\big(\pi_1(S), G\big)/G$ which form higher Teichmüller spaces, are preserved by this action. In the case when $G = \mathrm{PSU}(1,1)$ the action of the mapping class group on Teichmüller space is properly discontinuous and the quotient $\mathcal{M}(S)$ is the moduli space of Riemann surfaces.

Given a higher Teichmüller space it is natural to consider its quotient by the action of the mapping class group, to study how it relates to the moduli space of Riemann surface, and to investigate its possible compactifications. The first question here is whether the action of the mapping class group is properly discontinuous on higher Teichmüller spaces.

In order to answer this question the essential notion is that of a representation being well-displacing. For this let us introduce the translation lengths τ and τ_ρ of an element $\gamma \in \pi_1(S) - \{1\}$:

$$\begin{aligned} \tau(\gamma) &= \inf_{p \in \widetilde{S}} d(p, \gamma p) \\ \tau_\rho(\gamma) &= \inf_{z \in X} d_G(z, \rho(\gamma) z)\,, \end{aligned}$$

where d is the lift of a hyperbolic metric on S and d_G is a G-invariant Riemannian metric on the symmetric space X.

Definition 8.7. A representation $\rho : \pi_1(S) \to G$ is *well-displacing* if there exist constants $A, B > 0$ such that for all $\gamma \in \pi_1(S)$

$$A^{-1}\tau_\rho(\gamma) - B \leq \tau(\gamma) \leq A\tau_\rho(\gamma) + B.$$

The translation length τ depends on the choice of hyperbolic metric on S. For any two choices the translations lengths are comparable, thus the definition of well-displacing is independent of the chosen hyperbolic metric on S.

It is shown in [**?**, **?**, **?**] that representations in higher Teichmüller spaces are well-displacing. Then a simple argument (see e.g. [**?**, **?**]) shows that this implies that the action of the mapping class group on higher Teichmüller spaces is properly discontinuous.

Remark 8.8. The notion of well-displacing is related to the notion of quasiisometric embeddings. A representation $\rho : \pi_1(S) \to G$ is a quasiisometric embedding if there exist constants $A, B > 0$ such that for all $\gamma \in \pi_1(S)$

$$A^{-1} d_G\big(\rho(\gamma)z, z\big) - B \leq d(\gamma p, p) \leq A\, d_G\big(\rho(\gamma)z, z\big) + B,$$

for some $p \in \widetilde{S}$ and some $z \in X$.

Both notions can be defined more generally for representations of arbitrary finitely generated groups. The relation between the two notions is studied in [**?**]. Representations in higher Teichmüller spaces are also quasiisometric embeddings [**?**, **?**, **?**, **?**].

8.2.2 Relation to moduli space The notion of well-displacement also plays an important role when trying to obtain a mapping class group invariant projection from higher Teichmüller spaces to classical Teichmüller space. To describe this approach recall that given a representation $\rho : \pi_1(S) \to G$ and hyperbolic metric $h \in \mathrm{Hyp}(S)$, one can define the energy of a ρ-equivariant smooth map $f : \widetilde{S} \to X$ into the symmetric space $X = G/K$ as

$$E_\rho(f, h) := \int_S ||df||^2 \,\mathrm{dvol},$$

where $||df||(p)$, $p \in S$ is the norm of the linear map df_p with respect to the hyperbolic metric on S and the G-invariant Riemannian metric on X.

The map f is said to be harmonic if and only if it minimizes the energy in its ρ-equivariant homotopy class. Setting $E_\rho(h) := \inf_f E_\rho(f, h)$, where f ranges over all ρ-equivariant smooth maps $\widetilde{S} \to X$, we obtain a function

$$E_\rho : \mathcal{F}(S) = \mathrm{Diff}_0^+(S) \backslash \mathrm{Hyp}(S) \to \mathbb{R},$$

called the *energy functional* associated to the representation $\rho : \pi_1(S) \to G$. The energy functional is a smooth function on the Fricke space $\mathcal{F}(S)$. In the case when $G = \mathrm{PSU}(1, 1)$ and ρ is a discrete embedding, it is known that E_ρ has a unique minimum [**?**, **?**], namely the hyperbolic structure determined by ρ.

In the general case, one would like to construct a mapping class group invariant projection from higher Teichmüller spaces to classical Teichmüller space by showing that the energy functional has a unique minimum. As a first step we have

Theorem 8.9 ([?, Theorem 6.2.1]). *Let $\rho : \pi_1(S) \to G$ be a well-displacing representation, then the energy functional E_ρ is a proper function on $\mathcal{F}(S)$.*

In [?] Labourie describes an approach to realize the Hitchin component for $\mathrm{PSL}(n,\mathbb{R})$ as a vector bundle over Teichmüller space in a equivariant way with respect to the mapping class group. Recall for this that the isomorphism (see (6.1))

$$h_j : \mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})\big)/\,\mathrm{PSL}(n,\mathbb{R}) \to H^0\big(S, \oplus_{k=2}^n \Omega^k_{(S,j)}\big)\,,$$

where $H^0(S, \Omega^k_{(S,j)})$ is the vector space of holomorphic differentials (with respect to the fixed complex structure j on S) of degree k, is not mapping class group invariant. Consider the vector bundle $\mathcal{E}^n$ over Teichmüller space $\mathcal{T}(S)$ realized as space of complex structures on S, where the fiber over the complex structure j equals $H^0\big(S, \oplus_{k=3}^n \Omega^k_{(S,j)}\big)$. Then $\mathcal{E}^n$ has the same dimension as $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})\big)/\,\mathrm{PSL}(n,\mathbb{R})$. Labourie defines the Hitchin map

$$(j,\omega) \longmapsto h_j(0,\omega)\,,$$

where $j \in \mathcal{T}(S)$ is a complex structure and $\omega \in H^0\big(S, \oplus_{k=3}^n \Omega^k_{(S,j)}\big)$. This map is equivariant with respect to the mapping class group action. Labourie proves that it is surjective [?, Theorem 2.2.1] and conjectures that H is a homeomorphism, which would imply

Conjecture ([?, Conjecture 2.2.3]). *The quotient of the Hitchin component $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})\big)/\,\mathrm{PSL}(n,\mathbb{R})$ by the mapping class group is a vector bundle over the moduli space of Riemann surfaces with fiber being the space of holomorphic k-differentials $H^0\big(S, \oplus_{k=3}^n \Omega^k_S\big)$.*

In order to prove this conjecture it would be sufficient to show that for a Hitchin representation $\rho \in \mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})\big)/\,\mathrm{PSL}(n,\mathbb{R})$ the energy functional E_ρ has a nondegenerate minimum.

Conjecture 8.2.2 has been proved for $n = 2$ and $n = 3$. The proof for $\mathrm{PSL}(3,\mathbb{R})$ is independently due to Labourie [?] and Loftin [?]. They rely on the description of $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(3,\mathbb{R})\big)$ as deformation space of convex real projective structures due to Choi and Goldman, and use the theory of affine spheres developed in [?, ?] in order to prove

Theorem 8.10 ([?, ?]). *The quotient*

$$\mathrm{Map}(S)\backslash \mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(3,\mathbb{R})\big)/\,\mathrm{PSL}(3,\mathbb{R})$$

is a vector bundle over the moduli space of Riemann surface with fiber being the space of cubic holomorphic differentials on the surface.

For maximal representations the quotients by the mapping class group are expected to look more complicated due to the fact that

(1) the connected components consisting of maximal representations might have singularities, and
(2) the space of maximal representations has several connected components which need to be treated separately. (We will come back to this problem in § 8.4.)

8.2.3 Compactifications Mapping class group equivariant compactifications of higher Teichmüller spaces are partially understood.

A general construction to compactify the space of discrete, injective non-parabolic representations of a finitely generated group into G using the generalized marked length spectrum is given in [**?**]. This construction applies to give compactifications of higher Teichmüller spaces. Boundary points in this compactification can be interpreted as actions on $\mathbb{R}$-buildings [**?**].

For the Hitchin component $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(3,\mathbb{R})\big)/\,\mathrm{PSL}(3,\mathbb{R})$ the identification with the deformation space of convex real projective structures allows to obtain a better understanding of this compactification, see e.g. [**?**, **?**, **?**, **?**]. Through the study of degenerations of convex projective structures, Cooper et al. [**?**] obtain a description of boundary points as mixtures of measured laminations and special Finsler metrics (Hex metrics) on S.

Fock and Goncharov construct tropicalizations of the spaces of positive representations which they expect to provide (partial) completions [**?**] when S is noncompact. But except for the case when $G = \mathrm{PSL}(2,\mathbb{R})$ (treated in [**?**]), they do not define a topology of the union of space of positive representations and its tropicalized counterpart.

8.2.4 Crossratios Realizing $\partial\mathbb{D} \subset \mathbb{CP}^1$, the restriction of the classical crossratio function on $\mathbb{CP}^1$

$$c(x,y,t,z) = \frac{x-y}{x-t}\frac{z-t}{z-y}$$

gives a continuous real valued $\mathrm{PSU}(1,1)$-invariant function on

$$(\partial\mathbb{D})^{4*} := \left\{(x,y,z,t) \in (\partial\mathbb{D})^4 \,:\, x \neq t\, y \neq z\right\}.$$

This crossratio and several generalizations (see e.g. [**?**, **?**]) play an important role in the study of negatively curved manifolds and hyperbolic groups.

Given a hyperbolic element $\gamma \in \mathrm{PSU}(1,1)$ its period is defined as

$$l_c(\gamma) = \log c(\gamma^-, z, \gamma^+, \gamma z)\,,$$

where γ^+ is the unique attracting fixed point and γ^- the unique repelling fixed point of γ in $\partial\mathbb{D}$ and $z \in \partial\mathbb{D} - \{\gamma^\pm\}$ is arbitrary. The period of γ equals the translation length $\tau(\gamma) = \inf_{p\in\mathbb{D}} d_{\mathbb{D}}(p,\gamma p)$.

Given a discrete embedding $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$, let

$$c_\rho := \varphi^* c : (S^1)^{4*} \to \mathbb{R}$$

be the pullback of c by some ρ-equivariant boundary map, be the associated crossratio function. Then c_ρ contains all information about the marked length spectrum of S with respect to the hyperbolic metric defined by ρ. In particular, two discrete embeddings ρ_1, ρ_2 are conjugate if and only if $c_{\rho_1} = c_{\rho_2}$.

A generalized crossratio function is a $\pi_1(S)$-invariant continuous functions

$$(S^1)^{4*} = \left\{(x, y, z, t) \in (S^1)^4 \,:\, x \neq t\, y \neq z\right\} \to \mathbb{R}$$

satisfying the following relations [**?**, Introduction]:

(1) (Symmetry) $c(x, y, z, t) = c(z, t, x, y)$

(2) (Normalization)

$$c(x, y, z, t) = 0 \text{ if and only if } x = y \text{ or } z = t$$
$$c(x, y, z, t) = 1 \text{ if and only if } x = z \text{ or } y = t$$

(3) (Cocycle identity)

$$c(x, y, z, t) = c(x, y, z, w)c(x, w, z, t)$$
$$c(x, y, z, t) = c(x, y, w, t)c(w, y, z, t).$$

Among such functions crossratios arising from a discrete embedding $\pi_1(S) \to \mathrm{PSU}(1,1)$ are uniquely characterized by the functional equation $1 - c(x, y, z, t) = c(t, y, z, x)$.

The study of generalized crossratio functions associated to higher Teichmüller spaces has been pioneered by Labourie. In particular, he associates a generalized crossratio function to any Hitchin representation into $\mathrm{PSL}(n, \mathbb{R})$ and shows that crossratio functions arising from a Hitchin representation into $\mathrm{PSL}(n, \mathbb{R})$ are characterized by explicit functional equations [**?**].

In [**?**] Labourie and McShane establish generalized McShane identities for the crossratios associated to Hitchin representations into $\mathrm{PSL}(n, \mathbb{R})$.

Remark 8.11. Related crossratio functions of four partial flags consisting of a line and a hyperplane are used in the work of Fock and Goncharov [**?**] in order to construct explicit coordinates for the space of positive representations into $\mathrm{PSL}(n, \mathbb{R})$.

In the context of maximal representations crossratio functions have been defined and studied by Hartnick and Strubel [**?**]. They construct crossratio functions defined on a suitable subset $\check{S}^{4*}$ of the fourfold product of the Shilov boundary of any Hermitian symmetric space of tube type. They show that there is a unique such crossratio function which satisties some natural functorial properties. Given a maximal representation $\rho : \pi_1(S) \to G$ a concrete implementation of the continuous boundary map $\varphi : S^1 \to \check{S}$ (see Theorem 5.29) allows to pullback this crossratio function to a generalized crossratio function on $(S^1)^{4*}$. The well-displacing property of representations in higher Teichmüller spaces can be easily deduced from the existence of generalized crossratio functions. In all works investigating crossratio functions, the existence of boundary maps with special positivity properties (as discussed in § 7.1) play an important role.

8.3 Geometric structures

We already mentioned that Hitchin had asked in [**?**] about the geometric significance of Hitchin components, and one might raise the same question for maximal representation, even though the picture there seems to be more complicated due to the fact that the space of maximal representations has singularities and multiple components.

Interpreting higher Teichmüller spaces as deformation spaces of geometric structures is not just of interest in itself. Any such interpretation gives an important tool to study these spaces, their quotients by the mapping class group, their relations to the moduli space of Riemann surfaces as well as their compactifications. This is illustrated by the fact that the deeper understanding of these questions for the Hitchin component of $\mathrm{PSL}(3, \mathbb{R})$ relies on the Theorem by Choi and Goldman, which we already mentioned above:

Theorem 8.12 ([**?**, **?**]). *The Hitchin component*

$$\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(3, \mathbb{R})\big) / \,\mathrm{PSL}(3, \mathbb{R})$$

parametrizes convex real projective structures on S*.*

The original proof of this theorem relied on Goldman's work on convex projective structures on surfaces [**?**], which implied that the deformation space of these structures is an open domain in $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{PSL}(3, \mathbb{R})\big) / \,\mathrm{PSL}(3, \mathbb{R})$. Goldman and Choi [**?**] then proved that this subset is furthermore closed, establishing the above theorem.

In terms of the properties of Hitchin representations we have discussed so far, Theorem 8.12 is basically equivalent to the characterization of Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$ by the existence of a convex map from S^1 into $\mathbb{RP}^2$ (Theorem 7.1). We give a sketch of how Theorem 8.12 follows from Theorem 7.1 when $n = 3$.

Sketch of a proof of Theorem 8.12 assuming Theorem 7.1. A convex real projective structure on S is a pair (N, f), where N is the quotient Ω/Γ of a strictly convex domain Ω in $\mathbb{RP}^2$ by a discrete subgroup Γ of $\mathrm{PSL}(3,\mathbb{R})$, and $f : S \to N$ is a diffeomorphism.

Starting from a representation $\rho : \pi_1(S) \to \mathrm{PSL}(3,\mathbb{R})$ in the Hitchin component, let $\Omega_\xi \subset \mathbb{RP}^2$ be the strictly convex domain bounded by the convex curve $\xi(S^1) \subset \mathbb{RP}^2$. Then $\rho\big(\pi_1(S)\big)$ acts freely and properly discontinuously on Ω_ξ. The quotient $\Omega_\xi/\rho\big(\pi_1(S)\big)$ is a real projective convex manifold, diffeomorphic to S. Conversely given a real projective structure on S, we can ρ-equivariantly identify S^1 (identified with the boundary of $\pi_1(S)$) with the boundary of Ω and get a convex curve $\xi : S^1 \to \partial\Omega \subset \mathbb{RP}^2$. $\square$

Inspired by this proof and with Theorem 7.1 at hand for arbitrary n, one might try to follow a similar strategy on order to find geometric structures parametrized by the Hitchin component for $\mathrm{PSL}(n,\mathbb{R})$. This works for $n = 4$, where we obtain the following

Theorem 8.13 ([?]). *The Hitchin component for* $\mathrm{PSL}(4,\mathbb{R})$ *is naturally homeomorphic to the moduli space of properly convex foliated projective structures on* T^1S.

Properly convex foliated projective structures are locally homogeneous $\big(\mathrm{PSL}(4,\mathbb{R}), \mathbb{RP}^3\big)$-structures on T^1S satisfying the following additional conditions:

- every orbit of the geodesic flow is locally a projective line,
- every (weakly) stable leaf of the geodesic flow is locally a projective plane and the projective structure on the leaf obtained by restriction is convex.

Using the convex curve provided by Theorem 7.1 one can consider the corresponding discriminant surface $\Delta \subset \mathbb{RP}^3$, i.e. the union of all its tangent lines. The complement $\mathbb{RP}^3 - \Delta$ consists of two connected components, on both of which $\rho\big(\pi_1(S)\big)$ acts properly discontinuous. The quotient of one of the connected components by $\pi_1(S)$ is homeomorphic to T^1S, equipped with a properly convex foliated projective structure. The main work goes into establishing the converse direction, i.e. showing that the holonomy representation of a properly convex foliated projective structure on T^1S lies in the Hitchin component – this is rather tedious.

Remark 8.14. The above theorem implies that the Hitchin component for $\mathrm{PSp}(4,\mathbb{R})$ is naturally homeomorphic to the moduli space of properly convex foliated projective contact structures on the unit tangent bundle of S.

For $n \geq 5$ the above strategy seems to fail in general.

The first step in the strategy described above to find geometric structures which are parametrized by representations $\rho : \pi_1(S) \to G$ in higher Teichmüller spaces is to find domains of discontinuity for such representations in homogeneous spaces, more precisely in generalized flag varieties associated to G, on which $\pi_1(S)$ is supposed to act with compact quotient. This problem becomes more difficult the bigger G gets, since $\pi_1(S)$ is a group of cohomological dimension 2, whereas the dimension of the generalized flag varieties grows as G get bigger. So it comes a bit as a surprise that finding domains of discontinuity with compact quotient can be accomplished in the very general setting of Anosov representations.

Theorem 8.15 ([**?**, **?**]). *Let G be a semisimple Lie group and assume that no simple factor of G is locally isomorphic to* $\mathrm{PSL}(2,\mathbb{R})$*. Let $\rho : \pi_1(S) \to G$ be a (P_+, P_-)-Anosov representation. Let $P = MAN$ be the minimal parabolic subgroup of G. Then there exists an open non-empty set $\Omega_\rho \subset G/AN$, on which $\pi_1(S)$ acts freely, properly discontinuous and with compact quotient.*

Remark 8.16. The homogeneous space G/AN is the maximal compact quotient of G. In many cases the domain Ω_ρ descends to a domain of discontinuity in G/P.

Remark 8.17. Theorem 8.15 holds more general for Anosov representations of convex cocompact subgroups of Hadamard manifolds of strictly negative curvature or even of hyperbolic groups. The reader interested in the more general statement is referred to [**?**].

The main tool in order to define the domain of discontinuity Ω_ρ are the ρ-equivariant continuous boundary maps $\xi_\pm : S^1 \to G/P_\pm$ associated to the (P_+, P_-)-Anosov representation (see Proposition 8.5).

There is some evidence that – at least in the case of higher Teichmüller spaces – the quotients $\Omega_\rho/\rho\big(\pi_1(S)\big)$ are homeomorphic to the total spaces of bundles over S with compact fibers. This has been established for maximal representation into $\mathrm{Sp}(2n,\mathbb{R})$ as well as for Hitchin representations into $\mathrm{SL}(2n,\mathbb{R})$.

Theorem 8.18 ([**?**]).

1. *The Hitchin component for* $\mathrm{SL}(2n,\mathbb{R})$ *parametrizes real projective structures on a compact manifold M, which is topologically a* $\mathrm{O}(n)/\mathrm{O}(n-2)$*-bundle over the surface S.*
2. *Maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *parametrize real projective structures on a compact manifold M homeomorphic to an* $\mathrm{O}(n)/\mathrm{O}(n-2)$*-bundle over the surface S. Its isomorphism type depends on the connected component containing the representation.*

8.4 Topological invariants

The Hitchin component is by definition a single connected component, but the space of maximal representations is a priori only a union of connected components, and their might be more than one. In many cases the exact number of connected components of the space of maximal representations has been computed using methods from the theory of Higgs bundles [**?**, **?**, **?**, **?**, **?**]. And the most interesting family in terms of the number of connected components are maximal representations into symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$: there are 3×2^{2g} connected components when $n \geq 3$ [**?**] and $(3 \times 2^{2g} + 2g - 4)$ connected components when $n = 2$ [**?**].

Invariants to distinguish these connected components can be derived from the associated Higgs bundles, but topological invariants to distinguish the different connected components also arise from considering maximal representations as Anosov representations.

Recall that in the definition of Anosov structures one considers the flat G-bundle $G(\rho)$ over T^1S and the associated bundle $\mathcal{O}(\rho)$. The first part of the data of an Anosov structure is a section σ of $\mathcal{O}(\rho)$. Since $\mathcal{O}(\rho)$ is the G/H-bundle associated to $G(\rho)$ its sections are in one-to-one correspondence with reductions of the structure group of $G(\rho)$ from G to H. In general there is no canonical section, but in the case of Anosov structures we have

Proposition 8.19 ([**?**]). *If a section σ of $\mathcal{O}(\rho)$ with the properties required in Definition 8.1 exists, then it is unique.*

As a consequence an Anosov representation $\rho : \pi_1(S) \to G$ gives a canonical reduction of the G-principal bundle $G(\rho)$ to an H-principal bundle. This H-bundle is in general not flat; its characteristic classes give topological invariants of the Anosov representation ρ which live in $\mathrm{H}^*(T^1S)$.

In the situation of maximal representations $\rho : \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R})$, we have that $H = \mathrm{GL}(n,\mathbb{R})$, embedded into $\mathrm{Sp}(2n,\mathbb{R})$ as the stabilizer of two transverse Lagrangian subspaces. The topological invariants of significance are first and second Stiefel–Whitney classes, as well as an Euler class if $n = 2$.

Theorem 8.20 ([**?**]). *The topological invariants distinguish the connected components of $\mathrm{Hom}_{max}\big(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})\big) \setminus \mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})\big)$.*

Considering Hitchin representations as (P_{min}, P_{min}^{opp})-Anosov representations there is an additional first Stiefel–Whitney class, which distinguishes the connected components of $\mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})\big)$.

The invariants constructed using the Anosov property of maximal representations are in principle computable for a given representation $\rho : \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R})$. Explicit computations for various representations allows us to describe model representations in any connected component. This is of particular interest for $\mathrm{Sp}(4,\mathbb{R})$ as there are $2g-3$ connected components in which every representation is Zariski dense (see also [**?**]).

Besides the *irreducible Fuchsian representation* which were introduced to define the Hitchin component, there are two other kinds of model representations: A *twisted diagonal representation* is a maximal representation

$$\rho_\theta = (\iota \otimes \theta) : \pi_1(S) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(n) \subset \mathrm{Sp}(2n,\mathbb{R}),$$

where $\iota : \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ is a discrete embedding and $\theta : \pi_1(S) \to \mathrm{O}(n)$ is an orthogonal representation; $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(n)$ sits in $\mathrm{Sp}(2n,\mathbb{R})$ as the normalizer of the diagonal embedding

$$\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})^n \subset \mathrm{Sp}(2n,\mathbb{R}).$$

A *hybrid representation* is a maximal representation

$$\rho_k = \rho_1 * \rho_2 : S = S_1 \cup_\gamma S_2 \to \mathrm{Sp}(2n,\mathbb{R}),$$

$k = 3-2g, \cdots, -1$, which is obtained by amalgamation of an irreducible Fuchsian representation on $\pi_1(S_1)$ and a suitable deformation of an (untwisted) diagonal representation on $\pi_1(S_2)$. The subscript k indicates the Euler characteristic of S_1. The construction of hybrid representations relies on the additivity of the Toledo number and the Euler characteristic under gluing (see Proposition 5.10).

Theorem 8.21 ([?]). *When $n \geq 3$ any maximal representation $\rho : \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R})$ can be deformed either to an irreducible Fuchsian representation or to a twisted diagonal representation.*

When $n = 2$ there are $2g-3$ connected components $\mathcal{H}_k$, $k = 1, \cdots, 2g-3$ of $\mathrm{Hom}_{max}\big(\pi_1(S), \mathrm{Sp}(4,\mathbb{R})\big)$ in which every representation has Zariski dense image. Representations in $\mathcal{H}_k$ can be deformed to k-hybrid representations.

The information about model representations in each connected component can be used to obtain further information about the holonomies of maximal representations.

For representations in the Hitchin components, Theorem 6.5 implies that for every $\gamma \in \pi_1(S) - \{1\}$ the image $\rho(\gamma)$ is diagonalizable over $\mathbb{R}$ with distinct eigenvalues. This does not hold for the other components of maximal representations.

The Anosov property implies that for every $\gamma \in \pi_1(S) \setminus \{e\}$ the image $\rho(\gamma)$ is conjugate to an element in $\mathrm{GL}(n,\mathbb{R}) < \mathrm{Sp}(2n,\mathbb{R})$. More precisely, we have:

Theorem 8.22 ([?]). *Let $\mathbf{H}$ be a connected component of*

$$\mathrm{Hom}_{max}\big(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})\big) \setminus \mathrm{Hom}_{Hit}\big(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})\big),$$

and let $\gamma \in \pi_1(S) - \{1\}$ be an element corresponding to a simple curve. Then there exist

(1) *a representation $\rho \in \mathbf{H}$ such that the Jordan decomposition of $\rho(\gamma)$ in $\mathrm{GL}(n,\mathbb{R})$ has a nontrivial parabolic component.*

(2) *a representation* $\rho' \in \mathbf{H}$ *such that the Jordan decomposition of* $\rho(\gamma)$ *in* $\mathrm{GL}(n, \mathbb{R})$ *has a nontrivial elliptic component.*

This results indicates that understanding the structure of the space of maximal representations is much more complicated than understanding the structure of Hitchin components, since already the conjugacy classes in which the holonomy of one element can lie in might differ from connected component to connected component.

9 Open questions and further directions

In the previous sections we already mentioned some open questions regarding the quotients of higher Teichmüller spaces, their compactifications as well as their geometric significance. In this section we want to conclude our survey with mentioning some further directions in the study of higher Teichmüller spaces which to our knowledge have not yet been explored.

9.1 Positivity, causality and other groups

As we pointed out in § 7.1 an underlying common structure of higher Teichmüller spaces is that the homomorphisms in them admit equivariant boundary maps which satisfy some positivity or causality property.

The relation between positive triples in the full flag variety and the weaker notion of maximal triples in the space of Lagrangians discussed in § 7.2 is very special. It would be very interesting to discover weaker notions of positivity of k-tuples in (partial) flag varieties which then extends to other groups which are neither split real forms nor of Hermitian type. Such notions of positivity might lead to discovering higher Teichmüller spaces for other Lie groups G, which are again characterized by the existence of special boundary maps.

A first family of groups to look at could be $G = \mathrm{PO}(p, q)$, which is of Hermitian type if $(p, q) = (2, q)$ and a split real form if $(p, q) = (n, n + 1)$ or $(p, q) = (n, n)$.

Every time there is a notion of positivity or cyclic ordering, the images of boundary maps tend to be more regular, namely rectifiable circles. This contrasts with the case of quasifuchsian deformations into $\mathrm{PSL}(2, \mathbb{C})$ of compact surface groups in $\mathrm{PSL}(2, \mathbb{R})$, where in fact the limit set, or – what is the same – the image of the boundary map, is a topological circle with Hausdorff dimension larger than 1, unless the deformed group is Fuchsian.[3] This suggests

[3] This is due to [?]: see also the footnote on p. 76 of Fricke's address in Chicago in 1893, [?].

to study the deformations of the homomorphism

$$i \circ \rho : \pi_1(S) \to \mathbf{G}(\mathbb{C}),$$

where $i : G = \mathbf{G}(\mathbb{R})^\circ \to \mathbf{G}(\mathbb{C})$ is the natural inclusion and $\rho : \pi_1(S) \to G$ is either a maximal representation into a group of Hermitian type or a Hitchin representation into a real split Lie group. Observe that $i \circ \rho$ is Anosov for a suitable pair of parabolic subgroups and, as a result, small deformations of $i \circ \rho$ are as well.

9.2 Coordinates and quantizations for maximal representations

Fock and Goncharov describe explicit coordinate for the space of positive representations. For $\mathrm{PSL}(n, \mathbb{R})$ these coordinates have a particular nice form. Based on the explicit coordinate system they describe the cluster variety structure and quantizations of the space of positive representations.

It would be interesting to construct similar explicit coordinate systems for the space of maximal representations, in particular when $G = \mathrm{Sp}(2n, \mathbb{R})$. Theorem 8.22 gives a hint that constructing coordinates for the space of maximal representations is more involved. The structure of the coordinates also needs to be more complicated as they have to model the singularities of the space of maximal representations.

The additivity of the Toledo number on the other hand implies that the space of maximal representations of a compact surface S can be built out of the space of maximal representations of a pair of pants.

Having coordinates at hand, one might also ask for quantizations of the space of maximal representations or try to express the symplectic form on the space of maximal representations explicitly in coordinates.