Problems in Mean Curvature Flow

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Many of the following problems are classical, and many others have been told to me by various people over the years. Comments are welcome.

General

The following conjecture arises from the dimension-reducing theory of White.

1. Partial regularity conjecture. An embedded weak mean curvature of hypersurfaces $\mathbf{R}^{n+1}$ has a singular set of parabolic dimension at most $n-1$.

2. Nonsqueezing conjecture. Let M_0 be a smooth, embedded, compact initial hypersurface in $\mathbf{R}^{n+1}$ with weak mean curvature flow M_t. Then a higher multiplicity plane cannot occur as a blowup limit of M_t.

3. Uniqueness of tangent flow. Let N_t be a smooth. multiplicity-one self-similarly shrinking flow obtained as the limit of centered rescalings of a mean curvature flow. Then N_t is the full limit.

4. Positive mean curvature neighborhoods. Let M_t be a mean curvature flow and let (x,t) be a singularity of positive mean curvature type (sphere or cylinder). Then there is a spacetime neighborhood of (x,t) in which M_t has positive mean curvature.

5. Mean curvature flow of immersions. Develop a theory of mean curvature flow of immersions.

We call N_t a *limit flow* if it is the limit of a sequence of rescalings by factors λ_i^{-1} about a sequence of points (x_i, t_i). It is a *blowup limit* if $\lambda_i \to 0$.

6. Self-similarity for limit flows. Is a blowup limit always self-similarly shrinking, expanding, translating, or static?

7. Entrance law. (Griffeath) A random set possesses a (weak) mean curvature evolution, whose boundary has finite perimeter for $t > 0$, almost surely.

Generic properties

8. Generic positive curvature singularities. (Huisken) All singularities of a generic embedded mean curvature flow are spheres or cylinders.

Partial results by Ilmanen.

9. Generic point singularities. A generic weak mean curvature flow has only point singularities.

Flow of curves with triple junctions

10. Networks with triple junctions. Develop a theory of the flow by mean curvature of networks with triple junctions in the plane.

Flow of surfaces in $\mathbf{R}^3$

11. Optimal partial regularity in dimension 3. An embedded MCF in $\mathbf{R}^3$ satisfies $\dim_P \operatorname{sing} \mathcal{M} \leq 1$. The singular set consists of isolated points unless M_t is a tube that shrinks to a curve.

Here $\dim_P$ is the parabolic Hausdorff dimension.

12. No cylinder conjecture. Let N be an embedded shrinking soliton in $\mathbf{R}^3$, and suppose N is not the round cylinder. Can N have an end asymptotic to a cylinder?

13. Strict genus reduction conjecture. A shrinker N with mixed mean curvature has positive genus. The genus strictly decreases at any singularity modeled on N.

Special cases:

14. Wiggly plane. The only toplogical plane that is an embedded shrinker in $\mathbf{R}^3$ is the flat plane.

15. Planar domains. The only planar domain that is an embedded shrinker in $\mathbf{R}^3$ is the round cylinder.

16. Resolution of point singularities. Let the surface M_0 be possess an isolated singularity with a smooth tangent cone. Construct a a smooth evolution for a short time.

Special solutions in $\mathbf{R}^3$

17. Proof of existence of shrinkers. Prove the existence of the various shrinking solitons in $\mathbf{R}^3$ that have been found by computer:

monkey saddle with k holes,   punctured cube,   double cone with k tubes.

18. New shrinking solitons. Find new families of embedded shrinking solitons in $\mathbf{R}^3$.

This can be done conceptually or by computer.

19. Superposition problem. When can the union of two self-shrinkers in $\mathbf{R}^3$ be desingularized along the curve of intersection by Scherk surfaces with tiny holes, to produce an infinite family of smooth, embedded shrinking flows? For example:

$$\text{sphere} \cup \text{cylinder}, \quad \text{plane} \cup \text{cylinder?}$$

Angenent has suggested that these constructions might produce ends asymptotic to a round cylinder.

20. Stable shrinkers modulo symmetries. Besides the k-punctured (monkey) saddle family, are there other self-shrinking surfaces that are stable modulo symmetries?

21. Thin shrinking tubes. Can we construct a thin shrinking tube whose final set is a given analytic curve? Can we prove the blowup curve is always, say. C^∞, or analytic?

A *delta-wing* is a complete, convex, translating soliton of mean curvature flow in $\mathbf{R}^3$ that is not the bowl and not the Grim Reaper cross $\mathbf{R}$, if such a solution exists.

22. Shape of a delta-wing. Do matched asymptotics to deduce the shape of a delta-wing near infinity.

23. Heartbeat flow. Prove the heartbeat mean curvature flow exists.

The heartbeat curve is an immersed curve in the plane that consists of two oppositely turning half-Yin-Yang curves in $\mathbf{R}^2$ that join in the middle. It is infinite length, periodic and passes through a singularity once per half-cycle.

Variations on a theme

24. Codimension $(1,1)$ **minimal surfaces.** Develop a theory of minimal surfaces and mean curvature flow for spacelike submanifolds of codimension (1,1) surfaces in Lorentz manifolds.

25. Acceleration of curves by mean curvature. Develop a theory of timelike minimal surfaces in $R^{2,1}$.

26. IMCF with mixed curvature. Make a theory of inverse mean curvature flow of curves in the plane with curvature of mixed sign.