WEAKLY MAXIMAL REPRESENTATIONS OF SURFACE GROUPS

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Abstract. We introduce and study a new class of representations of surface groups into Lie groups of Hermitian type, called *weakly maximal* representations. They are defined in terms of invariants in bounded cohomology and extend considerably the scope of maximal representations studied in [12, 10, 8, 9, 27, 21, 19, 18, 6, 5, 4, 16]. We prove that weakly maximal representations are discrete and injective and describe the structure of the Zariski closure of the image. An interesting feature of these representations is that they admit an elementary topological characterization in terms of bi-invariant orderings. In particular if the target group is Hermitian of tube type, the ordering can be described in terms of the causal structure on the Shilov boundary.

— Research Announcement —

1. Introduction

This research announcement presents results obtained by the authors during the last two years concerning the class of so-called weakly maximal representations of surface groups into a Hermitian Lie group. A more detailed version of this note with full proofs is currently under preparation [1].

Given a compact oriented surface Σ of negative Euler characteristic, possibly with boundary, a general theme is to study the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)$ of the fundamental group of Σ into a semisimple Lie group G, and in particular to distinguish subsets of geometric significance, such as holonomy representations of geometric structures. Classical examples include the set of Fuchsian representations in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ or the set of quasi-Fuchsian representations in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{C}))$, where the target group is of real rank one. In recent years these studies have been extended to the case where G is of higher rank. Prominent examples of geometrically significant subsets of representation varieties for higher rank targets include Hitchin components [22, 13, 25], positive representations [15], maximal representations [12, 10, 8, 9, 27, 21, 18,

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6, 5, 4, 16] and Anosov representations [20, 25, 19]. Even though these subsets exhibit several common properties, which has lead to summarizing their study under the terminus *higher Teichmüller theory*, they are defined and investigated by very different methods.

The present article is concerned with an extension of higher Teichmüller theory in the Hermitian context. Recall that a semisimple Lie group G is called *Hermitian* if the associated symmetric space $\mathcal{X}$ admits a G-invariant Kähler form $\omega_{\mathcal{X}}$. This Kähler form can be used to define a continuous function $T : \mathrm{Hom}(\pi_1(\Sigma), G) \to \mathbb{R}$ on the representation variety; the invariant $T(\rho)$ is called the *Toledo number* of the representation ρ. If the surface Σ is closed then T is defined by the formula

$$T(\rho) := \frac{1}{2\pi} \cdot \int_{\Sigma} p_* f^* \omega_{\mathcal{X}},$$

where $f : \widetilde{\Sigma} \to \mathcal{X}$ is an arbitrary ρ-equivariant map and $p : \widetilde{\Sigma} \to \Sigma$ is the universal covering projection. For Σ with boundary a modification of this definition has been provided in [12]. In any case, the Toledo number is subject to a *Milnor-Wood type inequality* of the form

$$|T(\rho)| \leq ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|, \tag{1.1}$$

where $\kappa_G^{\mathrm{b}} \in H^2_{cb}(G;\mathbb{R})$ denotes the bounded Kähler class of G, i.e. the class corresponding to $\omega_{\mathcal{X}}$ under the isomorphisms $H^2_{cb}(G;\mathbb{R}) \cong H^2_c(G;\mathbb{R}) \cong \Omega^2(\mathcal{X})^G$, and $||\cdot||$ denotes the seminorm in continuous bounded cohomology (see [12]). The class of representations ρ with maximal Toledo invariant $T(\rho) = ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|$, or *maximal representations* for short, has been the main object of study in higher Teichmüller theory with Hermitian target groups ([12, 10, 8, 9, 27, 21, 19, 18, 6, 5, 4, 16]). Here we propose a generalization of maximal representations, which preserves many of their key properties. Our starting point is the observation that the inequality (1.1) can be refined into the chain of inequalities

$$|T(\rho)| \leq ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)| \leq ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|.$$

In particular, a representation is maximal iff it satisfies both $||\rho^*\kappa_G^{\mathrm{b}}|| = ||\kappa_G^{\mathrm{b}}||$ and $T(\rho) = ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|$. Representations satisfying $||\rho^*\kappa_G^{\mathrm{b}}|| = ||\kappa_G^{\mathrm{b}}||$ are called *tight*; these have been investigated in much greater generality in [11]. Here we are interested in representations satisfying the complementary property (see [26]):

Definition 1.1. A representation $\rho : \pi_1(\Sigma) \to G$ is *weakly maximal* if it satisfies

$$T(\rho) = ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|. \tag{1.2}$$

By definition a representation is maximal iff it is weakly maximal and tight. In the following three sections we will present results concerning

- structure theorems describing the range, kernel and Zariski closure of weakly maximal representations with nonzero Toledo invariant;

- a geometric interpretation of weakly maximal representations of nonzero Toledo invariant in terms of causal structures on Shilov boundaries;
- the relation between the space of weakly maximal representations and other prominent subsets of the representation variety.

These results demonstrate that weakly maximal representations form a very broad and geometrically significant class of representations which still share many desirable structural properties with maximal representations.

2. Structure theorems for weakly maximal representations

Various general structure theorems for maximal representations have been established by three of the present authors in [12]. We show that an essential part of these structure theorems can be established for weakly maximal representations. For the proofs it is important to understand the range of the Toledo number when restricted to weakly maximal representations. Let us assume that the Kähler form $\omega_{\mathcal{X}}$ has been normalized to have minimal holomorphic curvature -1. With this normalization we then have $||\kappa_G^{\mathrm{b}}|||\chi(\Sigma)| \in \mathbb{Z}$, hence maximal representations have integral Toledo number. Moreover, if Σ is closed, then $T(\rho) \in e_G^{-1} \cdot \mathbb{Z}$ for some integer e_G depending only on G. In particular, the range of T is finite. In strong contrast, if the surface is admitted to have a boundary, then $T(\rho)$ can take arbitrary values inside the closed interval $[-||\kappa_G^{\mathrm{b}}|||\chi(\Sigma)|, ||\kappa_G^{\mathrm{b}}|||\chi(\Sigma)|]$. It is therefore significant that for weakly maximal representations we can prove:

Theorem 2.1. *There is natural number n_G depending only on G, such that for every weakly maximal representation $\rho : \pi_1(\Sigma) \to G$ we have $n_G T(\rho) \in \mathbb{Z}$. In particular, T takes only finitely many values on weakly-maximal representations.*

Theorem 2.1 plays a crucial part in establishing the following basic properties of weakly maximal representations of nonzero Toledo invariant:

Theorem 2.2. *Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal representation and $T(\rho) \neq 0$. Then ρ is faithful with discrete image.*

An important step in the proof of Theorem 2.2 is the realization that a representation ρ is weakly maximal iff there exists $\lambda \geq 0$ such that

$$\rho^* \kappa_G^{\mathrm{b}} = \lambda \cdot \kappa_\Sigma^b, \tag{2.1}$$

where $\kappa_\Sigma^b \in H_b^2(\Gamma)$ is the bounded fundamental class of the surface Σ as introduced in [12]. By Theorem 2.1 the constant λ has in fact to be *rational*. This provides severe restrictions on the kernel and range of ρ.

Both Theorem 2.1 and Theorem 2.2 depend on understanding the Zariski closure of a weakly maximal representation. Unlike for maximal representations, the Zariski closure of a weakly maximal representation need not be reductive. To overcome this difficulty, we argue as follows. We first show that for a closed subgroup $L < G$ there exists a unique maximal normal subgroups of L on which

$\kappa_G^{\mathrm{b}}|_L$ vanishes. This subgroup is called the *Kähler radical* $\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$of L, and the quotient $L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ is automatically semisimple. While the Zariski closure of a weakly maximal representation can be fairly complicated, we have a rather good control over its quotient by its Kähler radical, provided the Toledo number is nonzero:

THEOREM 2.3. *Let $\rho : \pi_1(\Sigma) \to G$ be weakly maximal representation with $T(\rho) \neq 0$. Let $L < G$ be the Zariski closure of the image of ρ and set $H = L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$. Then*

1. *H is a semisimple Lie group of Hermitian type; all almost simple factor of H are of tube type.*
2. *The composition $\pi_1(\Sigma) \to L \to H$ is faithful with discrete image.*

REMARK 2.4. In the above theorems it is essential that the Toledo number is nonzero. However the class of weakly maximal representations with $T(\rho) = 0$ is also of interest. It includes in particular the set of representations where $\rho^*(\kappa_G^{\mathrm{b}}) = 0$. In the case when $G = \mathrm{PU}(1, n)$ such representations have been studied in [7].

3. Geometric description of weakly maximal representations

It turns out that techniques from [3] can be used to provide a geometric characterization of weakly maximal representations with nonzero Toledo invariant in terms of bi-invariant orders. To simplify the formulation we will only spell out the results in the case where the target group G is of *tube type*; this is justified by Theorem 2.3. We will also assume that G is *adjoint simple*.

We now fix an adjoint simple Hermitian Lie group G of tube type and denote by $\widehat{G} = \widetilde{G}/\pi_1(G)^{tor}$ the unique central $\mathbb{Z}$-extension of G. Then causal geometry gives rise to a bi-invariant partial order on $\widehat{G}$ (see [2] for a discussion of this and various related bi-invariant partial orders on Lie groups). A prototypical example arises from the action of $G = \mathrm{PU}(1, 1)$ on the boundary of the Poincaré disk $\mathbb{D}$; this action lifts to an action of the universal covering $\widehat{G} = \widetilde{\mathrm{PU}(1, 1)}$ on $\mathbb{R}$, hence induces a bi-invariant partial order on $\widehat{G}$ by setting

$$g \leq h :\Leftrightarrow \forall x \in \mathbb{R} : \; g.x \leq h.x.$$

In the general case one utilizes the fact that by the tube type assumption there exists a unique pair $\pm\mathcal{C}$ of G-invariant causal structures on the Shilov boundary $\check{S}$ of the bounded symmetric domain associated with G (see [24]). Here, by a causal structure $\mathcal{C}$ we mean a family of *closed* cones $\mathcal{C}_x \subset T_x\check{S}$, and invariance is understood in the sense that $g_*\mathcal{C}_x = \mathcal{C}_{gx}$. The causal structures $\pm\mathcal{C}$ lift to $\widehat{G}$-invariant causal structures on the universal covering $\check{R}$ of $\check{S}$, which in turn induce a pair of mutually inverse (closed) partial orders on $\check{R}$ via causal curves. Let us denote by $\preceq$ the partial order which is compatible with the orientation given by the Kähler class. We then obtain a bi-invariant partial order on $\widehat{G}$ by setting

$$g \leq_{\widehat{G}} h :\Leftrightarrow \forall x \in \check{R} :\ g.x \preceq h.x.$$

The *dominant set* $\widehat{G}^{++}$ (in the sense of [14, 3]) of this bi-invariant order is given by the formula

$$\widehat{G}^{++} := \{g \in \widehat{G} \,|\, \forall h \in G \exists n \in \mathbb{N} :\ g^n \geq_{\widehat{G}} h\},$$

We provide the following simple description in terms of the causal structure:

Theorem 3.1. *If $\widehat{G}$ is of tube type then*

$$\widehat{G}^{++} = \{g \in \widehat{G} \,|\, \forall x \in \check{R} :\ g.x \succ x\}.$$

We now provide an interpretation of weakly-maximal representations in terms of dominant sets. Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with n boundary components. We always assume that $\chi(\Sigma)_{g,n} < 0$ so that there exists a hyperbolization $\rho : \Gamma_{g,n} := \pi_1(\Sigma_{g,n}) \to \mathrm{PU}(1,1)$. If $n \geq 1$, then $\Gamma_{g,n}$ is a free group, hence ρ admits a lift $\widetilde{\rho} : \Gamma_{g,n} \to \widetilde{\mathrm{PU}(1,1)}$ whose restriction to the group of homologically trivial loops $\Lambda_{g,n} := [\Gamma_{g,n}, \Gamma_{g,n}]$ is unique. In particular, the translation number quasimorphism on $\widetilde{\mathrm{PU}(1,1)}$ pulls back to a quasimorphism $f_{\Sigma_{g,n}}$ on $\Lambda_{g,n}$. It turns out that this quasimorphism is independent of the choice of hyperbolization ρ; in fact it admits a topological description in terms of winding numbers [23]. In the case in which $n = 0$, one cannot perform this construction on $\Gamma_{g,0}$, but one has to pass to the central extension $\overline{\Gamma_{g,0}}$ that corresponds to the generator of $\mathrm{H}^2(\Gamma_{g,0}, \mathbb{Z})$ or, equivalently, can be realized as the fundamental group of the S^1-bundles over Σ_g of Euler number one. One then obtains in the same way as above a canonical quasimorphism $f_{\Sigma_{g,0}}$ on $\Lambda_{g,0} := [\overline{\Gamma}_{g,0}, \overline{\Gamma}_{g,0}]$. We emphasize that the quasimorphism $f_{\Sigma_{g,n}}$ depends on the topological surface $\Sigma_{g,n}$, not just the abstract group $\Gamma_{g,n}$.

Theorem 3.2. *Let G be an adjoint simple Hermitian Lie group of tube type and let $\widehat{G}$, $\widehat{G}^{++}$ as above. Let $\Sigma_{g,n}$ be a surface of negative Euler characteristic and $\Gamma_{g,n} := \pi_1(\Sigma_{g,n})$. Then a representation $\rho : \Gamma_{g,n} \to G$ is weakly maximal with $T(\rho) \neq 0$ iff for the unique lift $\widetilde{\rho} : \Lambda_{g,n} \to \widehat{G}$ there exists $N > 0$ such that*

$$f_{\Sigma_{g,n}}(\gamma) > N \Rightarrow \widetilde{\rho}(\gamma) \in \widehat{G}^{++} \quad (\gamma \in \Lambda_{g,n}). \tag{3.1}$$

Remark 3.3. If we define a family of partial orders $\leq_N$ on $\Lambda_{g,n}$ by

$$g <_N h :\Leftrightarrow f_{\Sigma_{g,n}}(g^{-1}h) > N,$$

and a partial order on $\widehat{G}$ by

$$g \leq_{++} h :\Leftrightarrow g^{-1}h \in \widehat{G}^{++}$$

then the conclusion can be rephrased by saying that $\widetilde{\rho}$ is order-preserving with respect to some $\leq_N$ and $\leq_{++}$.

4. Comparison to other classes of representations

Turning back to the general theme of studying subsets of the representation variety we describe basic properties of the set $\mathrm{Hom}_{wm}(\pi_1(\Sigma), G)$ of weakly maximal representations, and relate it to other geometrically meaningful subsets of $\mathrm{Hom}(\pi_1(\Sigma), G)$. We will denote by $\mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G) \subset \mathrm{Hom}(\pi_1(\Sigma), G)$ the subset of weakly maximal representations with nonzero Toledo number. Also we denote by $\mathrm{Hom}_{di}(\pi_1(\Sigma), G)$ the set of discrete and faithful representations. By Theorem 2.2 we have a chain of inclusions

$$\mathrm{Hom}_{max}(\pi_1(\Sigma), G) \subset \mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G) \subset \mathrm{Hom}_{di}(\pi_1(\Sigma), G). \tag{4.1}$$

The sets on the right [17] and on the left are closed; if $\partial\Sigma = \emptyset$ the left one is also open [12]. We are able to show:

Theorem 4.1. *The set* $\mathrm{Hom}_{wm}(\pi_1(\Sigma), G) \subset \mathrm{Hom}(\pi_1(\Sigma), G)$ *is closed.*

Combining this with Theorem 2.1 we then obtain:

Corollary 4.2. *The set* $\mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G) \subset \mathrm{Hom}(\pi_1(\Sigma), G)$ *is closed.*

Thus (4.1) is a chain of *closed* subsets of the representation variety. In the case where Σ is a closed surface we can refine this chain further: It has been established in [8, 10] that maximal representations are Shilov-Anosov in the sense of [20, 25]. Concerning the (open) set $\mathrm{Hom}_{\check{S}-An}(\pi_1(\Sigma), G))$ of all Shilov-Anosov representations we establish the following:

Theorem 4.3. *Assume that* $\partial\Sigma = \emptyset$ *and that* G *is a Lie group of tube type. Then*

$$\overline{\mathrm{Hom}_{\check{S}-An}(\pi_1(\Sigma), G)} \subset \mathrm{Hom}_{wm}(\pi_1(\Sigma), G). \tag{4.2}$$

For a *closed* surface Σ and a Hermitian group G of tube type we thus end up with the following diagram. Here we denote by $\mathrm{Hom}^*_\epsilon(\pi_1(\Sigma), G))$ the set of representations in $\mathrm{Hom}_\epsilon(\pi_1(\Sigma), G))$ of nonzero Toledo number. We also denote by $\mathrm{Hom}_{red}(\pi_1(\Sigma), G))$ the set of representations with reductive Zariski closure and by $\mathrm{Hom}_{Hitchin}(\pi_1(\Sigma), G))$ the Hitchin component in case G is locally isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$ (and the empty set otherwise). Then we have the following inclusions:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Hom}_{\check{S}-An} & \subset & \mathrm{Hom}_{wm} & & \\
 & & & & \cup & & \cup & & \\
\mathrm{Hom}_{Hitchin} & \subset & \mathrm{Hom}_{max} & \subset & \mathrm{Hom}^*_{\check{S}-An} & \subset & \mathrm{Hom}^*_{wm} & \subset & \mathrm{Hom}_{di} \\
 & & \cap & & & & & & \\
 & & \mathrm{Hom}_{tight} & \subset & \mathrm{Hom}_{red} & & & &
\end{array}$$

If Σ is allowed to have boundary, then the relations between the various subsets of the representation variety is more complicated.

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