A Dual Interpretation of the Gromov–Thurston Proof of Mostow Rigidity and Volume Rigidity for Representations of Hyperbolic Lattices

Michelle Bucher, Marc Burger, and Alessandra Iozzi

Abstract We use bounded cohomology to define a notion of volume of an $\mathrm{SO}(n,1)$-valued representation of a lattice $\Gamma < \mathrm{SO}(n,1)$ and, using this tool, we give a complete proof of the volume rigidity theorem of Francaviglia and Klaff (Geom. Dedicata 117, 111–124 (2006)) in this setting. Our approach gives in particular a proof of Thurston's version of Gromov's proof of Mostow Rigidity (also in the non-cocompact case), which is dual to the Gromov–Thurston proof using the simplicial volume invariant.

Keywords Rigidity · Real hyperbolic manifold · Bounded cohomology · Maximal representations

Mathematics Subject Classification (2010) 53C24 · 22E40 · 22E41

1 Introduction

Strong rigidity of lattices was proved in 1965 by Mostow [28] who, while searching for a geometric explanation of the deformation rigidity results obtained by Selberg [32], Calabi–Vesentini [14, 15] and Weil [35, 36], showed the remarkable fact that, under some conditions, topological data of a manifold determine its metric. Namely, he proved that if $M_i = \Gamma_i \backslash \mathbb{H}^n$, $i = 1, 2$ are compact quotients of real hyperbolic n-space and $n \geqslant 3$, then any homotopy equivalence $\varphi : M_1 \to M_2$ is, up to homotopy, induced by an isometry. Shortly thereafter, this was extended to the finite volume case by G. Prasad [29].

M. Bucher
Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, Case postale 64, 1211 Geneva 4, Switzerland
e-mail: Michelle.Bucher-Karlsson@unige.ch

M. Burger · A. Iozzi (✉)
Department Mathematik, ETH, Rämistrasse 101, 8092 Zurich, Switzerland
e-mail: iozzi@math.ethz.ch

M. Burger
e-mail: burger@math.ethz.ch

M.A. Picardello (ed.), *Trends in Harmonic Analysis*, Springer INdAM Series 3.
DOI 10.1007/978-88-470-2853-1_4. © Springer-Verlag Italia 2013

The methods introduced by Mostow emphasized the role of the quasi-isometries of $\widetilde{M}_i = \mathbb{H}^n$, their quasi-conformal extension to $\partial\mathbb{H}^n$, ergodicity phenomena of the Γ_i-action on $\partial\mathbb{H}^n$, as well as almost everywhere differentiability results *à la* Egorov.

In the 1970's, a new approach for rigidity in the real hyperbolic case was developed by Gromov. In this context he introduced ℓ^1-homology and the simplicial volume: techniques like smearing and straightening became important. This approach was then further developed by Thurston [33, Chap. 6] and one of its consequences is an extension to hyperbolic manifolds of Kneser's theorem for surfaces [25]. To wit, the computation of the simplicial volume $\|M\| = \mathrm{Vol}(M)/v_n$ implies, for a continuous map $f : M_1 \to M_2$ between compact real hyperbolic manifolds, that

$$\deg f \leqslant \frac{\mathrm{Vol}(M_2)}{\mathrm{Vol}(M_1)}.$$

If $\dim M_i \geqslant 3$, Thurston proved that equality holds if and only if f is homotopic to an isometric covering while the topological assertion in the case in which $\dim M_i = 2$ is Kneser's theorem [25].

The next step, in the spirit of Goldman's theorem [20]—what now goes under the theory of *maximal representations*—is to associate an invariant $\mathrm{Vol}(\rho)$ to an arbitrary representation

$$\rho : \pi_1(M) \to \mathrm{Isom}\big(\mathbb{H}^n\big)$$

of the fundamental group of M, satisfying a Milnor–Wood type inequality

$$\mathrm{Vol}(\rho) \leqslant \mathrm{Vol}(i).$$

The equality should be characterized as given by the "unique" lattice embedding i of $\pi_1(M)$, of course provided $\dim M \geqslant 3$. This was carried out in $\dim M = 3$ by Dunfield [17], following Toledo's modification of the Gromov–Thurston approach to rigidity [34].

If M is only of finite volume, a technical difficulty is the definition of the volume $\mathrm{Vol}(\rho)$ of a representation. Dunfield introduced for this purpose the notion of pseudodeveloping map and Francaviglia proved that the definition is independent of the choice of the pseudodeveloping map [18]. Then Francaviglia and Klaff [19] proved a "volume rigidity theorem" for representations

$$\rho : \pi_1(M) \to \mathrm{Isom}\big(\mathbb{H}^k\big),$$

where now k is not necessarily equal to $\dim M$. In their paper, the authors actually succeed in applying the technology developed by Besson–Courtois–Gallot in their seminal work on entropy rigidity [2]. An extension to representations of $\pi_1(M)$ into $\mathrm{Isom}(\mathbb{H}^n)$ for an arbitrary compact manifold M has been given by Besson–Courtois–Gallot [3].

Finally, Bader, Furman and Sauer proved a generalization of Mostow Rigidity for cocycles in the case of real hyperbolic lattices with some integrability condition, using, among others, bounded cohomology techniques, [1].

The aim of this paper is to give a complete proof of volume rigidity from the point of view of bounded cohomology, implementing a strategy first described in [24] and used in the work on maximal representations of surface groups [12, 13], as well as in the proof of Mostow Rigidity in dimension 3 in [11].

Our main contribution consists on the one hand in identifying the top dimensional bounded equivariant cohomology of the full group of isometries $\mathrm{Isom}(\mathbb{H}^n)$, and on the other in giving a new definition of the volume of a representation of $\pi_1(M)$, when M is not compact; this definition, that uses bounded relative cohomology, generalizes the one introduced in [13] for surfaces.

In an attempt to be pedagogical, throughout the paper we try to describe, in varying details, the proof of all results.

Let $\mathrm{Vol}_n(x_0, \ldots, x_n)$ denote the signed volume of the convex hull of the points $x_0, \ldots, x_n \in \overline{\mathbb{H}^n}$. Then Vol_n is a $G^+ := \mathrm{Isom}^+(\mathbb{H}^n)$-invariant cocycle on $\overline{\mathbb{H}^n}$ and hence defines a top dimensional cohomology class $\omega_n \in H_c^n(G^+, \mathbf{R})$. Let $i : \Gamma \hookrightarrow G^+$ be an embedding of Γ as a lattice in the group of orientation preserving isometries of $\mathbb{H}^n$ and let $\rho : \Gamma \to G^+$ be an arbitrary representation of Γ. Suppose first that Γ is torsion free. Recall that the cohomology of Γ is canonically isomorphic to the cohomology of the n-dimensional quotient manifold $M := i(\Gamma)\backslash\mathbb{H}^n$.

If M is compact, by Poincaré duality the cohomology groups $H^n(\Gamma, \mathbf{R}) \cong H^n(M, \mathbf{R})$ in top dimension are canonically isomorphic to $\mathbf{R}$, with the isomorphism given by the evaluation on the fundamental class $[M]$. We define the volume $\mathrm{Vol}(\rho)$ of ρ by

$$\mathrm{Vol}(\rho) = \langle \rho^*(\omega_n), [M] \rangle,$$

where $\rho^* : H_c^n(G^+, \mathbf{R}) \to H^n(\Gamma, \mathbf{R})$ denotes the pull-back via ρ. In particular the absolute value of the volume of the lattice embedding i is equal to the volume of the hyperbolic manifold M, $\mathrm{Vol}(M) = \langle i^*(\omega_n), [M] \rangle$.

If M is not compact, the above definition fails since $H^n(\Gamma, \mathbf{R}) \cong H^n(M, \mathbf{R}) = 0$. Thus we propose the following approach: since Vol_n is in fact a bounded cocycle, it defines a bounded class $\omega_n^b \in H_{b,c}^n(G^+, \mathbf{R})$ in the bounded cohomology of G^+ with trivial $\mathbf{R}$-coefficients. Thus associated to a homomorphism $\rho : \Gamma \to G^+$ we obtain $\rho^*(\omega_n^b) \in H_b^n(\Gamma, \mathbf{R})$; since $\widetilde{M} = \mathbb{H}^n$ is contractible, it follows easily that $H_b^n(\Gamma, \mathbf{R})$ is canonically isomorphic to the bounded singular cohomology $H_b^n(M, \mathbf{R})$ of the manifold M (this is true in much greater generality [5, 21], but it will not be used here). To proceed further, let $N \subset M$ be a compact core of M, that is the complement in M of a disjoint union of finitely many horocyclic neighborhoods E_i, $i = 1, \ldots, k$, of cusps. Those have amenable fundamental groups and thus the map $(N, \partial N) \to (M, \emptyset)$ induces an isomorphism in cohomology, $H_b^n(N, \partial N, \mathbf{R}) \cong H_b^n(M, \mathbf{R})$, by means of which we can consider $\rho^*(\omega_n^b)$ as a bounded relative class. Finally, the image of $\rho^*(\omega_n^b)$ via the comparison map $c : H_b^n(N, \partial N, \mathbf{R}) \to H^n(N, \partial N, \mathbf{R})$ is an ordinary relative class whose evaluation on the relative fundamental class $[N, \partial N]$ gives the definition of the volume of ρ,

$$\mathrm{Vol}(\rho) := \langle (c \circ \rho^*)(\omega_n^b), [N, \partial N] \rangle,$$

which turns out to be independent of the choice of the compact core N. When M is compact, we recover of course the invariant previously defined. We complete the definition in the case in which Γ has torsion by setting

$$\mathrm{Vol}(\rho) := \frac{\mathrm{Vol}(\rho|_\Lambda)}{[\Gamma : \Lambda]}$$

where $\Lambda < \Gamma$ is a torsion free subgroup of finite index.

Theorem 1.1 *Let $n \geqslant 3$. Let $i : \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ be a lattice embedding and let $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ be any representation. Then*

$$\big|\mathrm{Vol}(\rho)\big| \leqslant \big|\mathrm{Vol}(i)\big| = \mathrm{Vol}(M), \tag{1}$$

with equality if and only if ρ is conjugated to i by an isometry.

An analogous theorem, in the more general case of a representation $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^m)$ with $m \geqslant n$, has been proven by Francaviglia and Klaff [19] with a different definition of volume.

Taking in particular ρ to be another lattice embedding of Γ, we recover Mostow–Prasad Rigidity theorem for hyperbolic lattices:

Corollary 1.2 [28, 29] *Let Γ_1, Γ_2 be two isomorphic lattices in $\mathrm{Isom}^+(\mathbb{H}^n)$. Then there exists an isometry $g \in \mathrm{Isom}(\mathbb{H}^n)$ conjugating Γ_1 to Γ_2.*

As a consequence of Theorem 1.1, we also reprove Thurston's strict version of Gromov's degree inequality for hyperbolic manifolds. Note that this strict version generalizes Mostow Rigidity [33, Theorem 6.4]:

Corollary 1.3 [33, Theorem 6.4] *Let $f : M_1 \to M_2$ be a continuous proper map between two n-dimensional complete finite volume hyperbolic manifolds M_1 and M_2 with $n \geqslant 3$. Then*

$$\deg(f) \leqslant \frac{\mathrm{Vol}(M_2)}{\mathrm{Vol}(M_1)},$$

with equality if and only if f is homotopic to a local isometry.

Our proof of Theorem 1.1 follows closely the steps in the proof of Mostow Rigidity. In particular, the following result is the dual to the use of measure homology and smearing in [33]. We denote by $\varepsilon : G \to \{-1, 1\}$ the homomorphism defined by $\varepsilon(g) = 1$ if g is orientation preserving and $\varepsilon(g) = -1$ if g is orientation reversing.

Theorem 1.4 *Let $M = \Gamma\backslash\mathbb{H}^n$ be a finite volume real hyperbolic manifold. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation with non-elementary image and let $\varphi : \partial\mathbb{H}^n \to$*

$\partial\mathbb{H}^n$ be the corresponding equivariant measurable map. Then for every $(n+1)$-tuple of points $\xi_0, \ldots, \xi_n \in \partial\mathbb{H}^n$,

$$\int_{\Gamma\backslash \mathrm{Isom}(\mathbb{H}^n)} \varepsilon\left(\dot{g}^{-1}\right) \mathrm{Vol}_n\left(\varphi(\dot{g}\xi_0), \ldots, \varphi(\dot{g}\xi_n)\right) d\mu(\dot{g}) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n), \tag{2}$$

where μ is the invariant probability measure on $\Gamma\backslash \mathrm{Isom}(\mathbb{H}^n)$.

This allows us to deduce strong rigidity properties of the boundary map φ from the cohomological information about the boundary that, in turn, are sufficient to show the existence of an element $g \in \mathrm{Isom}^+(\mathbb{H}^n)$ conjugating ρ and i.

To establish the theorem, we first prove the almost everywhere validity of the formula in Theorem 1.4. Ideally, we would need to know that $H^n_{b,c}(G^+, \mathbf{R})$ is 1-dimensional and has no coboundaries in degree n in the appropriate cocomplex. However in general we do not know how to compute $H^n_{b,c}(G^+, \mathbf{R})$, except when $G^+ = \mathrm{Isom}^+(\mathbb{H}^2)$ or $\mathrm{Isom}^+(\mathbb{H}^3)$ and hence there is no direct way to prove the formula in (2). To circumvent this problem, we borrow from [7] (see also [9]) the essential observation that Vol_n is in fact a cocycle equivariant with respect to the full group of isometries $G = \mathrm{Isom}(\mathbb{H}^n)$, that is,

$$\mathrm{Vol}_n(gx_1, \ldots, gx_n) = \varepsilon(g)\, \mathrm{Vol}_n(x_1, \ldots, x_n).$$

This leads to consider $\mathbf{R}$ as a non-trivial coefficient module $\mathbf{R}_\varepsilon$ for G and in this context we prove that the comparison map

$$H^n_{b,c}(G, \mathbf{R}_\varepsilon) \xrightarrow{\cong} H^n_c(G, \mathbf{R}_\varepsilon)$$

is an isomorphism. By a slight abuse of notation, we denote again by $\omega^b_n \in H^n_{b,c}(G, \mathbf{R}_\varepsilon)$ and by $\omega_n \in H^n_c(G, \mathbf{R}_\varepsilon)$ the generator defined by Vol_n.

Using this identification and standard tools from the homological algebra approach to bounded cohomology, we obtain the almost everywhere validity of the formula in Theorem 1.4. Additional arguments involving Lusin's theorem are required to establish the formula pointwise. This is essential because one step of the proof (see the beginning of Sect. 4) consists in showing that, if there is the equality in (1), the map φ maps the vertices of almost every positively oriented maximal ideal simplex to vertices of positively (or negatively—one or the other, not both) oriented maximal ideal simplices. Since such vertices form a set of measure zero in the boundary, an almost everywhere statement would not be sufficient.

2 The Continuous Bounded Cohomology of $G = \mathrm{Isom}(\mathbb{H}^n)$

Denote by $G = \mathrm{Isom}(\mathbb{H}^n)$ the full isometry group of hyperbolic n-space, and by $G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$ its subgroup of index 2 consisting of orientation preserving isometries. As remarked in the introduction there are two natural G-module struc-

tures on $\mathbf{R}$: the trivial one, which we denote by $\mathbf{R}$, and the one given by multiplication with the homomorphism $\varepsilon : G \to G/G^+ \cong \{+1, -1\}$, which we denote by $\mathbf{R}_\varepsilon$.

Recall that if $q \in \mathbb{N}$, the continuous cohomology groups $H_c^q(G, \mathbf{R})$, respectively $H_c^q(G, \mathbf{R}_\varepsilon)$—or in short $H_c^\bullet(G, \mathbf{R}_{(\varepsilon)})$ for both—of G with coefficient in $\mathbf{R}_{(\varepsilon)}$, is by definition given as the cohomology of the cocomplex

$$C_c\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G = \big\{ f : G^{q+1} \to \mathbf{R}_{(\varepsilon)} \,\big|\, f \text{ is continuous and } \varepsilon(g) \cdot f(g_0, \ldots, g_q) = f(gg_0, \ldots, gg_q) \big\}$$

endowed with its usual homogeneous coboundary operator

$$\delta : C_c\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G \to C_c\big(G^{q+2}, \mathbf{R}_{(\varepsilon)}\big)^G$$

defined by

$$\delta f(g_0, \ldots, g_{q+1}) := \sum_{j=0}^{q+1} (-1)^j f(g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{q+1}).$$

This operator clearly restricts to the bounded cochains

$$C_{c,b}\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G = \Big\{ f \in C_c\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G \,\Big|\, \|f\|_\infty = \sup_{g_0, \ldots, g_q \in G} \big|f(g_0, \ldots, g_q)\big| < +\infty \Big\}$$

and the continuous bounded cohomology $H_{c,b}^q(G, \mathbf{R}_{(\varepsilon)})$ of G with coefficients in $\mathbf{R}_{(\varepsilon)}$ is the cohomology of this cocomplex. The inclusion

$$C_{c,b}\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G \subset C_c\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G$$

induces a comparison map

$$c : H_{c,b}^q(G, \mathbf{R}_{(\varepsilon)}) \longrightarrow H_c^q(G, \mathbf{R}_{(\varepsilon)}).$$

We call cochains in $C_{c,(b)}(G^{q+1}, \mathbf{R})^G$ invariant and cochains in $C_{c,(b)}(G^{q+1}, \mathbf{R}_\varepsilon)^G$ equivariant and apply this terminology to the cohomology classes as well. The sup norm on the complex of cochains induces a seminorm in cohomology

$$\|\beta\| = \inf\big\{ \|f\|_\infty \,\big|\, f \in C_{c,(b)}\big(G^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G, [f] = \beta \big\},$$

for $\beta \in H_{c,(b)}^q(G, \mathbf{R}_{(\varepsilon)})$.

The same definition gives the continuous (bounded) cohomology of any topological group acting either trivially on $\mathbf{R}$ or via a homomorphism into the multiplicative group $\{+1, -1\}$. A continuous representation $\rho : H \to G$ naturally induces pullbacks

$$H_{c,(b)}^\bullet(G, \mathbf{R}) \longrightarrow H_{c,(b)}^\bullet(H, \mathbf{R}) \quad \text{and} \quad H_{c,(b)}^\bullet(G, \mathbf{R}_\varepsilon) \longrightarrow H_{c,(b)}^\bullet(H, \mathbf{R}_\rho),$$

where $\mathbf{R}_\rho$ is the H-module $\mathbb{R}$ with the H-action given by the composition of $\rho : H \to G$ with $\varepsilon : G \to \{+1, -1\}$. Note that $\|\rho^*(\beta)\| \leqslant \|\beta\|$.

Since the restriction to G^+ of the G-action on $\mathbf{R}_{(\varepsilon)}$ is trivial, there is a restriction map in cohomology

$$H^\bullet_{c,(b)}(G, \mathbf{R}_{(\varepsilon)}) \longrightarrow H^\bullet_{c,(b)}(G^+, \mathbf{R}). \tag{3}$$

In fact, both the continuous and the continuous bounded cohomology groups can be computed isometrically on the hyperbolic n-space $\mathbb{H}^n$, as this space is isomorphic to the quotient of G or G^+ by a maximal compact subgroup. More precisely, set

$$C_{c,(b)}\big((\mathbb{H}^n)^{q+1}, \mathbf{R}_{(\varepsilon)}\big)^G = \big\{ f : (\mathbb{H}^n)^{q+1} \to \mathbf{R} \,\big|\, f \text{ is continuous (and bounded) and } \varepsilon(g) \cdot f(x_0, \ldots, x_q) = f(gx_0, \ldots, gx_q) \big\}$$

and endow it with its homogeneous coboundary operator. Then the cohomology of this cocomplex is isometrically isomorphic to the corresponding cohomology groups ([22, Chap. III, Prop. 2.3] and [27, Cor. 7.4.10] respectively).

It is now easy to describe the left inverses to the restriction map (3) induced by the inclusion. Indeed, at the cochain level, they are given by maps

$$p : C_{c,(b)}\big((\mathbb{H}^n)^{q+1}, \mathbf{R}\big)^{G^+} \longrightarrow C_{c,(b)}\big((\mathbb{H}^n)^{q+1}, \mathbf{R}\big)^G$$

and

$$\bar{p} : C_{c,(b)}\big((\mathbb{H}^n)^{q+1}, \mathbf{R}\big)^{G^+} \longrightarrow C_{c,(b)}\big((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon\big)^G$$

defined for $x_0, \ldots, x_q \in \mathbb{H}^n$ and $f \in C_{c,(b)}((\mathbb{H}^n)^{q+1}, \mathbf{R})^{G^+}$ by

$$p(f)(x_0, \ldots, x_q) = \frac{1}{2}\big(f(x_0, \ldots, x_q) + f(\tau x_0, \ldots, \tau x_q)\big),$$

$$\bar{p}(f)(x_0, \ldots, x_q) = \frac{1}{2}\big(f(x_0, \ldots, x_q) - f(\tau x_0, \ldots, \tau x_q)\big),$$

where $\tau \in G \smallsetminus G^+$ is any orientation reversing isometry. In fact, it easily follows from the G^+-invariance of f that $p(f)$ is invariant, $\bar{p}(f)$ is equivariant, and both $p(f)$ and $\bar{p}(f)$ are independent of τ in $G \smallsetminus G^+$. The following proposition is immediate:

Proposition 2.1 *The cochain map $(p, \bar{p})$ induces an isometric isomorphism*

$$H^\bullet_{c,(b)}(G^+, \mathbf{R}) \cong H^\bullet_{c,(b)}(G, \mathbf{R}) \oplus H^\bullet_{c,(b)}(G, \mathbf{R}_\varepsilon).$$

The continuous cohomology group $H^\bullet_c(G^+, \mathbf{R})$ is well understood since it can, via the van Est isomorphism [22, Corollary 7.2], be identified with the de Rham cohomology of the compact dual to $\mathbb{H}^n$, which is the n-sphere S^n. Thus it is generated by two cohomology classes: the constant class in degree 0, and the volume

form in degree n. Recall that the volume form ω_n can be represented by the cocycle $\mathrm{Vol}_n \in C_{c,b}((\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon)^G$ (respectively $\mathrm{Vol}_n \in L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon)^G$) given by

$$\mathrm{Vol}_n(x_0, \ldots, x_n) = \text{signed volume of the convex hull of } x_0, \ldots, x_n,$$

for $x_0, \ldots, x_n \in \mathbb{H}^n$, respectively $\partial\mathbb{H}^n$. Since the constant class in degree 0 is invariant, and the volume form is equivariant, using Proposition 2.1 we summarize this as follows:

$$H_c^0(G^+, \mathbf{R}) \cong H_c^0(G, \mathbf{R}) \cong \mathbf{R} \quad \text{and} \quad H_c^n(G^+, \mathbf{R}) \cong H_c^n(G, \mathbf{R}_\varepsilon) \cong \mathbf{R} \cong \langle \omega_n \rangle.$$

All other continuous cohomology groups are 0. On the bounded side, the cohomology groups are still widely unknown, though they are conjectured to be isomorphic to their unbounded counterparts. The comparison maps for G and G^+ are easily seen to be isomorphisms in degrees 2 and 3 [11]. We show that the comparison map for the *equivariant* cohomology of G is indeed an isometric isomorphism up to degree n, based on the simple Lemma 2.2. Before we prove it, it will be convenient to have yet two more cochain complexes to compute the continuous bounded cohomology groups. If $X = \mathbb{H}^n$ or $X = \partial\mathbb{H}^n$, consider the cochain space $L^\infty(X^{q+1}, \mathbf{R}_{(\varepsilon)})^G$ of G-invariant, resp. G-equivariant, essentially bounded measurable function classes endowed with its homogeneous coboundary operator. It is proven in [27, Cor. 7.5.9] that the cohomology of this cocomplex is isometrically isomorphic to the continuous bounded cohomology groups. Note that the volume cocycle Vol_n represents the same cohomology class viewed as continuous bounded or L^∞-cocycle on $\mathbb{H}^n$, as an L^∞-cocyle on $\partial\mathbb{H}^n$ or, by evaluation on $x \in \mathbb{H}^n$ or $x \in \partial\mathbb{H}^n$, as a continuous bounded or L^∞-cocycle on G.

Lemma 2.2 *For $q < n$ we have*

$$C_c\big((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon\big)^G = 0,$$
$$L^\infty\big((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon\big)^G = 0,$$
$$L^\infty\big((\partial\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon\big)^G = 0.$$

Proof Let $f : (\mathbb{H}^n)^{q+1} \to \mathbf{R}_\varepsilon$ or $f : (\partial\mathbb{H}^n)^{q+1} \to \mathbf{R}_\varepsilon$ be G-equivariant. The lemma relies on the simple observation that any $q + 1 \leqslant n$ points $x_0, \ldots, x_q$ either in $\mathbb{H}^n$ or in $\partial\mathbb{H}^n$ lie either on a hyperplane $P \subset \mathbb{H}^n$ or on the boundary of a hyperplane. Thus there exists an orientation reversing isometry $\tau \in G \smallsetminus G^+$ fixing $(x_0, \ldots, x_q)$ pointwise. Using the G-equivariance of f we conclude that

$$f(x_0, \ldots, x_q) = -f(\tau x_0, \ldots, \tau x_q) = -f(x_0, \ldots, x_q),$$

which implies $f \equiv 0$. □

It follows from the lemma that $H_{c,b}^q(G, \mathbf{R}_\varepsilon) \cong H_c^q(G, \mathbf{R}_\varepsilon) = 0$ for $q < n$. Furthermore, we can conclude that the comparison map for the equivariant cohomology of G is injective:

Proposition 2.3 *The comparison map induces an isometric isomorphism*

$$H^n_{c,b}(G, \mathbf{R}_\varepsilon) \xrightarrow{\cong} H^n_c(G, \mathbf{R}_\varepsilon).$$

Proof Since there are no cochains in degree $n-1$, there are no coboundaries in degree n and the cohomology groups $H^n_{c,b}(G, \mathbf{R}_\varepsilon)$ and $H^n_c(G, \mathbf{R}_\varepsilon)$ are equal to the corresponding spaces of cocycles. Thus, we have a commutative diagram

$$\begin{array}{ccc} H^n_{c,b}(G, \mathbf{R}_\varepsilon) & = & \mathrm{Ker}\{\delta : C_{c,b}((\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon)^G \to C_{c,b}((\mathbb{H}^n)^{n+2}, \mathbf{R}_\varepsilon)^G\} \\ \downarrow & & \downarrow \\ \mathbf{R} \cong H^n_c(G, \mathbf{R}_\varepsilon) & = & \mathrm{Ker}\{\delta : C_c((\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon)^G \to C_c((\mathbb{H}^n)^{n+2}, \mathbf{R}_\varepsilon)^G\}. \end{array}$$

The proposition follows from the fact that the lower right kernel is generated by the volume form ω_n which is represented by the bounded cocycle Vol_n, hence is in the image of the vertical right inclusion. □

Since there are no coboundaries in degree n in $C_c((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon)^G$, it follows that the cohomology norm of ω_n is equal to the norm of the unique cocycle representing it. In view of [23], its norm is equal to the volume v_n of an ideal regular simplex in $\mathbb{H}^n$.

Corollary 2.4 *The norm $\|\omega_n\|$ of the volume form $\omega_n \in H^n_c(G^+, \mathbf{R})$ is equal to the volume v_n of a regular ideal simplex in $\mathbb{H}^n$.*

As the cohomology norm $\|\omega_n\|$ is the proportionality constant between simplicial and Riemannian volume for closed hyperbolic manifolds [6, Theorem 2], the corollary gives a simple proof of the proportionality principle $\|M\| = \mathrm{Vol}(M)/v_n$ for closed hyperbolic manifolds, originally due to Gromov and Thurston.

3 Relative Cohomology

3.1 Notation and Definitions

As mentioned in the introduction, we consider a compact core N of the complete hyperbolic manifold M, that is a subset of M whose complement $M \setminus N$ in M is a disjoint union of finitely many geodesically convex cusps of M. If $q \geqslant 0$ and $\sigma : \Delta^q \to M$ denotes a singular simplex, where $\Delta^q = \{(t_0, \ldots, t_q) \in \mathbf{R}^{q+1} : \sum_{j=0}^q t_j = 1,\ t_j \geqslant 0 \text{ for all } j\}$ is a standard q-simplex, we recall that the (singular) cohomology $H^q(M, M \setminus N)$ of M relative to $M \setminus N$ is the cohomology of the cocomplex

$$C^q(M, M \setminus N) = \left\{ f \in C^q(M) \,\middle|\, f(\sigma) = 0 \text{ if } \mathrm{Im}(\sigma) \subset M \setminus N \right\}$$

endowed with its usual coboundary operator. (Here, $C^q(M)$ denotes the space of singular q-cochains on M.) We emphasize that all cohomology groups, singular or relative, are with $\mathbf{R}$ coefficients. The bounded relative cochains $C_b^q(M, M \setminus N)$ are those for which f is further assumed to be bounded, meaning that $\sup\{|f(\sigma)| \mid \sigma : \Delta^q \to M\}$ is finite. The coboundary restricts to bounded cochains and the cohomology of that cocomplex is the bounded cohomology of M relative to $M \setminus N$, which we denote by $H_b^\bullet(M, M \setminus N)$. The inclusion of cocomplexes induces a comparison map $c : H_b^\bullet(M, M \setminus N) \to H^\bullet(M, M \setminus N)$. Similarly, we could define the cohomology of N relative to its boundary ∂N and it is clear, by homotopy invariance, that $H_{(b)}^\bullet(N, \partial N) \cong H_{(b)}^\bullet(M, M \setminus N)$. We can identify the relative cochains on $(M, M \setminus N)$ with the Γ-invariant relative cochains $C^q(\mathbb{H}^n, U)^\Gamma$ on the universal cover $\mathbb{H}^n$ relative to the preimage $U = \pi^{-1}(M \setminus N)$ under the covering map $\pi : \mathbb{H}^n \to M$ of the finite union of horocyclic neighborhoods of cusps. We will identify $H_{(b)}^\bullet(N, \partial N)$ with the latter cohomology group. Note that U is a countable union of disjoint horoballs.

The inclusion $(M, \emptyset) \hookrightarrow (M, M \setminus N)$ induces a long exact sequence on both the unbounded and bounded cohomology groups

$$\cdots \longrightarrow H_{(b)}^{\bullet-1}(M \setminus N) \longrightarrow H_{(b)}^\bullet(M, M \setminus N) \longrightarrow H_{(b)}^\bullet(M) \longrightarrow H_{(b)}^\bullet(M \setminus N) \longrightarrow \cdots$$

Each connected component E_j of $M \setminus N$, $1 \leqslant j \leqslant k$, is a horocyclic neighborhood of a cusp, hence homeomorphic to the product of $\mathbf{R}$ with an $(n-1)$-manifold admitting a Euclidean metric; thus its universal covering is contractible and its fundamental group is virtually abelian (hence amenable). It follows that (see the introduction or [5, 21]) $H_b^\bullet(E_j) \cong H_b^\bullet(\pi_1(E_j)) = 0$ and hence $H_b^\bullet(M \setminus N) = 0$, proving that the inclusion $(M, \emptyset) \hookrightarrow (M, M \setminus N)$ induces an isomorphism on the bounded cohomology groups. Note that based on some techniques developed in [8] we can show that this isomorphism is isometric—a fact that we will not need in this note.

3.2 *Transfer Maps*

In the following we identify Γ with its image $i(\Gamma) < G^+$ under the lattice embedding $i : \Gamma \to G^+$. There exist natural transfer maps

$$H_b^\bullet(\Gamma) \xrightarrow{\text{trans}_\Gamma} H_{c,b}^\bullet(G, \mathbf{R}_\varepsilon) \quad \text{and} \quad H^\bullet(N, \partial N) \xrightarrow{\tau_{dR}} H_c^\bullet(G, \mathbf{R}_\varepsilon),$$

whose classical constructions we briefly recall here. The aim of this section will then be to establish the commutativity of the diagram (6) in Proposition 3.1. The proof is similar to that in [8], except that we replace the compact support cohomology by the relative cohomology, which leads to some simplifications. In fact, the same proof as in [13] (from where the use of relative bounded cohomology is borrowed) would have worked *verbatim* in this case, but we chose the other (and simpler) approach, to provide a "measure homology-free" proof.

3.2.1 The Transfer Map $\mathrm{trans}_\Gamma : H_b^\bullet(\Gamma) \to H_{c,b}^\bullet(G, \mathbf{R}_\varepsilon)$

We can define the transfer map at the cochain level either as a map

$$\mathrm{trans}_\Gamma : V_q^\Gamma \to V_q^G,$$

where V_q is one of $C_b((\mathbb{H}^n)^{q+1}, \mathbf{R})$, $L^\infty((\mathbb{H}^n)^{q+1}, \mathbf{R})$ or $L^\infty((\partial\mathbb{H}^n)^{q+1}, \mathbf{R})$. The definition is the same in all cases. Let thus c be a Γ-invariant cochain in V_q^Γ. Set

$$\mathrm{trans}_\Gamma(c)(x_0, \ldots, x_n) := \int_{\Gamma\backslash G} \varepsilon\left(\dot{g}^{-1}\right) \cdot c(\dot{g}x_0, \ldots, \dot{g}x_n)\, d\mu(\dot{g}), \tag{4}$$

where μ is the invariant probability measure on $\Gamma\backslash G$ normalized so that $\mu(\Gamma\backslash G) = 1$. Recall that $\Gamma < G^+$, so that $\varepsilon(\dot{g})$ is well defined. It is easy to check that the resulting cochain $\mathrm{trans}_\Gamma(c)$ is G-equivariant. Furthermore, the transfer map clearly commutes with the coboundary operator, and hence induces a cohomology map

$$H_b^\bullet(\Gamma) \xrightarrow{\mathrm{trans}_\Gamma} H_{c,b}^\bullet(G, \mathbf{R}_\varepsilon).$$

Note that if the cochain c is already G-equivariant, then $\mathrm{trans}_\Gamma(c) = c$, showing that trans_Γ is a left inverse of $i^* : H_{c,b}^\bullet(G, \mathbf{R}_\varepsilon) \to H_b^\bullet(\Gamma)$.

3.2.2 The Transfer Map $\tau_{dR} : H^\bullet(N, \partial N) \to H_c^\bullet(G, \mathbf{R}_\varepsilon)$

The relative de Rham cohomology $H_{dR}^\bullet(M, M \setminus N)$ is the cohomology of the cocomplex of differential forms $\Omega^q(M, M \setminus N)$ which vanish when restricted to $M \setminus N$. Then, as for usual cohomology, there is a de Rham Theorem

$$\Psi : H_{dR}^\bullet(M, M \setminus N) \xrightarrow{\cong} H^\bullet(M, M \setminus N) \cong H^\bullet(N, \partial N)$$

for relative cohomology. The isomorphism is given at the cochain level by integration. In order to integrate, we could either replace the singular cohomology by its smooth variant (i.e. take smooth singular simplices), or we prefer here to integrate the differential form on the straightened simplices. (The geodesic straightening of a continuous simplex is always smooth.) Thus, at the cochain level, the isomorphism is induced by the map

$$\Psi : \Omega^q(M, M \setminus N) \longrightarrow C^q(M, M \setminus N), \tag{5}$$

sending a differential form $\omega \in \Omega^q(M, M \setminus N) \cong \Omega^q(\mathbb{H}^n, U)^\Gamma$ to the singular cochain $\Psi(\omega)$ given by

$$\sigma \mapsto \int_{\pi_* \,\mathrm{straight}(x_0, \ldots, x_q)} \omega,$$

where $\pi : \mathbb{H}^n \to M$ is the canonical projection, the $x_i \in \mathbb{H}^n$ are the vertices of a lift of σ to $\mathbb{H}^n$, and $\mathrm{straight}(x_0, \ldots, x_q) : \Delta^q \to \mathbb{H}^n$ is the geodesic straightening. Ob-

serve that if σ is in U, then the straightened simplex is as well, since all components of U are geodesically convex.

The transfer map $\mathrm{trans}_{dR} : H^\bullet_{dR}(M, M \smallsetminus N) \to H^\bullet_c(G, \mathbb{R}_\varepsilon)$ is defined through the relative de Rham cohomology and the van Est isomorphism. At the cochain level the transfer

$$\mathrm{trans}_{dR} : \Omega^q\big(\mathbb{H}^n, U\big)^\Gamma \longrightarrow \Omega^q\big(\mathbb{H}^n, \mathbf{R}_\varepsilon\big)^G$$

is defined by sending the differential q-form $\alpha \in \Omega^q(\mathbb{H}^n)^\Gamma$ to the form

$$\mathrm{trans}_{dR}(\alpha) := \int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) \cdot \big(\dot{g}^*\alpha\big)\, d\mu(\dot{g}),$$

where μ is chosen as in (4). It is easy to check that the resulting differential form $\mathrm{trans}_{dR}(\alpha)$ is G-equivariant. Furthermore, the transfer map clearly commutes with the differential operator, and hence induces a cohomology map

$$\begin{array}{ccccc} H^\bullet(N, \partial N) & & & & H^\bullet_c(G, \mathbf{R}_\varepsilon) \\ \cong\uparrow & \Psi & & & \uparrow\cong \\ H^\bullet_{dR}(M, M \smallsetminus N) & \xrightarrow{\mathrm{trans}_{dR}} & H^\bullet\big(\Omega^\bullet\big(\mathbb{H}^n, \mathbf{R}_\varepsilon\big)^G\big) & \xrightarrow{=} & \Omega^\bullet\big(\mathbb{H}^n, \mathbf{R}_\varepsilon\big)^G, \end{array}$$

where the vertical arrow on the right is the van Est isomorphism and the horizontal arrow on the right follows from Cartan's lemma to the extent that any G-invariant differential form on $\mathbb{H}^n$ (or more generally on a symmetric space) is closed.

Let $\omega_{N,\partial N} \in H^n(M, M \smallsetminus N)$ be the unique class with $\langle \omega_{N.\partial N}, [N, \partial N]\rangle = \mathrm{Vol}(M)$. It is easy to check that

$$\mathrm{trans}_{dR}(\omega_{N,\partial N}) = \omega_n \in \Omega^n\big(\mathbb{H}^n, \mathbb{R}_\varepsilon\big)^G \cong H^n_c(G, \mathbf{R}_\varepsilon).$$

3.2.3 Commutativity of the Transfer Maps

Proposition 3.1 *The diagram*

$$\begin{array}{ccc} H^q_b(\Gamma) & & \\ \cong\uparrow & \searrow^{\mathrm{trans}_\Gamma} & \\ H^q_b(N, \partial N) & & H^q_{c,b}(G, \mathbf{R}_\varepsilon) \\ \downarrow c & & \downarrow c \\ H^q(N, \partial N) & \xrightarrow{\tau_{dR}} & H^q_c(G, \mathbf{R}_\varepsilon) \end{array} \tag{6}$$

commutes (here $\tau_{dR} = \mathrm{trans}_{dR} \circ \Psi^{-1}$*).*

Proof The idea of the proof is to subdivide the diagram (6) in smaller parts, by defining transfer maps directly on the bounded and unbounded relative singular cohomology of M and show that each of the following subdiagrams commute.

$$
\begin{array}{ccc}
H_b^q(\Gamma) & & \\
\cong\uparrow & \searrow^{\mathrm{trans}_\Gamma} & \\
H_b^q(N,\partial N) & \xrightarrow{\mathrm{trans}_b} & H_{c,b}^q(G,\mathbf{R}_\varepsilon) \\
\downarrow c & & c\downarrow \\
H^q(N,\partial N) & \xrightarrow{\mathrm{trans}} & H_c^q(G,\mathbf{R}_\varepsilon) \\
\cong\uparrow \Psi & & \Phi\uparrow\cong \\
H_{dR}^q(N,\partial N) & \xrightarrow{\mathrm{trans}_{dR}} & \Omega^q(\mathbb{H}^n,\mathbf{R}_\varepsilon)^G.
\end{array}
\tag{7}
$$

3.2.4 Definition of the Transfer Map for Relative Cohomology

In order to define a transfer map, we need to be able to integrate our cochain on translates of a singular simplex by elements of $\Gamma\backslash G$. This is only possible if the cochain is regular enough.

For $1 \leqslant i \leqslant k$, pick a point $b_i \in E_i$ in each horocyclic neighborhood of a cusp in M and $b_0 \in N$ in the compact core. Let $\beta' : M \to \{b_0, b_1, \ldots, b_k\}$ be the measurable map sending N to b_0 and each cusp E_i to b_i. Lift β' to a Γ-equivariant measurable map

$$\beta : \mathbb{H}^n \longrightarrow \pi^{-1}\big(\{b_0, b_1, \ldots, b_k\}\big) \subset \mathbb{H}^n$$

defined as follows. Choose lifts $\tilde{b}_0, \ldots, \tilde{b}_k$ of $b_0, \ldots, b_k$; for each $j = 1, \ldots, k$ choose a Borel fundamental domain $\mathscr{D}_j \ni \tilde{b}_j$ for the Γ-action on $\pi^{-1}(E_j)$ and choose a fundamental domain $\mathscr{D}_0 \ni \tilde{b}_0$ for the Γ-action on $\pi^{-1}(N)$. Now define $\beta(\gamma\mathscr{D}_j) := \gamma\tilde{b}_j$. In particular β maps each horoball into itself. Given $c \in C^q(\mathbb{H}^n, U)^\Gamma$, define

$$\beta^*(c) : \big(\mathbb{H}^n\big)^{q+1} \longrightarrow \mathbf{R}$$

by

$$\beta^*(c)(x_0, \ldots, x_q) = c\big(\mathrm{straight}\big(\beta(x_0), \ldots, \beta(x_q)\big)\big). \tag{8}$$

Remark that $\beta^*(c)$ is Γ-invariant, vanishes on tuples of points that lie in the same horoball in the disjoint union of horoballs $\pi^{-1}(E_i)$, and is independent of the chosen lift of β' (but not of the points $b_0, \ldots, b_k$). Thus, $\beta^*(c)$ is a cochain in $C^q(\mathbb{H}^n, U)^\Gamma$

which is now measurable, so that we can integrate it on translates of a given $(q+1)$-tuple of point. We define

$$\mathrm{trans}_\beta(c) : \left(\mathbb{H}^n\right)^{q+1} \longrightarrow \mathbf{R}$$

by

$$\mathrm{trans}_\beta(c)(x_0, \ldots, x_q) := \int_{\Gamma\backslash G} \varepsilon\left(\dot{g}^{-1}\right) \cdot \left(\beta^*(c)(\dot{g}x_0, \ldots, \dot{g}x_q)\right) d\mu(\dot{g}),$$

where μ is as in (4). It is easy to show that the integral is finite. Indeed, let D be the maximum of the distances between x_0 and x_i, for $i = 1, \ldots, q$. Then for $\dot{g} \in \Gamma\backslash G$ such that $\dot{g}x_0$ lies outside a D-neighborhood of the compact core N, each $\dot{g}x_i$ clearly lies outside N and hence $\beta^*(c)(\dot{g}x_0, \ldots, \dot{g}x_q)$ vanishes for such $\dot{g}$. It follows that the integrand vanishes outside a compact set, within which it takes only finitely many values. Furthermore, it follows from the Γ-invariance of c and $\beta(c)$ that $\mathrm{trans}_\beta(c)$ is G-invariant.

Since trans_β commutes with the coboundary operator, it induces a cohomology map

$$\mathrm{trans} : H^q(N, \partial N) \longrightarrow H_c^q(G, \mathbf{R}_\varepsilon).$$

As the transfer map trans_β restricts to a cochain map between the corresponding bounded cocomplexes, it also induces a map on the bounded cohomology groups

$$\mathrm{trans}_b : H_b^q(N, \partial N) \longrightarrow H_{c,b}^q(G, \mathbf{R}_\varepsilon),$$

and the commutativity of the middle diagram in (7) is now obvious.

3.2.5 Commutativity of the Lower Square

Denote by $\Phi : \Omega^q(\mathbb{H}^n, \mathbf{R}_\varepsilon) \longrightarrow L^\infty((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon)$ the map (analogous to the map Ψ defined in (5)) sending the differential form α to the cochain $\Phi(\alpha)$ mapping a $(q+1)$-tuple of points $(x_0, \ldots, x_q) \in (\mathbb{H}^n)^{q+1}$ to

$$\int_{\mathrm{straight}(x_0, \ldots, x_q)} \alpha.$$

The de Rham isomorphism is realized at the cochain level by precomposing Φ with the map sending a singular simplex in $\mathbb{H}^n$ to its vertices. To check the commutativity of the lower square, observe that

$$\mathrm{trans}_\beta \circ \Phi(\alpha)(x_0, \ldots, x_q) = \int_{\Gamma\backslash G} \varepsilon\left(\dot{g}^{-1}\right) \cdot \left(\int_{\mathrm{straight}(\beta(\dot{g}x_0), \ldots, \beta(\dot{g}x_q))} \alpha \right) d\mu(\dot{g}),$$

while

$$\Phi \circ \mathrm{trans}_{dR}(\alpha)(x_0, \ldots, x_q) = \int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) \cdot \left(\int_{\mathrm{straight}(\dot{g}x_0, \ldots, \dot{g}x_q)} \alpha \right) d\mu(\dot{g}).$$

If $d\alpha = 0$, the coboundary of the G-invariant cochain

$$(x_0, \ldots, x_{q-1}) \longmapsto \sum_{i=0}^{q-1} (-1)^i \int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) \cdot \left(\int_{\mathrm{straight}(\dot{g}x_0, \ldots, \dot{g}x_i, \beta(\dot{g}x_i), \ldots, \beta(\dot{g}x_{q-1}))} \alpha \right) d\mu(\dot{g})$$

is equal to the difference of the two given cocycles.

3.2.6 Commutativity of the Upper Triangle

Observe that the isomorphism $H_b^\bullet(M, M \setminus N) \cong H_b^\bullet(\Gamma)$ can be induced at the cochain level by the map $\beta^* : C_b^q(\mathbb{H}^n, U)^\Gamma \to L^\infty((\mathbb{H}^n)^{q+1}, \mathbf{R})^\Gamma$ defined in (8) (and for which we allow ourselves a slight abuse of notation). It is immediate that we now have commutativity of the upper triangle already at the cochain level,

$$\begin{array}{ccc} L^\infty\big((\mathbb{H}^n)^{q+1}, \mathbf{R}\big)^\Gamma & & \\ \uparrow{\scriptstyle \beta^*} & \searrow^{\mathrm{trans}_\Gamma} & \\ C_b^q\big(\mathbb{H}^n, U\big)^\Gamma & \xrightarrow[\mathrm{trans}_b]{} & L^\infty\big((\mathbb{H}^n)^{q+1}, \mathbf{R}_\varepsilon\big)^G. \end{array}$$

This finishes the proof of the proposition. □

3.3 *Properties of* Vol(ρ)

Lemma 3.2 *Let $i : \Gamma \hookrightarrow G$ be a lattice embedding. Then*

$$\mathrm{Vol}(i) = \mathrm{Vol}(M).$$

Proof Both sides are multiplicative with respect to finite index subgroups. We can hence without loss of generality suppose that Γ is torsion free. By definition, we have

$$\mathrm{Vol}(M) = \big\langle \omega_{N,\partial N}, [N, \partial N] \big\rangle,$$
$$\mathrm{Vol}(i) = \big\langle \big(c \circ i^*\big)\big(\omega_n^b\big), [N, \partial N] \big\rangle.$$

The desired equality would thus clearly follow from $\omega_{N,\partial N} = (c \circ i^*)(\omega_n^b)$. As the transfer map $\tau_{dR} : H^n(N, \partial N) \to H_c^n(G)$ is an isomorphism in top degree and sends $\omega_{N,\partial N}$ to ω_n, this is equivalent to

$$\omega_n = \tau_{dR}(\omega_{N,\partial N}) = \underbrace{\tau_{dR} \circ c}_{c \circ \mathrm{trans}_\Gamma} \circ i^*(\omega_n^b) = c \circ \mathrm{trans}_\Gamma \circ i^*(\omega_n^b) = c(\omega_n^b) = \omega_n,$$

where we have used the commutativity of the diagram (6) in Proposition 3.1 and the fact that $\mathrm{trans}_\Gamma \circ i^* = \mathrm{Id}$. □

Proposition 3.3 *Let $\rho : \Gamma \to G$ be a representation. The composition*

$$\mathbf{R} \cong H_{c,b}^n(G, \mathbf{R}_\varepsilon) \xrightarrow{\rho^*} H_b^n(\Gamma) \xrightarrow{\mathrm{trans}_\Gamma} H_{c,b}^n(G, \mathbf{R}_\varepsilon) \cong \mathbf{R}$$

is equal to $\lambda \cdot \mathrm{Id}$, where

$$|\lambda| = \frac{|\mathrm{Vol}(\rho)|}{\mathrm{Vol}(M)} \leqslant 1.$$

Proof As the quotient is left invariant by passing to finite index subgroups, we can without loss of generality suppose that Γ is torsion free. Let $\lambda \in \mathbf{R}$ be defined by

$$\mathrm{trans}_\Gamma \circ \rho^*(\omega_n^b) = \lambda \cdot \omega_n^b. \tag{9}$$

We apply the comparison map c to this equality and obtain

$$c \circ \mathrm{trans}_\Gamma \circ \rho^*(\omega_n^b) = \lambda \cdot c(\omega_n^b) = \lambda \cdot \omega_n = \lambda \cdot \tau_{dR}(\omega_{N,\partial N}).$$

The first expression of this line of equalities is equal to $\tau_{dR} \circ c \circ \rho^*(\omega_n^b)$ by the commutativity of the diagram (6). Since τ_{dR} is injective in top degree it follows that $(c \circ \rho^*)(\omega_n^b) = \lambda \cdot \omega_{N,\partial N}$. Evaluating on the fundamental class, we obtain

$$\mathrm{Vol}(\rho) = \langle (c \circ \rho^*)(\omega_n^b), [N, \partial N] \rangle = \lambda \cdot \langle \omega_{N,\partial N}, [N, \partial N] \rangle = \lambda \cdot \mathrm{Vol}(i) = \lambda \cdot \mathrm{Vol}(M).$$

For the inequality, we take the sup norms on both sides of (9), and get

$$|\lambda| = \frac{\| \mathrm{trans}_\Gamma \circ \rho^*(\omega_n^b) \|}{\| \omega_n^b \|} \leqslant 1,$$

where the inequality follows from the fact that all maps involved do not increase the norm. This finishes the proof of the proposition. □

4 On the Proof of Theorem 1.1

The simple inequality $|\mathrm{Vol}(\rho)| \leqslant |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$ follows from Proposition 3.3 and Lemma 3.2.

The proof is divided into three steps. The first step, which follows essentially Furstenberg's footsteps [37, Chap. 4], consists in exhibiting a ρ-equivariant measurable boundary map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$. In the second step we will establish that φ maps the vertices of almost every positively oriented ideal simplex to vertices of positively (or negatively—one or the other, not both) oriented ideal simplices. In the third and last step we show that φ has to be the extension of an isometry, which will provide the conjugation between ρ and i. The fact that $n \geqslant 3$ will only be used in the third step.

4.1 Step 1: The Equivariant Boundary Map

We need to define a measurable map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ such that

$$\varphi\big(i(\gamma)\cdot\xi\big) = \rho(\gamma)\cdot\varphi(\xi), \tag{10}$$

for every $\xi \in \partial\mathbb{H}^n$ and every $\gamma \in \Gamma$.

The construction of such boundary map is the sore point of many rigidity questions. In the rank one situation in which we are, the construction is well known and much easier, and is recalled here for completeness.

Since $\partial\mathbb{H}^n$ can be identified with $\mathrm{Isom}^+(\mathbb{H}^n)/P$, where $P < \mathrm{Isom}^+(\mathbb{H}^n)$ is a minimal parabolic, the action of Γ on $\partial\mathbb{H}^n$ is amenable. Thus there exists a Γ-equivariant measurable map $\varphi : \partial\mathbb{H}^n \to \mathscr{M}^1(\partial\mathbb{H}^n)$, where $\mathscr{M}^1(\partial\mathbb{H}^n)$ denotes the probability measures on $\partial\mathbb{H}^n$, [37]. We recall the proof here for the sake of the reader familiar with the notion of amenable group but not conversant with that of amenable action, although the result is by now classical.

Lemma 4.1 *Let G be a locally compact group, $\Gamma < G$ a lattice and P an amenable subgroup. Let X be a compact metrizable space with a Γ-action by homeomorphisms. Then there exists a Γ-equivariant boundary map $\varphi : G/P \to \mathscr{M}^1(X)$.*

Proof Let $C(X)$ be the space of continuous functions on X. The space

$$\mathrm{L}^1_\Gamma\big(G, C(X)\big) := \Big\{ f : G \to C(X) \,\Big|\, f \text{ is measurable, } \Gamma\text{-equivariant and } \int_{\Gamma\backslash G} \|f(\dot{g})\|_\infty \, d\mu(\dot{g}) < \infty \Big\},$$

is a separable Banach space whose dual is the space $\mathrm{L}^\infty_\Gamma(G, \mathscr{M}(X))$ of measurable Γ-equivariant essentially bounded maps from G into $\mathscr{M}(X)$, where $\mathscr{M}(X) = C(X)^*$ is the dual of $C(X)$. (Notice that since $C(X)$ is a separable Banach space, the concept of measurability of a function $G \to C(X)^*$ is the same as to whether $C(X)^*$ is endowed with the weak-* or the norm topology.) Then $\mathrm{L}^\infty_\Gamma(G, \mathscr{M}^1(X))$ is a convex compact subset of the unit ball of $\mathrm{L}^\infty_\Gamma(G, \mathscr{M}(X))$ that is right P-invariant. Since P is amenable, there exists a P-fixed point, that is nothing but the map $\varphi : G/P \to \mathscr{M}^1(X)$ we were looking for. □

We are going to associate to every $\mu \in \mathscr{M}^1(\partial\mathbb{H}^n)$ (in the image of φ) a point in $\partial\mathbb{H}^n$.

If the measure μ has only one atom of mass $\geqslant \frac{1}{2}$, then we associate to μ this atom. We will see that all other possibilities result in a contradiction.

If the measure μ has no atoms of mass greater than or equal to $\frac{1}{2}$, we can apply Douady and Earle's barycenter construction [16, Sect. 2] that to such a measure associates equivariantly a point $b_\mu \in \mathbb{H}^n$. By ergodicity of the Γ-action on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$, the distance $d := d(b_{\varphi(x)}, b_{\varphi(x')})$ between any two of these points is essentially constant. It follows that for a generic $x \in \partial\mathbb{H}^n$, there is a bounded orbit, contradicting the fact that the action is not elementary.

If on the other hand there is more than one atom whose mass is at least $\frac{1}{2}$, then the support of the measure must consist of two points (with an equally distributed measure). Denote by g_x the geodesic between the two points in the support of the measure $\varphi(x) \in \mathscr{M}^1(\partial\mathbb{H}^n)$. By ergodicity of the Γ-action on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$, the cardinality of the intersection $\operatorname{supp}(\varphi(x)) \cap \operatorname{supp}(\varphi(x'))$ must be almost everywhere constant and hence almost everywhere either equal to 0, 1 or 2.

If $|\operatorname{supp}(\varphi(x)) \cap \operatorname{supp}(\varphi(x'))| = 2$ for almost all $x, x' \in \partial\mathbb{H}^n$, then the geodesic g_x is Γ-invariant and hence the action is elementary.

If $|\operatorname{supp}(\varphi(x)) \cap \operatorname{supp}(\varphi(x'))| = 1$, then we have to distinguish two cases: either for almost every $x \in \partial\mathbb{H}^n$ there is a point $\xi \in \partial\mathbb{H}^n$ such that $\operatorname{supp}(\varphi(x)) \cap \operatorname{supp}(\varphi(x')) = \{\xi\}$ for almost all $x' \in \partial\mathbb{H}^n$, in which case again ξ would be Γ-invariant and the action elementary, or $\operatorname{supp}(\varphi(x)) \cup \operatorname{supp}(\varphi(x')) \cup \operatorname{supp}(\varphi(x''))$ consists of exactly three points for almost every $x', x'' \in \partial\mathbb{H}^n$. In this case the barycenter of the geodesic triangle with vertices in these three points is Γ-invariant and the action is, again, elementary.

Finally, if $|\operatorname{supp}(\varphi(x)) \cap \operatorname{supp}(\varphi(x'))| = 0$, let $D := d(g_x, g_{x'})$. By ergodicity on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$, d is essentially constant. Let $\gamma \in \rho(\Gamma)$ be a hyperbolic element whose fixed points are not the endpoints of g_x or $g_{x'}$. Then iterates of γ send a geodesic $g_{x'}$ into an arbitrarily small neighborhood of the attractive fixed point of γ, contradicting that g_x is at fixed distance from $g_{x'}$.

4.2 Step 2: Mapping Regular Simplices to Regular Simplices

The next step is to prove Theorem 1.4. Then if $\operatorname{Vol}(\rho) = \operatorname{Vol}(M)$, it will follow that the map φ in Step 1 sends almost all regular simplices to regular simplices.

From Proposition 3.3 we obtain that the composition of the induced map ρ^* and the transfer with respect to the lattice embedding i is equal to $\pm$ the identity on $H^n_{c,b}(G^+, \mathbf{R}_\varepsilon)$. In dimension 3, it follows from [4] that $H^3_{c,b}(\operatorname{Isom}^+(\mathbb{H}^3), \mathbf{R}) \cong \mathbf{R}$ and the proof can be formulated using trivial coefficients; this has been done in [11], which is the starting point of this paper. In higher dimension it is conjectured, but not known, that $H^n_{c,b}(G^+, \mathbf{R}) \cong \mathbf{R}$.

We can without loss of generality suppose that $\operatorname{trans}_\Gamma \circ \rho^*$ is equal to $+\operatorname{Id}$. Indeed, otherwise, we conjugate ρ by an orientation reversing isometry. We will now

show that the isomorphism realized at the cochain level leads to the equality (11), which is only an almost everywhere equality. Up to this point, the proof is elementary. The only difficulty in our proof is to show that the almost everywhere equality is a true equality, which we prove in Proposition 4.2. Note however that there are two cases in which Proposition 4.2 is immediate, namely 1) if φ is a homeomorphism, which is the case if Γ is cocompact and ρ is also a lattice embedding (which is the case of the classical Mostow Rigidity Theorem), and 2) if the dimension n equals 3. We give the alternative simple arguments below.

The bounded cohomology groups $H^n_{c,b}(G, \mathbf{R}_\varepsilon)$ and $H^n_b(\Gamma, \mathbf{R})$ can both be computed from the corresponding L^∞ equivariant cochains on $\partial\mathbb{H}^n$. The induced map $\rho^* : H^n_{c,b}(G, \mathbf{R}_\varepsilon) \to H^n_b(\Gamma, \mathbf{R})$ is represented by the pullback by φ, although it should be noted that the pullback in bounded cohomology cannot be implemented with respect to boundary maps in general, unless the class to pull back can be represented by a strict invariant Borel cocycle. This is our case for Vol_n and as a consequence, $\varphi^*(\mathrm{Vol}_n)$ is also a measurable Γ-invariant cocycle. It hence determines a cohomology class in $H^n_b(\Gamma)$ which, by [10, Corollary 3.7], represents $\rho^*(\omega_n)$.

The composition of maps $\mathrm{trans}_\Gamma \circ \rho^*$ is thus realized at the cochain level by

$$\begin{aligned} L^\infty\big((\partial\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon\big)^\Gamma &\longrightarrow L^\infty\big((\partial\mathbb{H}^n)^{n+1}, \mathbf{R}_\varepsilon\big)^G \\ v &\longmapsto \big\{(\xi_0, \ldots, \xi_n) \mapsto \textstyle\int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) v\big(\varphi(\dot{g}\xi_0, \ldots, \dot{g}\xi_n)\big)\, d\mu(\dot{g})\big\}. \end{aligned}$$

Since the composition $\mathrm{trans}_\Gamma \circ \rho^*$ is the multiplication by $\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}$ at the cohomology level and there are no coboundaries in degree n (Lemma 2.2), the above map sends the cocycle Vol_n to $\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n$. Thus, for almost every $\xi_0, \ldots, \xi_n \in \partial\mathbb{H}^n$ we have

$$\int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) \cdot \mathrm{Vol}_n\big(\varphi(\dot{g}\xi_0), \ldots, \varphi(\dot{g}\xi_n)\big)\, d\mu(\dot{g}) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n(\xi_0, \ldots, \xi_n). \quad (11)$$

Let $(\partial\mathbb{H}^n)^{(n+1)}$ be the G-invariant open subset of $(\partial\mathbb{H}^n)^{n+1}$ consisting of $(n+1)$-tuples of points $(\xi_0, \ldots, \xi_n)$ such that $\xi_i \neq \xi_j$ for all $i \neq j$. Observe that the volume cocycle Vol_n is continuous when restricted to $(\partial\mathbb{H}^n)^{(n+1)}$ and vanishes on $(\partial\mathbb{H}^n)^{n+1} \smallsetminus (\partial\mathbb{H}^n)^{(n+1)}$. Observe moreover that the volume of ideal simplices is a continuous extension of the volume of simplices with vertices in the interior B^n of the sphere $S^{n-1} = \partial\mathbb{H}^n$.

Proposition 4.2 *Let $i : \Gamma \to G$ be a lattice embedding, $\rho : \Gamma \to G$ a representation and $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ a Γ-equivariant measurable map. Identifying Γ with its image $i(\Gamma) < G$ via the lattice embedding, if*

$$\int_{\Gamma\backslash G} \varepsilon\big(\dot{g}^{-1}\big) \cdot \mathrm{Vol}_n\big(\varphi(\dot{g}\xi_0), \ldots, \varphi(\dot{g}\xi_n)\big)\, d\mu(\dot{g}) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n(\xi_0, \ldots, \xi_n)$$

for almost every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$, then the equality holds everywhere.

Before we proceed with the proof, let us observe that it immediately follows from the proposition that if ρ has maximal volume, then φ maps the vertices of almost every regular simplex to the vertices of a regular simplex of the same orientation, which is the conclusion of Step 2.

Proof if φ is a Homeomorphism Since φ is injective, both sides of the almost everywhere equality are continuous on $(\partial\mathbb{H}^n)^{(n+1)}$. Since they agree on a full measure subset of $(\partial\mathbb{H}^n)^{(n+1)}$, the equality holds on the whole of $(\partial\mathbb{H}^n)^{(n+1)}$. As for its complement, it is clear that if $\xi_i = \xi_j$ for $i \neq j$ then both sides of the equality vanish. □

Proof if $n = 3$ Both sides of the almost equality are defined on the whole of $(\partial\mathbb{H}^3)^4$, are cocycles on the whole of $(\partial\mathbb{H}^3)^4$, vanish on $(\partial\mathbb{H}^3)^4 \setminus (\partial\mathbb{H}^3)^{(4)}$ and are $\mathrm{Isom}^+(\mathbb{H}^3)$-invariant. Let $a, b : (\partial\mathbb{H}^3)^4 \to \mathbf{R}$ be two such functions and suppose that $a = b$ on a set of full measure. This means that for *almost every* $(\xi_0, \ldots, \xi_3) \in (\partial\mathbb{H}^3)^4$, we have $a(\xi_0, \ldots, \xi_3) = b(\xi_0, \ldots, \xi_3)$. Since $\mathrm{Isom}^+(\mathbb{H}^3)$ acts transitively on 3-tuples of distinct points in $\mathbb{H}^3$ and both a and b are $\mathrm{Isom}^+(\mathbb{H}^3)$-invariant, this means that for *every* $(\xi_0, \xi_1, \xi_2) \in (\partial\mathbb{H}^3)^{(3)}$ and almost every $\eta \in \partial\mathbb{H}^3$ the equality

$$a(\xi_0, \xi_1, \xi_2, \eta) = b(\xi_0, \xi_1, \xi_2, \eta)$$

holds. Let $\xi_0, \ldots, \xi_3 \in \partial\mathbb{H}^3$ be arbitrary. If $\xi_i = \xi_j$ for $i \neq j$, we have $a(\xi_0, \ldots, \xi_3) = b(\xi_0, \ldots, \xi_3)$ by assumption. Suppose $\xi_i \neq \xi_j$ whenever $i \neq j$. By the above, for every $i \in 0, \ldots, 3$ the equality

$$a(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_3, \eta) = b(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_3, \eta)$$

holds for η in a subset of full measure in $\partial\mathbb{H}^3$. Let η be in the (non empty) intersection of these four full measure subsets of $\partial\mathbb{H}^3$. We then have

$$\begin{aligned} a(\xi_0, \ldots, \xi_3) &= \sum_{i=0}^{3} (-1)^i a(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_3, \eta) \\ &= \sum_{i=0}^{3} (-1)^i b(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_3, \eta) = b(\xi_0, \ldots, \xi_3), \end{aligned}$$

where we have used the cocycle relations for a and b in the first and last equality respectively. □

Proof in the General Case Observe first of all that for all $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1} \setminus (\partial\mathbb{H}^n)^{(n+1)}$ the equality holds trivially.

Using the fact that $\partial\mathbb{H}^n \cong S^{n-1} \subset \mathbf{R}^n$, let us consider the function $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ as a function $\varphi : \partial\mathbb{H}^n \to \mathbf{R}^n$ and for $j = 1, \ldots, n$ denote by φ_j its coordinates. Since $\partial\mathbb{H}^n \cong G/P$, where P is a minimal parabolic, let ν be the quasi-invariant measure on $\partial\mathbb{H}^n$ obtained from the decomposition of the Haar measure μ_G with respect to the Haar measure μ_P on P, as in (17). According to Lusin's

theorem applied to the φ_j for $j = 1, \ldots, n$ (see for example [31, Theorem 2.24]), for every $\delta > 0$ there exist a measurable set $B_{\delta,i} \subset \partial\mathbb{H}^n$ with measure $\nu(B_{i,\delta}) \leqslant \delta$ and a continuous function $f'_{j,\delta} : \partial\mathbb{H}^n \to \mathbf{R}$ such that $\varphi_j \equiv f'_{j,\delta}$ on $\partial\mathbb{H}^n \smallsetminus B_{j,\delta}$. Set $f'_\delta := (f_{1,\delta}, \ldots, f_{n,\delta}) \to \mathbf{R}^n$ and consider the composition $f_\delta := r \circ f'_\delta$ with the retraction $r : \mathbf{R}^n \to \overline{B^n}$ to the closed unit ball $\overline{B^n} \subset \mathbf{R}^n$. Then, by setting $B_\delta := \bigcup_{j=1}^n B_{j,\delta}$, we see that φ coincides on $\partial\mathbb{H}^n \smallsetminus B_\delta$ with the continuous function $f_\delta : \partial\mathbb{H}^n \to \overline{B^n}$ and $\nu(B_\delta) \leqslant n\delta$.

Let $\mathscr{D} \subset G$ be a fundamental domain for the action of Γ on G. For every measurable subset $E \subset \mathscr{D}$, any measurable map $\psi : \partial\mathbb{H}^n \to \overline{B^n}$ and any point $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$, we use the notation

$$\mathscr{I}\big(\psi, E, (\xi_0, \ldots, \xi_n)\big) := \int_E \varepsilon\big(g^{-1}\big)\, \mathrm{Vol}_n\big(\psi(g\xi_0), \ldots, \psi(g\xi_n)\big)\, d\mu_G(g),$$

so that we need to show that if

$$\mathscr{I}\big(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\big) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\, \mathrm{Vol}_n(\xi_0, \ldots, \xi_n) \tag{12}$$

for almost every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$, then the equality holds everywhere.

Fix $\epsilon > 0$ and let $K_\epsilon \subset \mathscr{D}$ be a compact set such that $\mu_G(\mathscr{D} \smallsetminus K_\epsilon) < \epsilon$. The proof is broken up in several lemmas, that we state and use here, but whose proof we postpone.

Lemma 4.3 *With the above notations,*

$$\mu_G\big(\{g \in K_\epsilon : g\xi \in B_\delta\}\big) \leqslant \sigma_\epsilon(\delta), \tag{13}$$

where $\sigma_\epsilon(\delta)$ does not depend on $\xi \in \partial\mathbb{H}^n$ and $\sigma_\epsilon(\delta) \to 0$ when $\delta \to 0$.

Replacing φ with f_δ results in the following estimate for the integral.

Lemma 4.4 *With the notation as above, there exists a function $M_\epsilon(\delta)$ with the property that $\lim_{\delta \to 0} M_\epsilon(\delta) = 0$, such that*

$$\big|\mathscr{I}\big(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\big) - \mathscr{I}\big(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\big)\big| \leqslant M_\epsilon(\delta),$$

for all $(\xi_0, \ldots, \xi_{n+1}) \in (\partial\mathbb{H}^n)^{n+1}$.

Observe that, although

$$\big|\mathscr{I}\big(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\big) - \mathscr{I}\big(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\big)\big| < \epsilon \| \mathrm{Vol}_n \|, \tag{14}$$

for all $(\xi_0, \ldots, \xi_{n+1}) \in (\partial\mathbb{H}^n)^{(n+1)}$, the estimate

$$\left|\mathscr{I}\big(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\big) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\, \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right|$$

$$\leqslant \left|\mathscr{I}\left(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \mathscr{I}\left(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\right)\right|$$
$$+ \left|\mathscr{I}\left(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right| \leqslant \epsilon \| \mathrm{Vol}_n \|, \quad (15)$$

holds only for almost every $(\xi_0, \ldots, \xi_n) \in (\partial \mathbb{H}^n)^{(n+1)}$, since this is the case for (12).

From (15) and Lemma 4.4, it follows that

$$\left|\mathscr{I}\left(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right|$$
$$\leqslant \left|\mathscr{I}\left(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \mathscr{I}\left(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\right)\right|$$
$$+ \left|\mathscr{I}\left(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right|$$
$$< M_\epsilon(\delta) + \epsilon \| \mathrm{Vol}_n \|, \quad (16)$$

for almost every $(\xi_0, \ldots, \xi_n) \in (\partial \mathbb{H}^n)^{(n+1)}$.

The following lemma uses the continuity of f_δ to deduce that all of the almost everywhere equality that propagated from the use of (12) in (15), can indeed be observed to hold everywhere because of the use of Lusin theorem.

Lemma 4.5 *There exist a function* $L(\epsilon, \delta)$ *such that* $\lim_{\epsilon \to 0} \lim_{\delta \to 0} L(\epsilon, \delta) = 0$ *and*

$$\left|\mathscr{I}\left(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right| \leqslant L(\epsilon, \delta)$$

for all $(\xi_0, \ldots, \xi_n) \in (\partial \mathbb{H}^n)^{(n+1)}$.

From this, and from Lemma 4.4, and using once again (14), now all everywhere statements, we conclude that

$$\left|\mathscr{I}\left(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right|$$
$$\leqslant \left|\mathscr{I}\left(\varphi, \mathscr{D}, (\xi_0, \ldots, \xi_n)\right) - \mathscr{I}\left(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\right)\right|$$
$$+ \left|\mathscr{I}\left(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \mathscr{I}\left(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\right)\right|$$
$$+ \left|\mathscr{I}\left(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right|$$
$$< M_\epsilon(\delta) + L(\epsilon, \delta) + \epsilon \| \mathrm{Vol}_n \|,$$

for all $(\xi_0, \ldots, \xi_{n+1}) \in (\partial \mathbb{H}^n)^{n+1}$. This concludes the proof of Proposition 4.2, assuming the unproven lemmas. □

We now proceed to the proof of Lemmas 4.3, 4.4 and 4.5.

Proof of Lemma 4.3 Recall that $\partial\mathbb{H}^n = G/P$, where $P < G$ is a minimal parabolic and let $\eta : G/P \to G$ be a Borel section of the projection $G \to G/P$ such that $F := \eta(G/P)$ is relatively compact [26, Lemma 1.1]. Let $\widetilde{B}_\delta := \eta(B_\delta)$ and, if $\xi \in B_\delta$, set $\tilde{\xi} := \eta(\xi) \in \widetilde{B}_\delta$. On the other hand, if $g \in K_\epsilon$ and $g\xi \in B_\delta$, there exists $p \in P$ such that $g\tilde{\xi}p \in \widetilde{B}_\delta$ and, in fact, the p can be chosen to be in $P \cap F^{-1}(K_\epsilon)^{-1}F =: C_\epsilon$. Thus we have

$$\begin{aligned}\{g \in K_\epsilon : g\xi \in B_\delta\} &= \left\{g \in K_\epsilon : \text{there exists } p \in C_\epsilon \text{ with } g\tilde{\xi}p \in \widetilde{B}_\delta\right\} \\ &= \left\{g \in K_\epsilon \cap \widetilde{B}_\delta p^{-1}\tilde{\xi}^{-1} \text{ for some } p \in C_\epsilon\right\} \subset K_\epsilon \cap \widetilde{B}_\delta C_\epsilon^{-1}\tilde{\xi}^{-1},\end{aligned}$$

and hence

$$\mu_G\big(\{g \in K_\epsilon : g\xi \in B_\delta\}\big) \leqslant \mu_G\big(K_\epsilon\tilde{\xi}^{-1} \cap \widetilde{B}_\delta C_\epsilon^{-1}\big) \leqslant \mu_G\big(\widetilde{B}_\delta C_\epsilon^{-1}\big).$$

To estimate the measure, recall that there is a strictly positive continuous function $q : G \to \mathbf{R}^+$ and a positive measure ν on $\partial\mathbb{H}^n$ such that

$$\int_G f(g)q(g)\,d\mu_G(g) = \int_{\partial\mathbb{H}^n}\left(\int_P f(\dot{g}\xi)\,d\mu_P(\xi)\right)d\nu(\dot{g}), \tag{17}$$

for all continuous functions f on G with compact support, [30, Sect. 8.1].

We may assume that $\mu_G(\widetilde{B}_\delta C_\epsilon^{-1}) \neq 0$ (otherwise we are done). Then, since q is continuous and strictly positive and the integral is on a relatively compact set, there exists a constant $0 < \alpha < \infty$ such that

$$\alpha\mu_G\big(\widetilde{B}_\delta C_\epsilon^{-1}\big) = \int_{\partial\mathbb{H}^n}\left(\int_P \chi_{\widetilde{B}_\delta C_\epsilon^{-1}}(\dot{g}\xi)\,d\mu_P(\xi)\right)d\nu(\dot{g}).$$

But, by construction, if $g \in \widetilde{B}_\delta$, then $g\xi \in \widetilde{B}_\delta C_\epsilon^{-1}$ if and only if $\xi \in C_\epsilon^{-1}$, so that

$$\int_P \chi_{\widetilde{B}_\delta C_\epsilon^{-1}}(\dot{g}\xi)\,d\mu_P(\xi) = \mu_P\big(C_\epsilon^{-1}\big),$$

and hence

$$\alpha\mu_G\big(\widetilde{B}_\delta C_\epsilon^{-1}\big) = \nu(B_\delta)\mu_P\big(C_\epsilon^{-1}\big).$$

Since $\nu(B_\delta) < \delta$, the inequality (13) is proven with $\sigma_\epsilon(\delta) = \frac{1}{\alpha}\mu_P(C_\epsilon^{-1})\delta$. □

Proof of Lemma 4.4 Let us fix $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$. Then we have

$$\begin{aligned}&\big|\mathscr{I}\big(\varphi, K_\epsilon, (\xi_0, \ldots, \xi_n)\big) - \mathscr{I}\big(f_\delta, K_\epsilon, (\xi_0, \ldots, \xi_n)\big)\big| \\ &\quad\leqslant \big|\mathscr{I}\big(\varphi, K_{\epsilon,0}, (\xi_0, \ldots, \xi_n)\big) - \mathscr{I}\big(f_\delta, K_{\epsilon,0}, (\xi_0, \ldots, \xi_n)\big)\big| \\ &\qquad+ \big|\mathscr{I}\big(\varphi, K_{\epsilon,1}, (\xi_0, \ldots, \xi_n)\big) - \mathscr{I}\big(f_\delta, K_{\epsilon,1}, (\xi_0, \ldots, \xi_n)\big)\big|,\end{aligned}$$

where

$$K_{\epsilon,0} := \bigcap_{j=0}^{n} \{g \in K_\epsilon : g\xi_j \in \partial\mathbb{H}^n \smallsetminus B_\delta\} \quad \text{and} \quad K_{\epsilon,1} := K_\epsilon \smallsetminus K_{\epsilon,0}.$$

But $\varphi(g) = f_\delta(g)$ for all $g \in K_{\epsilon,0}$, hence the difference of the integrals on $K_{\epsilon,0}$ vanishes. Since

$$\mu_G(K_{\epsilon,1}) = \mu_G\left(K_\epsilon \cap \bigcup_{j=0}^{n} \{g \in K_\epsilon : g\xi_j \in B_\delta\}\right) \leqslant (n+1)\sigma_\epsilon(\delta),$$

we obtain the assertion with $M_\epsilon(\delta) := 2(n+1)\| \operatorname{Vol}_n \|\sigma_\epsilon(\delta)$. □

Proof of Lemma 4.5 If the volume were continuous on $(\partial\mathbb{H}^n)^{n+1}$ or if the function f_δ were injective, the assertion would be obvious.

Observe that φ is almost everywhere injective: in fact, by double ergodicity, the subset of $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ consisting of pairs (x, y) for which $\varphi(x) = \varphi(y)$ is a set of either zero or full measure and the latter would contradict non-elementarity of the action. Then on a set of full measure in $\partial\mathbb{H}^n \smallsetminus B_\delta$ the function f_δ is injective and hence $\operatorname{Vol}_n(f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n))$ is continuous provided the $f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n)$ are pairwise distinct.

So, for any $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$ we define

$$\mathscr{E}(\xi_0, \ldots, \xi_n) := \{g \in K_\epsilon : f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n) \text{ are pairwise distinct}\}.$$

Let $F \subset (B_\delta^c \times B_\delta^c)^{(2)}$ be the set of distinct pairs $(x, y) \in (B_\delta^c \times B_\delta^c)^{(2)}$ such that $f_\delta(x) = f_\delta(y)$. Then F is of measure zero, and given any $(\xi_0, \xi_1) \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ distinct, the set $\{g \in G : g(\xi_0, \xi_1) \in F\}$ is of μ_G-measure zero. This, together with Lemma 4.3, implies that

$$\mu_G\big(K_\epsilon \smallsetminus \mathscr{E}(\xi_0, \ldots, \xi_n)\big) \leqslant \mu_G\left(\bigcup_{j=0}^{n} \{g \in K_\epsilon : g\xi_j \in B_\delta\}\right) \leqslant (n+1)\sigma_\epsilon(\delta). \quad (18)$$

Let $\mathscr{S} \subset (\partial\mathbb{H}^n)^{(n+1)}$ be the set of full measure where the inequality (16) holds and let $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$. Since $\nu^{n+1}((\partial\mathbb{H}^n)^{(n+1)} \smallsetminus \mathscr{S}) = 0$, there exists a sequence of points $(\xi_0^{(k)}, \ldots, \xi_n^{(k)}) \in \mathscr{S}$ with $(\xi_0^{(k)}, \ldots, \xi_n^{(k)}) \to (\xi_0, \ldots, \xi_n)$. Then for every $g \in \mathscr{E}(\xi)$

$$\lim_{k\to\infty} \operatorname{Vol}_n\big(f_\delta\big(g\xi_0^{(k)}\big), \ldots, f_\delta\big(g\xi_n^{(k)}\big)\big) = \operatorname{Vol}_n\big(f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n)\big),$$

and, by the dominated convergence theorem applied to the sequence $h_k(g) := \operatorname{Vol}_n(f_\delta(g\xi_0^{(k)}), \ldots, f_\delta(g\xi_n^{(k)}))$, we deduce that

$$\lim_{k\to\infty} \mathscr{I}\big(f_\delta, \mathscr{E}(\xi_0, \ldots, \xi_n), \big(\xi_0^{(k)}, \ldots, \xi_n^{(k)}\big)\big) = \mathscr{I}\big(f_\delta, \mathscr{E}(\xi_0, \ldots, \xi_n), (\xi_0, \ldots, \xi_n)\big). \quad (19)$$

But then

$$\begin{aligned}
&\left|\mathscr{J}\left(f_\delta, K_\epsilon, (\xi_0,\dots,\xi_n)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n(\xi_0,\dots,\xi_n)\right| \\
&\quad\leqslant \left|\mathscr{J}\left(f_\delta, K_\epsilon, (\xi_0,\dots,\xi_n)\right) - \mathscr{J}\left(f_\delta, \mathscr{E}(\xi_0,\dots,\xi_n), (\xi_0,\dots,\xi_n)\right)\right| \\
&\qquad + \left|\mathscr{J}\left(f_\delta, \mathscr{E}(\xi_0,\dots,\xi_n), (\xi_0,\dots,\xi_n)\right) - \mathscr{J}\left(f_\delta, \mathscr{E}(\xi_0,\dots,\xi_n), \left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right)\right)\right| \\
&\qquad + \left|\mathscr{J}\left(f_\delta, \mathscr{E}(\xi_0,\dots,\xi_n), \left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right)\right) - \mathscr{J}\left(f_\delta, K_\epsilon, \left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right)\right)\right| \\
&\qquad + \left|\mathscr{J}\left(f_\delta, K_\epsilon, \left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right)\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n\left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right)\right| \\
&\qquad + \left|\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n\left(\xi_0^{(k)},\dots,\xi_n^{(k)}\right) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\,\mathrm{Vol}_n(\xi_0,\dots,\xi_n)\right|,
\end{aligned}$$

for all $(\xi_0,\dots,\xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$.

The first and third lines after the inequality sign are each $\leqslant (n+1)\|\,\mathrm{Vol}_n\,\|\sigma_\epsilon(\delta)$ because of (18); the second line after the equality is less than δ if k is large enough because of (19); the fourth line is $\leqslant M_\epsilon(\delta) + \epsilon\|\,\mathrm{Vol}_n\,\|$ by (16) since $(\xi_0^{(k)},\dots,\xi_n^{(k)}) \in \mathscr{S}$ and finally the last line is also less than δ if k if large enough. All of the estimate hold for all $(\xi_0,\dots,\xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$, and hence the assertion is proven with $L(\epsilon,\delta) := 2\delta + 2(n+1)\|\,\mathrm{Vol}_n\,\|\sigma_\epsilon(\delta) + M_\epsilon(\delta) + \epsilon\|\,\mathrm{Vol}_n\,\|$. □

4.3 *Step 3: The Boundary Map is an Isometry*

Suppose now that the equality $|\,\mathrm{Vol}(\rho)| = |\,\mathrm{Vol}(i)|$ holds. Then φ maps enough regular simplices to regular simplices. In this last step of the proof, we want to show that then φ is essentially an isometry, and this isometry will realize the conjugation between ρ and i.

In the case of a cocompact lattice $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ and a lattice embedding $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$, the limit map φ is continuous and the proof is very simple based on Lemma 4.6. This is the original setting of Gromov's proof of Mostow rigidity for compact hyperbolic manifolds.

If either the representation ρ is not assumed to be a lattice embedding, or if Γ is not cocompact, then the limit map φ is only measurable and one needs a measurable variant of Lemma 4.6 presented in Proposition 4.7 for $n \geqslant 4$. The case $n = 3$ was first proven by Thurston for his generalization (Corollary 1.3 here) of Gromov's proof of Mostow rigidity. It is largely admitted that the case $n = 3$ easily generalizes to $n \geqslant 4$, although we wish to point out that the proof is very much simpler for $n \geqslant 4$ based on the fact that the reflection group of a regular simplex is dense in the isometry group. For the proof of Proposition 4.7, we will omit the case $n = 3$ which is nicely written down in all necessary details by Dunfield [17, pp. 654–656], following the original [33, two last paragraphs of Sect. 6.4].

Let T denote the set of $(n+1)$-tuples of points in $\partial\mathbb{H}^n$ which are vertices of a regular simplex,

$$T = \left\{ \underline{\xi} = (\xi_0, \ldots, \xi_n) \in \left(\partial\mathbb{H}^n\right)^{n+1} \;\middle|\; \underline{\xi} \text{ are the vertices of an ideal regular simplex} \right\}.$$

We shall call an $(n+1)$-tuple in T a regular simplex. Note that the order of the vertices $\xi_0, \ldots, \xi_n$ induces an orientation on the simplex $\underline{\xi}$. For $\underline{\xi} \in T$, denote by $\Lambda_{\underline{\xi}} < \mathrm{Isom}(\mathbb{H}^n)$ the reflection group generated by the reflections in the faces of the simplex $\underline{\xi}$.

Lemma 4.6 *Let $n \geqslant 3$. Let $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in T$. Suppose that $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a map such that for every $\gamma \in \Lambda_{\underline{\xi}}$, the simplex with vertices $(\varphi(\gamma\xi_0), \ldots, \varphi(\gamma\xi_n))$ is regular and of the same orientation as $(\gamma\xi_0, \ldots, \gamma\xi_n) \in T$. Then there exists a unique isometry $h \in \mathrm{Isom}(\mathbb{H}^n)$ such that $h(\xi) = \varphi(\xi)$ for every $\xi \in \bigcup_{i=0}^n \Lambda_{\underline{\xi}}\xi_i$.*

Note that this lemma and its subsequent proposition are the only places in the proof where the assumption $n \geqslant 3$ is needed. The lemma is wrong for $n = 2$ since φ could be any orientation preserving homeomorphism of $\partial\mathbb{H}^2$.

Proof If $\underline{\xi} = (\xi_0, \ldots, \xi_n)$ and $(\varphi(\xi_0), \ldots, \varphi(\xi_n))$ belong to T, then there exists a unique isometry $h \in \mathrm{Isom}^+(\mathbb{H}^n)$ such that $h\xi_i = \varphi(\xi_i)$ for $i = 0, \ldots, n$. It remains to check that

$$h(\gamma\xi_i) = \varphi(\gamma\xi_i) \tag{20}$$

for every $\gamma \in \Lambda_{\underline{\xi}}$. Every $\gamma \in \Lambda_{\underline{\xi}}$ is a product $\gamma = r_k \cdot \ldots \cdot r_1$, where r_j is a reflection in a face of the regular simplex $r_{j-1} \cdot \ldots \cdot r_1(\underline{\xi})$. We prove the equality (20) by induction on k, the case $k = 0$ being true by assumption. Set $\eta_i = r_{k-1} \cdot \ldots \cdot r_1(\xi_i)$. By induction, we know that $h(\eta_i) = \varphi(\eta_i)$. We need to show that $h(r_k\eta_i) = \varphi(r_k\eta_i)$. The simplex $(\eta_0, \ldots, \eta_n)$ is regular and r_k is a reflection in one of its faces, say the face containing $\eta_1, \ldots, \eta_n$. Since $r_k\eta_i = \eta_i$ for $i = 1, \ldots, n$, it just remains to show that $h(r_k\eta_0) = \varphi(r_k\eta_0)$. The simplex $(r_k\eta_0, r_k\eta_1, \ldots, r_k\eta_n) = (r_k\eta_0, \eta_1, \ldots, \eta_n)$ is regular with opposite orientation to $(\eta_0, \eta_1, \ldots, \eta_n)$. This implies on the one hand that the simplex $(h(r_k\eta_0), h(\eta_1), \ldots, h(\eta_n))$ is regular with opposite orientation to $(h(\eta_0), h(\eta_1), \ldots, h(\eta_n))$, and on the other hand that the simplex $(\varphi(r_k\eta_0), \varphi(\eta_1), \ldots, \varphi(\eta_n))$ is regular with opposite orientation to $(\varphi(\eta_0), \ldots, \varphi(\eta_n))$. Since $(h(\eta_0), h(\eta_1), \ldots, h(\eta_n)) = (\varphi(\eta_0), \ldots, \varphi(\eta_n))$ and there is in dimension $n \geqslant 3$ only one regular simplex with face $h(\eta_1), \ldots, h(\eta_n)$ and opposite orientation to $(h(\eta_0), h(\eta_1), \ldots, h(\eta_n))$ it follows that $h(r_k\eta_0) = \varphi(r_k\eta_0)$. □

If φ were continuous, sending the vertices of all positively (respectively negatively) oriented ideal regular simplices to vertices of positively (resp. neg.) oriented ideal regular simplices, then it would immediately follow from the lemma that φ is equal to an isometry h on the orbits $\bigcup_{i=0}^n \Lambda_{\underline{\xi}}\xi_i$ of the vertices of one regular simplex under its reflection group. Since the set $\bigcup_{i=0}^n \Lambda_{\underline{\xi}}\xi_i$ is dense in $\partial\mathbb{H}^n$, the continuity of φ would imply that φ is equal to the isometry h on the whole $\partial\mathbb{H}^n$.

In the setting of the next proposition, we first need to show that there exist enough regular simplices for which φ maps every simplex of its orbit under reflections to a regular simplex. Second, we apply the lemma to obtain that φ is equal to an isometry on these orbits. Finally, we use ergodicity of the reflection groups to conclude that it is the same isometry for almost all regular simplices. As mentioned earlier, the proposition also holds for $n = 3$ (see [17, pp. 654-656] and [33, two last paragraphs of Sect. 6.4]), but in that case the proof is quite harder, since the reflection group of a regular simplex is discrete in $\mathrm{Isom}(\mathbb{H}^n)$ (indeed, one can tile $\mathbb{H}^3$ by regular ideal simplices) and in particular does not act ergodically on $\mathrm{Isom}(\mathbb{H}^n)$.

Proposition 4.7 *Let $n \geqslant 4$. Let $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be a measurable map sending the vertices of almost every positively, respectively negatively oriented regular ideal simplex to the vertices of a positively, resp. negatively, oriented regular ideal simplex. Then φ is equal almost everywhere to an isometry.*

Proof Let $T^\varphi \subset T$ denote the following subset of the set T of regular simplices:

$$T^\varphi = \big\{\underline{\xi} = (\xi_0, \ldots, \xi_n) \in T \,\big|\, \big(\varphi(\xi_0), \ldots, \varphi(\xi_n)\big) \text{ belongs to } T \text{ and has the same orientation as } (\xi_0, \ldots, \xi_n)\big\}.$$

By assumption, T^φ has full measure in T. Let $T^\varphi_\Lambda \subset T^\varphi$ be the subset consisting of those regular simplices for which all reflections by the reflection group $\Lambda_{\underline{\xi}}$ are in T^φ,

$$T^\varphi_\Lambda = \big\{\underline{\xi} \in T \,\big|\, \gamma\underline{\xi} \in T^\varphi \ \forall \gamma \in \Lambda_{\underline{\xi}}\big\}.$$

We claim that T^φ_Λ has full measure in T.

To prove the claim, we do the following identification. Since $G = \mathrm{Isom}(\mathbb{H}^n)$ acts simply transitively on the set T of (oriented) regular simplices, given a base point $\underline{\eta} = (\eta_0, \ldots, \eta_n) \in T$ we can identify G with T via the evaluation map

$$\begin{aligned} Ev_\eta : G &\longrightarrow T \\ g &\longmapsto g(\underline{\eta}). \end{aligned}$$

The subset T^φ is mapped to a subset $G^\varphi := (Ev_\eta)^{-1}(T^\varphi) \subset G$ via this correspondence. A regular simplex $\underline{\xi} = g(\underline{\eta})$ belongs to T^φ_Λ if and only if, by definition, $\gamma\underline{\xi} = \gamma g\underline{\eta}$ belongs to T^φ for every $\gamma \in \Lambda_{\underline{\xi}}$. Since $\Lambda_{\underline{\xi}} = g\Lambda_{\underline{\eta}}g^{-1}$, the latter condition is equivalent to $g\gamma_0\underline{\eta} \in T^\varphi$ for every $\gamma_0 \in \Lambda_{\underline{\eta}}$, or in other words, $g \in G^\varphi\gamma_0^{-1}$. The subset T^φ_Λ is thus mapped to

$$G^\varphi = Ev_{\underline{\eta}}^{-1}\big(T^\varphi_\Lambda\big) = \bigcap_{\gamma_0 \in \Lambda_{\underline{\eta}}} G^\varphi\gamma_0^{-1} \subset G$$

via the above correspondence. Since a countable intersection of full measure subsets has full measure, the claim is proved.

For every $\underline{\xi} \in T^{\varphi}_{\Lambda}$ and hence almost every $\underline{\xi} \in T$ there exists by Lemma 4.6 a unique isometry $h_{\underline{\xi}}$ such that $h_{\underline{\xi}}(\xi) = \varphi(\xi)$ on the orbit points $\xi \in \bigcup_{i=0}^{n} \Lambda_{\underline{\xi}} \xi_i$. By the uniqueness of the isometry, it is immediate that $h_{\gamma\underline{\xi}} = h_{\underline{\xi}}$ for every $\gamma \in \Lambda_{\underline{\xi}}$. We have thus a map $h : T \to \mathrm{Isom}(\mathbb{H}^n)$ given by $\underline{\xi} \mapsto h_{\underline{\xi}}$ defined on a full measure subset of T. Precomposing h by $Ev_{\underline{\eta}}$, it is straightforward that the left $\Lambda_{\underline{\xi}}$-invariance of h on $\Lambda_{\underline{\xi}}\underline{\xi}$ naturally translates to a global right invariance of $h \circ Ev_{\underline{\eta}}$ on G. Indeed, let $g \in G$ and $\gamma_0 \in \Lambda_{\underline{\eta}}$. We compute

$$h \circ Ev_{\underline{\eta}}(g \cdot \gamma_0) = h_{g\gamma_0\underline{\eta}} = h_{g\gamma_0 g^{-1} g\underline{\eta}} = h_{g\underline{\eta}} = h \circ Ev_{\underline{\eta}}(g),$$

where we have used the left $\Lambda_{g\underline{\eta}}$-invariance of h on the reflections of $g\eta$ in the third equality. (Recall, $g\gamma_0 g^{-1} \in g\Lambda_{\underline{\eta}} g^{-1} = \Lambda_{g\underline{\eta}}$.) Thus, $h \circ Ev_{\underline{\eta}} : G \to G$ is invariant under the right action of $\Lambda_{\underline{\eta}}$. Since the latter group is dense in G, it acts ergodically on G and $h \circ Ev_{\underline{\eta}}$ is essentially constant. This means that also h is essentially constant. Thus, for almost every regular simplex $\underline{\xi} \in T$, the evaluation of φ on any orbit point of the vertices of $\underline{\xi}$ under the reflection group $\Lambda_{\underline{\xi}}$ is equal to h. In particular, for almost every $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in T$ and also for almost every $\xi_0 \in \mathbb{H}^n$, we have $\varphi(\xi_0) = h(\xi_0)$, which finishes the proof of the proposition. □

We have now established that φ is essentially equal to the isometry $h \in \mathrm{Isom}(\mathbb{H}^n)$ on $\partial\mathbb{H}^n$. It remains to see that h realizes the conjugation between ρ and i. Indeed, replacing φ by h in (10) we have

$$\big(h \cdot i(\gamma)\big)(\xi) = \big(\rho(\gamma) \cdot h\big)(\xi),$$

for every $\xi \in \partial\mathbb{H}^n$ and $\gamma \in \Gamma$. Since all maps involved (h, $i(\gamma)$ and $\rho(\gamma)$) are isometries of $\mathbb{H}^n$ and two isometries induce the same map on $\partial\mathbb{H}^n$ if and only if they are equal it follows that

$$h \cdot i(\gamma) \cdot h^{-1} = \rho(\gamma)$$

for every $\gamma \in \Gamma$, which finishes the proof of the theorem.

Acknowledgements Michelle Bucher was supported by Swiss National Science Foundation project PP00P2-128309/1, Alessandra Iozzi was partially supported by Swiss National Science Foundation project 2000021-127016/2. The first and third named authors thank the Institute Mittag-Leffler in Djurholm, Sweden, for their warm hospitality during the preparation of this paper. Finally, we thank Beatrice Pozzetti for a thorough reading of the manuscript and numerous insightful remarks.

References

1. Bader, U., Furman, A., Sauer, R.: Integrable measure equivalence and rigidity of hyperbolic lattices. http://arxiv.org/abs/1006.5193v1
2. Besson, G., Courtois, G., Gallot, S.: Entropies et rigidités des espaces localement symétriques de courbure strictement négative. Geom. Funct. Anal. **5**, 731–799 (1995). MR 1354289 (96i:58136)

3. Besson, G., Courtois, G., Gallot, S.: Inégalités de Milnor-Wood géométriques. Comment. Math. Helv. **82**, 753–803 (2007). MR 2341839 (2009e:53055)
4. Bloch, S.J.: Higher Regulators, Algebraic K-theory, and Zeta Functions of Elliptic Curves. CRM Monograph Series, vol. 11. Am. Math. Soc., Providence (2000). MR 1760901 (2001i:11082)
5. Brooks, R.: Some remarks on bounded cohomology. In: Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, State Univ. New York, Stony Brook, NY, 1978. Ann. of Math. Stud., vol. 97, pp. 53–63. Princeton Univ. Press, Princeton (1981). MR 624804 (83a:57038)
6. Bucher-Karlsson, M.: The proportionality constant for the simplicial volume of locally symmetric spaces. Colloq. Math. **111**, 183–198 (2008). MR 2365796 (2008k:53105)
7. Bucher-Karlsson, M.: The simplicial volume of closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. J. Topol. **1**, 584–602 (2008). MR 2417444 (2009i:53025)
8. Bucher, M., Kim, I., Kim, S.: Proportionality principle for the simplicial volume of families of $\mathbf{Q}$-rank 1 locally symmetric spaces. arXiv:1209.4697
9. Bucher, M., Monod, N.: The norm of the Euler class. Math. Ann. **353**, 523–544 (2012)
10. Burger, M., Iozzi, A.: Boundary maps in bounded cohomology. Geom. Funct. Anal. **12**(2), 281–292 (2002). MR 1911668 (2003d:53065b). Appendix to: Burger, M., Monod, M.: Geom. Funct. Anal. **12**, 219–280 (2002). MR 1911660 (2003d:53065a)
11. Burger, M., Iozzi, A.: A useful formula from bounded cohomology. In: Géométries à courbure négative ou nulle, groupes discrets et rigidités. Sémin. Congr., vol. 18, pp. 243–292. Soc. Math. France, Paris (2009). MR 2655315 (2011m:22017)
12. Burger, M., Iozzi, A., Wienhard, A.: Surface group representations with maximal Toledo invariant. C. R. Math. Acad. Sci. Paris **336**(5), 387–390 (2003). MR 1979350 (2004e:53076)
13. Burger, M., Iozzi, A., Wienhard, A.: Surface group representations with maximal Toledo invariant. Ann. Math. (2) **172**(1), 517–566 (2010). MR 2680425
14. Calabi, E., Vesentini, E.: Sur les variétés complexes compactes localement symétriques. Bull. Soc. Math. Fr. **87**, 311–317 (1959). MR 0111057 (22 #1922a)
15. Calabi, E., Vesentini, E.: On compact, locally symmetric Kähler manifolds. Ann. Math. **71**, 472–507 (1960). MR 0111058 (22 #1922b)
16. Douady, A., Earle, C.J.: Conformally natural extension of homeomorphisms of the circle. Acta Math. **157**, 23–48 (1986). MR 857678 (87j:30041)
17. Dunfield, N.M.: Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds. Invent. Math. **136**, 623–657 (1999). MR 1695208 (2000d:57022)
18. Francaviglia, S.: Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds. Int. Math. Res. Not. **9**, 425–459 (2004). MR 2040346 (2004m:57032)
19. Francaviglia, S., Klaff, B.: Maximal volume representations are Fuchsian. Geom. Dedic. **117**, 111–124 (2006). MR 2231161 (2007d:51019)
20. Goldman, W.M.: Discontinuous groups and the Euler class. ProQuest LLC, Ann Arbor, MI, 1980. Ph. D. Thesis, University of California, Berkeley. MR 2630832
21. Gromov, M.: Volume and bounded cohomology. Inst. Hautes Études Sci. Publ. Math. **56**, 5–99 (1982). MR 686042 (84h:53053)
22. Guichardet, A.: Cohomologie des Groupes Topologiques et des Algèbres de Lie. Textes Mathématiques [Mathematical Texts], vol. 2. CEDIC, Paris (1980). MR 644979 (83f:22004)
23. Haagerup, U., Munkholm, H.J.: Simplices of maximal volume in hyperbolic n-space. Acta Math. **147**, 1–11 (1981). MR 631085 (82j:53116)
24. Iozzi, A.: Bounded cohomology, boundary maps, and rigidity of representations into $\mathrm{Homeo}_+(\mathbf{S}^1)$ and $\mathrm{SU}(1,n)$. In: Rigidity in Dynamics and Geometry, Cambridge, 2000, pp. 237–260. Springer, Berlin (2002). MR 1919404 (2003g:22008)
25. Kneser, H.: Die kleinste Bedeckungszahl innerhalb einer Klasse von Flächenabbildungen. Math. Ann. **103**, 347–358 (1930). MR 1512626
26. Mackey, G.W.: Induced representations of locally compact groups. I. Ann. Math. (2) **55**, 101–139 (1952). MR 0044536 (13,434a)

27. Monod, N.: Continuous Bounded Cohomology of Locally Compact Groups. Lect. Notes in Mathematics, vol. 1758. Springer, Berlin (2001). MR 1840942 (2002h:46121)
28. Mostow, G.D.: Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. Inst. Hautes Études Sci. Publ. Math. **34**, 53–104 (1968). MR 0236383 (38 #4679)
29. Prasad, G.: Strong rigidity of $\mathbf{Q}$-rank 1 lattices. Invent. Math. **21**, 255–286 (1973). MR 0385005 (52 #5875)
30. Reiter, H., Stegeman, J.D.: Classical Harmonic Analysis and Locally Compact Groups, London Mathematical Society Monographs, New Series, vol. 22, 2nd edn. Clarendon, Oxford University Press, New York (2000). MR 1802924 (2002d:43005)
31. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987). MR 924157 (88k:00002)
32. Selberg, A.: On discontinuous groups in higher-dimensional symmetric spaces. In: Contributions to Function Theory, Internat. Colloq. Function Theory, Bombay, 1960, pp. 147–164. Tata Institute of Fundamental Research, Bombay (1960). MR 0130324 (24 #A188)
33. Thurston, W.: Geometry and Topology of 3-Manifolds. Notes from Princeton University, Princeton (1978)
34. Toledo, D.: Representations of surface groups in complex hyperbolic space. J. Differ. Geom. **29**, 125–133 (1989). MR 978081 (90a:57016)
35. Weil, A.: On discrete subgroups of Lie groups. Ann. Math. (2) **72**, 369–384 (1960). MR 0137792 (25 #1241)
36. Weil, A.: On discrete subgroups of Lie groups. II. Ann. Math. (2) **75**, 578–602 (1962). MR 0137793 (25 #1242)
37. Zimmer, R.J.: Ergodic Theory and Semisimple Groups. Monographs in Mathematics, vol. 81. Birkhäuser, Basel (1984). MR 776417 (86j:22014)