A new compactification of the Drinfeld period domain over a finite field

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Abstract

We study a compactification of the Drinfeld period domain over a finite field which arises naturally in the context of Drinfeld moduli spaces. This compactification is in some sense dual to the compactification by projective space. It is normal but singular at the boundary. We construct a desingularization and obtain a smooth modular compactification of the Drinfeld period domain with a natural stratification, simple functorial description, and the boundary a divisor with normal crossings in the strongest sense.

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1 Introduction

Let $d \geq 1$ be an integer and let V be a d-dimensional vector space over a finite field $\mathbb{F}_q$ with q elements. Denote by S_V the symmetric algebra on V over $\mathbb{F}_q$ and by $\mathrm{Frac}(S_V)$ its field of fractions. Thus S_V is non-canonically isomorphic to the polynomial ring over $\mathbb{F}_q$ in d variables, but refraining from the choice of a basis for V will make the constructions and statements of the article more lucid and canonical.

Let R_V and RS_V be the $\mathbb{F}_q$-subalgebras of $\mathrm{Frac}(S_V)$ defined as follows:

$$R_V := \mathbb{F}_q\left[\frac{1}{v}\;\middle|\; v \in V \smallsetminus \{0\}\right]$$

$$RS_V := \mathbb{F}_q\left[v, \frac{1}{v}\;\middle|\; v \in V \smallsetminus \{0\}\right]$$

We make S_V and RS_V into graded rings by defining $\deg_{S_V}(v) = \deg_{RS_V}(v) := 1$ and $\deg_{RS_V}\left(\frac{1}{v}\right) := -1$. We make R_V into a graded ring by defining $\deg_{R_V}\left(\frac{1}{v}\right) := 1$. This definition will turn out to be more convenient for the remainder of the article.

Associated to these rings we define the following schemes over $\mathbb{F}_q$:

$$P_V := \mathrm{Proj}\,(S_V)$$

$$Q_V := \mathrm{Proj}\,(R_V)$$

$$\Omega_V := \mathrm{Spec}\,((RS_V)_0)$$

Thus P_V is non-canonically isomorphic to projective space $\mathbb{P}^{d-1}_{\mathbb{F}_q}$. The scheme Ω_V is equal to the open affine complement of the union of all $\mathbb{F}_q$-rational hyperplanes in P_V, as well as the open affine subscheme of Q_V obtained by inverting all homogeneous elements of degree 1 in R_V. It is an important example of a period domain over a finite field - the Drinfeld period domain. See for example Rapoport [8], Orlik [9] and Orlik-Rapoport [10].

The scheme Ω_V is an analogue over $\mathbb{F}_q$ of Drinfeld's period domain over a nonarchimedean local field F. The latter is defined as the complement in projective $(d-1)$-space of the union of all F-rational hyperplanes, and makes sense only as a rigid analytic space, not as an algebraic variety. When F has equal positive characteristic, this period domain plays a central role in the analytic description of the moduli space of Drinfeld modules. See for example Deligne-Husemoller [11] and Drinfeld [12]. The special case of Drinfeld modules of rank d with respect to the ring $\mathbb{F}_q[t]$ and with a level structure of level (t) leads naturally to the scheme Ω_V defined above. The natural (Satake, or Baily-Borel) compactification of this Drinfeld moduli space turns out to be essentially Q_V, and not P_V. As Q_V is singular (see section 4) we are thus led to the natural problem of constructing a good desingularization of Q_V, which is the main goal of this article.

The details of the relation of Ω_V, Q_V and its desingularization with Drinfeld moduli spaces will not be discussed.

We now describe the main results of the article. The study of the projective coordinate ring R_V of Q_V is carried out in sections 2 and 3. We determine a presentation of R_V with generators and relations (Corollary 2.5), prove that R_V is integrally closed (Proposition 2.12), and determine its Hilbert polynomial (Propositions 2.6, 2.9, 2.10). Although R_V is neither regular nor factorial (Proposition 2.11), we show that up to a Frobenius power, the ring R_V is isomorphic to the symmetric algebra S_{V^*} on the dual vector space V^* over $\mathbb{F}_q$ (see Theorem 3.1 for a precise statement). From this we again deduce that R_V is integrally closed.

In section 4 we study the projective variety $Q_V = \mathrm{Proj}(R_V)$. Applying the ring-theoretic results of sections 2 and 3, we deduce that Q_V is a normal scheme, and compute its degree (Corollary 4.1). We construct a natural and well-behaved stratification of Q_V where the strata are indexed by nonzero subspaces of the vector space V (Theorem 4.6, Remark 4.7). This stratification is in some sense dual to the usual stratification of projective space P_V (Proposition 4.2), where the strata are indexed by nonzero quotients of V. We discuss the birational equivalence of Q_V and P_V (Propositions 4.8, 4.9). Finally we prove that the singular locus of Q_V is equal to the union of all strata of codimension at least 2 (Theorem 4.10).

Before turning our attention to the construction of a good desingularization of Q_V in arbitrary dimension, we study the special cases where $d = 2$ and $d = 3$ in section 5. If $d = 2$, the schemes Q_V and P_V are isomorphic smooth projective curves, and in fact $Q_V = \mathrm{Proj}(R_V)$ is the q-uple embedding of the projective line P_{V^*} (Proposition 5.1). If $d = 3$, the schemes Q_V and P_V are non-isomorphic surfaces. We prove that P_V and Q_V become isomorphic after blowing up both surfaces in every zero-dimensional stratum (Theorem 5.2).

Section 6 forms the heart of the article. Here we construct a desingularization B_V of Q_V in arbitrary dimension. Throughout the section we work exclusively with functors of points. Thus we begin by determining the functors represented by P_V, Q_V and Ω_V (Corollaries 6.3, 6.4). We define B_V functorially and show that it is indeed representable by a projective variety over $\mathbb{F}_q$ (Proposition 6.5, Corollary 6.14). We then construct a stratification for B_V where the strata are indexed by filtrations of the vector space V (Theorem 6.9). The stratum corresponding to the trivial filtration is open and dense in B_V and isomorphic to Ω_V (Proposition 6.7). The stratification enjoys several natural and beautiful geometric properties (Proposition 6.6, Corollary 6.8, Remark 6.11, Corollary 6.15). We prove that B_V is smooth (Proposition 6.13, Corollary 6.14) and that the boundary $B_V \setminus \Omega_V$ is a divisor with normal crossings in the strongest sense (Proposition 6.16). Finally, we construct projective morphisms to P_V and Q_V which are isomorphisms on Ω_V. Therefore B_V is a desingularization of Q_V (Corollary 6.17).

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2 Structure of R_V

In this section we study the rings R_V and RS_V defined in the introduction. Our first goals are to find $\mathbb{F}_q$-bases for R_V and RS_V, and a presentation of R_V with generators and relations. Not surprisingly, it is convenient to treat these questions simultaneously. Having found an $\mathbb{F}_q$-basis for R_V, we deduce formulas for the Hilbert function and Hilbert polynomial of R_V. At the end of the section we give an ad-hoc proof of the fact that R_V is integrally closed.

Let $A_V := \mathbb{F}_q\left[X_v \mid 0 \neq v \in V\right]$ denote the polynomial ring over $\mathbb{F}_q$ in the indeterminates X_v for all nonzero vectors v in V. Denote by κ_V the degree-preserving surjection of graded rings

$$\kappa_V : A_V \twoheadrightarrow R_V$$
$$X_v \mapsto \frac{1}{v} .$$

Let $J_V \subset A_V$ be the homogeneous ideal generated by all homogeneous elements of the form

$$X_v - \alpha X_{\alpha v}$$

for all $0 \neq v \in V$ and $\alpha \in \mathbb{F}_q^{\times}$, and

$$X_v X_{v'} + X_{v'} X_{v''} + X_v X_{v''}$$

for all $v, v', v'' \in V \setminus \{0\}$ such that $v + v' + v'' = 0$.

It is easy to see that the ideal J_V is contained in the kernel of κ_V. In Corollary 2.5 we will see that J_V is actually equal to the kernel of κ_V. Thus we will obtain a presentation of R_V with generators and relations as $R_V \cong A_V/J_V$.

For the entire section, we choose an $\mathbb{F}_q$-basis $e_1, \ldots, e_d$ of V. We call a monomial in A_V *reduced with respect to this basis* if it is of the special form

$$\prod_{i=1}^{d} \left(X_{e_i + \sum_{j=1}^{i-1} \alpha_{ij} e_j} \right)^{r_i}$$

for some $r_i \in \mathbb{Z}^{\geq 0}$ and $\alpha_{ij} \in \mathbb{F}_q$. Let K_V denote the $\mathbb{F}_q$-linear subspace of A_V generated by all reduced monomials.

Proposition 2.1 *The $\mathbb{F}_q$-vector space A_V is the sum of the linear subspaces J_V and K_V.*

Proof. We have to show that every monomial in A_V lies in the sum $J_V + K_V$. By the definition of J_V, we have the following two relations in the quotient A_V/J_V :

(1) $\quad X_{\alpha v} = \frac{1}{\alpha} X_v \quad$ for some $\alpha \in \mathbb{F}_q$ and $0 \neq v \in V$

(2) $\quad X_v X_{v'} = X_{v'} X_{v-v'} - X_v X_{v-v'} \quad$ for some $v \neq v' \in V \setminus \{0\}$

We will show that every monomial in A_V can be transformed into an element of K_V by using these relations finitely many times.

For any nonzero vector $v = \sum_{i=1}^d \alpha_i e_i$ in V we define

$$\mathrm{width}(v) := \max\{i \in \{1, \ldots, d\} \mid \alpha_i \neq 0\}.$$

For any monomial $f = \prod_{i=1}^n X_{v_i}$ in A_V we define

$$\mathrm{width}(f) := \sum_{i=1}^n \mathrm{width}(v_i).$$

We carry out the proof by induction on the width w of the monomial f, starting with $w = 0$. By the usual convention that the empty sum is equal to 0 and the empty product is equal to 1, the case $w = 0$ implies $f = 1$ and thus $f \in K_V$.

Now assume that the statement is true for all monomials of width less than w, and let $f = \prod_{i=1}^n X_{v_i}$ have width w. Using relation (1), it suffices to consider the case where every $v_i = \sum_{j=1}^d \alpha_{ij} e_j$ is "monic" in the sense that $\alpha_{ij} = 1$ for $j = \mathrm{width}(v_i)$. If f is not already reduced, there exist indices j and k such that $v_j \neq v_k$, but $\mathrm{width}(v_j) = \mathrm{width}(v_k)$. Since v_j and v_k are "monic", this implies that $\mathrm{width}(v_j - v_k) < \mathrm{width}(v_j)$. Using relation (2) we can replace the factor $X_{v_j} X_{v_k}$ by $X_{v_k} X_{v_j - v_k} - X_{v_j} X_{v_j - v_k}$, so that

$$f = \left(\prod_{i \neq j,k} X_{v_i}\right) X_{v_k} X_{v_j - v_k} - \left(\prod_{i \neq j,k} X_{v_i}\right) X_{v_j} X_{v_j - v_k}.$$

By construction, both summands have width less than w. Thus the induction hypothesis implies that both summands lie in $J_V + K_V$, hence also f. **q.e.d.**

Proposition 2.2 *The system (S) of all elements of RS_V of the following form is an $\mathbb{F}_q$-basis of RS_V:*

$$\prod_{i=1}^d \left(e_i + \sum_{j=1}^{i-1} \alpha_{ij} e_j\right)^{r_i}$$

for some $r_i \in \mathbb{Z}$, $\alpha_{ij} \in \mathbb{F}_q$, and $\alpha_{ij} = 0$ for all j if $r_i \geq 0$.

For the second part of the proof of Proposition 2.2 we will need the following well-known lemma, whose proof we include for lack of a suitable reference.

Lemma 2.3 *Let L be a field and let $L(T)$ denote the field of rational functions in one indeterminate over L. Then the system of elements*

$$\big((T-a)^n \mid a \in L,\ n \in \mathbb{Z},\ \textit{and } a = 0 \textit{ if } n \geq 0\big)$$

of $L(T)$ is linearly independent over L.

Proof. Assume there exists a non-trivial linear combination

$$\sum_{a,n} \alpha_{(a,n)}(T-a)^n = 0.$$

Since the subsystem $(T^n \mid n \geq 0)$ of the system under consideration is already L-linearly independent, there must exist an element $b \in L$ and an integer $n < 0$ such that $\alpha_{(b,n)} \neq 0$. Fix such an element b and let m denote the minimum of all integers $n < 0$ with $\alpha_{(b,n)} \neq 0$. We multiply the above linear combination with the least common multiple of all the denominators. Then every summand is divisible by $(T-b)$ except for the summand corresponding to the index (b,m). Thus we obtain an equation of the form

$$(T-b) \cdot f(T) + \alpha_{(b,m)} g(T) = 0,$$

where f and g are nonzero polynomials in $L[T]$ and $g(b) \neq 0$. Substituting b for T yields the desired contradiction. **q.e.d.**

Proof of Proposition 2.2. First we show that the system (S) generates RS_V over $\mathbb{F}_q$. Set

$$V(i) := \sum_{j=1}^{i} \mathbb{F}_q e_j.$$

Since the ideal J_V is contained in the kernel of κ_V and since κ_V is surjective, Proposition 2.1 implies that $\kappa_V(K_V) = R_V$. Unfolding the definitions, we see that the composition of κ_V with the inclusion $R_V \hookrightarrow RS_V$ maps the system of all reduced monomials of A_V onto the subsystem of (S) consisting of all elements

$$\prod_{i=1}^{d} \left(e_i + \sum_{j=1}^{i-1} \alpha_{ij} e_j \right)^{r_i}$$

with $r_i \leq 0$ for all i.

Therefore, every element of RS_V can be written as a linear combination of elements x of the form

$$x = \frac{\prod_{i=1}^{d} e_i^{\ell_i}}{\prod_{i=1}^{d} (e_i + v_i)^{r_i}}$$

for some $v_i \in V(i-1)$, $r_i \geq 0$, $\ell_i \geq 0$.

After cancelling we can assume that whenever $v_i = 0$, either $r_i = 0$ or $\ell_i = 0$. If x is not already an element of the system (S), there exists an index i such that $r_i > 0$ and $\ell_i > 0$. Denote by n the largest such index i. We will now show that x can be written as a linear combination of elements of the same form, but with smaller value of n for each summand. Once this is shown, one can apply the same procedure to each of the summands. By performing at most d iteration steps, x can be written as a linear combination of elements of the above form with the additional property that for every index i, either $r_i = 0$ or $\ell_i = 0$, i.e., as a linear combination of elements of the system (S).

In order to find the desired linear combination of x, note that

$$\frac{e_n^{\ell_n}}{(e_n + v_n)^{r_n}} = \frac{(e_n + v_n - v_n)^{\ell_n}}{(e_n + v_n)^{r_n}} = \sum_{k=0}^{\ell_n} \binom{\ell_n}{k} (e_n + v_n)^{\ell_n - r_n - k} v_n^k.$$

Now since $v_n \in V(n-1)$, the k-th power v_n^k on the right hand side can be written as a linear combination of products of basis vectors e_i with $1 \leq i \leq n-1$. Thus multiplying this equation by the missing factors

$$\frac{\prod_{i \neq n} e_i^{\ell_i}}{\prod_{i \neq n} (e_i + v_i)^{r_i}}$$

and subsequent cancelling yields the desired linear combination of x. Thus we have shown that the system (S) generates RS_V over $\mathbb{F}_q$.

Now we come to the linear independence of the system (S). We proceed by induction on d. The case $d = 1$ is an immediate consequence of Lemma 2.3. We now assume that the statement holds for $(d-1)$ indeterminates. Let T be a finite indexing set and assume that

$$\sum_{t \in T} \gamma_t b_t = 0,$$

where $\gamma_t \in \mathbb{F}_q$ and where

$$b_t := \prod_{i=1}^{d} \left(e_i + \sum_{j=1}^{i-1} \alpha_{ijt} e_j \right)^{r_{it}}$$

is an element of the system (S) under consideration. To simplify notation, denote $v_{it} := \sum_{j=1}^{i-1} \alpha_{ijt} e_j$.

In order to apply the induction hypothesis, we wish to arrange the summands of the finite sum $\sum_{t \in T} \gamma_t b_t$ in a convenient way. As the index t runs over the indexing set T, the pair (r_{dt}, v_{dt}) takes values in the set $\mathbb{Z} \times \mathbb{F}_q(e_1, \ldots, e_{d-1})$. More formally, we consider the map

$$T \longrightarrow \mathbb{Z} \times \mathbb{F}_q(e_1, \ldots, e_{d-1})$$

$$t \longmapsto (r_{dt}, v_{dt}) .$$

We partition T into the non-empty fibers of this map: For $n \in \mathbb{Z}$ and $f \in \mathbb{F}_q(e_1, \ldots, e_{d-1})$ define $T_{(n,f)} := \{t \in T \mid (r_{dt}, v_{dt}) = (n, f)\}$. Note that for $n \geq 0$, the set $T_{(n,0)}$ is the only potentially non-empty set among the $T_{(n,f)}$. For notational purposes, define

$$C_{(n,f)} := \sum_{t \in T_{(n,f)}} \gamma_t \cdot \left(\prod_{i=1}^{d-1} (e_i + v_{it})^{r_{it}} \right).$$

We calculate

$$\begin{aligned} 0 &= \sum_{t \in T} \gamma_t b_t \\ &= \sum_{(n,f)} \sum_{t \in T_{(n,f)}} \gamma_t \cdot (e_d + f)^n \cdot \left(\prod_{i=1}^{d-1} (e_i + v_{it})^{r_{it}} \right) \\ &= \sum_{(n,f)} C_{(n,f)} \cdot (e_d + f)^n . \end{aligned}$$

Since $C_{(n,f)}$ lies in $\mathbb{F}_q(e_1, \ldots, e_{d-1})$ for all pairs (n, f), we can apply Lemma 2.3 with ground field $L = \mathbb{F}_q(e_1, \ldots, e_{d-1})$ and coefficients $C_{(n,f)}$. We conclude that $C_{(n,f)} = 0$ for all (n, f).

Note that every element $C_{(n,f)} \in \mathbb{F}_q(e_1, \ldots, e_{d-1})$ has precisely the form of an element of the system (S) under consideration, only in one indeterminate less. Thus we can apply the induction hypothesis to each $C_{(n,f)}$ and conclude that $\gamma_t = 0$ for all $t \in T$. This finishes the proof of the linear independence of the system (S). **q.e.d.**

Corollary 2.4 *The system consisting of all elements of R_V of the following form is an $\mathbb{F}_q$-basis of R_V:*

$$\prod_{i=1}^{d} \left(e_i + \sum_{j=1}^{i-1} \alpha_{ij} e_j \right)^{r_i}$$

for some $r_i \in \mathbb{Z}^{\leq 0}$, $\alpha_{ij} \in \mathbb{F}_q$, and $\alpha_{ij} = 0$ for all j if $r_i = 0$.

Proof. We have already seen that $J_V \subset \ker(\kappa_V)$. Since $\kappa_V : A_V \twoheadrightarrow R_V$ is surjective, Proposition 2.1 implies that $R_V = \kappa_V(K_V)$. Since κ_V maps the system of all reduced monomials in A_V to precisely the system under consideration, this shows that the system generates R_V as an $\mathbb{F}_q$-vector space. The linear independence follows readily from Proposition 2.2. **q.e.d.**

Corollary 2.5 *The kernel of κ_V is equal to the ideal J_V. Thus we have found a presentation of R_V by generators and relations as*

$$R_V \cong A_V / J_V.$$

Our choice of grading for R_V in the introduction implies that this isomorphism is actually a degree-preserving isomorphism of graded $\mathbb{F}_q$-algebras.

Proof. It follows from Corollary 2.4 that the restriction $\kappa_V|_{K_V} : K_V \to R_V$ is an isomorphism of $\mathbb{F}_q$-vector spaces, and that the $\mathbb{F}_q$-vector space A_V decomposes as $A_V = J_V \oplus K_V$. Thus $\ker(\kappa_V) \subset J_V$. The converse inclusion $J_V \subset \ker(\kappa_V)$ has already been observed at the beginning of the section. **q.e.d.**

Using the basis for R_V of Corollary 2.4 we will now study the Hilbert function and the Hilbert polynomial of R_V. We refer the reader to Matsumura [1], section 13, and Bruns-Herzog [2], chapter 4, for background material.

Recall from the introduction that we define $\deg_{R_V}(\frac{1}{v}) := 1$, so that R_V becomes a $\mathbb{Z}^{\geq 0}$-graded ring. We denote the Hilbert function of R_V by H_d. This notation is well-chosen since the isomorphism class of R_V only depends on the dimension d of the vector space V. Our first goal is to derive a recursion formula for the Hilbert function.

Proposition 2.6 *The value of the Hilbert function H_d at an integer $n \in \mathbb{Z}^{\geq 0}$ can be computed from the Hilbert function H_{d-1} by the formula*

$$H_d(n) = H_{d-1}(n) + q^{d-1} \cdot \sum_{k=0}^{n-1} H_{d-1}(k).$$

Proof. Set $V(d-1) := \sum_{i=1}^{d-1} \mathbb{F}_q e_i$ and fix an integer $n \in \mathbb{Z}^{\geq 0}$. Then $H_d(n)$ is precisely the number of basis elements

$$\prod_{i=1}^{d} \left(e_i + \sum_{j=1}^{i-1} \alpha_{ij} e_j \right)^{r_i}$$

of degree n, where $r_i \leq 0$ and $\alpha_{ij} \in \mathbb{F}_q$ as in Corollary 2.4. We count the number of such basis elements of degree n for fixed values of the exponent r_d. Then the sum of these numbers will be equal to $H_d(n)$.

Any basis element of degree n decomposes uniquely into a product

$$b \cdot \left(e_d + \sum_{j=1}^{d-1} \alpha_{dj} e_j \right)^{r_d},$$

where b is a basis element of $R_{V(d-1)}$ of degree $n - r_d$. Thus if $r_d = 0$, the number of possibilities is equal to $H_{d-1}(n)$. If $r_d < 0$, the scalars α_{dj} can be chosen arbitrarily in $\mathbb{F}_q$ for all $j = 1, \ldots, d-1$. Hence there are q^{d-1} possibilities for the second factor and $H_{d-1}(n - r_d)$ possibilities for the first factor. Thus

$$H_d(n) = H_{d-1}(n) + \sum_{r_d=1}^{n} q^{d-1} \cdot H_{d-1}(n - r_d) = H_{d-1}(n) + q^{d-1} \cdot \sum_{k=0}^{n-1} H_{d-1}(k).$$

q.e.d.

Remark 2.7 *Using the formula of Proposition 2.6 above, one easily computes the Hilbert function of R_V for small values of d. The following formulas hold for all integers $n \in \mathbb{Z}^{\geq 0}$.*

$$\begin{aligned} H_1(n) &= 1 \\ H_2(n) &= qn + 1 \\ H_3(n) &= \left(\frac{q^3}{2}\right) n^2 + \left(-\frac{3}{2}q^3 + q^2 + q\right) n + \left(q^3 + 1\right) \end{aligned}$$

We now study the Hilbert polynomial of R_V. We will need the following well-known lemma.

Lemma 2.8 *For integers $m \geq 1$ and $n \geq 0$ denote by $s_m(n)$ the m-th power sum*

$$s_m(n) := \sum_{k=1}^{n} k^m.$$

Then $s_m(n)$ is a polynomial in n of degree $(m+1)$ and with leading coefficient $\frac{1}{m+1}$.

Lemma 2.8 can easily be proven directly by standard arguments using the difference function $\Delta(n) := s_m(n+1) - s_m(n) = n^m$. Alternatively, see Conway-Guy [7] for an explicit formula for the coefficients of the polynomial $s_m(n)$ in terms of Bernoulli numbers.

We denote by P_d the Hilbert polynomial of R_V, i.e., the unique polynomial $P_d \in \mathbb{Q}[n]$ such that $P_d(n) = H_d(n)$ for all $n \gg 0$. Due to an obstruction in cohomology, one cannot in general expect that the Hilbert function and the Hilbert polynomial agree for all $n \geq 0$. This is however true for a polynomial ring, and, as we will now see, also for the ring R_V.

Proposition 2.9 *(i) $P_d(n) = H_d(n)$ for all $n \geq 0$.*

(ii) The Hilbert polynomial of R_V satisfies the same recursion formula as the Hilbert function, i.e.,

$$P_d(n) = P_{d-1}(n) + q^{d-1} \cdot \sum_{k=0}^{n-1} P_{d-1}(k)$$

for all integers $n \in \mathbb{Z}^{\geq 0}$.

Proof. We prove (i) by induction on d. For the case $d = 1$ see Remark 2.7 above. If the statement is true for $(d-1)$, Proposition 2.6 above implies that

$$H_d(n) = P_{d-1}(n) + q^{d-1} \cdot \sum_{k=0}^{n-1} P_{d-1}(k)$$

for all $n \geq 0$. Now since $P_{d-1}(n)$ is a polynomial in n, Lemma 2.8 implies that the right hand side of the equation is already a polynomial in n. This polynomial agrees with the Hilbert polynomial P_d for all $n \gg 0$, and therefore even for all $n \in \mathbb{Z}$. Thus $H_d(n) = P_d(n)$ for all $n \geq 0$.

The statement of (ii) follows directly from (i) and Proposition 2.6. **q.e.d.**

Proposition 2.10 *The Hilbert polynomial of R_V has degree $(d-1)$, and its leading coefficient is equal to*

$$\frac{1}{(d-1)!}\, q^{\frac{(d-1)(d-2)}{2}} .$$

Proof. We proceed by induction on d. For the case $d = 1$ we again refer to Remark 2.7 and part (i) of Proposition 2.9 above. Now assume the statement is true for $(d-1)$, i.e.,

$$P_{d-1}(n) = \sum_{i=0}^{d-2} a_i n^i$$

for some $a_i \in \mathbb{Q}$ and where

$$a_{d-2} = \frac{1}{(d-2)!}\, q^{\frac{(d-2)(d-3)}{2}} .$$

Using (ii) of Proposition 2.9 above, we calculate

$$P_d(n) = \sum_{i=0}^{d-2} a_i n^i + q^{d-1} \sum_{k=0}^{n-1} \sum_{i=0}^{d-2} a_i k^i =$$

$$= \sum_{i=0}^{d-2} a_i n^i + \sum_{i=0}^{d-2} q^{d-1} a_i s_i(n-1).$$

Note that the polynomials $s_i(n)$ and $s_i(n-1) \in \mathbb{Q}[n]$ have the same degree and the same leading coefficient. Thus from Lemma 2.8 we see that $P_d(n)$ has degree $(d-1)$ and that its leading coefficient is equal to

$$\frac{1}{d-1}\, q^{d-1} a_{d-2} = \frac{1}{(d-1)!}\, q^{(d-1)+\frac{(d-2)(d-3)}{2}} = \frac{1}{(d-1)!}\, q^{\frac{(d-1)(d-2)}{2}} .$$

q.e.d.

The following proposition collects some elementary properties of the ring R_V.

Proposition 2.11 *(i) The Krull dimension of R_V is equal to d.*

(ii) If $d \geq 2$, the ring R_V is not factorial.

(iii) If $d \geq 2$, the ring R_V is not regular at the augmentation ideal $\mathfrak{m} := \bigoplus_{i\geq 1} R_{V,i}$.

Proof. The ring R_V is a finitely generated $\mathbb{F}_q$-algebra and an integral domain. Thus the Krull dimension of R_V is equal to the transcendence degree of its field of fractions $\mathrm{Frac}(R_V)$ over $\mathbb{F}_q$. Since $\mathrm{Frac}(R_V)$ is isomorphic to a function field over $\mathbb{F}_q$ in d indeterminates, the transcendence degree of $\mathrm{Frac}(R_V)$ is equal to d. This proves (i). Alternatively, one could appeal to Proposition 2.10, where it was proven that the degree of the Hilbert polynomial of R_V equals $(d-1)$. This again shows that the Krull dimension of R_V is equal to d.

We now prove (ii). If $d \geq 2$, we can choose pairwise linearly independent vectors v, v', v'' in V such that $v + v' + v'' = 0$. The elements $\frac{1}{v}$, $\frac{1}{v'}$, $\frac{1}{v''} \in R_V$ are irreducible since they are homogeneous of degree 1. The equality

$$\frac{1}{v}\left(\frac{1}{v'} + \frac{1}{v''}\right) = -\frac{1}{v'}\frac{1}{v''}$$

shows that $\frac{1}{v}$ divides the product $\frac{1}{v'}\frac{1}{v''}$, but it does not divide any of the two factors. Thus $\frac{1}{v}$ is irreducible but not prime, so R_V cannot be factorial.

To prove (iii), we need to show that if $d \geq 2$, the local ring $R_{V,\mathfrak{m}}$ at the maximal ideal $\mathfrak{m} \subset R_V$ is not regular. Since R_V is a finitely generated algebra over a field and an integral domain, every maximal ideal of R_V has the same height. Therefore part (i) of the proposition implies that the Krull dimension of $(R_V)_\mathfrak{m}$ is equal to d. Because $\mathfrak{m}$ is a maximal ideal, it suffices to show that $\dim_{\mathbb{F}_q} \mathfrak{m}/\mathfrak{m}^2 > d$.

The $\mathbb{F}_q$-algebra R_V is generated in degree 1 over $R_{V,0} = \mathbb{F}_q$. Thus

$$\dim_{\mathbb{F}_q} \mathfrak{m}/\mathfrak{m}^2 = H_d(1),$$

where H_d denotes the Hilbert function of R_V as in Proposition 2.6 above. From the recursion formula in Proposition 2.6 we see that

$$H_d(1) = H_{d-1}(1) + q^{d-1}.$$

By iterating and using that $H_1(1) = 1$, we conclude that $H_d(1) = \sum_{i=0}^{d-1} q^i$. If $d \geq 2$, this number is greater than d. **q.e.d.**

We finish the section with an ad-hoc proof of the fact that the domain R_V is integrally closed. We will give a more conceptual proof in section 3 below.

Proposition 2.12 *The integral domain R_V is integrally closed.*

Proof. It suffices to show that R_V is integrally closed in the domain RS_V, which is a localization of the regular ring S_V and thus integrally closed in $\mathrm{Frac}(S_V)$.

Given an element b of the basis for RS_V defined in Proposition 2.2, we define a pair of non-negative integers $\ell(b) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ as follows. If b is contained in R_V, set $\ell(b) := (0,0)$. If b is not contained in R_V, there exists an integer i such that $r_i > 0$. Then we set $n := \max\{i \mid r_i > 0\}$ and define

$$\ell(b) := (n, r_n) \ \in \ \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}.$$

Fix a total order on $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ by defining

$$(x,y) > (x',y') \ :\Longleftrightarrow \ \text{either } x > x' \text{ or if } x = x', \text{ then } y > y'.$$

For an arbitrary element $x = \sum_i \beta_i b_i \in RS_V$ for pairwise distinct basis elements $b_i \in RS_V$ and scalars $\beta_i \in \mathbb{F}_q^{\times}$, define

$$\ell(x) := \max_i \ell(b_i).$$

Lemma 2.13 *(i) Let $x, x' \in RS_V$. Then $\ell(x + x') \leq \max(\ell(x), \ell(x'))$.*

(ii) Let $x \in RS_V$ and $a \in R_V$. Then $\ell(ax) \leq \ell(x)$.

(iii) Let $x \in RS_V \smallsetminus R_V$ and $n \in \mathbb{Z}^{>0}$. Then $\ell(x^n) > \ell(x^{n-1})$.

Proof of Lemma 2.13. Part (i) is clear from how we extended the definition of ℓ from the special case of a basis element to an arbitrary element of RS_V above.

It follows from the algorithm used in the first part of the proof of Proposition 2.2 that statement (ii) is true if a and x are basis elements. For arbitrary $a = \sum_i \alpha_i a_i$ and $x = \sum_j \beta_j b_j$ with basis elements a_i, b_j and scalars α_i, $\beta_j \in \mathbb{F}_q^{\times}$, we calculate

$$\ell(ax) \overset{(i)}{\leq} \max_{i,j} \ell(a_i b_j) \leq \max_j \ell(b_j) = \ell(x).$$

The statement of (iii) is clear if x is a basis element. If $x = \sum \beta_i b_i$ is arbitrary, fix an index k such that $\ell(x) = \ell(b_k)$. Then we conclude that $\ell(x^n) = \ell(b_k^n) > \ell(b_k^{n-1}) = \ell(x^{n-1})$.

q.e.d.

Assume that $x \in RS_V \smallsetminus R_V$ is integral over R_V. Fix an equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

for some $n > 0$ and some $a_i \in R_V$. Then

$$\ell\left(x^n\right) = \ell\left(\sum_{i=0}^{n-1} a_i x^i\right) \overset{(i)}{\leq} \max_{i \leq n-1} \ell\left(a_i x^i\right) \overset{(ii)}{\leq} \max_{i \leq n-1} \ell\left(x^i\right) \overset{(iii)}{=} \ell\left(x^{n-1}\right) \overset{(iii)}{<} \ell\left(x^n\right),$$

contradiction. **q.e.d.**

3 Relating R_V and S_{V^*}

It is the goal of this section to relate the rings R_V and S_{V^*}, where V^* denotes the dual vector space of V. We will show that R_V and S_{V^*} are "isomorphic up to Frobenius" in a sense made precise by Theorem 3.1 below.

The majority of this section is devoted to the proof of Theorem 3.1. As an application, we will use the theorem to give a more conceptual proof of the fact that R_V is integrally closed.

We begin by fixing some notation. For a linear form $w \in V^*$ and a vector $v \in V$ we define $\langle w, v \rangle := w(v)$. Furthermore, we define the Frobenius homomorphisms

$$F_{R_V} : \ R_V \hookrightarrow R_V, \quad x \longmapsto x^{q^{d-1}}$$

and

$$F_{S_{V^*}} : \ S_{V^*} \hookrightarrow S_{V^*}, \quad x \longmapsto x^{q^{d-1}}.$$

Theorem 3.1 *(i) There exists a unique $\mathbb{F}_q$-algebra homomorphism $\psi : S_{V^*} \to R_V$ such that*

$$w \longmapsto \sum_{\substack{v \in V \\ \langle w,v \rangle = 1}} \frac{1}{v} \ .$$

(ii) There exists a unique $\mathbb{F}_q$-algebra homomorphism $\varphi : R_V \to S_{V^*}$ such that

$$\frac{1}{v} \longmapsto \prod_{\substack{w \in V^* \\ \langle w,v \rangle = 1}} w \ .$$

(iii) $\varphi \circ \psi \ = \ F_{S_{V^*}}$

(iv) $\psi \circ \varphi \ = \ F_{R_V}$

Proof of 3.1(i). We need to show that the formula for $\psi(w)$ is $\mathbb{F}_q$-linear as w ranges over the vector space V^*. This is immediate once we write $\psi(w)$ in the form

$$\psi(w) = \sum_{\substack{v \in V \\ \langle w,v \rangle = 1}} \frac{1}{v} \ = \sum_{v \in (V \smallsetminus \{0\})/\mathbb{F}_q^\times} \frac{\langle w, v \rangle}{v} \ .$$

q.e.d.

Proof of 3.1(ii).

We first construct an $\mathbb{F}_q$-algebra homomorphism $\eta : S_V \to \mathrm{Frac}(S_{V^*})$ such that

$$0 \neq v \longmapsto \prod_{\substack{w \in V^* \\ \langle w,v\rangle = 1}} \frac{1}{w} .$$

In order to show that such a homomorphism η exists, we need to verify that

(a) $\eta(\alpha v) = \alpha \eta(v)$

(b) $\eta(v + v') = \eta(v) + \eta(v')$

for all $\alpha \in \mathbb{F}_q^\times$ and all $0 \neq v, v' \in V$.

To prove (a), we calculate

$$\eta(\alpha v) = \prod_{\substack{w \in V^* \\ \langle w,\alpha v\rangle = 1}} \frac{1}{w} = \prod_{\substack{w \in V^* \\ \langle w,v\rangle = 1}} \frac{\alpha}{w} = \alpha^{q^{d-1}} \cdot \prod_{\substack{w \in V^* \\ \langle w,v\rangle = 1}} \frac{1}{w} = \alpha \eta(v).$$

Equation (b) follows from equation (a) if v and v' are linearly dependent. If v and v' are linearly independent, we have to prove that the formula

$$\prod_{\substack{w \in V^* \\ \langle w,v+v'\rangle = 1}} \frac{1}{w} = \prod_{\substack{w \in V^* \\ \langle w,v\rangle = 1}} \frac{1}{w} + \prod_{\substack{w \in V^* \\ \langle w,v'\rangle = 1}} \frac{1}{w}$$

holds in $\mathrm{Frac}(S_{V^*})$.

After choosing an appropriate basis for V we can identify V with $\mathbb{F}_q^d$ and v, v' with the standard basis elements $(1, 0, \dots, 0), (0, 1, 0, \dots, 0) \in \mathbb{F}_q^d$. Denote by $X_1, \dots, X_d \in \left(\mathbb{F}_q^d\right)^*$ the dual basis of the standard basis of $\mathbb{F}_q^d$. Then we have to prove that the formula

$$\prod_{\substack{\underline{\alpha} \in \mathbb{F}_q^d \\ \alpha_1 + \alpha_2 = 1}} \frac{1}{\sum_{i=1}^d \alpha_i X_i} = \prod_{\substack{\underline{\alpha} \in \mathbb{F}_q^d \\ \alpha_1 = 1}} \frac{1}{\sum_{i=1}^d \alpha_i X_i} + \prod_{\substack{\underline{\alpha} \in \mathbb{F}_q^d \\ \alpha_2 = 1}} \frac{1}{\sum_{i=1}^d \alpha_i X_i}$$

holds in the field of rational functions $\mathbb{F}_q(X_1, \dots, X_d)$.

We proceed by induction on the number of indeterminates d. For $d = 2$ we have to show that

$$\prod_{\alpha \in \mathbb{F}_q} \frac{1}{\alpha X_1 + (1-\alpha) X_2} = \prod_{\alpha \in \mathbb{F}_q} \frac{1}{X_1 + \alpha X_2} + \prod_{\alpha \in \mathbb{F}_q} \frac{1}{\alpha X_1 + X_2} .$$

Multiplying this equation by the factor $X_1 \cdot \prod_{\alpha \in \mathbb{F}_q} (X_2 - \alpha X_1)$ we obtain the equivalent equation

$$\left(\prod_{\substack{\alpha \in \mathbb{F}_q \\ \alpha \neq 1}} \frac{1}{1-\alpha} \right) (X_2 - X_1) = X_1 + \left(\prod_{\substack{\alpha \in \mathbb{F}_q \\ \alpha \neq 0}} \frac{1}{\alpha} \right) X_2.$$

This equation in turn follows from the observation that

$$\prod_{\substack{\alpha\in\mathbb{F}_q\\ \alpha\neq 1}} \frac{1}{1-\alpha} = -1 = \prod_{\substack{\alpha\in\mathbb{F}_q\\ \alpha\neq 0}} \frac{1}{\alpha}\ .$$

We now assume that the statement is true for $(d-1)$ indeterminates $T_1,\dots,T_{d-1}$. We will deduce the statement for the d indeterminates $X_1,\dots,X_d$ by substituting the expressions $\left(X_i^q - X_d^{q-1}X_i\right)$ for T_i for all $i=1,\dots,d-1$.
Let ℓ be a linear form over $\mathbb{F}_q$ in two variables α_1,α_2. We calculate

$$\begin{aligned}
\prod_{\substack{\underline{\alpha}\in\mathbb{F}_q^{d-1}\\ \ell(\alpha_1,\alpha_2)=1}} \frac{1}{\sum_{i=1}^{d-1}\alpha_i T_i} &= \prod_{\substack{\underline{\alpha}\in\mathbb{F}_q^{d-1}\\ \ell(\alpha_1,\alpha_2)=1}} \frac{1}{\sum_{i=1}^{d-1}\alpha_i\left(X_i^q - X_d^{q-1}X_i\right)}\\
&= \prod_{\substack{\underline{\alpha}\in\mathbb{F}_q^{d-1}\\ \ell(\alpha_1,\alpha_2)=1}} \frac{1}{\left(\sum_{i=1}^{d-1}\alpha_i X_i\right)^q - X_d^{q-1}\left(\sum_{i=1}^{d-1}\alpha_i X_i\right)}\\
&= \prod_{\substack{\underline{\alpha}\in\mathbb{F}_q^{d-1}\\ \ell(\alpha_1,\alpha_2)=1}} \prod_{\beta\in\mathbb{F}_q} \frac{1}{\sum_{i=1}^{d-1}\alpha_i X_i + \beta X_d}\\
&= \prod_{\substack{\underline{\alpha}\in\mathbb{F}_q^{d}\\ \ell(\alpha_1,\alpha_2)=1}} \frac{1}{\sum_{i=1}^{d}\alpha_i X_i}\ .
\end{aligned}$$

By using this calculation for each of the three linear forms

$$\begin{aligned}
\ell' :&\quad (\alpha_1,\alpha_2)\longmapsto \alpha_1+\alpha_2\ ,\\
\ell'' :&\quad (\alpha_1,\alpha_2)\longmapsto \alpha_1\ ,\\
\ell''' :&\quad (\alpha_1,\alpha_2)\longmapsto \alpha_2\ ,
\end{aligned}$$

we deduce the statement for the d variables $X_1,\dots,X_d$ from the statement in the $(d-1)$ variables $T_1,\dots,T_{d-1}$. This finishes the proof of part (b) above.

We have therefore shown that the $\mathbb{F}_q$-algebra homomorphism $\eta : S_V \to \mathrm{Frac}(S_{V^*})$ is well-defined. Since $\eta(v)\neq 0$ for all $0\neq v\in V$, the map η extends uniquely from S_V to RS_V. By restricting this extension from RS_V to its subring R_V we obtain the desired map φ.
q.e.d.

Proof of 3.1(iii). We have to show that for any $0\neq w_0\in V^*$ the equation

$$\sum_{\substack{v\in V\\ \langle w_0,v\rangle=1}} \prod_{\substack{w\in V^*\\ \langle w,v\rangle=1}} w = w_0^{q^{d-1}}$$

holds in S_{V^*}.

We choose a basis $X_1,\ldots,X_d$ for V^* such that $X_d = w_0$. By fixing the corresponding dual basis for V, we can identify V with $\mathbb{F}_q^d$ and the dual basis of $X_1,\ldots,X_d$ with the standard basis of $\mathbb{F}_q^d$. Then we need to show that the equation

$$\sum_{\substack{\underline{\alpha}\in\mathbb{F}_q^d\\ \alpha_d=1}} \prod_{\substack{\underline{\beta}\in\mathbb{F}_q^d\\ \langle\underline{\alpha},\underline{\beta}\rangle=1}} \left(\sum_{i=1}^{d}\beta_i X_i\right) = X_d^{q^{d-1}}$$

holds in the field of rational functions $\mathbb{F}_q(X_1,\ldots,X_d)$. Equivalently, we have to prove the equation

$$(E)\qquad \sum_{\underline{\alpha}\in\mathbb{F}_q^{d-1}} \prod_{\underline{\beta}\in\mathbb{F}_q^{d-1}} \left(X_d + \sum_{i=1}^{d-1}\beta_i\left(X_i-\alpha_i X_d\right)\right) = X_d^{q^{d-1}}.$$

We proceed by induction on the number of variables d. The case $d=1$ is clear. Since we will use the statement for two indeterminates in the induction step below, we treat the case $d=2$ next. We calculate

$$\begin{aligned}
\sum_{\alpha\in\mathbb{F}_q}\prod_{\beta\in\mathbb{F}_q}\left(X_2+\beta\left(X_1+\alpha X_2\right)\right) &= \sum_{\alpha\in\mathbb{F}_q}\left(X_2^q-\left(X_1-\alpha X_2\right)^{q-1}\cdot X_2\right)\\
&= \sum_{\alpha\in\mathbb{F}_q} X_2^q - X_2\cdot\sum_{\alpha\in\mathbb{F}_q}\frac{X_1^q-\alpha^q X_2^q}{X_1-\alpha X_2}\\
&= qX_2^q - X_2\cdot\sum_{\alpha\in\mathbb{F}_q}\sum_{i=0}^{q-1}X_1^{q-i-1}\cdot\left(-\alpha X_2\right)^i\\
&= -X_2\cdot\sum_{i=0}^{q-1}\left(X_1^{q-i-1}\cdot X_2^i\cdot\sum_{\alpha\in\mathbb{F}_q}\alpha^i\right)\\
&= X_2^q,
\end{aligned}$$

where for the last equality we used that

$$\sum_{\alpha\in\mathbb{F}_q}\alpha^i = \begin{cases} 0 & \text{if } i\neq q-1.\\ -1 & \text{if } i=q-1.\end{cases}$$

Here we define $0^0 := 1$, in accordance to our calculation above.

We now assume that equation (E) holds for the $(d-1)$ variables $T_1,\ldots,T_{d-1}$, i.e.

$$\sum_{\underline{\alpha}\in\mathbb{F}_q^{d-2}} \prod_{\underline{\beta}\in\mathbb{F}_q^{d-2}} \left(T_{d-1} + \sum_{i=1}^{d-2}\beta_i\left(T_i-\alpha_i T_{d-1}\right)\right) \;=\; T_{d-1}^{q^{d-2}}.$$

Given a scalar $\alpha_{d-1} \in \mathbb{F}_q$, we substitute the expression

$$X_d^q - X_d \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1}$$

for T_{d-1}, and the expression

$$X_i^q - X_i \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1}$$

for T_i for $i = 1, \ldots, d-2$.

Thus for each scalar $\alpha_{d-1} \in \mathbb{F}_q$ we obtain an equation in the variables $X_1, \ldots, X_d$. We denote by (E') the sum of all of these equations.

The right hand side of (E') is equal to

$$\sum_{\alpha_{d-1} \in \mathbb{F}_q} \left(X_d^q - X_d \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1} \right)^{q^{d-2}} .$$

By the calculation for the case $d = 2$ above, this expression is equal to $(X_d^q)^{q^{d-2}} = X_d^{q^{d-1}}$.

The left hand side of (E') is equal to

$$\begin{aligned}
& \sum_{\underline{\alpha} \in \mathbb{F}_q^{d-1}} \prod_{\underline{\beta} \in \mathbb{F}_q^{d-2}} \Bigg(\Big(X_d^q - X_d \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1} \Big) \\
& + \sum_{i=1}^{d-2} \beta_i \Big(X_i^q - \alpha_i X_d^q - X_i \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1} + \alpha_i X_d \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1} \Big) \Bigg) \\
= \; & \sum_{\underline{\alpha} \in \mathbb{F}_q^{d-1}} \prod_{\underline{\beta} \in \mathbb{F}_q^{d-2}} \Bigg(\Big(X_d + \sum_{i=1}^{d-2} \beta_i \, (X_i - \alpha_i X_d) \Big)^q \\
& - \Big(X_d + \sum_{i=1}^{d-2} \beta_i \, (X_i - \alpha_i X_d) \Big) \cdot (X_{d-1} - \alpha_{d-1} X_d)^{q-1} \Bigg) \\
= \; & \sum_{\underline{\alpha} \in \mathbb{F}_q^{d-1}} \prod_{\underline{\beta} \in \mathbb{F}_q^{d-1}} \left(X_d + \sum_{i=1}^{d-1} \beta_i \, (X_i - \alpha_i X_d) \right) .
\end{aligned}$$

Thus we have deduced the desired formula in the d variables $X_1, \ldots, X_d$. **q.e.d.**

Proof of 3.1(iv).
Using (iii) we see that

$$\varphi \circ \big((\psi \circ \varphi) - F_{R_V} \big) = (F_{S_{V^*}} \circ \varphi) - (\varphi \circ F_{R_V}) = 0.$$

Thus once we show that φ is injective, the assertion of (iv) will follow from (iii).

To prove the injectivity of φ we first fix appropriate S_{V^*}-algebra structures on the rings of interest for the proof. Consider R_V and $\mathrm{Frac}(R_V)$ as S_{V^*}-algebras via the map $\psi : S_{V^*} \hookrightarrow R_V$. Consider $\mathrm{Frac}(S_{V^*})$ as an S_{V^*}-algebra via the inclusion $S_{V^*} \hookrightarrow \mathrm{Frac}(S_{V^*})$. Finally, denote by $S_{V^*}^F$ the ring S_{V^*} endowed with the S_{V^*}-algebra structure induced by the Frobenius map $F_{S_{V^*}}$.

The following diagram of S_{V^*}-algebras commutes because of the above choices of S_{V^*}-algebra structures.

$$\begin{tikzcd}
& R_V \arrow[r, "\varphi"] \arrow[d, hook] & S_{V^*}^F \arrow[d] \\
\mathrm{Frac}\, S_{V^*} \arrow[r, hook] \arrow[dr, hook, "\mathrm{Frac}\,\psi"'] & R_V \otimes_{S_{V^*}} \mathrm{Frac}\, S_{V^*} \arrow[r, "j"] \arrow[d, hook, "i"] & S_{V^*}^F \otimes_{S_{V^*}} \mathrm{Frac}\, S_{V^*} \\
& \mathrm{Frac}\, R_V &
\end{tikzcd}$$

Here we denote by $\mathrm{Frac}(\psi)$ the field homomorphism induced by $\psi : S_{V^*} \hookrightarrow R_V$. The map j is induced by $\varphi : R_V \hookrightarrow S_{V^*}^F$ and by the identity map of $\mathrm{Frac}(S_{V^*})$. The map i is induced by the inclusion $R_V \hookrightarrow \mathrm{Frac}(R_V)$ and by $\mathrm{Frac}(\psi)$. Since the diagram commutes, it suffices to show that j is injective.

The fields $\mathrm{Frac}(R_V)$ and $\mathrm{Frac}(S_{V^*})$ have the same transcendence degree d over $\mathbb{F}_q$, so that $\mathrm{Frac}(R_V)$ is algebraic over $\mathrm{Frac}(S_{V^*})$ via the map $\mathrm{Frac}(\psi)$. Therefore $\mathrm{Frac}(R_V)$ is integral over the domain $R_V \otimes_{S_{V^*}} \mathrm{Frac}(S_{V^*})$ via the injection i. Thus the domain $R_V \otimes_{S_{V^*}} \mathrm{Frac}(S_{V^*})$ has to be a field as well, and j must be injective. **q.e.d.**

Using Theorem 3.1 and the basis for RS_V from Proposition 2.2, we will now give a more transparent proof of the fact that R_V is integrally closed.

Proof of Proposition 2.12.

Let $x \in \mathrm{Frac}(R_V)$ be integral over R_V. As in the proof given in section 2 we conclude that $x \in RS_V$. Being injective, the maps φ and ψ of Proposition 3.1 extend to the fields of fractions of R_V and S_{V^*}. By abuse of notation we refer to these extensions as φ and ψ again. Since x is integral over R_V, its image $\varphi(x) \in \mathrm{Frac}(S_{V^*})$ is integral over the integrally closed domain S_{V^*}. Thus $\varphi(x)$ already lies in S_{V^*} and its image $\psi(\varphi(x)) = x^{q^{d-1}}$ is an element of R_V.

Thus it suffices to show that an element $y \in RS_V$ lies in the subring R_V if and only if the Frobenius power $y^{q^{d-1}}$ lies in R_V. If y is an element of the basis for RS_V constructed in Proposition 2.2, the Frobenius power $y^{q^{d-1}}$ is again a basis element, and the statement about y follows directly from the shape of the basis under consideration. Since the Frobenius map is a homomorphism, we can reduce the general case to the case of a basis element by considering arbitrary linear combinations. **q.e.d.**

4 Structure of Q_V

We now study the projective variety $Q_V = \mathrm{Proj}(R_V)$. We begin with some basic properties of Q_V which follow directly from the results of section 2. We then construct stratifications for Q_V and for the projective space $P_V = \mathrm{Proj}(S_V)$ which are "dual" in a sense made precise by Proposition 4.2 and Proposition 4.6 below. Finally, we determine the singular locus of Q_V and discuss the birational equivalence $Q_V \supset \Omega_V \subset P_V$.

Corollary 4.1 *(i) The dimension of Q_V is $(d-1)$.*

(ii) Q_V is projectively normal.

(iii) The degree of Q_V is equal to $q^{\frac{(d-1)(d-2)}{2}}$.

Proof. Part (i) follows directly from Proposition 2.11, (i). Part (ii) is merely a repetition of Proposition 2.12. Part (iii) follows from part (i) together with Proposition 2.10. **q.e.d.**

We now construct the aforementioned stratifications, beginning with the more familiar case of P_V.

A surjection of $\mathbb{F}_q$-vector spaces $\sigma : V \twoheadrightarrow V'' \neq 0$ induces a degree-preserving surjection of graded $\mathbb{F}_q$-algebras $S_V \twoheadrightarrow S_{V''}$ with kernel $(v \mid v \in \ker(\sigma)) \subset S_V$. Thus we see that for any proper subspace $V' \subsetneq V$, the scheme $P_{V/V'}$ is the closed subscheme of P_V corresponding to the homogeneous ideal $(v \mid v \in V') \subset S_V$. The scheme $\Omega_{V/V'}$ is the locally closed subscheme of P_V obtained by intersecting the closed subscheme $P_{V/V'}$ with the open subscheme of P_V on which the homogeneous element $\prod_{v \in V \smallsetminus V'} v$ of S_V does not vanish.

Theorem 4.2 (Stratification of P_V) *The underlying set of the scheme P_V is the disjoint union*

$$P_V \;=\; \dot{\bigcup_{V' \subsetneq V}} \Omega_{V/V'}\,.$$

Proof. The disjointness is clear from the description of the strata $\Omega_{V/V'}$ as locally closed subschemes of P_V. In order to see that every point of P_V lies in one of the $\Omega_{V/V'}$, let $x \in P_V$ and observe that the set $V_x := \{v \in V \mid v(x) = 0\}$ is a linear subspace of V. Then x lies in $\Omega_{V/V_x} \subset P_V$ by definition of V_x. **q.e.d.**

Remark 4.3 *The closure of a stratum $\Omega_{V/V'} \subset P_V$ is again a union of strata:*

$$\overline{\Omega_{V/V'}} \;=\; P_{V/V'} \;=\; \dot{\bigcup_{V' \subset W \subsetneq V}} \Omega_{V/W} \;\subset\; P_V$$

We now proceed in a similar fashion to construct a stratification for the scheme Q_V.

Lemma 4.4 *Let $0 \neq V' \subset V$ be a nonzero linear subspace of V. Then there exists a degree-preserving surjection of graded $\mathbb{F}_q$-algebras $\rho : R_V \twoheadrightarrow R_{V'}$ such that*

$$\frac{1}{v} \longmapsto \begin{cases} \frac{1}{v} & \textit{if } v \in V' \setminus \{0\}. \\ 0 & \textit{if } v \in V \setminus V'. \end{cases}$$

Furthermore, the kernel of ρ is equal to the ideal $\left(\frac{1}{v} \mid v \in V \setminus V'\right)$.

Proof. We claim that the surjection

$$\pi : A_V := \mathbb{F}_q \left[X_v \mid 0 \neq v \in V\right] \longrightarrow\!\!\!\!\!\rightarrow R_{V'}$$

$$X_v \longmapsto \begin{cases} \frac{1}{v} & \text{if } v \in V' \setminus \{0\}. \\ 0 & \text{if } v \in V \setminus V'. \end{cases}$$

has the kernel $\mathfrak{a} := J_V + (X_v \mid v \in V \setminus V')$. This suffices to prove the lemma because of the presentation $R_V = A_V/J_V$ of Proposition 2.5.

The inclusion $\ker(\pi) \subset \mathfrak{a}$ is immediate from the presentation $R_{V'} = A_{V'}/J_{V'}$. We now prove the converse inclusion. It is easy to see that all elements X_v of $\mathfrak{a}$ for some $v \in V \setminus V'$ and all elements of $\mathfrak{a}$ of the form $X_v - \alpha X_{\alpha v}$ for some $0 \neq v \in V$ and some $\alpha \in \mathbb{F}_q^\times$ already lie in the kernel of π. Thus we only need to show that for three nonzero vectors $v, v', v'' \in V$ with $v + v' + v'' = 0$, the element $f := X_v X_{v'} + X_{v'} X_{v''} + X_v X_{v''} \in \mathfrak{a}$ lies in the kernel of π as well. This is clear if at least two of the three vectors lie in $V \setminus V'$. If not, then all three vectors must lie in V', so that $f \in J_{V'}$ and therefore $\pi(f) = 0$ as well. **q.e.d.**

Corollary 4.5 *Let $0 \neq V' \subset V$ be a nonzero linear subspace of V. Then the scheme $Q_{V'}$ is the closed subscheme of Q_V corresponding to the homogeneous ideal $\left(\frac{1}{v} \mid v \in V \setminus V'\right) \subset R_V$. The scheme $\Omega_{V'}$ is the locally closed subscheme of Q_V obtained by intersecting the closed subscheme $Q_{V'}$ with the open subscheme of Q_V on which the homogeneous element $\prod_{0 \neq v \in V'} \frac{1}{v}$ of R_V does not vanish.*

For any three nonzero $\mathbb{F}_q$-vector spaces $0 \neq V'' \subset V' \subset V$, the induced triangle of closed immersions

$$\begin{tikzcd} Q_{V''} \arrow[rr, hook] \arrow[dr, hook] & & Q_{V'} \arrow[dl, hook'] \\ & Q_V & \end{tikzcd}$$

commutes.

Proof. The statements about $Q_{V'}$ and $\Omega_{V'}$ are immediate from Lemma 4.4 above. The last statement follows from the commutativity of the triangle

$$\begin{tikzcd} R_{V''} & & R_{V'} \arrow[ll, two heads] \\ & R_V \arrow[ul, two heads] \arrow[ur, two heads] & \end{tikzcd}$$

q.e.d.

Theorem 4.6 (Stratification of Q_V) *The underlying set of the scheme Q_V is the disjoint union*

$$Q_V \;=\; \dot{\bigcup_{0\neq V'\subset V}} \Omega_{V'}\,.$$

Proof. As in the proof of Proposition 4.2 above, the disjointness is a direct consequence of the description of the strata $\Omega_{V'}$ as locally closed subschemes of Q_V. To show that any point x of Q_V lies in one of the strata $\Omega_{V'}$, we define the set $V_x := \{v \in V \smallsetminus \{0\} \mid \frac{1}{v}(x) \neq 0\} \cup \{0\}$. We claim that V_x is a nonzero linear subspace of V. To see this, we need to show that if $\frac{1}{v}(x) \neq 0$ and $\frac{1}{w}(x) \neq 0$ for some $v, w \in V$, then also $\frac{1}{v+w}(x) \neq 0$. This is immediate from the equality

$$\frac{1}{v}\cdot\frac{1}{w} = \frac{1}{v+w}\cdot\left(\frac{1}{v}+\frac{1}{w}\right).$$

Thus x lies in the stratum $\Omega_{V_x} \subset Q_V$ by definition of the subspace V_x. **q.e.d.**

Remark 4.7 *The closure of a stratum $\Omega_{V'} \subset Q_V$ is again a union of strata:*

$$\overline{\Omega_{V'}} \;=\; Q_{V'} \;=\; \dot{\bigcup_{0\neq W\subset V'}} \Omega_W \;\subset\; Q_V$$

Comparing the two stratifications 4.2 and 4.6 of P_V and Q_V, respectively, we observe that the sets of isomorphism classes of strata occurring in the two stratifications are identical. The disparity of the schemes P_V and Q_V is reflected in the functoriality of the indexing sets: The strata of P_V are indexed by nonzero quotients of V, whereas the strata of Q_V are indexed by nonzero subspaces of V. Since P_V and Q_V both contain the non-empty open stratum Ω_V, they are birationally equivalent.

Before studying this birational equivalence we introduce convenient open affine covers of P_V and Q_V. More precisely, for each stratum of P_V and of Q_V we construct the smallest open affine neighborhood which is itself a union of strata. We denote the structure sheaves of P_V and Q_V by $\mathcal{O}_{P_V}$ and by $\mathcal{O}_{Q_V}$, respectively.

We begin with the projective space P_V. For a proper subspace $V' \subsetneqq V$, define

$$\mathcal{U}_{V/V'} \;:=\; \dot{\bigcup_{W\subset V'}} \Omega_{V/W} \;\subset\; P_V .$$

We call $\mathcal{U}_{V/V'}$ *the strata neighborhood of* $\Omega_{V'}$ *in* P_V. Note that $\mathcal{U}_{V/V'}$ is indeed the open affine subscheme of P_V on which the homogeneous element $\prod_{v\in V\smallsetminus V'} v$ of S_V does not vanish. Therefore the affine coordinate ring of $\mathcal{U}_{V/V'}$ is equal to

$$\mathcal{O}_{P_V}\left(\mathcal{U}_{V/V'}\right) \;=\; \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v\in V, w\in V\smallsetminus V'\right].$$

We proceed analogously for the scheme Q_V. For a nonzero subspace $0\neq V'\subset V$, define

$$\mathcal{V}_{V'} \;:=\; \dot{\bigcup_{V'\subset W}} \Omega_{W} \;\subset\; Q_V .$$

We call $\mathcal{V}_{V'}$ *the strata neighborhood of* $\Omega_{V'}$ *in* Q_V. It is indeed the open affine subscheme of Q_V on which the homogeneous element $\prod_{0\neq v\in V'} \frac{1}{v}$ of R_V does not vanish. Thus the affine coordinate ring of $\mathcal{V}_{V'}$ is equal to

$$\mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right) \;=\; \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v\in V', w\in V\smallsetminus \{0\}\right].$$

Proposition 4.8 *The morphism $P_V\supset\Omega_V\to Q_V$ identifying the open stratum Ω_V in P_V with the open stratum Ω_V in Q_V can be extended uniquely to the union of all strata of P_V of codimension ≤ 1. This extension map collapses each 1-codimensional stratum $\Omega_{V/V'}$ of P_V to the corresponding 0-dimensional closed stratum $\Omega_{V'}$ of Q_V.*

Proof. Let $V'\subsetneqq V$ be a 1-dimensional proper subspace of V. Recall that

$$\begin{aligned}
\mathcal{O}_{P_V}\left(\mathcal{U}_{V/V'}\right) &= \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v\in V, w\in V\smallsetminus V'\right]\\
\mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right) &= \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v\in V', w\in V\smallsetminus \{0\}\right]\\
(RS_V)_0 &= \mathcal{O}_{Q_V}\left(\Omega_V\right) = \mathcal{O}_{P_V}\left(\Omega_V\right) = \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v\in V, w\in V\smallsetminus \{0\}\right]
\end{aligned}$$

We prove the existence of an extension to the strata neighborhood $\mathcal{U}_{V/V'}\subset P_V$ by providing a dotted arrow which makes the following diagram of affine coordinate rings commute:

$$\begin{tikzcd}
\mathcal{O}_{P_V}\left(\mathcal{U}_{V/V'}\right) \arrow[dr, hook] & & \mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right) \arrow[ll, dotted] \arrow[dl, hook'] \\
& (RS_V)_0 &
\end{tikzcd}$$

Since the dimension of V' is equal to 1, the quotient $\frac{v}{w}$ lies in $\mathbb{F}_q$ for any two non-zero vectors $v, w \in V'$. Thus

$$\mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right) \;=\; \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V', w \in V \smallsetminus V'\right].$$

Hence we can define the desired dotted arrow to be the inclusion

$$t : \mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right) \hookrightarrow \mathcal{O}_{P_V}\left(\mathcal{U}_{V/V'}\right).$$

We now show that the 1-codimensional stratum $\Omega_{V/V'}$ of P_V is mapped onto the 0-dimensional closed stratum $\Omega_{V'}$ of Q_V under the corresponding map of affine schemes. The ideal I of the closed subscheme $\Omega_{V/V'} \hookrightarrow \mathcal{U}_{V/V'}$ is equal to

$$I \;=\; \left(\frac{v}{w} \;\middle|\; v \in V', w \in V \smallsetminus V'\right) \;\subset\; \mathcal{O}_{P_V}\left(\mathcal{U}_{V/V'}\right).$$

Therefore, its inverse image

$$t^{-1}(I) \;=\; \left(\frac{v}{w} \;\middle|\; v \in V', w \in V \smallsetminus V'\right) \;\subset\; \mathcal{O}_{Q_V}\left(\mathcal{V}_{V'}\right)$$

is precisely the ideal of the closed subscheme $\Omega_{V'} \hookrightarrow \mathcal{V}_{V'}$, as desired.

Since Q_V is separated and P_V is reduced, any extension of the morphism $P_V \supset \Omega_V \to Q_V$ to an open subset containing $\Omega_V \subset P_V$ is unique. Thus all the extensions obtained by varying the 1-dimensional proper subspace $V' \subsetneqq V$ can be glued together. **q.e.d.**

The following proposition can be proved in exactly the same fashion:

Proposition 4.9 *The morphism $Q_V \supset \Omega_V \to P_V$ identifying the open stratum Ω_V in Q_V with the open stratum Ω_V in P_V can be extended uniquely to the union of all strata of Q_V of codimension ≤ 1. This extension map collapses each 1-codimensional stratum $\Omega_{V'}$ of Q_V to the corresponding 0-dimensional closed stratum $\Omega_{V/V'}$ of P_V.*

Theorem 4.10 *The singular locus of Q_V consists of all strata of codimension at least 2:*

$$Q_V^{sing} \;=\; \dot{\bigcup_{\dim(V/V'\geq 2)}} \Omega_{V'}.$$

Proof. Let $0 \neq V' \subset V$ be a non-zero linear subspace of V. Choose a linear subspace $V'' \subset V$ such that $V = V' \oplus V''$. Our goal is to construct a morphism of schemes over $\mathbb{F}_q$

$$\theta : \mathcal{V}_{V'} \longrightarrow \Omega_{V'} \times \operatorname{Spec} R_{V''}$$

which restricts to an isomorphism of a neighborhood of the closed subscheme $\Omega_{V'}$ in $\mathcal{V}_{V'}$ onto a neighborhood of the closed subscheme $\Omega_{V'} \times \{0\}$ in $\Omega_{V'} \times \operatorname{Spec} R_{V''}$. Here we denote by $\{0\}$ the vertex of the affine cone $\operatorname{Spec} R_{V''}$. This will link the problem of determining whether the points of $\Omega_{V'}$ are singular in Q_V to the singularity of the ring $R_{V''}$, which has already been treated in Proposition 2.11.

We now construct the morphism θ. Fix a nonzero vector $v_0' \in V'$. From the presentation of $R_{V''}$ in Proposition 2.5 it is easy to see that there exists a unique $\mathbb{F}_q$-algebra homomorphism

$$R_{V''} = \mathbb{F}_q\left[\frac{1}{v''} \;\middle|\; v'' \in V'' \smallsetminus \{0\}\right] \;\longrightarrow\; \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V', w \in V \smallsetminus \{0\}\right] = \mathcal{O}_{Q_V}(\mathcal{V}_{V'})$$

such that

$$\frac{1}{v''} \mapsto \frac{v_0'}{v''}\ .$$

It is clear that this map is injective. Furthermore, there exists a natural inclusion

$$(RS_{V'})_0 = \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v, w \in V' \smallsetminus \{0\}\right] \;\hookrightarrow\; \mathcal{O}_{Q_V}(\mathcal{V}_{V'})\,.$$

After identifying the rings $(RS_{V'})_0$ and $R_{V''}$ with their images in $\mathcal{O}_{Q_V}(\mathcal{V}_{V'})$ under these injections, we observe that their intersection is trivial in the sense that $(RS_{V'})_0 \cap R_{V''} = \mathbb{F}_q \subset \mathcal{O}_{Q_V}(\mathcal{V}_{V'})$. Thus the induced $\mathbb{F}_q$-algebra homomorphism

$$\varepsilon : (RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''} \;\hookrightarrow\; \mathcal{O}_{Q_V}(\mathcal{V}_{V'})$$

is injective as well. To simplify notation we identify the ring $(RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''}$ with its image

$$\mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v, w \in V' \smallsetminus \{0\}\right]\left[\frac{v_0'}{v''} \;\middle|\; v'' \in V'' \smallsetminus \{0\}\right]$$

in $\mathcal{O}_{Q_V}(\mathcal{V}_{V'})$. The homomorphism ε induces the desired map of affine schemes

$$\theta :\ \mathcal{V}_{V'} \;\longrightarrow\; \Omega_{V'} \times \operatorname{Spec} R_{V''}.$$

We now show that θ satisfies the property stated at the beginning of the proof. The closed subscheme $\Omega_{V'} \hookrightarrow \mathcal{V}_{V'}$ corresponds to the ideal

$$I := \left(\frac{v}{w} \;\middle|\; v \in V', w \in V \smallsetminus V'\right)$$

in $\mathcal{O}_{Q_V}(\mathcal{V}_{V'})$. The inverse image

$$\varepsilon^{-1}(I) = \left(\frac{v}{w} \;\middle|\; v \in V', w \in V'' \smallsetminus \{0\}\right)$$

in the ring

$$(RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''} \;=\; \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v, w \in V' \smallsetminus \{0\}\right]\left[\frac{v_0'}{v''} \;\middle|\; v'' \in V'' \smallsetminus \{0\}\right]$$

is precisely the ideal corresponding to the closed subscheme $\Omega_{V'} \times \{0\} \hookrightarrow \Omega_{V'} \times \operatorname{Spec} R_{V''}$. Therefore θ restricts to an isomorphism of the closed subscheme $\Omega_{V'} \hookrightarrow \mathcal{V}_{V'}$ onto the closed subscheme $\Omega_{V'} \times \{0\} \hookrightarrow \Omega_{V'} \times \operatorname{Spec} R_{V''}$.

We claim that the injective $\mathbb{F}_q$-algebra homomorphism

$$\varepsilon_s : \left((RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''}\right)_s \longhookrightarrow \left(\mathcal{O}_{Q_V}(\mathcal{V}_{V'})\right)_s$$

obtained by localizing ε with respect to the homogeneous element

$$s := \prod_{\substack{0 \neq v' \in V' \\ 0 \neq v'' \in V''}} \left(\frac{v_0'}{v'} + \frac{v_0'}{v''}\right) \;\in\; (RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''}$$

is surjective and thus an isomorphism. To see this, we have to show that for nonzero vectors $u, u' \in V'$ and $u'' \in V''$, the element $\frac{u}{u'+u''}$ of $\left(\mathcal{O}_{Q_V}(\mathcal{V}_{V'})\right)_s$ already lies in the localization $\left((RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''}\right)_s$. This follows from the equality

$$\frac{u}{u'+u''} \cdot \left(\frac{v_0'}{u'} + \frac{v_0'}{u''}\right) = \frac{u}{u'} \cdot \frac{v_0'}{u''}$$

since the element $\left(\frac{v_0'}{u'} + \frac{v_0'}{u''}\right)$ is invertible in the localization $\left((RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''}\right)_s$.

Since $\theta : \mathcal{V}_{V'} \to \Omega_{V'} \times \operatorname{Spec} R_{V''}$ is the map of affine schemes corresponding to the ring homomorphism $\varepsilon : (RS_{V'})_0 \otimes_{\mathbb{F}_q} R_{V''} \hookrightarrow \mathcal{O}_{Q_V}(\mathcal{V}_{V'})$, the fact that the localized homomorphism ε_s is an isomorphism implies that θ is an isomorphism away from the zero-loci of s in $\mathcal{V}_{V'}$ and $\Omega_{V'} \times \operatorname{Spec} R_{V''}$. Therefore, our next goal is to show that the zero-locus of s in $\mathcal{V}_{V'}$ is disjoint from $\Omega_{V'}$, and that the zero-locus of s in $\Omega_{V'} \times \operatorname{Spec} R_{V''}$ is disjoint from $\Omega_{V'} \times \{0\}$. We need to show that given nonzero vectors $v' \in V'$ and $v'' \in V''$, the homogeneous element $\left(\frac{v_0'}{v'} + \frac{v_0'}{v''}\right)$ vanishes nowhere on $\Omega_{V'} \subset \mathcal{V}_{V'}$ and $\Omega_{V'} \times \{0\} \subset \Omega_{V'} \times \operatorname{Spec} R_{V''}$. For a point $x \in \Omega_{V'} \subset \mathcal{V}_{V'}$ or $x \in \Omega_{V'} \times \{0\} \subset \Omega_{V'} \times \operatorname{Spec} R_{V''}$, we calculate

$$\left(\frac{v_0'}{v'} + \frac{v_0'}{v''}\right)(x) = \frac{v_0'}{v'}(x) + 0 \neq 0$$

by the definition of $\Omega_{V'}$, as desired.

We have now shown that the map θ induces an isomorphism of a neighborhood of $\Omega_{V'}$ in $\mathcal{V}_{V'}$ with a neighborhood of $\Omega_{V'} \times \{0\}$ in $\Omega_{V'} \times \operatorname{Spec} R_{V''}$, and that this isomorphism is an extension of the isomorphism $\Omega_{V'} \cong \Omega_{V'} \times \{0\}$. This implies that θ yields a bijection

$$\left(Q_V^{\mathrm{sing}} \cap \Omega_{V'}\right) \;\cong\; (\Omega_{V'} \times \operatorname{Spec} R_{V''})^{\mathrm{sing}} \cap (\Omega_{V'} \times \{0\})\,.$$

Since $\Omega_{V'}$ is smooth, we note that

$$(\Omega_{V'} \times \operatorname{Spec} R_{V''})^{\mathrm{sing}} \;=\; \Omega_{V'} \times (\operatorname{Spec} R_{V''})^{\mathrm{sing}}\,.$$

It was already shown in Proposition 2.11 that the vertex $\{0\}$ of $\operatorname{Spec} R_{V''}$ is a singular point if and only if $\dim V'' \geq 2$. Thus the subset $\Omega_{V'} \subset Q_V$ consists of only non-singular points if $\dim V/V' \leq 1$, and of only singular points if $\dim V/V' \geq 2$. **q.e.d.**

5 Low-dimensional examples

In this section we study the special cases $d = 2$ and $d = 3$. If $d = 2$, the varieties P_V and Q_V are isomorphic curves. If $d = 3$, they are non-isomorphic surfaces. In this case, we will prove that the blow-up of P_V in every closed stratum is isomorphic to the blow-up of Q_V in every closed stratum. In particular, the blow-up of Q_V (or of P_V) is a desingularization of Q_V in this special case.

We first assume that $d = 2$. Then the curve Q_V is non-singular according to Theorem 4.10. In fact, Propositions 4.8 and 4.9 imply that the curves Q_V and P_V are even isomorphic in this case. Knowing that Q_V is a smooth curve, this of course also follows from the general fact that up to isomorphism there is a unique smooth projective curve in every birational equivalence class.

In a similar vein, we now show that the map φ of Theorem 3.1 yields an isomorphism of Q_V with the q-uple embedding of P_{V^*}.

Proposition 5.1 *Let $d = 2$. Then the map $\varphi : R_V \to S_{V^*}$ of Theorem 3.1 induces an isomorphism of graded $\mathbb{F}_q$-algebras*

$$R_V \xrightarrow{\cong} S_{V^*}^{(q)} := \bigoplus_{i \geq 0} S_{V^*,\, qi} \ .$$

Thus the projective curve $Q_V = \mathrm{Proj}(R_V)$ is the q-uple embedding of the projective line $P_{V^*} = \mathrm{Proj}(S_{V^*})$.

Proof. The degree of the homogeneous element $\varphi(\frac{1}{v}) \in S_{V^*}$ is equal to q for any nonzero vector $v \in V$. Thus φ factors through a degree-preserving map $R_V \to S_{V^*}^{(q)}$. From Theorem 3.1 we already know that this map is injective. We now prove that it is also surjective. Since the map is degree-preserving, $\mathbb{F}_q$-linear and injective, it suffices to show that the graded $\mathbb{F}_q$-algebras R_V and $S_{V^*}^{(q)}$ have the same Hilbert function. The Hilbert function H_2 of R_V was determined to be $H_2(n) = qn + 1$ in Remark 2.7 above. Since $\dim(V^*) = d = 2$, this is precisely the Hilbert function of $S_{V^*}^{(q)}$. **q.e.d.**

For the remainder of the section let $d = 3$. Thus P_V and Q_V are surfaces. According to Theorem 4.10 the singular locus of Q_V is the union of all 0-dimensional strata. A 0-dimensional stratum of Q_V or P_V consists of precisely one closed point.

In this situation, Propositions 4.8 states that the morphism $P_V \supset \Omega_V \to Q_V$ can be extended to the union of all strata of P_V of codimension ≤ 1 by collapsing each 1-dimensional stratum of P_V to the corresponding 0-dimensional stratum of Q_V. Proposition 4.9 provides the analogous statement for the morphism $Q_V \supset \Omega_V \to P_V$. We will now show that by blowing up both surfaces in all 0-dimensional strata one obtains isomorphic objects. For background material on blowing up we refer the reader to Hartshorne [5], II.7, Eisenbud [3], 5.2., and Eisenbud-Harris [4], IV.2.

Denote by $\widetilde{P_V} \to P_V$ and by $\widetilde{Q_V} \to Q_V$ the blow-ups of P_V and of Q_V, respectively, in all the 0-dimensional strata. Thus the open stratum $P_V \supset \Omega_V \subset Q_V$ is also a dense open subset of $\widetilde{P_V}$ and $\widetilde{Q_V}$.

Theorem 5.2 *Let $d = 3$. Then the identity morphism $P_V \supset \Omega_V = \Omega_V \subset Q_V$ extends uniquely to an isomorphism of the blow-ups $\widetilde{P_V} \cong \widetilde{Q_V}$. In particular, $\widetilde{Q_V}$ is a desingularization of Q_V.*

Proof. Let $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V$ be a complete flag of V. We will construct open affine subschemes $\mathcal{A}_{V_1,V_2}$ of $\widetilde{P_V}$ and $\mathcal{B}_{V_1,V_2}$ of $\widetilde{Q_V}$ which contain Ω_V and are isomorphic via a map extending the identity on Ω_V. We will show that if $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V$ ranges over all complete flags of V, the open sets $\mathcal{A}_{V_1,V_2}$ cover $\widetilde{P_V}$ and the open sets $\mathcal{B}_{V_1,V_2}$ cover $\widetilde{Q_V}$. Since the blow-ups $\widetilde{P_V}$ and $\widetilde{Q_V}$ are separable and reduced, this implies that the identity map $P_V \supset \Omega_V = \Omega_V \subset Q_V$ extends uniquely to an isomorphism $\widetilde{P_V} \cong \widetilde{Q_V}$.

We begin with the construction of the affine open subscheme $\mathcal{A}_{V_1,V_2}$ of $\widetilde{P_V}$. Denote by $\widetilde{\mathcal{U}_{V/V_2}}$ the inverse image of the strata neighborhood $\mathcal{U}_{V/V_2}$ under the projection map $\widetilde{P_V} \to P_V$. Denote by $A := \mathbb{F}_q\left[\frac{v}{w} \mid v \in V, w \in V \smallsetminus V_2\right]$ the affine coordinate ring of $\mathcal{U}_{V/V_2}$ and by $I := \left(\frac{v}{w} \mid v \in V_2, w \in V \smallsetminus V_2\right)$ the ideal in A corresponding to the closed point Ω_{V/V_2} of $\mathcal{U}_{V/V_2}$. Thus

$$\widetilde{\mathcal{U}_{V/V_2}} = \operatorname{Proj}\left(\operatorname{Bl}_I A\right),$$

where $\operatorname{Bl}_I A := A \oplus I \oplus I^2 \oplus \ldots$ denotes the blow-up algebra of A with respect to I.

Inverting the homogeneous element

$$f := \prod_{\substack{v \in V_2 \smallsetminus V_1 \\ w \in V \smallsetminus V_2}} \frac{v}{w} \in I^k \subset \operatorname{Bl}_I A$$

of degree $k := \#(V_2 \smallsetminus V_1) \cdot \#(V \smallsetminus V_2)$ in $\operatorname{Bl}_I A$ yields the desired open affine subscheme $\mathcal{A}_{V_1,V_2}$ of $\widetilde{P_V}$, with affine coordinate ring equal to $(\operatorname{Bl}_I A)\left[\frac{1}{f}\right]_0$. This affine coordinate ring is equal to

$$\begin{aligned} & \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V,\ w \in V \smallsetminus V_2\right]\left[\frac{v}{w} \cdot \frac{w'}{v'} \;\middle|\; v \in V_2,\ v' \in V_2 \smallsetminus V_1,\ w, w' \in V \smallsetminus V_2\right] \\ = \;& \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V_2,\ w \in V \smallsetminus V_1\right], \end{aligned}$$

where the last equality is a consequence of the fact that $\dim V_2 = 2$:

The second ring is clearly contained in the first ring. For the converse inclusion, one sees easily that it suffices to show that for given vectors $v \in V$ and $w \in V \smallsetminus V_2$, the quotient $\frac{v}{w}$ is an element of the second ring. To see this, note that the vector space V decomposes as $V = V_2 \oplus \mathbb{F}_q w$ since $\dim V_2 = 2 = \dim V - 1$. Hence there exist elements $v_2 \in V_2$ and $\alpha \in \mathbb{F}_q$ such that $v = v_2 + \alpha w$. Thus $\frac{v}{w} = \frac{v_2}{w} + \alpha$ is indeed an element of the second ring.

We now proceed analogously with the construction of the affine open subscheme $\mathcal{B}_{V_1,V_2}$ of $\widetilde{Q_V}$. Denote by $\widetilde{\mathcal{V}_{V_1}}$ the inverse image of the strata neighborhood $\mathcal{V}_{V_1}$ under the projection map $\widetilde{Q_V} \to Q_V$. Denote by $B := \mathbb{F}_q\left[\frac{v}{w} \mid v \in V_1, w \in V \smallsetminus \{0\}\right]$ the affine coordinate ring of $\mathcal{V}_{V_1}$ and by $J := \left(\frac{v}{w} \mid v \in V_1, w \in V \smallsetminus V_1\right)$ the ideal in B corresponding to the closed point Ω_{V_1} of $\mathcal{V}_{V_1}$. Thus

$$\widetilde{\mathcal{V}_{V_1}} = \operatorname{Proj}\left(\operatorname{Bl}_J B\right).$$

Inverting the homogeneous element

$$g := \prod_{\substack{v \in V_1 \smallsetminus \{0\} \\ w \in V_2 \smallsetminus V_1}} \frac{v}{w} \in J^\ell \subset \operatorname{Bl}_J B$$

of degree $\ell := \#(V_1 \smallsetminus \{0\}) \cdot \#(V_2 \smallsetminus V_1)$ in $\operatorname{Bl}_J B$ yields the desired open affine subscheme $\mathcal{B}_{V_1,V_2}$ of $\widetilde{Q_V}$, with affine coordinate ring equal to $(\operatorname{Bl}_J B)\left[\frac{1}{g}\right]_0$. Using that any two nonzero vectors of the 1-dimensional vector space V_1 only differ by multiplication with a scalar in $\mathbb{F}_q^\times$, we see that this affine coordinate ring is equal to

$$\begin{aligned}
& \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V_1, w \in V \smallsetminus \{0\}\right]\left[\frac{v}{w} \cdot \frac{w'}{v'} \;\middle|\; v, v' \in V_1 \smallsetminus \{0\},\ w \in V \smallsetminus V_1,\ w' \in V_2 \smallsetminus V_1\right] \\
= \;& \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V_1, w \in V \smallsetminus V_1\right]\left[\frac{w'}{w} \;\middle|\; w \in V \smallsetminus V_1,\ w' \in V_2 \smallsetminus V_1\right] \\
= \;& \mathbb{F}_q\left[\frac{v}{w} \;\middle|\; v \in V_2,\ w \in V \smallsetminus V_1\right].
\end{aligned}$$

Thus the open subscheme $\mathcal{A}_{V_1,V_2} \subset \widetilde{P_V}$ is isomorphic to the open subscheme $\mathcal{B}_{V_1,V_2} \subset \widetilde{Q_V}$ via a map extending the identity on Ω_V.

It remains to show that the constructed open subsets of the blow-ups are indeed coverings. We begin with the blow-up $\widetilde{P_V}$. For a fixed 2-dimensional subspace V_2 of V, we prove that if V_1 ranges over all 1-dimensional subspaces of V_2, the open sets $\mathcal{A}_{V_1,V_2}$ cover $\widetilde{\mathcal{U}_{V/V_2}}$. Let $\mathfrak{p}$ be a homogeneous prime ideal of $\operatorname{Bl}_I A$ such that for every 1-dimensional subspace V_1 of V_2, the homogeneous element

$$f_{V_1} := \prod_{\substack{v \in V_2 \smallsetminus V_1 \\ w \in V \smallsetminus V_2}} \frac{v}{w} \in I^k \subset \operatorname{Bl}_I A$$

of degree $k = \#(V_2 \smallsetminus V_1) \cdot \#(V \smallsetminus V_2)$ lies in $\mathfrak{p}$. We have to show that $\mathfrak{p}$ contains the augmentation ideal $I \oplus I^2 \oplus \ldots$ of $\mathrm{Bl}_I A$.

Denote by $V_\mathfrak{p}$ the set consisting of all vectors $v \in V_2$ for which there exists a vector $w \in V \smallsetminus V_2$ such that the homogeneous element $\frac{v}{w} \in I^1 \subset \mathrm{Bl}_I A$ of degree 1 lies in $\mathfrak{p}$. The set $V_\mathfrak{p}$ is in fact a subspace of V.

We have to show that $V_\mathfrak{p}$ is equal to V_2. Choose a 1-dimensional subspace V_1 of V_2. Since f_{V_1} lies in $\mathfrak{p}$, there exists a nonzero vector $v' \in V_2 \smallsetminus V_1$ such that $v' \in V_\mathfrak{p}$. Set $V_1' := \mathbb{F}_q v'$. Then since $f_{V_1'}$ lies in $\mathfrak{p}$, there exists a nonzero vector $v'' \in V_2 \smallsetminus V_1'$ such that $v'' \in V_\mathfrak{p}$. By construction, the vectors v' and v'' are linearly independent. Thus $V_\mathfrak{p}$ is equal to V_2.

We now prove the analogous result for the blow-up $\widetilde{Q_V}$. For a fixed 1-dimensional subspace V_1 of V, we prove that if V_2 ranges over all 2-dimensional subspaces of V containing V_1, the open sets $\mathcal{B}_{V_1,V_2}$ cover $\widetilde{\mathcal{V}_{V_1}}$. Let $\mathfrak{q}$ be a homogeneous prime ideal of $\mathrm{Bl}_J B$ such that for every 2-dimensional subspace V_2 of V containing V_1, the homogeneous element

$$g_{V_2} := \prod_{\substack{v \in V_1 \smallsetminus \{0\} \\ w \in V_2 \smallsetminus V_1}} \frac{v}{w} \in I^\ell \subset \mathrm{Bl}_J B$$

of degree $\ell = \#(V_1 \smallsetminus \{0\}) \cdot \#(V_2 \smallsetminus V_1)$ lies in $\mathfrak{q}$. We have to show that $\mathfrak{q}$ contains the augmentation ideal $J \oplus J^2 \oplus \ldots$ of $\mathrm{Bl}_J B$.

Choose a generator v_1 of V_1. Denote by $V_\mathfrak{q}$ the set consisting of $0 \in V$ and of all nonzero vectors $w \in V$ such that the homogeneous element $\frac{v_1}{w} \in J^1 \subset \mathrm{Bl}_J B$ of degree 1 does not lie in $\mathfrak{q}$. It is easy to see that the set $V_\mathfrak{q}$ is in fact a subspace of V. Since $V_\mathfrak{q}$ contains V_1, the dimension of $V_\mathfrak{q}$ is at least 1. We have to show that the dimension of $V_\mathfrak{q}$ is in fact equal to one.

Choose any 2-dimensional subspace V_2 of V containing V_1. Then since g_{V_2} lies in $\mathfrak{q}$, there exists a nonzero vector $w \in V_2 \smallsetminus V_1$ which does not lie in $V_\mathfrak{q}$. Hence the dimension of $V_\mathfrak{q}$ is at most 2. If it was equal to 2, the fact that the homogeneous element $g_{V_\mathfrak{q}}$ lies in $\mathfrak{q}$ yields a contradiction to the fact that $\mathfrak{q}$ is a prime ideal. Thus the dimension of $V_\mathfrak{q}$ is equal to 1, as desired. **q.e.d.**

6 Desingularization of Q_V in arbitrary dimension

This section forms the heart of the article. We construct a desingularization B_V of Q_V as follows. We first define B_V as a functor, motivated by a functorial interpretation of blowing up and by the results of section 5. We then prove that this functor is representable by a projective variety over $\mathbb{F}_q$ which contains Ω_V as a dense open subscheme. We exhibit a natural stratification for B_V, show that B_V is nonsingular, and prove that the boundary $B_V \setminus \Omega_V$ is a divisor with normal crossings in the strongest sense. Finally, we construct morphisms to P_V and Q_V which are isomorphisms on Ω_V.

We begin by describing the functors of points of P_V, Q_V and Ω_V. For basic results regarding open and closed subfunctors and representability in algebraic geometry we refer the reader to Eisenbud-Harris [4], chapter VI, and Grothendieck [6], EGA 0: 8.1, EGA I: 3.4.

For convenience and future reference we first collect some basic open and closed conditions in the following lemma. We say that a morphism of sheaves or a section of a sheaf vanishes at a point of a scheme if it vanishes after pulling back to the residue field at that point.

Lemma 6.1 *Let X be a scheme. Let $\mathcal{F}$ be a locally free coherent sheaf on X and let $f \in \Gamma(X, \mathcal{F})$ be a global section of $\mathcal{F}$. Furthermore, let $\varphi : \mathcal{G} \to \mathcal{H}$ be a morphism of locally free coherent sheaves on X.*

(i) The set of points of X on which f vanishes is a closed subset of X.

(ii) The set of points of X on which φ vanishes is a closed subset of X.

(iii) The set of points of X on which φ is an isomorphism is an open subset of X.

Proof. All statements are local, so we can assume that $X = \mathrm{Spec}(A)$ is affine and $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are free of finite rank. Suppose f is given by the coordinates $a_1, \ldots, a_n \in A$. Then the zero locus of f is equal to the zero locus of the ideal generated by the a_i in A. This proves (i). Part (ii) follows from (i) if we set $\mathcal{F} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$, the sheaf of morphisms from $\mathcal{H}$ to $\mathcal{G}$, and $f := \varphi \in \Gamma(X, \mathcal{F})$. To prove (iii), we can assume that $\mathcal{G}$ and $\mathcal{H}$ have the same finite rank, so that φ is given by a matrix with coefficients in A. Then φ is an isomorphism away from the zero locus of the determinant of this matrix, which by (i) is a closed subset of X. **q.e.d.**

We now recall a description of the functor of points of an arbitrary projective scheme over an affine base, generalizing the well-known characterization of projective n-space as the functor which to a scheme T associates the set of invertible quotients of $\mathcal{O}_T^{\oplus n+1}$.

Given an invertible sheaf $\mathcal{L}$ on a scheme T, we call a collection of global sections $s_1, \ldots, s_n \in \Gamma(T, \mathcal{L})$ of $\mathcal{L}$ *generating* if for each $t \in T$ the images $(s_1)_t, \ldots, (s_n)_t$ in the stalk $\mathcal{L}_t$ generate the stalk as an $\mathcal{O}_{T,t}$-module. Of course this is equivalent to requiring that the sections $s_1, \ldots, s_n$ do not vanish simultaneously at any point of T, or to requiring that the induced morphism of sheaves $\mathcal{O}_T^{\oplus n} \to \mathcal{L}$ is a surjection.

Remark 6.2 *Let A be a ring and let $f_1, \ldots, f_r$ be homogeneous polynomials in $A[X_0, \ldots, X_n]$. Define*

$$X := \operatorname{Proj}\left([X_0, \ldots, X_n] \,/\, (f_1, \ldots, f_r)\right).$$

Then the functor of points of X is isomorphic to the contravariant functor from the category of A-schemes to the category of sets which to an A-scheme T associates the set of all equivalence classes of the following data: An invertible sheaf $\mathcal{L}$ on T, together with generating global sections $s_0, \ldots, s_n \in \Gamma(T, \mathcal{L})$, such that $f_i(s_0, \ldots, s_n) = 0$ in $\Gamma\left(T, \mathcal{L}^{\otimes \deg f_i}\right)$ for all i. Two such pairs of invertible sheaves with global sections $(\mathcal{L}, s_0, \ldots, s_n)$, $(\mathcal{L}', s_0', \ldots, s_n')$ are defined to be equivalent if there exists an isomorphism $\mathcal{L} \cong \mathcal{L}'$ which identifies s_i with s_i' for all i.

The proof is a straightforward adaption of the well-known special case of projective space. See for example [5], chapter II, Theorem 7.1.

Corollary 6.3 *The scheme P_V represents the functor which associates to an $\mathbb{F}_q$-scheme T the set of all equivalence classes of pairs $(\mathcal{L}, \varphi)$ consisting of an invertible sheaf $\mathcal{L}$ together with a surjection $\varphi : \mathcal{O}_T \otimes_{\mathbb{F}_q} V \twoheadrightarrow \mathcal{L}$.*
The open subscheme Ω_V of P_V corresponds to the open subfunctor obtained by additionally requiring that for any nonzero vector $0 \neq v \in V$, the section $\varphi(1 \otimes v) \in \Gamma(T, \mathcal{L})$ vanishes nowhere on T.

Corollary 6.4 *The scheme Q_V represents the functor which associates to an $\mathbb{F}_q$-scheme T the set of all equivalence classes of pairs $(\mathcal{G}, \lambda)$ consisting of an invertible sheaf $\mathcal{G}$ together with a map of sets $\lambda : V \setminus \{0\} \to \Gamma(T, \mathcal{G})$ such that*

(i) the set of global sections $\{\lambda(v) \mid 0 \neq v \in V\}$ generates $\mathcal{G}$

(ii) $\lambda(\alpha v) = \frac{1}{\alpha}\lambda(v)$ for all $0 \neq v \in V$ and $\alpha \in \mathbb{F}_q$

(iii) $\lambda(v)\cdot\lambda(v') = \lambda(v)\cdot\lambda(v+v') + \lambda(v')\cdot\lambda(v+v')$ in $\Gamma(T, \mathcal{G}^{\otimes 2})$ for all linearly independent vectors $v, v' \in V$.

The open subscheme Ω_V of Q_V corresponds to the open subfunctor obtained by additionally requiring that for any nonzero vector $0 \neq v \in V$, the section $\lambda(v) \in \Gamma(T, \mathcal{G})$ vanishes nowhere on T.

Proof. Apply Remark 6.2 above to the presentation of R_V in Corollary 2.5. **q.e.d.**

The birational equivalence of P_V and Q_V takes the following shape on the level of functors. Given a T-valued point $(\mathcal{G}, \lambda)$ of the open subfunctor of Q_V in Corollary 6.4, any section $\lambda(v)$ trivializes the invertible sheaf $\mathcal{G}$, so we can assume that $\mathcal{G} = \mathcal{O}_T$. For a nonzero vector

$0 \neq v \in V$ define $\varphi(1 \otimes v) := \frac{1}{\lambda(v)}$. Then the properties (i) and (ii) of λ above imply that we can extend φ to a morphism of sheaves $\varphi : \mathcal{O}_T \otimes_{\mathbb{F}_q} V \twoheadrightarrow \mathcal{L} := \mathcal{O}_T$ as in Corollary 6.3. This yields a natural transformation from the open subfunctor of Q_V in Corollary 6.4 to the open subfunctor of P_V in Corollary 6.3, and the inverse map is constructed analogously.

We now come to the definition of the contravariant functor B_V from the category of schemes over $\mathbb{F}_q$ to the category of sets. To an $\mathbb{F}_q$-scheme T, we associate the set of all equivalence classes of the following objects: For every nonzero subspace $0 \neq V' \subset V$ an invertible sheaf $\mathcal{L}_{V'}$ on T, together with a surjection

$$\varphi_{V'} : \ \mathcal{O}_T \otimes_{\mathbb{F}_q} V' \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L}_{V'} \ ,$$

and for every inclusion of nonzero subspaces $0 \neq V'' \subset V' \subset V$ a morphism

$$\psi_{V'}^{V''} : \ \mathcal{L}_{V''} \longrightarrow \mathcal{L}_{V'} \ ,$$

such that the restriction of $\varphi_{V'}$ to the subsheaf $\mathcal{O}_T \otimes_{\mathbb{F}_q} V'' \ \subset \ \mathcal{O}_T \otimes_{\mathbb{F}_q} V'$ is equal to the composition $\psi_{V'}^{V''} \circ \varphi_{V''}$. In other words, we require the diagram

$$\begin{tikzcd}
\mathcal{O}_T \otimes_{\mathbb{F}_q} V' \arrow[r, two heads, "\varphi_{V'}"] & \mathcal{L}_{V'} \\
\mathcal{O}_T \otimes_{\mathbb{F}_q} V'' \arrow[u, hook] \arrow[r, two heads, "\varphi_{V''}"'] & \mathcal{L}_{V''} \arrow[u, "\psi_{V'}^{V''}"']
\end{tikzcd}$$

to commute.

For the sake of readability we denote such an object as a triple $(\mathcal{L}, \varphi, \psi)$. Two such objects $(\mathcal{L}, \varphi, \psi)$ and $(\tilde{\mathcal{L}}, \tilde{\varphi}, \tilde{\psi})$ are defined to be equivalent if for every nonzero subspace $V' \subset V$ there exists an isomorphism of invertible sheaves $\mathcal{L}_{V'} \cong \tilde{\mathcal{L}}_{V'}$ which is compatible with the surjections $\varphi_{V'}$ and $\tilde{\varphi}_{V'}$.

Since all morphisms φ in this definition are surjective, the maps ψ making the corresponding diagrams commute are unique. In particular, for any chain of nonzero subspaces

$$0 \neq V_1 \subset V_2 \subset V_3 \subset V$$

the maps ψ automatically satisfy the cocycle condition

$$\psi_{V_2}^{V_1} \ \circ \ \psi_{V_3}^{V_2} \ = \ \psi_{V_3}^{V_1} \ .$$

Similarly, if two objects $(\mathcal{L}, \varphi, \psi)$ and $(\tilde{\mathcal{L}}, \tilde{\varphi}, \tilde{\psi})$ are equivalent, the corresponding isomorphisms $\mathcal{L} \cong \tilde{\mathcal{L}}$ are automatically compatible with the maps ψ and $\tilde{\psi}$ as well.

We finish the construction of the functor B_V by associating to a morphism $f : T \to T'$ of $\mathbb{F}_q$-schemes the map of sets $f^* : B_V(T') \to B_V(T)$ obtained by pulling back all of the above data along f. We now study this functor in more detail.

Proposition 6.5 *The functor B_V is representable by a projective scheme over $\mathbb{F}_q$.*

Proof. We show that B_V is isomorphic to a closed subfunctor of the functor represented by the product of projective spaces $\prod_{0\neq V'\subset V} P_{V'}$. This implies both the representability of B_V and the projectivity of the representing scheme.

A T-valued point of the product $\prod_{V'} P_{V'}$ is given by a collection of invertible quotients

$$\big(\varphi_{V'}:\ \mathcal{O}_T\otimes_{\mathbb{F}_q} V' \twoheadrightarrow \mathcal{L}_{V'}\big)_{0\neq V'\subset V}.$$

Fix an inclusion of nonzero subspaces $0\neq V''\subset V'$ of V. Then since $\varphi_{V'}$ is an epimorphism, there exists at most one dotted arrow $\psi_{V'}^{V''}$ making the diagram

$$\begin{tikzcd}
\mathcal{O}_T\otimes_{\mathbb{F}_q} V' \arrow[r, two heads, "\varphi_{V'}"] & \mathcal{L}_{V'} \\
\mathcal{O}_T\otimes_{\mathbb{F}_q} V'' \arrow[u, hook] \arrow[r, two heads, "\varphi_{V''}"'] & \mathcal{L}_{V''} \arrow[u, dotted, "\psi_{V'}^{V''}"']
\end{tikzcd}$$

commute. Thus B_V is isomorphic to the subfunctor of $\prod_{V'} P_{V'}$ which associates to an $\mathbb{F}_q$-scheme T the set of all collections of invertible quotients

$$\big(\varphi_{V'}:\ \mathcal{O}_T\otimes_{\mathbb{F}_q} V' \twoheadrightarrow \mathcal{L}_{V'}\big)_{0\neq V'\subset V}$$

satisfying the extra condition that for any inclusion of nonzero subspaces $0\neq V''\subset V'\subset V$, there exists a morphism $\psi_{V'}^{V''}$ making the above diagram commute.

To finish the proof we have to show that this extra condition is indeed a closed condition. It suffices to prove this for a single fixed inclusion of nonzero subspaces $0\neq V''\subset V'\subset V$. Denote by $i_{V''}$ the inclusion of the kernel of $\varphi_{V''}$ into $\mathcal{O}_T\otimes_{\mathbb{F}_q} V''$. Then the diagram

$$\begin{tikzcd}
 & & \mathcal{O}_T\otimes_{\mathbb{F}_q} V' \arrow[r, "\varphi_{V'}"] & \mathcal{L}_{V'} \arrow[r] & 0 \\
0 \arrow[r] & \ker(\varphi_{V''}) \arrow[r, "i_{V''}"'] & \mathcal{O}_T\otimes_{\mathbb{F}_q} V'' \arrow[u, hook] \arrow[r, "\varphi_{V''}"'] & \mathcal{L}_{V''} \arrow[u, dotted, "\psi_{V'}^{V''}"'] \arrow[r] & 0
\end{tikzcd}$$

shows that the existence of a dotted arrow $\psi_{V'}^{V''}$ is equivalent to the condition that the composite morphism $\varphi_{V'}\circ i_{V''}$ vanishes. Since the sheaf $\ker(\varphi_{V''})$ is again locally free, this is indeed a closed condition by Lemma 6.1, (ii). **q.e.d.**

By the usual abuse of notation we make no notational distinction between the functor B_V and the scheme representing it.

Our next goal is to construct the aforementioned natural stratification of B_V. Fix a filtration $\mathcal{F} = (V = V_0 \supsetneq \ldots \supsetneq V_{r-1} \supsetneq V_r = 0)$ of V.

We define a subfunctor $\mathcal{U}_\mathcal{F}$ of B_V by imposing the following condition on the set of T-valued points. For every inclusion of nonzero subspaces $0 \neq V'' \subset V'$ of V with the property that there exists no index i such that $V'' \subset V_i \subsetneq V'$, we require the morphism $\psi_{V'}^{V''}$ to be an isomorphism. Thus $\mathcal{U}_\mathcal{F}$ is an open subfunctor of B_V by Lemma 6.1, (iii), and representable by an open subscheme of B_V, which we denote by $\mathcal{U}_\mathcal{F}$ as well.

Similarly, we define a subfunctor $\mathcal{Z}_\mathcal{F}$ of B_V by imposing the following condition: For every inclusion of nonzero subspaces $0 \neq V'' \subset V'$ of V with the property that there exists an index i such that $V'' \subset V_i \subsetneq V'$, we require the morphism $\psi_{V'}^{V''}$ to be equal to zero. It follows from Lemma 6.1, (ii), that $\mathcal{Z}_\mathcal{F}$ is a closed subfunctor, and therefore representable by a closed subscheme of B_V, which we denote by $\mathcal{Z}_\mathcal{F}$ as well.

Finally, we define a subfunctor $\mathcal{S}_\mathcal{F}$ of B_V by imposing both conditions simultaneously: Given an inclusion of nonzero subspaces $0 \neq V'' \subset V'$ of V, we require that $\psi_{V'}^{V''}$ is equal to zero if there exists an index i such that $V'' \subset V_i \subsetneq V'$, and an isomorphism in all other cases. Thus $\mathcal{S}_\mathcal{F}$ is a locally closed subfunctor of B_V, and set-theoretically the equation

$$\mathcal{S}_\mathcal{F}(T) \;=\; \mathcal{U}_\mathcal{F}(T) \;\cap\; \mathcal{Z}_\mathcal{F}(T)$$

holds for every $\mathbb{F}_q$-scheme T. We denote the corresponding locally closed subscheme by $\mathcal{S}_\mathcal{F}$ as well.

Proposition 6.6 *Let $\mathcal{F} = (V = V_0 \supsetneq \ldots \supsetneq V_{r-1} \supsetneq V_r = 0)$ be a filtration of V. Then the scheme $\mathcal{Z}_\mathcal{F}$ decomposes as a product as follows:*

$$\mathcal{Z}_\mathcal{F} \;\cong\; B_{V_0/V_1} \times B_{V_1/V_2} \times \cdots \times B_{V_{r-1}/V_r}$$

Proof. We construct the isomorphism on the level of functors. Let T be an $\mathbb{F}_q$-scheme and let $(\mathcal{L}, \varphi, \psi)$ be a T-valued point of $\mathcal{Z}_\mathcal{F}$. For every integer $i = 0, \ldots, r-1$ we construct a T-valued point $(\mathcal{M}, \rho, \zeta)$ of $B_{V_i/V_{i+1}}$ as follows. Let W be a subspace of V such that $V_{i+1} \subsetneq W \subset V_i$. Set $\mathcal{M}_{W/V_{i+1}} := \mathcal{L}_W$. By definition of the subfunctor $\mathcal{Z}_\mathcal{F}$, the morphism $\psi_W^{V_{i+1}}$ is equal to zero. It follows from the compatibility of the maps φ and ψ in the definition of B_V that φ_W vanishes on the subsheaf $\mathcal{O}_T \otimes_{\mathbb{F}_q} V_{i+1} \subset \mathcal{O}_T \otimes_{\mathbb{F}_q} W$ and thus descends to a morphism

$$\rho_{W/V_{i+1}} :\; \mathcal{O}_T \otimes_{\mathbb{F}_q} W/V_{i+1} \twoheadrightarrow \mathcal{M}_{W/V_{i+1}}\;.$$

Finally, for an inclusion of subspaces $V_{i+1} \subsetneq U \subset W \subset V_i$, define $\zeta_{W/V_{i+1}}^{U/V_{i+1}} := \psi_W^U$. This construction yields a natural transformation of functors $\mu : \mathcal{Z}_\mathcal{F} \to B_{V_0/V_1} \times \cdots \times B_{V_{r-1}/V_r}$.

We now define a natural transformation in the converse direction. Assume that for every integer $i = 0, \ldots, r-1$ we are given a T-valued point $(\mathcal{M}(i), \rho(i), \zeta(i))$ of $B_{V_i/V_{i+1}}$. We construct a T-valued point $(\mathcal{L}, \varphi, \psi)$ of $\mathcal{Z}_\mathcal{F}$ as follows. Let W be a nonzero subspace of V, and let j be the unique integer such that $W \subset V_j$ and $W \not\subset V_{j+1}$. Set $\mathcal{L}_W := \mathcal{M}(j)_{(W+V_{j+1})/V_{j+1}}$ and define φ_W as the composite surjection

$$\varphi_W : \quad \mathcal{O}_T \otimes_{\mathbb{F}_q} W \;\longrightarrow\!\!\!\!\rightarrow\; \mathcal{O}_T \otimes_{\mathbb{F}_q} (W + V_{j+1})/V_{j+1} \;\longrightarrow\!\!\!\!\rightarrow\; \mathcal{M}(j)_{(W+V_{j+1})/V_{j+1}} \,,$$

where the second map is given by $\rho(j)_{(W+V_{j+1})/V_{j+1}}$.

Given an inclusion of nonzero subspaces $U \subset W$ of V, let k denote the unique integer such that $U \subset V_k$ and $U \not\subset V_{k+1}$, and let j denote the unique integer such that $W \subset V_j$ and $W \not\subset V_{j+1}$. If $k > j$, define the morphism ψ^U_W to be zero. If $k = j$, set $\psi^U_W := \zeta(j)^{(U+V_{j+1})/V_{j+1}}_{(W+V_{j+1})/V_{j+1}}$.

We now check that the triple $(\mathcal{L}, \varphi, \psi)$ indeed defines a T-valued point of $\mathcal{Z}_\mathcal{F}$. Let U, W, k, j be defined as in the last paragraph. We need to check that the restriction of φ_W to the subsheaf $\mathcal{O}_T \otimes_{\mathbb{F}_q} U \subset \mathcal{O}_T \otimes_{\mathbb{F}_q} W$ is equal to the composition $\psi^U_W \circ \varphi_U$. If $k = j$, this follows from the corresponding property of the triple $(\mathcal{M}(j), \rho(j), \zeta(j))$. If $k > j$ we observe that $U \subset V_{j+1} \subset W$. Thus both morphisms are equal to zero.

Therefore the triple $(\mathcal{L}, \varphi, \psi)$ defines a T-valued point of B_V. By definition of the morphisms ψ it is clear that $\psi^{V''}_{V'}$ is equal to zero whenever there exists an index i such that $V'' \subset V_i \subsetneq V'$. Hence the triple $(\mathcal{L}, \varphi, \psi)$ indeed defines a T-valued point of the subfunctor $\mathcal{Z}_\mathcal{F}$ of B_V. We have thus constructed a natural transformation $\nu : \; B_{V_0/V_1} \times \cdots \times B_{V_{r-1}/V_r} \to \mathcal{Z}_\mathcal{F}$ in the converse direction.

To complete the proof we need to show that the natural transformations μ and ν are inverse to each other. It follows directly from the construction that $\mu \circ \nu = \mathrm{id}$. We now prove that $\nu \circ \mu = \mathrm{id}$. Let T be an $\mathbb{F}_q$-scheme, let $(\mathcal{L}, \varphi, \psi)$ be a T-valued point of $\mathcal{Z}_\mathcal{F}$, and denote by $(\tilde{\mathcal{L}}, \tilde{\varphi}, \tilde{\psi})$ its image under the composition $\nu \circ \mu = \mathrm{id}$. We show that $(\mathcal{L}, \varphi, \psi)$ and $(\tilde{\mathcal{L}}, \tilde{\varphi}, \tilde{\psi})$ are equivalent triples. Fix a nonzero subspace W of V. We have to prove that $\varphi_W : \mathcal{O}_T \otimes_{\mathbb{F}_q} W \twoheadrightarrow \mathcal{L}_W$ and $\tilde{\varphi}_W : \mathcal{O}_T \otimes_{\mathbb{F}_q} W \twoheadrightarrow \tilde{\mathcal{L}}_W$ are equivalent quotients.

Let j denote the unique integer such that $W \subset V_j$ and $W \not\subset V_{j+1}$. By chasing through the construction of the natural transformations μ and ν, one verifies that $\tilde{\mathcal{L}}_W = \mathcal{L}_{W+V_{j+1}}$ and that $\tilde{\varphi}_W$ is equal to the composition

$$\tilde{\varphi}_W : \quad \mathcal{O}_T \otimes_{\mathbb{F}_q} W \;\hookrightarrow\; \mathcal{O}_T \otimes_{\mathbb{F}_q} (W + V_{j+1}) \;\overset{\varphi_{(W+V_{j+1})}}{\twoheadrightarrow}\; \mathcal{L}_{(W+V_{j+1})} \,.$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_T \otimes_{\mathbb{F}_q} (W + V_{j+1}) \arrow[r, two heads, "\varphi_{(W+V_{j+1})}"] & \mathcal{L}_{(W+V_{j+1})} \\
\mathcal{O}_T \otimes_{\mathbb{F}_q} W \arrow[u, hook] \arrow[ur, two heads, "\tilde{\varphi}_W"'] \arrow[r, two heads, "\varphi_W"'] & \mathcal{L}_W \arrow[u, "\psi^W_{W+V_{j+1}}"']
\end{tikzcd}$$

The commutativity of the lower triangle implies that $\psi^W_{W+V_{j+1}}$ is surjective and thus an isomorphism. Therefore the quotients $(\mathcal{L}_W, \varphi_W)$ and $(\tilde{\mathcal{L}}_W, \tilde{\varphi}_W)$ are equivalent. This concludes the proof of the proposition. **q.e.d.**

Proposition 6.7 *Let $(V \supsetneq 0)$ be the trivial filtration of V. Then $\mathcal{U}_{(V \supsetneq 0)} = \mathcal{S}_{(V \supsetneq 0)}$ and*

$$\mathcal{S}_{(V \supsetneq 0)} \;\cong\; \Omega_V \;.$$

Proof. The first statement is clear. In order to prove that $\mathcal{S}_{(V \supsetneq 0)} \cong \Omega_V$, we use the functorial interpretation of Ω_V obtained in Proposition 6.3 above. The functor $\mathcal{S}_{(V \supsetneq 0)}$ is the open subfunctor of B_V defined by the condition that all morphisms ψ are isomorphisms. Thus for an $\mathbb{F}_q$-scheme T, any triple $(\mathcal{L}, \varphi, \psi)$ in $\mathcal{S}_{(V \supsetneq 0)}(T)$ can be recovered up to equivalence of triples from the quotient $\varphi_V : \mathcal{O}_T \otimes_{\mathbb{F}_q} V \twoheadrightarrow \mathcal{L}_V$. Every such quotient φ_V satisfies the condition that for any nonzero vector $0 \neq v \in V$, the section $\varphi_V(1 \otimes v) \in \Gamma(T, \mathcal{L}_V)$ vanishes nowhere on T. Conversely, it is clear that every quotient φ_V satisfying this condition can be extended to a triple $(\mathcal{L}, \varphi, \psi)$ in $\mathcal{S}_{(V \supsetneq 0)}(T)$. **q.e.d.**

Corollary 6.8 *Let $\mathcal{F} = (V = V_0 \supsetneq \ldots \supsetneq V_{r-1} \supsetneq V_r = 0)$ be a filtration of V. Then the scheme $\mathcal{S}_\mathcal{F}$ decomposes as a product as follows:*

$$\mathcal{S}_\mathcal{F} \;\cong\; \Omega_{V_0/V_1} \times \Omega_{V_1/V_2} \times \cdots \times \Omega_{V_{r-1}/V_r}$$

Proof. Under the isomorphism of Proposition 6.6, the open subfunctor $\mathcal{S}_\mathcal{F}$ of $\mathcal{Z}_\mathcal{F}$ corresponds to the open subfunctor $S_{(V_0/V_1 \,\supsetneq\, 0)} \times \cdots \times S_{(V_{r-1}/V_r \,\supsetneq\, 0)}$ of $B_{V_0/V_1} \times \cdots \times B_{V_{r-1}/V_r}$. Then the statement follows from Proposition 6.7. **q.e.d.**

Theorem 6.9 (Stratification of B_V) *The underlying set of the scheme B_V is the disjoint union*

$$B_V \;=\; \dot{\bigcup_{\mathcal{F}}} \mathcal{S}_\mathcal{F} \,,$$

where the indexing set consists of all filtrations $\mathcal{F}$ of the vector space V.

Proof. We need to show that $B_V(K)$ is the disjoint union of the subsets $\mathcal{S}_\mathcal{F}(K)$ for every extension field K of $\mathbb{F}_q$. The disjointness is clear from the definition of the functors $\mathcal{S}_\mathcal{F}$. Let $(\mathcal{L}, \varphi, \psi)$ be a K-valued point of B_V. Since K is a field, we can assume that every invertible sheaf $\mathcal{L}$ of the triple $(\mathcal{L}, \varphi, \psi)$ is equal to K. Furthermore, every morphism ψ is either an isomorphism or equal to zero. We now provide an algorithm to construct a filtration $\mathcal{F}$ such that $(\mathcal{L}, \varphi, \psi)$ is an element of $\mathcal{S}_\mathcal{F}(K)$.

Set $V_0 := V$. If $\psi_V^{V'}$ is an isomorphism for all nonzero subspaces V' of V, define $V_1 := 0$. Thus $\mathcal{F}$ is the trivial filtration $(V \supsetneqq 0)$ in this case. If not all of the $\psi_V^{V'}$ are isomorphisms, let W be a nonzero subspace of V of maximal dimension such that $\psi_V^W = 0$. We claim that W is uniquely determined by this property.

More generally, we show that any nonzero subspace V' with the property that $\psi_V^{V'} = 0$ must already be contained in W: If V' is not contained in W, then W is a proper subspace of $W + V'$. Then it follows from the maximality of W that $\psi_{W+V'}^W = 0$ and $\psi_{W+V'}^{V'} = 0$. This in turn implies that $\varphi_{W+V'} = 0$, a contradiction. Thus V' must have been contained in W, and W is unique. Set $V_1 := W$.

We can now repeat the above step with V replaced by V_1. Iterating this procedure yields a filtration $\mathcal{F} = (V = V_0 \supsetneqq V_1 \supsetneqq \cdots \supsetneqq V_r = 0)$ of V with the property that $\psi_{V'}^{V''} = 0$ if and only if there exists an index i such that $V'' \subset V_i \subsetneq V'$. Thus the triple $(\mathcal{L}, \varphi, \psi)$ is an element of $\mathcal{S}_{\mathcal{F}}(K)$. **q.e.d.**

In analogy to sections 4 and 5 we call the locally closed subschemes $\mathcal{S}_{\mathcal{F}}$ *strata* and their open neighborhoods $\mathcal{U}_{\mathcal{F}}$ *strata neighborhoods*. Theorem 6.9 above shows that the $\mathcal{U}_{\mathcal{F}}$ form an open cover of B_V.

As a corollary of Proposition 6.6 and Theorem 6.9, we give a description of the set of K-valued points of B_V.

Corollary 6.10 *Let K be an extension field of $\mathbb{F}_q$. Then there exists a natural bijection between the set $B_V(K)$ of K-valued points of B_V and the set of pairs $(\mathcal{F}, x)$ consisting of a filtration*

$$\mathcal{F} \;=\; (V = V_0 \supsetneqq V_1 \supsetneqq \ldots \supsetneqq V_{r-1} \supsetneqq V_r = 0)$$

and an element

$$x \;\in\; \Omega_{V_0/V_1}(K) \times \ldots \times \Omega_{V_{r-1}/V_r}(K).$$

Let $\mathcal{F}$ and $\mathcal{F}'$ be filtrations of V. We define $\mathcal{F} \cap \mathcal{F}'$ to be the filtration of V which consists of precisely those subspaces which occur in both $\mathcal{F}$ and $\mathcal{F}'$. We use the notation $\mathcal{F}' \subset \mathcal{F}$ to indicate that $\mathcal{F}'$ can be obtained from $\mathcal{F}$ by deleting some of the filtration steps. The following statements follow directly from Theorem 6.9 above.

Remark 6.11

(i)
$$\mathcal{U}_{\mathcal{F}} \;=\; \dot{\bigcup_{\mathcal{F}' \subset \mathcal{F}}} \mathcal{S}_{\mathcal{F}'}$$

(ii)
$$\mathcal{Z}_{\mathcal{F}} \;=\; \dot{\bigcup_{\mathcal{F} \subset \mathcal{F}'}} \mathcal{S}_{\mathcal{F}'}$$

(*iii*)
$$\mathcal{F}' \subset \mathcal{F} \iff \mathcal{U}_{\mathcal{F}'} \subset \mathcal{U}_{\mathcal{F}}$$

(*iv*)
$$\mathcal{U}_{\mathcal{F}\cap\mathcal{F}'} = \mathcal{U}_{\mathcal{F}} \cap \mathcal{U}_{\mathcal{F}'}$$

Our next goal is to show that B_V is a smooth projective variety and that the boundary $B_V \smallsetminus \Omega_V$ is a divisor with normal crossings in the sense that it is Zariski-locally isomorphic to the embedding of a union of coordinate planes into affine space.

Fix a filtration $\mathcal{F} = (V = V_0 \supsetneqq \ldots \supsetneqq V_r = 0)$ of V. For every integer $i = 1, \ldots, r-1$ we fix a subspace $W_i \subset V_{i-1}$ such that $V_{i-1} = V_i \oplus W_i$, together with a nonzero vector w_i in W_i. In addition we fix a nonzero vector w_r in V_{r-1}.

Lemma 6.12 *The functor $\mathcal{U}_{\mathcal{F}}$ is isomorphic to the functor that associates to an $\mathbb{F}_q$-scheme T the set of commutative diagrams of the form*

$$\begin{tikzcd}
\mathcal{O}_T \otimes_{\mathbb{F}_q} V_0 \arrow[r, two heads, "\varphi_{V_0}"] & \mathcal{O}_T \\
\mathcal{O}_T \otimes_{\mathbb{F}_q} V_1 \arrow[r, two heads, "\varphi_{V_1}"] \arrow[u, hook] & \mathcal{O}_T \arrow[u, "\psi_{V_0}^{V_1}"'] \\
\mathcal{O}_T \otimes_{\mathbb{F}_q} V_2 \arrow[r, two heads, "\varphi_{V_2}"] \arrow[u, hook] & \mathcal{O}_T \arrow[u, "\psi_{V_1}^{V_2}"'] \\
\mathcal{O}_T \otimes_{\mathbb{F}_q} V_{r-1} \arrow[r, two heads, "\varphi_{V_{r-1}}"] \arrow[u, dashed] & \mathcal{O}_T \arrow[u, dashed]
\end{tikzcd}$$

with the following properties:

(i) For every integer $i = 0, \ldots, r-1$ and every vector $v \in V_i \smallsetminus V_{i+1}$ the section $\varphi_{V_i}(1 \otimes v)$ in $\Gamma(T, \mathcal{O}_T)$ vanishes nowhere on T.

(ii) For every integer $i = 1, \ldots, r$, the section $\varphi_{V_{i-1}}(w_i)$ is equal to 1 in $\Gamma(T, \mathcal{O}_T)$.

Proof. Let T be an $\mathbb{F}_q$-scheme and let $(\mathcal{L}, \varphi, \psi)$ be a T-valued point of $\mathcal{U}_{\mathcal{F}}$. We first show that for every $i = 0, \ldots, r-1$ the invertible sheaf $\mathcal{L}_{V_i}$ is trivial. Choose a vector $v \in V_i \smallsetminus V_{i+1}$ and let W denote the $\mathbb{F}_q$-span of v. Then since W is one-dimensional, the corresponding invertible sheaf $\mathcal{L}_W$ is trivialized by the map φ_W. By the definition of $\mathcal{U}_{\mathcal{F}}$, the morphism $\psi_{V_i}^{W}$ is an isomorphism. Thus $\mathcal{L}_{V_i}$ is trivial as well. By using the equivalence relation on the set of triples $(\mathcal{L}, \varphi, \psi)$ we can assume that $\mathcal{L}_{V_i} = \mathcal{O}_T$.

It follows directly from the definition of $\mathcal{U}_{\mathcal{F}}$ that a triple $(\mathcal{L}, \varphi, \psi)$ in $\mathcal{U}_{\mathcal{F}}(T)$ can be reconstructed up to equivalence of triples from the subdiagram $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ pictured above, and that every such subdiagram satisfies property (i). Conversely, every diagram $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ satisfying property (i) can be extended to a triple $(\mathcal{L}, \varphi, \psi)$ in $\mathcal{U}_{\mathcal{F}}(T)$. Thus $\mathcal{U}_{\mathcal{F}}$ is isomorphic to the functor that associates to an $\mathbb{F}_q$-scheme T the set of equivalence classes of diagrams $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ satisfying property (i). Additionally requiring property (ii) above is then equivalent to the choice of a representative for each equivalence class of diagrams.

q.e.d.

We use the description of $\mathcal{U}_{\mathcal{F}}$ obtained in Lemma 6.12 in the following definition and in Proposition 6.13 below.

Define a natural transformation

$$\tau : \mathcal{U}_{\mathcal{F}} \longrightarrow \mathbb{A}_{\mathbb{F}_q}^{r-1} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$$

as follows. Let T be an $\mathbb{F}_q$-scheme and let $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ be a T-valued point of $\mathcal{U}_{\mathcal{F}}$. Then for every $i = 0, \ldots, r-2$ the morphism $\psi_{V_i}^{V_{i+1}} : \mathcal{O}_T \to \mathcal{O}_T$ yields a T-valued point of $\mathbb{A}_{\mathbb{F}_q}^1$ since

$$\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T, \mathcal{O}_T) \;\cong\; \Gamma(T, \mathcal{O}_T) \;\cong\; \mathbb{A}_{\mathbb{F}_q}^1(T).$$

Furthermore, it follows from property (i) in Lemma 6.12 above that for every $i = 0, \ldots, r-2$ the surjection

$$\varphi_{V_i}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{i+1}} : \; \mathcal{O}_T \otimes_{\mathbb{F}_q} W_{i+1} \twoheadrightarrow \mathcal{O}_T$$

yields a T-valued point of $\Omega_{W_{i+1}}$.

Thus we define the image of $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ under τ to be the T-valued point

$$\left(\psi_{V_0}^{V_1}, \; \ldots \; , \psi_{V_{r-2}}^{V_{r-1}}, \; \varphi_{V_0}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_1}, \; \ldots \; , \; \varphi_{V_{r-2}}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{r-2}}, \; \varphi_{V_{r-1}}\right)$$

of the product scheme $\mathbb{A}_{\mathbb{F}_q}^{r-1} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$.

Proposition 6.13 *The natural transformation*

$$\tau : \mathcal{U}_{\mathcal{F}} \longrightarrow \mathbb{A}_{\mathbb{F}_q}^{r-1} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$$

is injective. Furthermore, the image of τ is an open subfunctor of the product functor $\mathbb{A}_{\mathbb{F}_q}^{r-1} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$. In particular, the scheme $\mathcal{U}_{\mathcal{F}}$ is isomorphic to an open subscheme of affine space $\mathbb{A}_{\mathbb{F}_q}^{d-1}$.

Proof. We first prove that τ is injective. Let T be an $\mathbb{F}_q$-scheme and let $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ be a T-valued point of $\mathcal{U}_{\mathcal{F}}$. It suffices to show that every morphisms φ_{V_j} is uniquely determined by the image of $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ under τ. We proceed by downwards induction on j. The statement is clear for $j = r-1$. For arbitrary j, the morphism φ_{V_j} can be reconstructed from the composition $\psi_{V_j}^{V_{j+1}} \circ \varphi_{V_{j+1}}$ and the restriction $\varphi_{V_j}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{j+1}}$ as follows. Given a nonzero vector $v \in V_j$, there exist unique vectors $v_{j+1} \in V_{j+1}$ and $u_{j+1} \in W_{j+1}$ such that $v = v_{j+1} + u_{j+1}$. Then it follows from the commutativity of the diagram $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ that

$$(*) \qquad \varphi_{V_j}(1 \otimes v) \;=\; \left(\psi_{V_j}^{V_{j+1}} \circ \varphi_{V_{j+1}}\right)(1 \otimes v_{j+1}) \;+\; \varphi_{V_j}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{j+1}}(1 \otimes u_{j+1}).$$

Thus by induction we conclude that φ_{V_j} is uniquely determined by the image of $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ under τ. This shows that τ is injective.

We now determine the image of τ. From equation $(*)$ above we see that every T-valued point of $\mathbb{A}^{r-1}_{\mathbb{F}_q} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$ gives rise to a commutative diagram $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$. However, the morphisms φ_{V_i} might lack property (i) of Lemma 6.12, so that the diagram $(\varphi_{V_i}, \psi_{V_i}^{V_{i+1}})$ constructed in this way is not necessarily a T-valued point of $\mathcal{U}_{\mathcal{F}}$. Thus the image of τ is the subfunctor of $\mathbb{A}^{r-1}_{\mathbb{F}_q} \times \Omega_{W_1} \times \Omega_{W_2} \times \cdots \times \Omega_{W_{r-1}} \times \Omega_{V_{r-1}}$ defined by requiring that every morphism φ_{V_i} constructed inductively via equation $(*)$ possesses property (i).

More explicitly, the image of τ is the subfunctor defined by the following condition: Given any integer $j = 0, \ldots, r-2$ and any collection of vectors $v_{r-1} \in V_{r-1}$, $u_{r-1} \in W_{r-1}$, $u_{r-2} \in W_{r-2}$, $\ldots$, $u_{j+1} \in W_{j+1}$, not all equal to zero, we require that the global section

$$\begin{aligned}
&\varphi_{V_j}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{j+1}}(1 \otimes u_{j+1}) \\
+\;& \psi_{V_j}^{V_{j+1}} \circ \varphi_{V_{j+1}}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{j+2}}(1 \otimes u_{j+2}) \\
+\;& \psi_{V_j}^{V_{j+1}} \circ \psi_{V_{j+1}}^{V_{j+2}} \circ \varphi_{V_{j+2}}|_{\mathcal{O}_T \otimes_{\mathbb{F}_q} W_{j+3}}(1 \otimes u_{j+3}) \\
&\vdots \\
+\;& \psi_{V_j}^{V_{j+1}} \circ \ldots \circ \psi_{V_{r-2}}^{V_{r-1}} \circ \varphi_{V_{r-1}}(1 \otimes v_{r-1})
\end{aligned}$$

in $\Gamma(T, \mathcal{O}_T)$ vanishes nowhere on T.
This condition is an open condition by Proposition 6.1. **q.e.d.**

Corollary 6.14 *The scheme B_V is a smooth projective variety.*

Proof. In Proposition 6.5 we have already shown that B_V is projective. Proposition 6.13 above implies that every open subscheme $\mathcal{U}_{\mathcal{F}}$ of B_V is irreducible. Furthermore, every $\mathcal{U}_{\mathcal{F}}$ contains the open stratum $\mathcal{S}_{(V \supsetneq 0)} \cong \Omega_V$. Therefore the fact that the $\mathcal{U}_{\mathcal{F}}$ form a cover of B_V implies that Ω_V is dense in B_V, and therefore B_V is irreducible. The smoothness of B_V follows from Proposition 6.13 and the fact that the open subschemes $\mathcal{U}_{\mathcal{F}}$ cover B_V. **q.e.d.**

Corollary 6.15 *The closure of a stratum $\mathcal{S}_\mathcal{F}$ in B_V is again a union of strata and carries a natural subscheme structure:*

$$\overline{\mathcal{S}_\mathcal{F}} \;=\; \dot{\bigcup_{\mathcal{F}\subset\mathcal{F}'}} \mathcal{S}_{\mathcal{F}'} \;=\; \mathcal{Z}_\mathcal{F}$$

Proof. Proposition 6.6 and Proposition 6.14 together imply that the scheme $\mathcal{Z}_\mathcal{F}$ is irreducible. Thus the non-empty open subscheme $\mathcal{S}_\mathcal{F}$ of $\mathcal{Z}_\mathcal{F}$ is dense in $\mathcal{Z}_\mathcal{F}$, and the claim follows from Remark 6.11, (*ii*). **q.e.d.**

Corollary 6.16 *The boundary $B_V \setminus \Omega_V \subset B_V$ is a divisor with normal crossings in the sense that it is Zariski-locally isomorphic to the embedding of a union of coordinate planes into affine space.*

Proof. It suffices to verify the claim on every open subscheme $\mathcal{U}_\mathcal{F}$ of B_V. We use the characterization of $\mathcal{U}_\mathcal{F}$ in Lemma 6.12. The boundary $\mathcal{U}_\mathcal{F} \setminus \Omega_V$ represents the closed subfunctor of $\mathcal{U}_\mathcal{F}$ defined by the condition that at least one of the morphisms $\psi_{V_i}^{V_{i+1}}$ is equal to zero. Thus it is clear from the definition of the natural transformation τ above that the embedding of $\mathcal{U}_\mathcal{F} \setminus \Omega_V$ into $\mathcal{U}_\mathcal{F}$ is isomorphic to an embedding of a union of coordinate planes into an open subset of affine space $\mathbb{A}^{d-1}_{\mathbb{F}_q}$. **q.e.d.**

Finally, we construct morphisms from B_V to P_V and Q_V. Define a morphism from B_V to P_V on the level of functors by mapping a triple $(\mathcal{L}, \varphi, \psi)$ to the quotient $\varphi_V : \mathcal{O}_T \otimes_{\mathbb{F}_q} V \twoheadrightarrow \mathcal{L}_V$. It is clear from Proposition 6.3 that this morphism induces an isomorphism on Ω_V.

Similarly, we use the functorial description of Q_V in Proposition 6.4 to define a morphism from B_V to Q_V. Given an $\mathbb{F}_q$-scheme T and a triple $(\mathcal{L}, \varphi, \psi)$ in $B_V(T)$, construct a pair $(\mathcal{G}, \lambda)$ in $Q_V(T)$ as follows.

The collection of invertible sheaves $(\mathcal{L}_{V'})_{0\neq V'\subset V}$ forms an inverse system via the morphisms ψ. Thus there exists the inverse limit sheaf $\varprojlim \mathcal{L}_{V'}$ on T. In the special case that the triple $(\mathcal{L}, \varphi, \psi)$ lies in $\mathcal{U}_\mathcal{F}(T)$, it is clear that $\varprojlim \mathcal{L}_{V'}$ is again invertible. In the general case, we conclude that $\varprojlim \mathcal{L}_{V'}$ is locally isomorphic to an invertible sheaf (and thus itself invertible) since the open subschemes $\mathcal{U}_\mathcal{F}$ cover B_V. We define $\mathcal{G}$ as the dual $\big(\varprojlim \mathcal{L}_{V'}\big)^\vee := \mathcal{H}om_{\mathcal{O}_T}\big(\varprojlim \mathcal{L}_{V'}, \mathcal{O}_T\big)$ of $\varprojlim \mathcal{L}_{V'}$.

Let v be a nonzero vector in V and denote by $\mathbb{F}_q v$ the one-dimensional subspace spanned by v. We define the global section $\lambda(v)$ of $\mathcal{G}$ as the composition

$$\lambda(v) :\; \varprojlim \mathcal{L}_{V'} \;\longrightarrow\; \mathcal{L}_{\mathbb{F}_q v} \;\xrightarrow{\varphi_{\mathbb{F}_q v}^{-1}}\; \mathcal{O}_T \otimes_{\mathbb{F}_q} \mathbb{F}_q v \;\xrightarrow{\cong}\; \mathcal{O}_T.$$

Then it follows directly from the construction that $\lambda(v)$ satisfies properties (*ii*) and (*iii*) of Proposition 6.4, and that the collection of global sections $(\lambda(v))_{0\neq v\in V}$ generates $\mathcal{G}$. We have thus defined a morphism from B_V to Q_V. The description of Ω_V as a subfunctor of B_V in Proposition 6.7 and as a subfunctor of Q_V in Proposition 6.4 implies that this morphism induces an isomorphism on Ω_V. Since both B_V and Q_V are projective schemes over $\mathbb{F}_q$, this morphism is projective as well.

Corollary 6.17 *The projective variety B_V is a desingularization of Q_V.*

A classical theorem of surface theory (see for example [5], chapter V, Corollary 5.4) states that every birational morphism of nonsingular projective surfaces can be factored into finitely many monoidal transformations. From this theorem one can easily deduce that if $d = 3$, the desingularization B_V coincides with the blowups $\widetilde{P_V} = \widetilde{Q_V}$ constructed in section 5.

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