ISOMETRIC PROPERTIES OF RELATIVE BOUNDED COHOMOLOGY

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Abstract. We show that the isomorphism induced by the inclusion of pairs $(X, \emptyset) \subset (X, Y)$ between the relative bounded cohomology of (X, Y) and the bounded cohomology of X is isometric in degree at least 2 if the fundamental group of each connected component of Y is amenable.

1. Introduction

In the mid seventies, Gromov introduced the bounded cohomology of a space and showed that it vanishes in all degrees $n \geq 1$ for simply connected CW-complexes [9]. Brooks pointed out that this implies that the bounded cohomology of a space is isomorphic to the one of its fundamental group [1]. In his note he also made the first step towards the relative homological algebra approach of the bounded cohomology of groups. Ivanov then developed this approach (with trivial coefficients) [10], incorporating the seminorm into the theory. This lead to the final form of Gromov's theorem, namely that for a countable CW-complex the bounded cohomology is isometrically isomorphic to the bounded cohomology of its fundamental group. We emphasize that, here and in the sequel, the coefficients are the trivial module $\mathbb{R}$.

Bounded cohomology can be defined for pairs (X, Y) of spaces, that is Y is a subspace of the space X, and there is an exact sequence

$$\dots \longrightarrow \mathrm{H}_{\mathrm{b}}^{n-1}(Y) \xrightarrow{\delta_n} \mathrm{H}_{\mathrm{b}}^{n}(X, Y) \xrightarrow{j_n} \mathrm{H}_{\mathrm{b}}^{n}(X) \xrightarrow{i_n} \mathrm{H}_{\mathrm{b}}^{n}(Y) \longrightarrow \dots ,$$

where j_n is induced by the inclusion of the corresponding cochain complexes, i_n is induced by the restriction map and δ_n is the connecting homomorphism.

A striking consequence of this long exact sequence arises when we assume that each connected component of Y has amenable fundamental group. Indeed, as observed by Trauber in the 70's, one of the characteristic features of bounded group cohomology is that it vanishes for amenable groups in degree $n \geq 1$. This implies that j_n is an isomorphism of vector spaces for $n \geq 2$. In low degree, the

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isomorphism does not hold. Instead, it follows from $\mathrm{H}_\mathrm{b}^1(X)=0$ that we have an exact sequence

$$\mathrm{H}_\mathrm{b}^0(X,Y) \xhookrightarrow{j_0} \mathrm{H}_\mathrm{b}^0(X) \xrightarrow{i_0} \mathrm{H}_\mathrm{b}^0(Y) \xrightarrow{\delta_1} \mathrm{H}_\mathrm{b}^1(X,Y)\,.$$

If X is path connected then $\mathrm{H}_\mathrm{b}^0(X,Y)=0$ and $\mathrm{H}_\mathrm{b}^0(X)=\mathbb{R}$, while $\mathrm{H}_\mathrm{b}^0(Y)=\ell^\infty(\pi_0(Y))$. Here and in the sequel, $\ell^\infty(S)$ is the Banach space of all bounded real valued functions on the set S.

A sore point in this theory has been the question whether, under the above hypotheses, j_n is isometric. Our goal is to prove the following:

Theorem 1. *Let $X\supset Y$ be a pair of countable CW-complexes. Assume that each connected component of Y has amenable fundamental group. Then the morphism obtained from the inclusion*

$$j_n:\mathrm{H}_\mathrm{b}^n(X,Y)\longrightarrow \mathrm{H}_\mathrm{b}^n(X)$$

is an isometric isomorphism for every $n\geq 2$.

We briefly illustrate the use of this result to obtain Milnor–Wood type inequalities. Let M be a compact oriented manifold with boundary ∂M, and G a topological group with a given continuous bounded class $\kappa^\mathrm{b}\in\mathrm{H}_\mathrm{cb}^n(G,\mathbb{R})$, where $n=\dim M$; assume that every connected component of ∂M has amenable fundamental group. Using Gromov's isomorphism, for every homomorphism $\rho:\pi_1(M)\to G$ we obtain by pullback a class

$$\rho^*(\kappa^\mathrm{b})\in\mathrm{H}_\mathrm{b}^n(\pi_1(M))\cong\mathrm{H}_\mathrm{b}^n(M)$$

in the bounded cohomology of M and, if $n\geq 2$, a bounded relative class

$$j_n^{-1}(\rho^*(\kappa^\mathrm{b}))\in\mathrm{H}_\mathrm{b}^n(M,\partial M)\,,$$

whose evaluation on the relative fundamental class gives an invariant

$$\mathrm{T}(\rho,M):=\langle j_n^{-1}(\rho^*(\kappa^\mathrm{b})),[M,\partial M]\rangle\,,$$

which generalizes the Toledo invariant for surfaces introduced in [4], as well as the volume of a representation defined in [2]. The following is an immediate corollary of Theorem 1 and the fact that the pullback is norm decreasing:

Corollary 2. *There is the inequality*

$$|\,\mathrm{T}(\rho,M)|\leq\|\kappa^\mathrm{b}\|\,\|[M,\partial M]\|_1\,,$$

where $\|[M,\partial M]\|_1$ denotes the norm of the relative fundamental class in ℓ^1-homology and $\|\kappa^\mathrm{b}\|$ is the canonical norm of the bounded continuous class κ^b in $\mathrm{H}_\mathrm{cb}^n(G,\mathbb{R})$.

This inequality is particularly useful if one knows the values of both norms occurring on the right hand side. This happens for example in the two following notable cases.

In the case of degree 2, if G is a connected semisimple Lie group of non-compact type and with finite center, whose associated symmetric spaces G/K is Hermitian and $\kappa^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ is the bounded Kähler class (defined in [3]). The canonical norm of κ^{b} has been computed in [6, 5] and is given by

$$\|\kappa^{\mathrm{b}}\| = \frac{\mathrm{rank}(G)}{2}$$

and, since M is a compact surface possibly without boundary, the equality

$$\|[M,\partial M]\|_1 = 2|\chi(M)|$$

is easily established.

In the real hyperbolic case $G = \mathrm{SO}(n,1)^\circ$, the volume form on real hyperbolic space $\mathcal{H}^n_{\mathbb{R}}$ gives rise to a bounded continuous class $\kappa^{\mathrm{b}} \in H^n_{cb}(G,\mathbb{R})$. Moreover if M is the compact core of a finite volume quotient N of $\mathcal{H}^n_{\mathbb{R}}$, then

$$\|[M,\partial M]\|_1 = \frac{\mathrm{Vol}(N)}{v_n} \qquad \text{and} \qquad \|\kappa^{\mathrm{b}}\| = v_n\,,$$

where v_n is the volume of the ideal regular simplex in $\mathcal{H}^n_{\mathbb{R}}$. The norm of the volume class is computed in [9, 13]. In the latter reference, Thurston gives, in the 3-dimensional real hyperbolic case, a proof of the proportionality between the relative simplicial volume and the volume for the norm coming from measure homology, but it is only much later [7] that this norm is indeed shown to be equal to the relative simplicial volume. In the meantime, a direct proof valid in all dimensions of the proportionality principle for hyperbolic manifolds in the relative case has been given in [8].

2. Resolutions in bounded cohomology

Let X be a space, where here and in the sequel by a space we will always mean a countable CW-complex. We denote by $\mathrm{C}^n_{\mathrm{b}}(X)$ the complex of bounded real valued n-cochains on X and, if $Y \subset X$ is a subspace, by $\mathrm{C}^n_{\mathrm{b}}(X,Y)$ the subcomplex of those bounded cochains that vanish on simplices with image contained in Y. All these spaces of cochains are endowed with the ℓ^∞-norm and the corresponding cohomology groups are equipped with the corresponding quotient seminorm[1].

For our purposes, it is important to observe that the universal covering map $p : \widetilde{X} \to X$ induces an isometric identification of the complex $\mathrm{C}^n_{\mathrm{b}}(X)$ with the complex $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X})^\Gamma$ of $\Gamma := \pi_1(X)$-invariant bounded cochains on $\widetilde{X}$. Similarly, if $Y' := p^{-1}(Y)$, we obtain an isometric identification of the complex $\mathrm{C}^n_{\mathrm{b}}(X,Y)$ with the complex $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X},Y')^\Gamma$ of Γ-invariants of $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X},Y')$.

[1]This definition of seminorm is the one introduced by Gromov [9, Section 4.1]. The seminorm defined by Park [12] was shown to be different from Gromov's by Frigerio and Pagliantini [7].

The main ingredient in the proof of Theorem 1, which is also essential in the proof of Gromov's theorem, is the result of Ivanov [10] that the complex of Γ-invariants of

$$\mathbb{R} \longrightarrow \mathrm{C}^0_{\mathrm{b}}(\widetilde{X}) \longrightarrow \mathrm{C}^1_{\mathrm{b}}(\widetilde{X}) \longrightarrow \cdots$$

computes the bounded cohomology of Γ. In fact, we will use the more precise statement that the latter complex is a strong resolution of $\mathbb{R}$ by relatively injective Γ-Banach modules (see [10] for the definitions of strong resolutions and relatively injective modules).

By standard homological algebra techniques [10], it follows from the fact that $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X})$ is a strong resolution by Γ-modules and $\ell^\infty(\Gamma^{\bullet+1})$ is a cochain complex (even a strong resolution) by relatively injective Γ-modules that there exists a Γ-morphism of complexes

$$g_n : \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}) \longrightarrow \ell^\infty(\Gamma^{n+1}) \tag{$\Diamond$}$$

extending the identity, and such that g_n is contracting, i.e. $\|g_n\| \leq 1$, for $n \geq 0$. This map induces Ivanov's isometric isomorphism $\mathrm{H}^\bullet_{\mathrm{b}}(X) \to \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma)$.

The second result we need lies at the basis of the fact that the bounded cohomology of Γ can be computed isometrically from the complex of bounded functions on any amenable Γ-space. We will need only a particular case of the isomorphism, which is the existence of a contracting map between the complexes $\ell^\infty(\Gamma^{n+1})$ and the space of alternating bounded functions $\ell^\infty_{\mathrm{alt}}(S^{n+1})$ when S is a discrete amenable Γ-space. This is a very special case of [11], for which we present a direct proof.

Proposition 3 ([11, Theorem 7.2.1]). *Assume that Γ is a group acting on a set S such that all stabilizers are amenable subgroups of Γ. Then for $n \geq 0$ there is a Γ-morphism of complexes*

$$\mu_n : \ell^\infty(\Gamma^{n+1}) \longrightarrow \ell^\infty_{\mathrm{alt}}(S^{n+1})$$

extending $\mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ that is contracting.

Proof. Alternation gives a contracting Γ-morphism of complexes

$$\ell^\infty(S^{n+1}) \longrightarrow \ell^\infty_{\mathrm{alt}}(S^{n+1}),$$

so that it suffices to construct a contracting Γ-morphism of complexes

$$\mu_n : \ell^\infty(\Gamma^{n+1}) \longrightarrow \ell^\infty(S^{n+1}).$$

We first construct μ_0 and then inductively μ_n, for $n \geq 1$. Identify S with a disjoint union $\sqcup_{i\in I}\Gamma/\Gamma_i$ of right cosets, where $\Gamma_i < \Gamma$ is amenable and let $\lambda_i \in \ell^\infty(\Gamma_i)^*$ be a left Γ_i-invariant mean, for every $i \in I$. We define $\mu_0 : \ell^\infty(\Gamma) \to \ell^\infty(S)$ for $f \in \ell^\infty(\Gamma)$ by setting $\mu_0(f)(\gamma\Gamma_i)$ to be the Γ_i-invariant mean λ_i of the bounded function $\Gamma_i \to \mathbb{R}$ defined by $\eta \mapsto f(\gamma\eta)$. Clearly $\mu_0(\mathbb{1}_\Gamma) = \mathbb{1}_S$, so that μ_0 extends $\mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $\|\mu_0\| \leq 1$.

Assume now that we have defined $\mu_{n-1} : \ell^\infty(\Gamma^n) \to \ell^\infty(S^n)$. Then we define μ_n as the composition of the following maps:

$$\begin{tikzcd}
\ell^\infty(\Gamma^{n+1}) \arrow[r, "="] \arrow[dddd, dashed, "\mu_n"'] & \ell^\infty(\Gamma \times \Gamma^n) \arrow[r, "\cong"] & \ell^\infty(\Gamma, \ell^\infty(\Gamma^n)) \arrow[d] \\
 & & \ell^\infty(\Gamma, \ell^\infty(S^n)) \arrow[d, "\cong"] \\
 & & \ell^\infty(S^n, \ell^\infty(\Gamma)) \arrow[d] \\
 & & \\
\ell^\infty(S^{n+1}) & \ell^\infty(S \times S^n) \arrow[l, "="'] & \ell^\infty(S^n, \ell^\infty(S)) \arrow[l]
\end{tikzcd},$$

where $\cong$ denotes a Banach space isomorphism, while the first vertical arrow is induced by $\mu_{n-1} : \ell^\infty(\Gamma^n) \to \ell^\infty(S^n)$ and the third by $\mu_0 : \ell^\infty(\Gamma) \to \ell^\infty(S)$. Since all morphisms involved are contracting and equivariant for suitable Γ-actions, the same holds for μ_n. Finally one verifies that $(\mu_n)_{n\geq 0}$ is a morphism of complexes. $\square$

3. Proof of Theorem 1

Let, as above, $p : \widetilde{X} \to X$ be the universal covering map, $\Gamma := \pi_1(X)$ and $Y = \sqcup_{i\in I} C_i$ the decomposition of Y into a union of connected components. If $\check{C}_i$ is a choice of a connected component of $p^{-1}(C_i)$ and Γ_i denotes the stabilizer of $\check{C}_i$ in Γ then

$$p^{-1}(C_i) = \bigsqcup_{\gamma\in\Gamma/\Gamma_i} \gamma\check{C}_i\,.$$

Let $\mathcal{F} \subset \widetilde{X} \smallsetminus Y'$ be a fundamental domain for the Γ-action on $\widetilde{X} \smallsetminus Y'$, where $Y' = p^{-1}(Y)$ as before. Define the Γ-equivariant map

$$r : \widetilde{X} \to S := \Gamma \sqcup \bigsqcup_{i\in I} \Gamma/\Gamma_i$$

as follows:

$$r(\gamma x) := \begin{cases} \gamma \in \Gamma & \text{if } x \in \mathcal{F}, \\ \gamma\Gamma_i \in \Gamma/\Gamma_i & \text{if } x \in \check{C}_i\,. \end{cases}$$

For every $n \geq 0$ define

$$r_n : \ell^\infty_{\mathrm{alt}}(S^{n+1}) \longrightarrow \mathrm{C}^n_{\mathrm{b}}(\widetilde{X})$$

by

$$r_n(c)(\sigma) = c(r(\sigma_0), \dots, r(\sigma_n))\,,$$

where $c \in \ell^\infty_{\mathrm{alt}}(S^{n+1})$ and $\sigma_0, ..., \sigma_n \in \widetilde{X}$ are the vertices of a singular simplex $\sigma : \Delta^n \to \widetilde{X}$. Clearly $(r_n)_{n\geq 0}$ is a Γ-morphism of complexes extending the identity on $\mathbb{R}$ and $\|r_n\| \leq 1$ for all $n \geq 0$.

Observe that if $n \geq 1$ and $\sigma(\Delta^n) \subset Y'$, then there are $i \in I$ and $\gamma \in \Gamma$ such that $\sigma(\Delta^n) \subset \gamma \check{C}_i$. Thus

$$r(\sigma_0) = \cdots = r(\sigma_n) = \gamma\Gamma_i$$

and thus

$$r_n(c)(\sigma) = c(\gamma\Gamma_i, \dots, \gamma\Gamma_i) = 0\,,$$

since c is alternating. This implies that the image of r_n is in $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X}, Y')$. Thus we can write $r_n = j_n \circ r'_n$, where $j_n : \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}, Y') \hookrightarrow \mathrm{C}^n_{\mathrm{b}}(\widetilde{X})$ is the inclusion and $r'_n : \ell^\infty_{\mathrm{alt}}(S^{n+1}) \to \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}, Y')$ is a norm decreasing Γ-morphism that induces a norm non-increasing map[2] in cohomology

$$\mathrm{H}(r'_n) : \mathrm{H}^n(\ell^\infty_{\mathrm{alt}}(S^{\bullet+1})^\Gamma) \longrightarrow \mathrm{H}^n_{\mathrm{b}}(X, Y)\ ,$$

for $n \geq 1$.

Using the map g_n defined in ($\diamondsuit$) and the map μ_n provided by Proposition 3 since, for all i, the group Γ_i is a quotient of $\pi_1(C_i)$, and hence amenable, we have the following diagram

$$\begin{tikzcd}
\mathrm{C}^n_{\mathrm{b}}(\widetilde{X}) \arrow[r, "g_n"] \arrow[rrrd, dashed, "\text{extends } \mathrm{Id}_{\mathbb{R}}"'] & \ell^\infty(\Gamma^{n+1}) \arrow[r, "\mu_n"] & \ell^\infty_{\mathrm{alt}}(S^{n+1}) \arrow[r, "r'_n", "\text{for } n\geq 1"'] \arrow[rd, "r_n"] & \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}, Y') \arrow[d, "j_n"] \\
 & & & \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}),
\end{tikzcd}$$

where the dotted map is the composition $r_n \circ \mu_n \circ g_n$ which is a Γ-morphism of strong resolutions by relatively injective modules extending the identity, and hence induces the identity on $\mathrm{H}^n_{\mathrm{b}}(X) = \mathrm{H}^n(\mathrm{C}^\bullet_{\mathrm{b}}(\widetilde{X})^\Gamma)$.

We proceed now to show that, for $n \geq 2$, the map

$$\mathrm{H}(j_n) : \mathrm{H}^n_{\mathrm{b}}(X, Y) \longrightarrow \mathrm{H}^n_{\mathrm{b}}(X)$$

induced by j_n is an isometric embedding in cohomology. In view of the long exact sequence for pairs in bounded cohomology and the fact that $\mathrm{H}^\bullet_{\mathrm{b}}(Y) = 0$ in degree greater than 1, we already know that $\mathrm{H}(j_n)$ is an isomorphism. From the above it follows that

$$\mathrm{H}(j_n)\mathrm{H}(r'_n \circ \mu_n \circ g_n) = \mathrm{Id}_{\mathrm{H}^n_{\mathrm{b}}(X)}\,.$$

Let $y \in \mathrm{H}^n_{\mathrm{b}}(X, Y)$ and set $x = \mathrm{H}(j_n)(y)$. Then $\mathrm{H}(j_n)(\mathrm{H}(r'_n \circ \mu_n \circ g_n)(x)) = x$ and, as $\mathrm{H}(j_n)$ is injective, we get

$$y = \mathrm{H}(r'_n \circ \mu_n \circ g_n)(x)\,.$$

[2]To avoid confusion, we use here a different notation for the cochain map and the induced cohomology map. This is contrary to our notation in the introduction.

Since the maps $\mathrm{H}(j_n)$ and $\mathrm{H}(r'_n \circ \mu_n \circ g_n)$ are norm nonincreasing it follows that

$$\|x\| = \|\mathrm{H}(j_n)(y)\| \leq \|y\| \quad \text{and} \quad \|y\| = \|\mathrm{H}(r'_n \circ \mu_n \circ g_n)(x)\| \leq \|x\|$$

so that $\|\mathrm{H}(j_n)(y)\| = \|x\| = \|y\|$ and hence $\mathrm{H}(j_n)$ is norm preserving.

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