Morse Theory and its Applications

A thesis submitted to attain the degree of

MASTER OF SCIENCES

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presented by

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supervised by

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Abstract

In this thesis, I explain the basics of Morse theory, a mathematical field in differential topology. Through Morse theory we can understand the shapes of manifolds using smooth functions. I start with the basic definitions and build up the differential geometry background needed to define Morse functions. One important result is the Morse lemma, which shows how a Morse function can be expressed using its second derivative. Additionally, I demonstrate how a Morse function captures the CW complex structure of a manifold. I also explore how, in certain conditions, Morse functions define chain complexes, whose homology groups are isomorphic to cellular homology. Finally, I discuss an application of Morse theory to mesh parametrizations in computer science.

Acknowledgements

I would like to show my appreciation to my supervisor Dr. Sara Kališnik Hintz for suggesting this interesting topic and for providing guidance and helpful feedback at every stage of the thesis.

Contents

Chapter 1

Introduction

A *manifold* is a topological space, which locally greatly resembles Euclidean space. Concretely, this means that every point in the manifold admits an open neighborhood U and a homeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$, which we call *chart*.

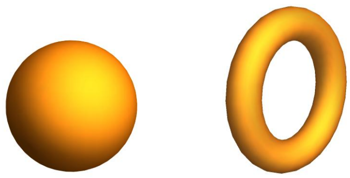
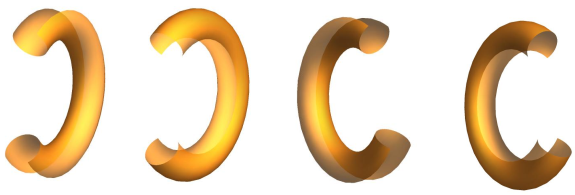


Figure 1.1: The sphere and the torus are manifolds.

The torus can be described with the following 4 charts, each homeomorphic to $\mathbb{R}^2$, which are illustrated in Figure 1.2.



(a) A_1 (b) A_2 (c) A_3 (d) A_4

Figure 1.2: Visualization of 4 charts on a torus.

Since manifolds locally behave like Euclidean spaces, we can equip them with a differentiable structure, which permits us to perform calculus on maps between two manifolds. In that context, we expand the concept of *total derivatives* and *directional derivatives*. To do that, we attach to every point a space called *tangent space*, which captures all possible directions in which one can tangentially

move through the point. The elements of this space are called *tangent vectors*. For a function defined on a manifold, the tangent vectors can be used to measure the rate at which the function value changes when moving in a certain direction. If this measurement exists for every point and direction, we say that the function is *differentiable*. This material is covered in depth in Chapter 2.
Morse theory is a mathematical branch within differential topology, which explores the topology of manifolds with the help of differentiable functions. This is done by analyzing the *critical points* of a differentiable function, which are per definition the points where the derivative vanishes, so the points where the functions behavior changes. We associate to every critical point the *Hessian matrix*, which contains information about how the function behaves around the critical point. If the Hessian matrix is invertible, the critical point is called *nondegenerate*. The nondegeneracy of the critical points is crucial as it allows us to classify the critical points as local maxima, local minima or saddle points. The number of negative eigenvalues of the Hessian matrix is called *index*. Intuitively, it describes the number of linear independent directions in which the function decreases. The height function on the torus is a Morse function with one maximum, one minimum and two saddle points. The fundamental concepts and definitions from the theory of Morse functions will be discussed in Chapter 3.

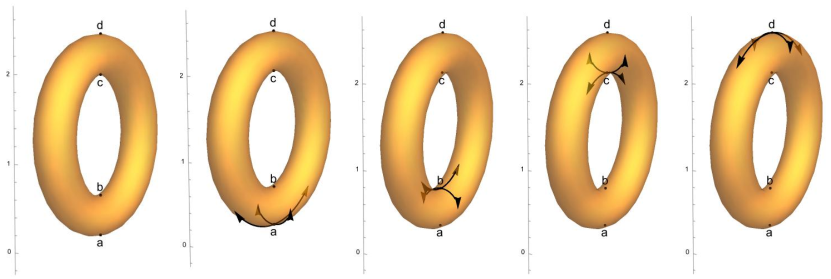


Figure 1.3: Height function on the standing torus with 4 critical points and visualization of the *index* of the 4 critical points.

In Chapter 4, we discuss *pseudo-gradient fields* and their properties. A pseudo-gradient field is a generalization of the gradient known from calculus in $\mathbb{R}^n$ in the sense that near critical points it coincides with the gradient and away from critical points it points 'approximately' in the same direction as the gradient. Each pseudo-gradient defines flow lines, which are curves that follow the pseudo-gradient. One example of a flow line is in Figure 1.4 on the left, the red curve which emerges from c and goes to b. For every critical point p we can define the *stable*, respectively *unstable* manifold, which consists of all points whose flow lines end up in p, respectively emerge, from p.

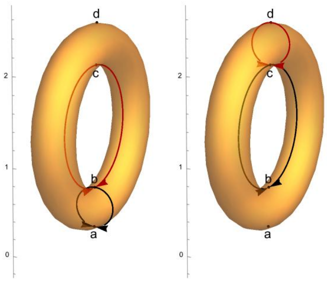


Figure 1.4: Stable manifolds (red) and unstable manifolds (black) of the critical points b and c of the height function on the standing torus.

The unstable manifolds determine the topology of the manifold. Concretely, we examine the *sublevel sets* $V^r = f^{-1}((-\infty, r])$, $r \in \mathbb{R}$. It turns out that the topology of the sublevel sets does not change except when we cross a *critical value* $r \in \mathbb{R}$, which is per definition the image of a critical point p. At this point, a cell (corresponding to the unstable manifold of p) of dimension $\text{Ind}(p)$ is attached to V^r.

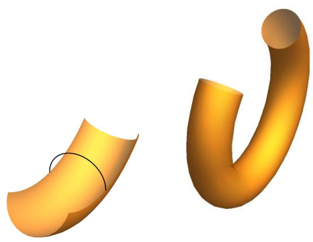


Figure 1.5: A cell of dimension 1 is attached to $V^{f(b)}$, which has the same homotopy type as the torus with a piece of the tube cut out.

If the flow lines of all the stable and unstable manifolds meet *transversally* according to a pseudo-gradient field X, we say that the pseudo-gradient field satisfies the *Smale condition*. Intuitively, two manifolds intersect transversally, when they are not 'parallel' at their intersection. In that case, we associate to the Morse function a *chain complex* and calculate its *homology groups*. In addition, the Morse function induces a well-defined cellular decomposition, called *Morse-Smale complex*, of the manifold by intersecting all the stable and unstable manifolds. The Morse homology groups allow us to gain further insights into the geometry of the manifold, such as the number of connected components. We delve into this topic in Chapter 5.

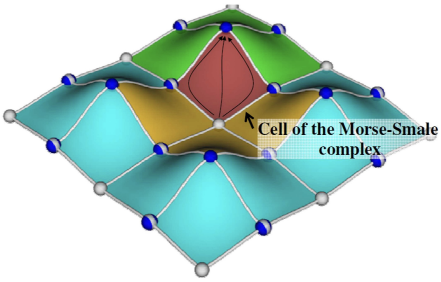


Figure 1.6: Part the of Morse-Smale complex of the height function of a surface. [Sma]

Morse-Smale complexes have various practical applications in several fields of applied mathematics and computer science, one of which is *mesh parametrization*. A *mesh* is a discretization of a geometric domain into small simple shapes, such as triangles or quadrilaterals in two dimensions and tetrahedra or hexahedra in three. We recommend [MB] as additional reading material on meshes.

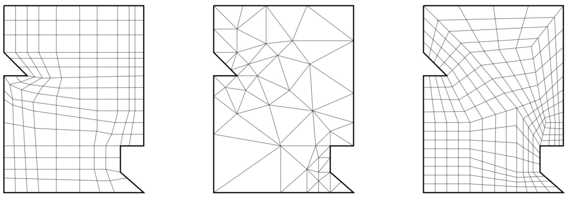
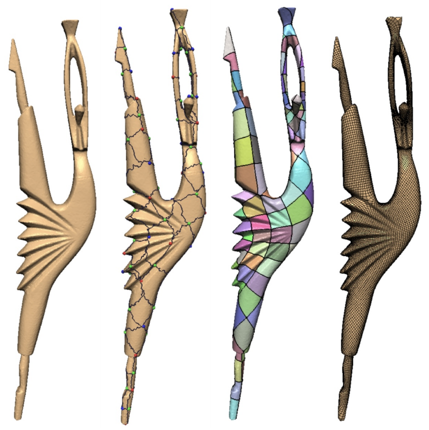


Figure 1.7: Three sample meshes. [MB]

Surface remeshing is a process that optimizes a mesh to improve its geometrical properties in order to obtain a regular grid. Raw surface input meshes are often generated from laser scanning, isosurface extraction or other methods which often suffer from irregularities and noise. In Chapter 6, we focus on a particular remeshing process called *Spectral Surface Quadrangulation* [Don+06]. The term *spectral* refers to the study of *eigenfunctions* of operators, which play a crucial role in this particular remeshing process. The idea is to examine a specific *Laplacian eigenfunction* of the input mesh, which benefits in general from well-spaced critical points over the surface. An iterative relaxation algorithm is then used to simultaneously improve the Morse-Smale complex while computing a globally smooth parametrization. This parametrization is used to generate the final semi-regular grid of well-shaped quadrilaterals.



(a) (b) (c) (d)

Figure 1.8: (a): Surface, (b): Morse-Smale complex, (c): Improved Morse-Smale complex, (d): Final semi-regular grid. [Don+06]

Chapter 2

Differentiable Manifolds

In this chapter we give a introduction to *manifolds.* Intuitively, one might think of a manifold as a topological space, which locally resembles Euclidean space. Subspaces of manifolds are called *submanifolds.* We introduce the main definitions and we have a look at how to endow manifolds with an additional structure, which allows us to perform calculus on maps between manifolds. All the definitions and propositions in this section stem from [ADE13],[Ser24] and [Zü20].

2.1 Manifolds, Submanifolds and Maps

Definition 2.1 (**N-dimensional topological Manifold, Chart, and Atlas).**

1. *An* n**-dimensional topological manifold** V *is a separable topological space such that for every point* $p \in V$ *there exists an open neighborhood* $U \subset V$ *of* p *and a homeomorphism*

$$\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$$

   In other words, every point is contained in an open subset homeomorphic to an open subset of $\mathbb{R}^n$.

2. *A pair* (U, φ)*, where* U *is an open subset of* V *containing* $x \in V$ *and* $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$ *is a homeomorphism, is a* **chart***.*

3. *A system of charts* $\phi = \{(\varphi_\alpha, U_\alpha)\}_{\alpha\in I}$*, where* I *is any index set, forms an* **atlas** *of the topological manifold* V *if* $\bigcup_{\alpha\in I} U_\alpha = V$.

The main example in this thesis is going to be the standing torus, which is a 2-dimensional manifold.

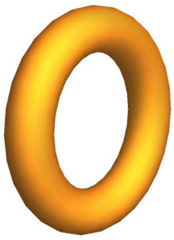


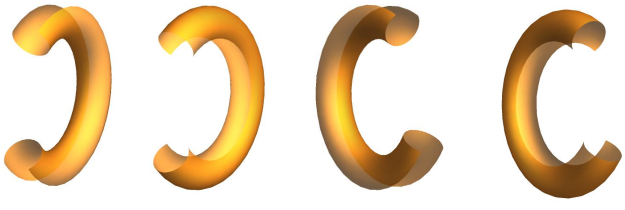
Figure 2.1: Standing torus.

Example 2.1. *The standing torus is a 2-dimensional manifold. Concretly, T consists of the points (x, y, z) so that*

$$x = R_2 \sin(\phi)$$
$$y = (R_1 + R_2 \cos(\phi)) \cos(\varphi)$$
$$z = (R_1 + R_2 \cos(\phi)) \sin(\varphi) + R_1 + R_2$$

with $\phi, \varphi \in [0, 2\pi)$ and $0 < R_2 < R_1$, where R_1 is the inner radius and R_2 is the radius of the outer tube. The angle φ represents the rotation of the torus' axis of revolution and ϕ the rotation around the tube. We can use the parametrization above to construct a concrete atlas. Note that the set $[0, 2\pi) \times [0, 2\pi)$ is not open in $\mathbb{R}^2$, so we need at least 2 charts. Making use of the fact that $\sin$ and $\cos$ are both 2π periodic functions, we use the parametrization from above to define the following 4 charts, which form indeed an atlas:

1. $\psi_1 : (-\frac{3\pi}{4}, \frac{3\pi}{4}) \times (\frac{\pi}{4}, \frac{7\pi}{4}) \to A_1 \subset \mathbb{R}^3$
   $(\varphi, \phi) \mapsto (R_2 \sin(\phi), (R_1 + R_2 \cos(\phi)) \cos(\varphi), (R_1 + R_2 \cos(\phi)) \sin(\varphi) + R_1 + R_2)$,
2. $\psi_2 : (-\frac{3\pi}{4}, \frac{3\pi}{4}) \times (-\frac{3\pi}{4}, \frac{3\pi}{4}) \to A_2 \subset \mathbb{R}^3$
   $(\varphi, \phi) \mapsto (R_2 \sin(\phi), (R_1 + R_2 \cos(\phi)) \cos(\varphi), (R_1 + R_2 \cos(\phi)) \sin(\varphi) + R_1 + R_2)$,
3. $\psi_3 : (\frac{\pi}{4}, \frac{7\pi}{4}) \times (\frac{\pi}{4}, \frac{7\pi}{4}) \to A_3 \subset \mathbb{R}^3$
   $(\varphi, \phi) \mapsto (R_2 \sin(\phi), (R_1 + R_2 \cos(\phi)) \cos(\varphi), (R_1 + R_2 \cos(\phi)) \sin(\varphi) + R_1 + R_2)$,
4. $\psi_4 : (\frac{\pi}{4}, \frac{7\pi}{4}) \times (-\frac{3\pi}{4}, \frac{3\pi}{4}) \to A_4 \subset \mathbb{R}^3$
   $(\varphi, \phi) \mapsto (R_2 \sin(\phi), (R_1 + R_2 \cos(\phi)) \cos(\varphi), (R_1 + R_2 \cos(\phi)) \sin(\varphi) + R_1 + R_2)$.



(a) A_1 (b) A_2 (c) A_3 (d) A_4

Figure 2.2: Visualization of the 4 charts.

It is also possible to define manifolds with *boundary.*

Definition 2.2 (**Manifold with Boundary**). *An $n-$***dimensional manifold with boundary** *V is a separable topological space such that for every point $p \in V$ there exists an open neighborhood $U \subset V$ of p and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$ or a homeomorphism $\varphi\colon U \to \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.*

Remark 2.1. *The interior of an n-dimensional manifold with boundary is an n-dimensional manifold without boundary and the boundary is an $n-1$-dimensional manifold without boundary.*

Example 2.2. *A disc is a 2-dimensional manifold with boundary. A ball (sphere plus interior) is a 3-dimensional manifold with boundary.*

As already mentioned, manifolds can be equipped with an additional structure. For us, the most interesting class will be the class of the differentiable manifolds, for which we can in some sense apply the usual concepts of calculus. First, recall the definition of a diffeomorphism in $\mathbb{R}^n$ for $n \geq 0$:

Definition 2.3 (**C^r-Diffeomorphism**). *A map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ with $n, m \geq 0$ is a* **C^r-diffeomorphism** *if*

1. *f is bijective.*
2. *f and f^{-1} are r times continuously differentiable.*

Example 2.3. *Consider the map*

$$\begin{aligned} f\colon \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} x+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

As f is a bijective linear map, it is clearly a C^r- diffeomorphism for every $r \in \mathbb{N} \cup \{\infty\}$. In particular, every coordinate change is a C^r-diffeomorphism for every $r \in \mathbb{N} \cup \{\infty\}$.

Example 2.4. *The map*

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$

is a C^r-diffeomorphism for every $r \in \mathbb{N} \cup \{\infty\}$.

After having established the notion of diffeomorphisms, we are now able to endow a manifold V with a globally defined differentiable structure.

Definition 2.4 (**Transition Map, and C^r-Atlas**). *Let I be any index set.*

1. *For $\alpha, \beta \in I$, the homeomorphism*

$$\varphi_{\beta\alpha}\colon = \varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is called the **transition map** *between φ_α and φ_β.*

2. *For $1 \leq r \leq \infty$, we say that the atlas $\{\varphi_\alpha\}_{\alpha \in I}$ is a C^r-**atlas** of V if every transition map $\varphi_{\beta\alpha}$ is a C^r map. In particular, since $(\varphi_{\beta\alpha})^{-1} = \varphi_{\alpha\beta}$, it follows then that every transition map is a C^r-diffeomorphism.*

The *transition maps* provide a way of comparing two charts of an atlas. In the anticipation of constructing r-differentiable maps between two manifolds, we require the transition maps to be r-differentiable.

Definition 2.5 (**Maximal differentiable Structure, and differentiable Manifold of Class** C^r)**.** *For $1 \leq r \leq \infty$:*

1. *A* **differentiable structure of class** C^r *on a topological manifold is maximal C^r atlas, that is, a C^r atlas not contained in a bigger one.*

2. *A* **differentiable manifold of class** C^r *or a C^r* **manifold** *is a topological manifold equipped with a C^r structure. In particular, a* smooth manifold *is a C^∞-manifold.*

The idea behind the concept of a *maximal atlas* is that it is in fact unique. Basically, the C^r-manifolds are the manifolds for which it is possible to define C^r maps between manifolds.

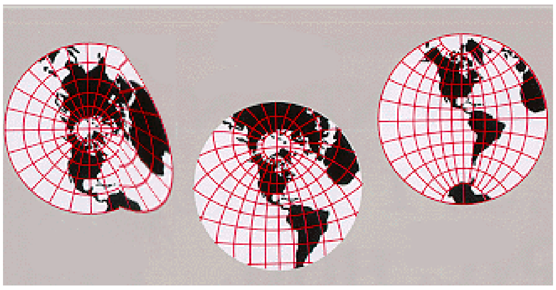


Figure 2.3: Example of a non differentiable atlas of charts for the globe. [Wik23a]

We will now have a look at maps between manifolds. The main goal is to extend the concept of *differentiable maps* in $\mathbb{R}^n$ to manifolds.

Definition 2.6 (C^r-**map between Manifolds**)**.** *Let V, W be C^r-manifolds where $1 \leq r \leq \infty$. A* **map** *$f: V \to W$ is called* $\mathbf{C^r}$*, if for every point $p \in V$, there exist a chart (φ, U) of V with $p \in U$ and a chart $(\phi, \tilde{U})$ of W with $f(U) \subset \tilde{U}$ such that the map*

$$\phi \circ f \circ \varphi^{-1} : \varphi(U) \to \tilde{U}$$

is C^r.

Remark 2.2. *For this definition to make sense we really need the manifolds to be of class C^r. Indeed, let (φ_α, U) and (φ_β, U') be two charts of V with $p \in U$ and let $(\phi_\alpha, \tilde{U})$ and $(\phi_\beta, \tilde{U}')$ let be two charts of W, which contain $f(U)$ respectively $f(\tilde{U}')$. Then we have that:*

$$\phi_\alpha \circ f \circ \varphi_\alpha = \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\phi_{\alpha\beta}} \circ \phi_\beta \circ f \circ \varphi_\beta^{-1} \circ \underbrace{\varphi_\beta \circ \varphi_\alpha^{-1}}_{\varphi_{\beta\alpha}}$$

As by assumption $\phi_\alpha \circ f \circ \varphi_\alpha$ is r times differentiable, it must hold that $\phi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$ are r times differentiable, i.e. V and W have to be C^r-manifolds.

Remark 2.3. *C^r-manifolds with C^r-maps form a category.*

2.2 Tangent Vectors and Tangent Maps

Having defined differentiable maps between manifolds, we are now able to expand the concept of *total derivatives* and *directional derivatives* in $\mathbb{R}^n$ to those maps. We start by introducing the concept of the *tangent space*. Intuitively, for a differentiable manifold V, we want to attach to every point $x \in V$, a tangent space, a real vector space which contains all the possible directions in which one can tangentially move through x. One can think of the tangent space as a generalization of *tangent lines* to curves in 2-dimensional spaces and *tangent planes* to surfaces in 3-dimensional spaces. We call the elements of this space *tangent vectors* at x. The dimension of this vector space is the same as the one of the manifold itself.

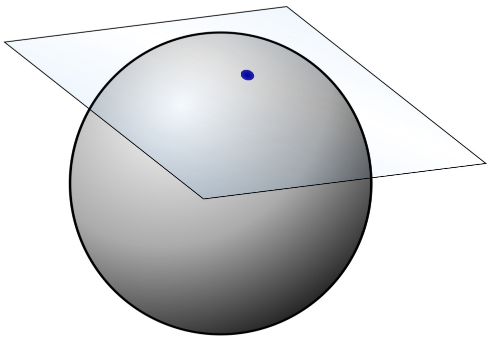


Figure 2.4: Tangent space illustrated as affine plane for a point on the sphere. [Wik23b]

Formally, the *tangent space* of a manifold V is defined as follows:

Definition 2.7 (Tangent Space). *A* **vector tangent to the n-dimensional manifold V at** *x is an equivalence class α of curves c on V passing trough x, namely*

$$c\colon (-\epsilon, \epsilon) \to V$$

such that $c(0) = x$ and $\alpha = [c]$. The equivalence relation is defined as follows: If c_1 and c_2 are curves passing trough x, then:

$$c_1 \sim c_2 \text{ if and only if } (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

for a chart (U, φ) of x. We denote the set of all vectors tangent at x as T_xV.

Remark 2.4. *This definition is well-defined in the sense that the equivalence relation does not depend on the chart φ. Let (U', ψ) be a different chart of x,* *then we obtain:*

$$
\begin{aligned}
(\psi \circ c_1)'(0) &= (\psi \circ \varphi^{-1} \circ \varphi \circ c_1)'(0) \\
&= D_{(\varphi\circ c_1)'(0)}(\psi \circ \varphi^{-1}) \circ (\varphi \circ c_1)'(0) \\
&= D_{(\varphi\circ c_2)'(0)}(\psi \circ \varphi^{-1}) \circ (\varphi \circ c_2)'(0) \\
&= (\psi \circ c_2)'(0).
\end{aligned}
$$

If we fix a chart (U, φ), we can associate to every tangent vector α unique vector $v \in \mathbb{R}^n$, namely the velocity vector of the curve $\varphi \circ c$ at the point 0, where $[c] = \alpha$. This is well-defined by Remark 2.4 and $\varphi \circ c$ is a curve in $\mathbb{R}^n$, where we are allowed to use the standard derivation rules. It follows that the tangent space T_xV at a point $x \in V$ can be naturally identified with $\mathbb{R}^n$:
For a chart (U, φ) of x, we define the map

$$d\varphi_x \colon T_xV \to \mathbb{R}^n, \quad \alpha = [c] \mapsto (\varphi \circ c)'(0).$$

As the map $d\varphi_x$ is bijective, we have a one-to-one correspondence between T_xV and $\mathbb{R}^n$.

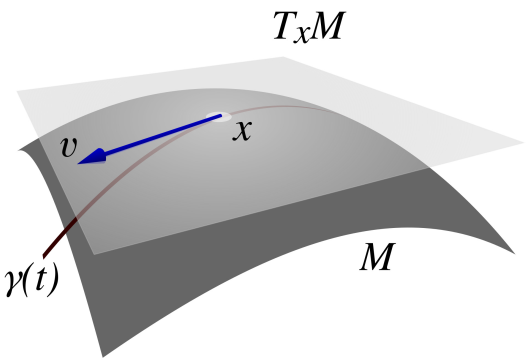


Figure 2.5: Illustration of correspondence between T_xV and $\mathbb{R}^n$. [Wik23b]

We have now developed the necessary tools to generalize the concept of total derivatives for functions $f \colon V \to W$ for manifolds V, W.

Definition 2.8 (**Derivative**)**.** *For every $x \in V$, the* **derivative** *df_x is the function*

$$
\begin{aligned}
df_x \colon T_xV &\to T_{f(x)}W \\
[c] &\mapsto [f \circ c]
\end{aligned}
$$

Example 2.5. *Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $x \in \mathbb{R}^n$, identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ and $T_{f(x)}\mathbb{R}^m$ with $\mathbb{R}^m$.*

$$
\begin{aligned}
df_x \colon \mathbb{R}^n \to \mathbb{R}^m \quad v = c'(0) \approx [c] \mapsto [f \circ c] \approx & D_0(f \circ c) \\
&= J_f(c(0))c'(0) \\
&= J_f(x)v \\
&= Df(x)v
\end{aligned}
$$

where $J_f(x)$ is the Jacobian matrix of f at the point x. In particular, for the special case $V = \mathbb{R}^n$, df_x coincides the total derivative $Df(x)$.

Definition 2.9 (**Submersion, and immersion**). *Let V and W be differentiable manifolds and $f : V \to W$ be a differentiable map.*

1. *f is called a* **submersion** *if for all $x \in V$ the differential df_x is surjective.*

2. *f is called an* **immersion** *if for all $x \in V$ the differential df_x is injective.*

Definition 2.10 (**Embedding**). *Let V and W be smooth manifolds, an* embedding *is an injective continuos immersion $f : V \to W$.*

2.3 Submanifolds

In this thesis, submanifolds of $\mathbb{R}^n$ play an important role. A submanifold of $\mathbb{R}^n$ is a subspace V that locally greatly resembles a linear subspace of $\mathbb{R}^n$.

Definition 2.11 (**Submanifold**). *A* **d-dimensional submanifold V of class C^k** *is a subset of $\mathbb{R}^n$ such that for every point $p \in V$ there is a neighborhood $U \subset \mathbb{R}^n$ and a C^k-diffeomorphism*

$$\varphi : U \to \varphi(U) \subset \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$$

of $\mathbb{R}^n$ such that

$$\varphi(V \cap U) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}) \text{ with } 0 \in \varphi(U).$$

Example 2.6. *The sphere and the torus are both 2 dimensional submanifolds of $\mathbb{R}^3$.*

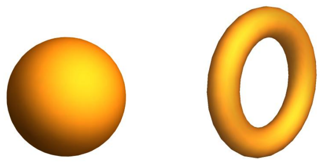


Figure 2.6: The sphere and the torus are 2 dimensional submanifolds.

There are several ways to express submanifolds. This will be formalized in Theorem 2.3 (Submanifold theorem). The following two theorems are useful not just for the proof of Theorem 2.3, but also for further applications.

Theorem 2.1 (**Inverse Function Theorem**). *Suppose that $W \subset \mathbb{R}^n$ is an open set, let $f : W \to \mathbb{R}^n$ be a smooth function, $p \in W$, $f(p) = 0$ and $Df(p)$ is bijective. Then there exist open neighborhoods $V \subset W$ of p and $U \subset \mathbb{R}^n$ of 0 such that $f|_V$ is a diffeomorphism from V onto U.*

Theorem 2.2 (**Implicit Function Theorem**). *Let $0 < d < n$, $k \geq 1$ be integers, $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}^{n-d}$ be a differentiable function. We write a point in $\mathbb{R}^d \times \mathbb{R}^{n-d}$ as (x, y) with $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{n-d}$. Assume that we have $(x_0, y_0) \in U$ with $g(x_0, y_0) = 0$ such that the $(n-d) \times (n-d)$ matrix*

$$J_y f(x_0, y_0) := \frac{\partial f_j}{\partial y_i}(x_0, y_0)_{1 \leq j \leq n-d, d+1, \leq i \leq n}$$

is invertible. Then, there exist $W \subset \mathbb{R}^d$ *and* $\tilde{W} \subset \mathbb{R}^{n-d}$ *open such that* $W \times \tilde{W} \subset U$ *and a function* $h\colon W \to \tilde{W}$ *such that for all* $(x, y) \in W \times \tilde{W} \subset U$ *, it holds*

$$f(x, y) = 0 \iff y = h(x).$$

Proofs of the theorems can be found in [Ser24] on p.78-79 and p.80-81.

Theorem 2.3 (Submanifold theorem). *Let* $V \subset \mathbb{R}^n$*. Then, the following properties are equivalent:*

1. *V is a submanifold of dimension d of $\mathbb{R}^n$.*
2. *Every point x of V has an open neighborhood U in $\mathbb{R}^n$ such that there exists a submersion $g\colon U \to \mathbb{R}^{n-d}$ with $U \cap V = g^{-1}(0)$.*
3. *Every point x of V has an open neighborhood U in $\mathbb{R}^n$ such that there exists a neighborhood Ω of 0 in $\mathbb{R}^d$ and an immersion $h\colon \Omega \to \mathbb{R}^n$ that is a homeomorphism from Ω onto $U \cap V$.*

Proof. (1) $\implies$ (2) : By assumption, we have for every point $x \in V$ a neighborhood $U \subset \mathbb{R}^n$ and a chart $\varphi\colon U \to \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$. Define $g\colon U \to \mathbb{R}^{n-d}$ $g(x) = (\varphi_{d+1}(x), \ldots, \varphi_n(x))$. $dg_x = Dg(x)$ consists of $n-d$ columns of the invertible matrix $d\varphi_x = D\varphi(x)$, thus dg_x has rank $n-d$ and is surjective.
(2) $\implies$ (3) : As $Dg(x)$ has rank $n-d$, $Dg(x)$ contains $n-d$ linearly independent columns. We may assume that those are the last $n-d$ columns, if not we might just permute the variables. the $(n-d) \times (n-d)$ matrix

$$(\partial_i g_j)(x)_{1 \leq j \leq n-d, d+1 \leq i \leq n}$$

is invertible. g satisfies the assumptions of Theorem 2.2 (Implicit function theorem) and there exists $W \subset \mathbb{R}^d$, $\tilde{W} \subset \mathbb{R}^{n-d}$ and $k\colon W \to \tilde{W} \subset \mathbb{R}^{n-d}$ such that $\forall x \in W \times \tilde{W}$ it holds:

$$\begin{aligned}
&(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \in V \\
\iff &g((x_1, \ldots, x_d, x_{d+1}, \ldots, x_n)) = 0 \\
\iff &(x_{d+1}, \ldots, x_d) = k(x_1, \ldots, x_d).
\end{aligned}$$

We have $V \cap \tilde{U} = \text{graph}(k)$ and we can define $h\colon W \to \mathbb{R}^n$, $h(x) = (x, k(x))$ and $h(W) = \text{graph}(k) = V \cap \tilde{U}$. It follows from construction that h is an immersion.
(3) $\implies$ (1) : [Ser24] p.84

□

In other words, a submanifold can be described locally by equations (where the number of equations is the codimension) or by a parametrization (where the number of parameters is the dimension).

Example 2.7. *The standing torus from Example* 2.1 *is a submanifold.*

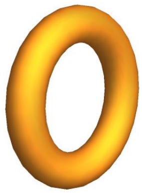


Figure 2.7: Standing torus.

Every compact manifold ca be embedded as submanifold of a Euclidean space. This is useful as we can apply theorems about submanifolds to any compact manifold.

Theorem 2.4 (**Whitney embedding theorem**). *Let V be a compact manifold, then there exists an embedding of V as a submanifold of a Euclidean space $\mathbb{R}^n$.*

Chapter 3

Morse Functions

The goal of this chapter is to introduce the basic notions in the theory of *Morse functions*. We start by providing the definition of Morse functions and give two concrete examples, namely the height function on the standing torus and on the sphere S^2. We then show that every manifold admits a Morse function. In the end, we provide a full proof of the *Morse lemma*, which is one of the most fundamental lemmas in Morse theory. All the manifolds and functions we consider are of class C^∞. All the definitions and propositions in this section stem from [ADE13],[RLC06] and [Zü20].

3.1 Critical Points and Hessians

In this section, we give the definition of a *critical point* and of a *Hessian* for a map defined on a manifold.

Definition 3.1 (**Critical Point, Critical Value**). *Let V be a manifold and let $f\colon V \to \mathbb{R}$ be a function.*

1. *A* **critical point** *of f is a point p such that $(df)_p = 0$.*
2. *A* **critical value** *of f is a point $x \in \mathbb{R}$ that is the image of a critical point.*

Critical values have some fundamental properties. One is that the set of all critical values has measure 0 in $\mathbb{R}$. Another is that the preimage of a regular value is an $n-m$-dimensional submanifold of V.

Theorem 3.1 (**Sard's Theorem (for real-valued Functions)**). *Let V be a manifold and let $f\colon V \to \mathbb{R}$ be a function, then the set of critical values of f has measure zero in $\mathbb{R}$.*

Theorem 3.2 (**Regular value Theorem**). *Let V and W be manifolds of dimension n and m, let $f\colon V \to W$ and let α be a regular value of f. Then, $f^{-1}(\alpha)$ is a submanifold of $n-m$-dimensional submanifold of V.*

Proofs of the theorems can be found in [Zü20] on p. 44-46 and p.47. We continue by giving an example on how to calculate *critical points* of a function. Concretly, we are going to look at the height function on the standing torus (Example 2.1).

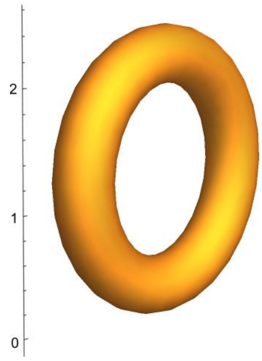


Figure 3.1: Height function of the standing torus with inner radius 1 and radius of the tube 0.25.

Example 3.1. *Consider the standing torus T. As as submanifolds of $\mathbb{R}^3$, T is parametrized as:*

$$\begin{aligned} x &= R_2\sin(\phi) \\ y &= (R_1 + R_2\cos(\phi))\cos(\varphi) \\ z &= (R_1 + R_2\cos(\phi))\sin(\varphi) + R_1 + R_2 \end{aligned}$$

with $\phi, \varphi \in [0, 2\pi)$ and $0 < R_2 < R_1$, where R_1 is the inner radius and R_2 is the radius of the tube. φ represents the rotation of the torus' axis of revolution and ϕ the rotation around the tube. The height function on this torus is:

$$\begin{aligned} f\colon T &\to \mathbb{R} \\ (\varphi, \phi) &\mapsto (R_1 + R_2\cos(\phi))\sin(\varphi) + R_1 + R_2. \end{aligned}$$

The differential at (φ, ϕ) is

$$(df)_{(\varphi,\phi)} = ((R_1 + R_2\cos(\phi))\cos(\varphi), -R_2\sin(\varphi)\sin(\phi)).$$

Thus, the critical points $p = (\varphi, \phi)$ fullfill the following equations:

$$R_1\cos(\varphi) + R_2\cos(\varphi)\cos(\phi) = 0 \text{ and } -R_2\sin(\varphi)\sin(\phi) = 0.$$

By solving the equations, we find that that critical points are:

$$a = (\frac{3\pi}{2}, 0),\ b = (\frac{3\pi}{2}, \pi),\ c = (\frac{\pi}{2}, \pi),\ d = (\frac{\pi}{2}, 0).$$

The following image illustrates the critical points of the height function on the torus:

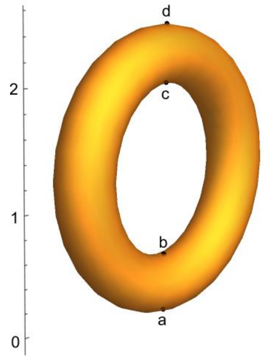


Figure 3.2: Critical points a, b, c, d of the height function of the torus.

To understand the nature of the critical points, we introduce a further concept, namely the *Hessian*. It is closely related to the *Hessian matrix* known from calculus. The Hessian contains information about how a function behaves around critical points.

Definition 3.2 (**Hessian**). *Let V be a n-dimensional manifold and let $p \in V$ be a critical point, let (U, φ) be a chart around p. For vectors $v, w \in T_pV$, let $(v_1, \dots, v_n) = d\varphi_p(v)$ and $(w_1, \dots, w_n) = d\varphi_p(w)$. Then, the* **Hessian** *of f at p, using the chart (U, φ), is the bilinear form $T_xV \times T_xV \to \mathbb{R}$ given by the formula:*

$$\mathrm{Hess}_p(f)(v, w) = \sum_{i,j=1}^{n} \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(p) v_i w_j.$$

The Hessian of a function f defined on a manifold V at a point p depends on the coordinate chart φ. However, we will only be interested in the Hessian at critical points p, for which it turns out that the Hessian is independent of the chosen chart. This result is formalized in the following proposition.

Proposition 3.1. *When p is a critical point for $f \colon V \to \mathbb{R}$, the Hessian at p is independent of the coordinate chart.*

Proof. Let (U, φ) and (V, ϕ) be two coordinates charts of V around p. To simplify the notation, we will write $(x_1, \dots, x_n)$ for φ and $(y_1, \dots, y_n)$ for ϕ. Also, let $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in T_pV$ interpreted in the coordinate chart φ, so in other words, it holds that $(v_1, \dots, v_n) = d\varphi_p(v)$ and $(w_1, \dots, w_n) = d\varphi_p(w)$ for some $v, w \in V$. Additionally, we define the function Q

$$Q := \phi_p \circ \varphi_p^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n.$$

We have $d\phi_p(v) = dQ(v_1, \dots, v_n)$ and similarly $d\phi_p(w) = dQ(w_1, \dots, w_n)$. Then, the goal is to show:

$$\sum_{i,j=1}^{n} \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i x_j} v_i w_j = \sum_{i,j=1}^{n} \frac{\partial^2 (f \circ \phi^{-1})}{\partial y_i y_j} dQ(v_1, \dots, v_n)_i dQ(w_1, \dots, w_n)_j$$

$$
\begin{aligned}
\sum_{i,j=1}^{n} \frac{\partial^2 (f\circ\varphi^{-1})}{\partial x_i x_j} v_i w_j &\overset{(*)}{=} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial (f\circ\phi^{-1}\circ Q)}{\partial x_j}\right) v_i w_j \\
&\overset{(C)}{=} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \frac{\partial (f\circ\phi^{-1})}{\partial y_k}\frac{\partial y_k}{\partial x_j}\right) v_i w_j \\
&\overset{(P)}{=} \sum_{i,j=1}^{n} (\sum_{k=1}^{n} \frac{\partial (f\circ\phi^{-1})}{\partial x_i \partial y_k}\frac{\partial y_k}{\partial x_j} v_i w_j \\
&\qquad \underbrace{+\sum_{k=1}^{n} \frac{\partial (f\circ\phi^{-1})}{\partial y_k}\frac{\partial y_k}{\partial x_i \partial x_j}) v_i w_j}_{0 \text{ since } p \text{ is a critical point}} \\
&\overset{(P)}{=} \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n}\sum_{h=1}^{n} \frac{\partial y_h}{\partial x_i}\frac{\partial (f\circ\phi^{-1})}{\partial y_k y_h}\frac{\partial y_k}{\partial x_j} v_i w_j\right) \\
&\overset{(**)}{=} \sum_{k,h=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial y_h}{\partial x_i} v_i \left(\sum_{j=1}^{n} \frac{\partial y_k}{\partial x_j} w_j \frac{\partial (f\circ\phi^{-1})}{\partial y_k \partial y_h}\right)\right) \\
&\overset{(C)}{=} \sum_{k,h=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial y_h}{\partial x_i} v_i dQ(w_1,\dots,w_n)_k \frac{\partial (f\circ\phi^{-1})}{\partial y_k \partial y_h}\right) \\
&\overset{(C)}{=} \sum_{k,h=1}^{n} \frac{\partial (f\circ\phi^{-1})}{\partial y_k \partial y_h} dQ(v_1,\dots,v_n)_h dQ(w_1,\dots,w_n)_k .
\end{aligned}
$$

where (C) denotes the chain rule, (P) denotes the product rule, in $(*)$ we just replaced φ^{-1} with $\phi^{-1}\circ Q$ and $(**)$ follows from distributivity. Thus, the Hessian is independent of the chart. □

Definition 3.3 (**Degenerate Critical Point**). *Let $f\colon V \to \mathbb{R}$ be a function and p a critical point. We call p a* **degenerate critical point** *of f if $\mathrm{Hess}_p(f)$ is degenerate as a bilinear form. In the same manner, we call p a* nondegenerate critical point *of f if $\mathrm{Hess}_p(f)$ is degenerate as a bilinear form.*

Remark 3.1. *Note that this definition is well-defined as $\mathrm{Hess}_p(f)$ does not depend on the coordinate chart for critical points as we have shown in Proposition* 3.1.

Remark 3.2. *If $\mathrm{Hess}_p(f)$ is nondegenerate and as $\mathbb{R}^n$ is a finite n-dimensional vector space , we can choose any basis of $\mathbb{R}^n$ and write $\mathrm{Hess}_p(f)$ as matrix M using this basis such that*

$$
\mathrm{Hess}_p(f)(v,w) = v^T M w,
$$

where v and w are identified with their corresponding vectors in $\mathbb{R}^n$. In particular, it holds then that $\mathrm{Hess}_p(f)$ is nondegenerate if and only if $det(M) \neq 0$. In addition, as M is symmetric, there exists a basis in which the matrix M is diagonal.

Notation 3.1. *From now on, we write $\mathrm{Hess}_p(f)$ for the matrix M.*

Example 3.2. *Coming back to Example* 3.1*, namely the height function on the torus, we see that all the critical points are indeed nondegenerate. The Hessian (in the coordinates (φ, ϕ)) is of the following form:*

$$\mathrm{Hess}_{(\varphi,\phi)}(f) = \begin{pmatrix} -R_1 \sin(\varphi) - R_2 \sin(\varphi)\cos(\phi) & -R_2 \cos(\varphi)\sin(\phi) \\ -R_2 \cos(\varphi)\sin(\phi) & -R_2 \sin(\varphi)\cos(\phi) \end{pmatrix}.$$

Then, by inserting the coordinates of the critical points (φ, ϕ), we can easily see that $\det(\mathrm{Hess}_{(\varphi,\phi)}(f)) \neq 0$ for all critical points. For example, for the critical point $(\varphi, \phi) = (\frac{3\pi}{2}, 0) = a$, it holds that:

$$\mathrm{Hess}_{(\frac{3\pi}{2},0)}(f) = \begin{pmatrix} R_1 + R_2 & 0 \\ 0 & R_2 \end{pmatrix}.$$

Obviously, $\det(\mathrm{Hess}_{(\frac{3\pi}{2},0)}(f)) \neq 0$ since $R_1, R_2 > 0$. The calculations for the other critical points work in the same way.

3.2 Existence and Genericness of Morse Functions

We have now developed all the necessary tools to define the *Morse functions*. We start by giving the definition and then we look at two examples. At the end of this section we show that every compact manifold admits a *Morse function* and that every continuous function can be uniformly approximated by Morse functions.

Definition 3.4 (**Morse Function**)**.** *A differentiable function $f : V \to \mathbb{R}$ is a* **Morse function** *if all its critical points are nondegenerate.*

Example 3.3. *The height function on the torus is a* Morse function*.*

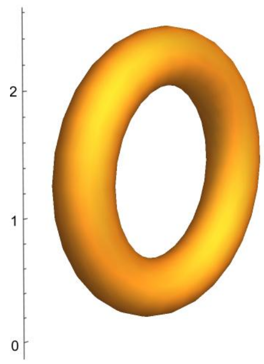


Figure 3.3: Height function of the standing torus with inner radius 1 and radius of the outer tube 0.25.

We can easily show that the height function on the sphere S^2 is also a Morse function.

Example 3.4. *Let $V = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and consider the height function*

$$\begin{aligned} f : V &\to \mathbb{R} \\ (x, y, z) &\mapsto z \end{aligned}$$

Consider the local coordinates $(x, y, \pm\sqrt{1 - x^2 - y^2})$ (upper and lower hemispheres), $(\pm\sqrt{1 - y^2 - z^2}, y, z)$ (left and right hemispheres) and $(x, \pm\sqrt{1 - x^2 - z^2}, z)$ (front and back hemispheres). Then, we obtain:

$$\begin{aligned} df_{(x,y)} &= \left(\frac{-\alpha x}{\sqrt{1 - x^2 - y^2}}, \frac{-\alpha y}{\sqrt{1 - x^2 - y^2}} \right) \quad \text{with } \alpha \in \{1, -1\} \\ df_{(y,z)} &= (0, \pm 1) \\ df_{(x,z)} &= (0, \pm 1). \end{aligned}$$

We have two critical points: $u = (0, 0, 1)$ and $v = (0, 0, 1)$. The Hessian has the following form:

$$\mathrm{Hess}_{(0,0)}(f) = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

which is nondegenerate for $\alpha \in \{1, -1\}$. Thus, the height functions on S^2 is indeed a Morse function.

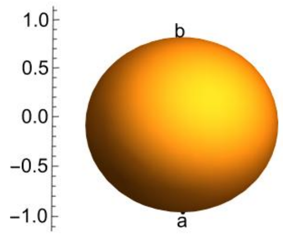


Figure 3.4: Critical points a, b of the height function of the sphere.

The next proposition tells us that every submanifold of $\mathbb{R}^n$ admits a Morse function.

Proposition 3.2. *Let $V \subset \mathbb{R}^n$ be a submanifold. For almost every point $p \in \mathbb{R}^n$, the function*

$$f_p : V \to \mathbb{R} \quad , x \mapsto |x - p|^2$$

is a Morse function.

Proof. Let $x \in V$, as V is a submanifold of $\mathbb{R}^n$, there is a neighborhood U of x, which can be described by a parametrization in d variables:

$$(u_1, \ldots, u_d) \mapsto x(u_1, \ldots, u_d),$$

where d is the dimension of V. As V is a submanifold of $\mathbb{R}^n$, the tangent space T_xV can be naturally embedded into $\mathbb{R}^n$ for $x \in V$. So, let $v \in T_xV \subset \mathbb{R}^n$, then we have that:

$$(df_p)_x(v) = 2(x - p) \cdot v.$$

In particular, we can deduce that x is a critical point if and only if $T_xV \perp x-p$. In addition, for x to be nondegenerate, it must hold that $\det(\mathrm{Hess}_x(f_p)) \neq 0$. So, in summary, the nondegenerate critical points $x \in V$ of the function f_p are determined by the following conditions:

1. $T_xV \perp x-p$,

2. $\det(\mathrm{Hess}_x(f_p)) \neq 0$.

By calculating the Hessian (in the coordinates $(u_1, \ldots, u_d)$), we obtain:

$$\mathrm{Hess}_x(f_p) = \left(\frac{\partial^2 f_p}{\partial u_i \partial u_j}\right)_{1\leq i,j\leq d} = \left(2\left(\frac{\partial x}{\partial u_j}\cdot\frac{\partial x}{\partial u_i}\right) + (x-p)\cdot\frac{\partial^2 x}{\partial u_i \partial u_j}\right)_{1\leq i,j\leq d}.$$

The idea is to use Theorem 3.1 (Sard's Theorem). So, we need to find a function for which the set of points $p \in V$, such that f_p is not a Morse function, are exactly the critical values. So, let

$$\begin{aligned} H &= \{p \in V : f_p \text{ is not a Morse function}\} \\ &= \{p \in V : f_p \text{ has a critical point } x \text{ such that } \det(\mathrm{Hess}_x(f_p)) = 0\}. \end{aligned}$$

The proposition follows now from the following lemma:

Lemma 3.1. *Let V be a manifold, let $M = \{(x,v) \in V \times \mathbb{R}^n : T_xV \perp v\} \subset V \times \mathbb{R}^n$ and define the function*

$$\begin{aligned} E\colon M &\to \mathbb{R}^n \\ (x,v) &\mapsto x+v. \end{aligned}$$

Then, the following holds:

1. *M is a submanifold of $V \times \mathbb{R}^n$.*

2. *The point $p = E(x,v) \in \mathbb{R}^n$ is a critical value of E if and only if* $\det(\mathrm{Hess}_x(f_p)) = 0$.

A proof of the lemma can be found in [ADE13] on p.10. The first part of Lemma 3.1 ensures that M is a manifold and as E is clearly a C^∞-function, we are allowed to apply Theorem 3.1 (Sard's theorem) on the function E. It follows from the second part of Lemma 3.1 and Theorem 3.1 (Sard's theorem) that H as measure 0. $\square$

Remark 3.3. *As every compact manifold can be embedded as a submanifold of $\mathbb{R}^n$ (Theorem 2.4), it follows from Proposition 3.2 that every compact manifold admits a Morse function.*

We use now Proposition 3.2 to show that every C^∞-function defined on a manifold V can be uniformly approximated by Morse functions.

Proposition 3.3. *Let V be a manifold that can be embedded as a submanifold into a Euclidean space and let $f\colon V \to \mathbb{R}$ be a C^∞ function. Let k be an integer. Then f and all its derivatives of order $\leq k$ can be uniformly approximated by Morse functions on every compact subset.*

Proof. Let $n \in \mathbb{N}$ such that V can be embedded into $\mathbb{R}^{n-1}$. Then, we can construct a further embedding $h\colon V \to \mathbb{R}^n$ as:

$$h(x) = (f(x), h_2(x), .., h_n(x)).$$

Let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ and let $p = (-k, 0, \dots, 0) + \epsilon$. By Proposition 3.2, for almost every ϵ, f_p is a Morse function. Consequently, the function

$$g_k(x) = \frac{f_p(h(x)) - k^2}{2k}$$

is also a Morse function. Additionally, we see that $g_k(x)$ has the following form:

$$\begin{aligned} g_k(x) &= \frac{1}{2k}\left((f(x) + k - \epsilon_1(k))^2 + \sum_{i=2}^{n} (h_i(x) - \epsilon_i(k))^2 - k^2 \right) \\ &= \frac{1}{2k}\Bigg(f(x)^2 + 2f(x)k + k^2 - 2\epsilon_1(k)f(x) - 2k\epsilon_1(k) + \epsilon_1(k)^2 + \sum_{i=2}^{n} h_i^2(x) + \sum_{i=2}^{n} \epsilon_i(k)^2 \\ &\quad - \sum_{i=2}^{n} 2\epsilon_i(k)h_i(x) - k^2 \Bigg) \\ &= f(x) + \frac{f(x)^2 + \sum_{i=2}^{n} h_i^2(x)}{2k} - \frac{\epsilon_1(k)f(x) + \sum_{i=2}^{n} \epsilon_i(k)h_i(x)}{k} + \frac{\sum_{i=1}^{n} \epsilon_i(k)^2}{2k} - \epsilon_1(k) \end{aligned}$$

By restricting f and $(g_k)_{k\in\mathbb{N}}$ on an arbitrary compact subset E, we obtain:

$$\begin{aligned} &\lim_{k\to\infty} \sup_{x\in E} |g_k(x) - f(x)| \\ &= \lim_{k\to\infty} \sup_{x\in E} \left| \frac{f(x)^2 + \sum_{i=2}^{n} h_i(x)}{2k} - \frac{\epsilon_1(k)f(x) + \sum_{i=2}^{n} \epsilon_i(k)h_i(x)}{k} + \frac{\sum_{i=1}^{n} \epsilon_i^2(k)}{2k} - \epsilon_1(k) \right| \\ &\le \lim_{k\to\infty} \sup_{x\in E} \left| \frac{B^2 + A}{2k} - \frac{\epsilon_1(k)B + A\sum_{i=2}^{n} \epsilon_i(k)}{k} + \frac{\sum_{i=1}^{n} \epsilon_i^2(k)}{2k} - \epsilon_1(k) \right| \\ &\le \lim_{k\to\infty} \sup_{x\in E} \left(\left| \frac{B^2 + A}{2k} \right| + \left| \frac{\epsilon_1(k)B}{k} \right| + \left| \frac{A\sum_{i=2}^{n} \epsilon_i(k)}{k} \right| + \left| \frac{\sum_{i=1}^{n} \epsilon_i^2(k)}{2k} \right| + |\epsilon_1(k)| \right) \\ &\overset{(*)}{\to} 0 \end{aligned}$$

$(*)$: by choosing the $g_k's$ such that $\lim_{k\to\infty} \sum_{i=1}^{n} |\epsilon_i(k)| \to 0$, $A, B \in \mathbb{R}$ such that $\sum_{i=2}^{n} h_i(x) \le A$, $f(x) \le B$ for every $x \in V$, which exist as V is compact.

□

3.3 The Morse Lemma

In this section, we are going to have a look at one of the main lemmas in Morse theory, namely the *Morse Lemma*. It gives us insight into what happens near critical points of a Morse function. We know that in general, in the neighborhood of a critical point, a function can be closely approximated by its second-order derivative. The *Morse Lemma* guarantees something stronger, namely that for the right chart the *Morse function* even agrees with this approximation.

Theorem 3.3 (**Morse Lemma**). *Let p be a nondegenerate critical point of the function $f\colon V \to \mathbb{R}$. There exist a neighborhood U of p and a diffeomorphism $\varphi\colon (U,p) \to (\mathbb{R}^n, 0)$ such that*

$$f \circ \varphi^{-1}(x_1, \ldots, x_n) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

Remark 3.4. *The integer i that appears in the statement of the Morse lemma is called* the index *of the critical point. It equals the number of negative eigenvalues of* $\mathrm{Hess}_p(f)$*. Intuitively, it describes the number of linear independent directions in which the function f decreases.*

Example 3.5. *Again, coming back to Example* 2.1*, it is easy to see that a has index 0, b and c have index 1 and d has index 2. This can be seen by considering*

$$\mathrm{Hess}_{(\varphi,\phi)}(f) = \begin{pmatrix} -R_1\sin(\varphi) - R_2\sin(\varphi)\cos(\phi) & -R_2\cos(\varphi)\sin(\phi) \\ -R_2\cos(\varphi)\sin(\phi) & -R_2\sin(\varphi)\cos(\phi) \end{pmatrix}$$

at the critical points and calculating the eigenvalues. For example, for the critical point $(\varphi,\phi) = (\frac{3\pi}{2}, 0) = a$, it holds that:

$$\mathrm{Hess}_{(\frac{3\pi}{2},0)}(f) = \begin{pmatrix} R_1 + R_2 & 0 \\ 0 & R_2 \end{pmatrix},$$

where $R_1 + R_2, R_2 > 0$ are the two positive eigenvalues It follows that $\mathrm{Ind}(a) = 0$ *and a is a minimum. In particular, there exists a open neighborhood and a diffeomorphism $\varphi\colon \big(U, (\frac{3\pi}{2}, 0)\big) \to (\mathbb{R}^2, 0)$ such that $f \circ \varphi^{-1}(x_1, x_2) = f(a) + x_1^2 + x_2^2$. The calculations for the other critical points work in the same way.*

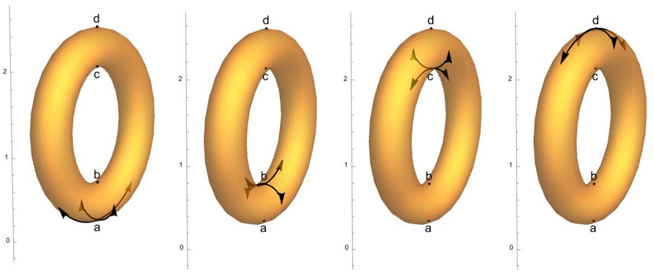


Figure 3.5: Visualization of the *index* of the 4 critical points.

We continue with the proof of the *Morse lemma.*

Proof. We can assume that $V = \mathbb{R}^n$ and $p = 0$ as there is always a chart which sends p to $0 \in \mathbb{R}^n$. In addition, as every real-valued symmetric matrix is diagonalizable and a change of basis is a diffeomorphism, we may assume that $\mathrm{Hess}_p(f)$ is diagonal.
We do a proof by induction on the dimension of the manifold. We start with the **base case** $\mathbf{n = 1}$.
By the Taylor formula, we know that f can be written in the following form:

$$f(x) = f(0) + \frac{1}{2}f''(0)x^2 + \epsilon(x)x^2 = f(0) \pm ax^2(1 + \epsilon(x)),$$

where $a \in \mathbb{R}_{\geq 0}$ and $\epsilon(x) = \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2 dt$.
We then set $x_1 = \varphi(x) = x\sqrt{a(1+\epsilon(x))}$. As $\varphi'(0) = \sqrt{a} \neq 0$, it follows by Theorem 2.1 (Inverse function theorem) that φ is a local diffeomorphism. In particular, we have that

$$f \circ \varphi^{-1}(x_1) = f(x) = f(0) \pm x_1^2.$$

Thus, $f \circ \varphi^{-1}$ has indeed the desired form.
We continue with the **induction step**.
We assume that the lemma holds for $n-1$ and we want to show that it implies the lemma for n. To do that, we write $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and consider the function $f(x, y)$ as a function $f_y(x)$ in of real variable and $y \in \mathbb{R}^{n-1}$ as a parameter. As in the case $n = 1$, the Taylor formula allows us to write $f_y(x)$ in the following form:

$$f(x, y) = f_y(x) = f_y(0) + f_y'(0)x + \frac{1}{2}f_y''(0)x^2 + x^2\epsilon(x, y).$$

As $\mathrm{Hess}_p(f)$ is diagonal by assumption, $f_y''(0) \neq 0$ as $f_y''(0)$ is the column respectively row ($\mathrm{Hess}_p(f)$ is symmetric) corresponding to the variable x and $\mathrm{Hess}_p(f)$ does not contain a 0 row or column.
$(*)$ If $f_y'(0)$ is zero, we can proceed as in the case $n = 1$. In particular, the map

$$\varphi\colon (x, y) \mapsto (x_1 = x\sqrt{a(y)(1+\epsilon(x, y))}, y_1 = y)$$

is the desired local diffeomorphism. Again, Theorem 2.1 (Inverse function theorem) guarantees that, as $(x\sqrt{a(y)(1+\epsilon(x, y))})'(0) \neq 0$, φ is a local diffeomorphism and again we have that

$$f \circ \varphi^{-1}(x_1, y_1) = \pm x_1^2 + f(0, y_1).$$

Then, as $f(0, y_1)$ can be viewed as a function defined on $\mathbb{R}^{n-1}$, the result follows by induction.
The idea is now to show that there is always a chart in which $f_y'(0) = 0$. To do that, we are going to use Theorem 2.2 (Implicit function theorem). First, note that the critical points of f_y are exactly the solutions of the equation

$$\frac{\partial f}{\partial x}(x, y) = 0.$$

In addition, as we assumed that $Hess_p(f)$ is diagonal, it holds that

$$\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0.$$

As a consequence, Theorem 2.2 (Implicit function theorem) (applied to the function $\frac{\partial f}{\partial x}\colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$) tells us that in a neighborhood of $(0, 0)$, the solutions $x \in \mathbb{R}$ can be expressed as the image of a function $\varphi \in C^\infty(\mathbb{R}^{n-1})$ defined in the neighborhood of $0 \in \mathbb{R}^{n-1}$. Explicitly, it holds in a neighborhood of $0 \in \mathbb{R}^{n-1}$ that $x = \varphi(y)$. As $\mathrm{Hess}_p(f)$ is diagonal, the derivative of $\frac{\partial f}{\partial x}$ with respect to y is 0. Consequently, $(d\varphi)_0 = 0$.
We define now the local diffeomorphism

$$\Phi(x, y) = (x + \varphi(y), y).$$

The differential of Φ at 0 is the identity. We define the function $\tilde{f} = f \circ \Phi$ and obtain that $\frac{\partial \tilde{f}}{\partial x}(0, y) = 0$ and $d^2\tilde{f}_{(0,0)} = (d^2 f)_{(0,0)}$. So $\tilde{f}'_y(0) = 0$ and proceed as above $(*)$. □

The following corollary is a direct consequence of the *Morse lemma*:

Corollary 3.1. *The nondegenerate critical points of a Morse function* $f\colon V \to \mathbb{R}$ *are isolated.*

Proof. Let p be a nondegenerate critical point. By the Morse lemma, we know that there exists a neighborhood U of p and a diffeomorphism $\varphi\colon (U, p) \to (\mathbb{R}^n, 0)$ such that

$$f \circ \varphi^{-1}(x_1, \ldots, x_n) = f(x) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

In these coordinates

$$df_p = (\pm 2x_k)_{1 \le k \le n}$$

with $p = (x_1, \ldots, x_n)$. In particular, $df_p \equiv 0$ if and only if $p = (0, \ldots, 0)$. It follows that $\varphi^{-1}((0, \ldots, 0)) = p$ is the only critical point in U. □

Chapter 4

Pseudo-Gradients

The main goal of this chapter is to introduce *pseudo-gradient fields*. They are useful as their trajectories connect the critical points of a Morse function. This allows us to introduce the *stable* and *unstable manifolds*, which will be the first step towards defining *Morse complexes* and *Morse homology*. At the end of the chapter, we will have a closer look at the *Smale condition*, which guarantees that two critical points with consecutive indices are only connected by a finite number of trajectories. We will then have developed all necessary tools to introduce *Morse complexes* and *Morse homology*. The manifolds and functions we consider in this chapter are of class C^∞. All the definitions, propositions and theorems of this chapter stem from [ADE13] and [Lat94].

4.1 Gradients and Flow Lines

Definition 4.1 (**Gradient**). *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. The* **gradient grad f** *is defined as follows:*

$$\text{grad } f\colon \mathbb{R}^n \to \mathbb{R}^n, \quad (\text{grad } f)(x) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})(x)$$

where the coordinates are considered in the canonical basis.

Remark 4.1. *Let $\langle\ ,\ \rangle$ be the usual Euclidean inner product in $\mathbb{R}^n$. It is also possible to define grad f as the unique vector field with the following property:*

$$\langle (\text{grad } f)(x), Y\rangle = (df)_x(Y).$$

In other words, grad f *is the unique vector field such that for $x, Y \in \mathbb{R}^n$ the directional derivative in direction Y at the point x equals $\langle (\text{grad } f)(x), Y\rangle$.*

Definition 4.2 (**Flow Line**). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, a* **flow line** *is a curve*

$$\varphi^s(x)\colon I \to \mathbb{R}^n, \qquad s \mapsto \varphi^s(x)$$

for an open interval $I \subset \mathbb{R}^n$ that satisfies the differential equation

$$\begin{cases} \frac{\partial}{\partial s}\varphi^s(x) & = -(\text{grad } f)(\varphi^s(x)) \\ \varphi^0(x) & = x \end{cases}.$$

Gradients and flow lines have the following properties.

Proposition 4.1. *1.* grad f *vanishes exactly at the critical points of the function f.*

2. The function f is decreasing along the flow lines in the field -grad f*:*

$$\frac{\partial}{\partial s}(f(\varphi^s(x))) = -|(\text{grad } f)(\varphi^s(x))|^2 < 0.$$

Proof. The point x is a critical point if and only if $(df)_x(Y) = 0 = \langle(\text{grad } f)(x), Y\rangle$ for every $Y \in \mathbb{R}^n$, which implies $(\text{grad } f)(x){=}0$. For the second part:

$$\begin{aligned}\frac{\partial}{\partial s}(f(\varphi^s(x))) &\overset{(*)}{=} (df)_{\varphi^s(x)}\left(\frac{\partial}{\partial s}\varphi^s(x)\right)\\ &\overset{(**)}{=} \left\langle(\text{grad } f)(\varphi^s(x)), \frac{\partial}{\partial s}\varphi^s(x)\right\rangle\\ &\overset{(***)}{=} \langle(\text{grad } f)(\varphi^s(x)), -(\text{grad } f)(\varphi^s(x))\rangle\\ &= -|(\text{grad } f)(\varphi^s(x))|^2 < 0,\end{aligned}$$

where $(*)$ follows from the chain rule, $(**)$ follows from Remark 4.1 and $(***)$ from Definition 4.2. □

In the next step, we introduce *pseudo-gradient fields*, which are a generalization of the gradient (Definition 4.1) in the sense that near critical points they coincide with it and away from critical points they point 'approximately' in the same direction. In addition, they do not rely on an inner product.

Definition 4.3 (**Morse Chart, and Morse Neighborhood**)**.** *Let V be a manifold and $f\colon V \to \mathbb{R}$ a Morse function. A chart (U, φ) of a critical point p such that*

$$f \circ \varphi^{-1}(x_1, \dots, x_n) = f(p) - \sum_{j=1}^{\text{Ind}(p)} x_j^2 + \sum_{j=\text{Ind}(p)+1} x_j^2$$

is called a **Morse chart***. The set U is called a* **Morse neighborhood***.*

Definition 4.4 (**Pseudo-gradient field**)**.** *Let $f\colon V \to \mathbb{R}$ be a Morse function on an n-dimensional manifold V. A* **pseudo-gradient field** *adapted to f is vector field $X\colon V \to \bigsqcup_{x\in V} T_xV$ with $X_x = X(x) \in T_xV$ such that:*

1. *We have $(df)_x(X_x) \leq 0$, where equality holds if and only if x is a critical point.*
2. *In a Morse chart in the neighborhood of a critical point X coincides with the negative gradient for the canonical metric on $\mathbb{R}^n$.*

4.2 Morse Charts

The goal of this section is to describe a specific kind of Morse neighborhoods, which are bounded by level sets and trajectories according to a pseudo-gradient field. Let $p \in V$ be a critical point with index i. We start by introducing some notation:

- $V_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0,\ j \geq i+1\}$;
- $V_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0,\ j \leq i\}$;
- $Q \colon \mathbb{R}^n = V_- \oplus V_+ \to \mathbb{R}$ such that $Q(x) = -|x_-|^2 + |x_+|^2$ with $x = x_- + x_+$;
- Let $\epsilon, \eta > 0$, $\tilde{U}(\epsilon, \eta) = \{x \in \mathbb{R}^n : -\epsilon \leq Q(x) \leq \epsilon$ and $|x_-||x_+| \leq \sqrt{\eta(\epsilon + \eta)}\}$;
- $\partial_+ U = \{x \in U : Q(x) = \epsilon$ and $|x_-| \leq \sqrt{\eta}\}$;
- $\partial_- U = \{x \in \tilde{U} : Q(x) = -\epsilon$ and $|x_+| \leq \sqrt{\eta}\}$;
- $\partial_0 U = \{x \in \partial U : |x_-||x_+| = \sqrt{\eta(\epsilon + \eta)}\}$.

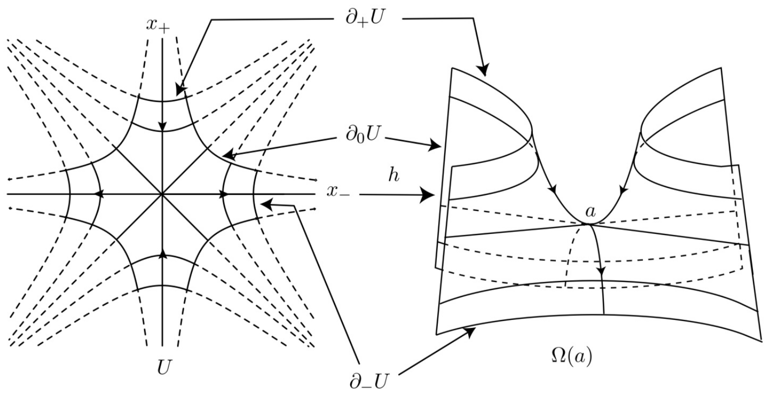


Figure 4.1: A Morse chart. [ADE13]

The set V_- contains the coordinates in which the function decreases, similarly V_+ describes the coordinates in which the function increases. With the notation above, we have that for a critical point p and a Morse chart (U, φ) of p:

$$f \circ \varphi^{-1}(x) = f(p) + Q(x).$$

Let us have a closer look at $U(\epsilon, \eta) \subset \mathbb{R}^n$ and make sense of Figure 4.1. The parameter ϵ describes the level sets, which bound the Morse neighborhood. Indeed, for $Q(x) = \epsilon$ and $Q(x) = -\epsilon$, it holds that $1 = \frac{|x_+|^2}{\epsilon} - \frac{|x_-|^2}{\epsilon}$ and $1 = \frac{|x_-|^2}{\epsilon} - \frac{|x_+|^2}{\epsilon}$, which are equations for hyperbolas with transverse axis on the x-axis and vertices $(\pm\epsilon, 0)$, respectively on the y-axis with vertices $(0, \pm\epsilon)$. This coincides with Figure 4.1, where $\partial_+ U$ and $\partial_- U$ have the form of hyperbolas. Those hyperbolas correspond to the level sets $f(p) + \epsilon$ and $f(p) - \epsilon$ respectively. As we want the Morse neighborhood to be bounded, it makes sense to require that $|x_+|^2 \leq \eta$ and $|x_-|^2 \leq \eta$ for some small $\eta \in \mathbb{R}$. As $-(\text{grad } Q)(x_-, x_+) = 2(x_-, x_+)$, it is clear that in the Morse chart, the gradient lines are of the form $|x_+|^2 = \frac{a}{|x_-|^2}$ with $a \in \mathbb{R}$. Then by solving the equations, we obtain the corresponding 4 hyperbolas, which describe $\partial_0 U$. From now on, we are going to use the following notation: for a critical point $p \in V$, we denote:

- $\Omega(p) = \varphi^{-1}(U(\epsilon, \eta))$ for some $\epsilon, \eta > 0$;

- $\partial_+\Omega(p) = \varphi^{-1}(\partial_+ U)$;
- $\partial_-\Omega(p) = \varphi^{-1}(\partial_- U)$;
- $\partial_0\Omega(p) = \varphi^{-1}(\partial_0 U)$.

where (Ω, φ) is a Morse chart such that $\varphi(\Omega) = U(\epsilon, \eta)$ for some $\epsilon, \eta > 0$.

Remark 4.2. *By the definition of the gradient $\mathbb{R}^n$, the trajectories of the gradient are orthogonal to the level sets of the function. This holds as for every point x in a certain level set and every vector v tangent to a level set we have that:*

$$0 = (df)_x(v) = \langle (\text{grad } f)(x), v \rangle.$$

Thus, $(\text{grad } f)(x)$ *is indeed orthogonal to v and so are the trajectories. It makes therefore sense that $\partial_+ U, \partial_- U$ and $\partial_0 U$ form two orthogonal families of equilateral hyperbolas.*

4.3 Existence of Pseudo-Gradients

In this section, we prove that for any compact manifolds V and Morse function $f \colon V \to \mathbb{R}$, there exists a *pseudo-gradient field*. This is a simple consequence of the existence of *partitions of unity*.

Definition 4.5 (**Partition of Unity**)**.** *Let X be a topological space and let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite covering. A* **partition of unity** *subordinate to $\mathscr{U}$ is a family of real valued functions $(f_\alpha)_{\alpha \in A}$ with the following properties for all $x \in X$:*

1. $0 \leq f_\alpha(x) \leq 1 \quad \forall \alpha \in A$;
2. $\sum_{\alpha \in A} f_\alpha(x) = 1$;
3. $\overline{\{x \in X \colon f_\alpha(x) > 0\}} \subset U_\alpha \quad \forall \alpha \in A$.

Partitions of unity are useful because they allow us to extend local constructions to the whole space. One can show that every compact manifold admits a *partition of unity* ([ADE13] p.537).

Proposition 4.2. *Pseudo-gradient fields exist for all Morse function on all compact manifolds.*

Proof. The proof is constructive, which means that we are going to construct a concrete *pseudo-gradient* for a Morse function $f \colon V \to \mathbb{R}$. We denote by $c_1, \ldots, c_r$ the critical points of f on V and by $(U_1, h_1), \ldots, (U_r, h_r)$ the corresponding Morse charts in the neighborhoods of these points. There are only finitely many critical points as the critical points of a Morse function on a compact manifold V are isolated. Let $\Omega_j = h_j(U_j) \in V$ be the open images and we add more open sets such that we obtain a finite open cover $(\Omega_j)_{1 \leq j \leq N}$ of V. In addition, we assume that the added open sets do not contain any critical points. We start by defining a vector field X_j on every open set Ω_j for every $1 \leq j \leq N$.

We then use a *partition of unity* to extend those vector fields to vector fields $\tilde{X}_j$ defined on V. So let $1 \leq j \leq N$ and let $x \in \Omega_j$, we define:

$$X_j(x) = -\underbrace{(T_{h_j^{-1}(x)} h_j)\underbrace{(\text{grad } (f \circ h_j)(h_j^{-1}(x))}_{\in \mathbb{R}^n}}_{\in T_x V}.$$

So, $X_j(x)$ is just the corresponding vector in $-T_{h_j^{-1}(x)}V$ to (grad $(f \circ h_j)(h_j^{-1}(x))$.
It follows from construction that $(df)_x(X_j(x)) \leq 0$ for every $x \in \Omega_j$ and that X_j is zero exactly at c_j as *grad* $(f \circ h_j)(h_j^{-1}(x))$ is exactly zero at $h_j^{-1}(0)$.
In the next step, we use a *partition of unity* $(\varphi_j)_j$ associated with the cover $(\Omega_j)_j$ to extend the local vector fields X_j. We do that by defining:

$$\tilde{X}_j(x) = \begin{cases} \varphi_j(x)X_j(x) & x \in \Omega_j \\ 0 & otherwise \end{cases}.$$

We then set

$$X = \sum_{j=1}^{N} \tilde{X}_j.$$

It holds now that

$$(df)_x(X_x) = \sum_{j=1}^{N} (df)_x((\tilde{X}_j)_x) \leq 0.$$

This inequality is an equality exactly when $\varphi_j(x)X_j(x) = 0$ for every $1 \leq j \leq N$, which when that x is a critical point.

□

4.4 Stable and Unstable Manifolds

In this section, we introduce the definitions of *stable* and *unstable* manifolds. As already mentioned, these are going to play a fundamental role in the development of *Morse complexes.* To start, let V be a manifold, $f: V \to \mathbb{R}$ a Morse function with a critical point p. Denote by φ^s the flow of a pseudo-gradient X. Then we define:

Definition 4.6 (**Stable, and unstable Manifold**).
1. *The* **stable manifold** $W^s(p) = \{x \in V : \lim_{s \to +\infty} \varphi^s(x) = p\}$
2. *The* **unstable manifold** $W^u(p) = \{x \in V : \lim_{s \to -\infty} \varphi^s(x) = p\}$

Intuitively, for a critical point p, $W^s(p)$ contains all the points whose flow lines 'end up' at p, and $W^u(p)$ contains all the points whose flow lines 'start' at p.

Example 4.1. *Consider again the height function on the standing torus and the pseudo-gradient field is simply the gradient for the metric induced by that on* $\mathbb{R}^3$. $W^s(a)$ *consists of the whole torus but the point* d *and the circle passing through* b *and* c *and* $W^u(a) = \{a\}$. $W^u(d)$ *consists of the whole torus but the circle passing through* b *and* c *and* $W^s(d) = \{d\}$. *For the critical points* b *and* c*, consider Figure 4.2.*

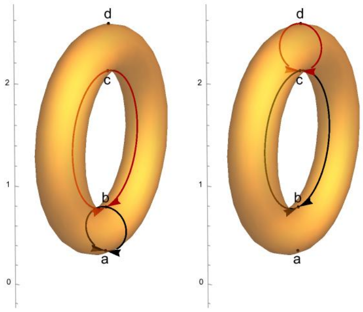


Figure 4.2: Stable manifolds (red) and unstable manifolds (black) of the critical points b and c of the height function on the standing torus.

It turns out, that the *stable manifolds* and *unstable manifolds* of a critical point p are manifolds itself. Furthermore, the dimension of the *unstable manifold* coincides with the index $\text{Ind}(p)$ of the critical point p. This is formalized in the following proposition:

Proposition 4.3. *The stable and unstable manifolds of the critical point a are submanifolds of V that are diffeomorphic to open disks. Moreover, we have*

$$\dim W^u(a) = \text{codim} W^s(a) = \text{Ind}(a),$$

where $\text{Ind}(a)$ *denotes the index of the point a as a critical point of f.*

Proof. To prove this proposition, we use the notion introduced earlier for *Morse charts* (Section 4.2). Let $a \in V$ be a critical point. Let $(U = U(\epsilon, \eta), h)$ be a Morse chart as described in Section 4.2. $W^s(a)$ can then be identified with

$$(h(\partial_+ U \cap V_+) \times \mathbb{R}) \cup \{0\},$$

as for every point $x \in W^s(a)$, there exists exactly one $t \in \mathbb{R}$ and one $c \in h(\partial_+ U \cap V_+)$ such that $\varphi^t(x) = c$. Let k be the index of a, then $h(\partial_+ U \cap V_+)$ is a sphere of dimension $n - k - 1$, in particular a manifold, as it is exactly the image under h of the sphere $|x_+|^2 = \epsilon$ in V_+. Thus,

$$W^s(a) \approx (S^{n-k-1} \times \mathbb{R}) \cup \{0\}.$$

As we see, $W^s(a)$ can be obtained by compactifying $(S^{n-k-1} \times \mathbb{R})$ by adding the point 0, which is diffeomorphic to an an open disc of dimension $n - k$.

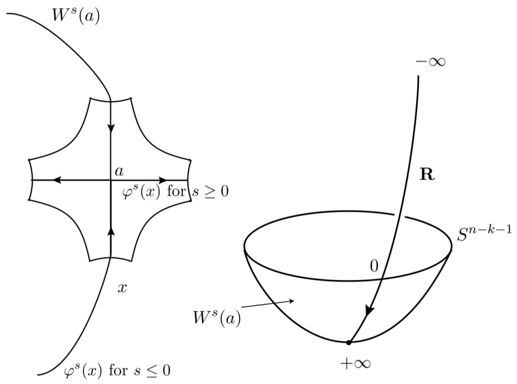


Figure 4.3: $W^s(a)$. [ADE13]

□

The most important property of the flow lines of a pseudo-gradient field X is that they all connect critical points of the function f. This means that all the flow lines come from a critical point and go toward another critical point.

Proposition 4.4. *Suppose that V is a compact manifold. Let $\lambda\colon \mathbb{R} \to V$ be a trajectory of the pseudo-gradient field X.Then there exist critical points c and d of f such that*

$$\lim_{s\to-\infty} \lambda(s) = c \text{ and } \lim_{s\to+\infty} \lambda(s) = d.$$

Proof. We prove that $\lim_{s\to+\infty} \lambda(s) = d$. Choose for every critical point c a Morse chart $\Omega(c)$ and consider the union $\Omega = \bigcup_{c\in\mathrm{Crit}(f)} \Omega(c)$. Suppose now that the assumption does not hold. This means that there exists a time t_0 such that $\lambda(t) \notin \Omega$ for every $t \geq t_0$. It holds in particular that $\lambda(t) \notin \mathrm{Crit}(f)$ for $t \geq t_0$, thus $(df)_{\lambda(t)}(X_{\lambda(t)}) < 0$ and there exists $\epsilon > 0$ such that $(df)_{\lambda(t)}(X_{\lambda(t)}) < -\epsilon$ for every $t \geq t_0$. But then we obtain for every $t \geq t_0$:

$$f(\lambda(t)) - f(\lambda(t_0)) = \int_t^{t_0} d(f\circ\lambda)_u du = \int_t^{t_0} (df)_{\lambda(u)} X_{\lambda(u)} du < -\epsilon(t-t_0)$$

But then, it follows that

$$\lim_{s\to+\infty} f(\lambda(s)) < \lim_{s\to+\infty} = \epsilon s = -\infty,$$

which is impossible since V is compact. The proof of $\lim_{s\to-\infty} = \lambda(s) = c$ works the same way by setting $f = -f$. □

We continue with looking at the *sublevel sets* of a Morse function f.

Definition 4.7 (Sublevel set). *Let $a \in \mathbb{R}$, then the sublevel set of for a is*

$$V^a = f^{-1}(]-\infty, a]).$$

Remark 4.3. *By Theorem 3.2 (Regular value theorem), it follows that V^a is a manifold with boundary when a is a critical value.*

We will see in the next theorem that the topology of the level sets does not change as long as we do not cross a critical value.

Theorem 4.1. *Let a and b be two real numbers such that f does not have any critical value in the interval $[a,b]$. We suppose that $f^{-1}([a,b])$ is compact. Then V^b is diffeomorphic to V^a.*

Proof. The idea of the proof is to construct a vector field Y such that for every $x \in [a,b]$, f decreases constantly when x is pushed along the flow line corresponding to Y. To do that, let X be a pseudo-gradient of f. We define $Y\colon V \to \bigsqcup_{x\in V} T_xV$, such that

$$\begin{cases} -\frac{X_x}{(df)_x(X_x)} & x \in f^{-1}([a,b]) \\ 0 & \text{outside of a compact neighborhood of } f^{-1}([a,b]). \end{cases}$$

By construction, Y has compact support and therefore the flow ϕ^s is well-defined for every $s \in \mathbb{R}$. Now, note that for every $x \in V$ and $\phi^s(x) \in f^{-1}([a,b])$ we have:

$$\begin{aligned} \frac{\partial}{\partial s} f \circ \phi^s(x) &= (df)_{\phi^s(x)}(\frac{\partial}{\partial s}\phi^s(x)) \\ &= (df)_{\phi^s(x)}(Y_{\phi^s(x)}) \\ &= -1. \end{aligned}$$

Thus, we obtain that for $\phi^s(x) \in f^{-1}([a,b])$:

$$f \circ \phi^s(x) = -s + f(x).$$

In particular, ϕ^{b-a} is a diffeomorphism of V such that $\phi^{b-a}(V^b) = V^a$. □

Remark 4.4. *The map*

$$\begin{aligned} r\colon V^b \times [0,1] &\to V^a \\ (x,s) &\mapsto \begin{cases} x & f(x) \le a \\ \phi^{s(f(x)-a)}(x) & a \le f(x) \le b \end{cases} \end{aligned}$$

is a deformation retract of V^b onto V^a.

As an application of Theorem 4.1, we will have a look at *Reeb's theorem*:

Corollary 4.1 (Reeb's theorem)**.** *Let V be a compact manifold. Suppose that there exists a Morse function on V that has only two critical points. Then V is homeomorphic to a sphere.*

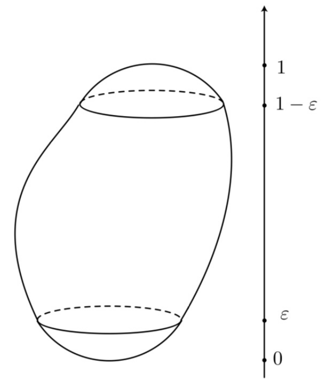


Figure 4.4: Reeb's theorem. [ADE13]

Proof. As V is compact and f has only two critical points, those two points have to be a minimum a and a maximum b of f. Also, we may assume that $f(V) = [0, 1]$. Then, by the Morse Lemma, it holds that there exist neighborhoods U_a and U_b as well as diffeomorphisms $\varphi_a\colon U_a \to \mathbb{R}^n$ and $\varphi_b\colon U_b \to \mathbb{R}^n$ such that

$$f \circ \varphi_a^{-1}(x_1, \dots, x_n) = \sum_{j=1}^{n} x_j^2$$

and

$$f \circ \varphi_b^{-1}(x_1, \dots, x_n) = 1 - \sum_{j=1}^{n} x_j^2.$$

It is now easy to see that we can choose $\epsilon > 0$ small such that $f^{-1}([0, \epsilon])$ and $f^{-1}([1-\epsilon, 1])$ are homeomorphic to discs $D^n \subset \mathbb{R}^n$. Indeed,

$$f^{-1}([0, \epsilon]) \approx \{x \in \mathbb{R}^n\colon \sum_{j=1}^{n} x_j^2 \leq \epsilon\} \approx D^n,$$

$$f^{-1}([1-\epsilon, 1]) \approx \{x \in \mathbb{R}^n\colon \underbrace{1 - \epsilon \leq 1 - \sum_{j=1}^{n} x_j^2 \leq 1}_{\iff \epsilon \geq \sum_{j=1}^{n} x_j^2 \geq 0}\} \approx D^n$$

By assumption, there are no critical values in the interval $[0, 1]$, so by Theorem 4.1 $f^{-1}([0, \epsilon])$ is diffeomorphic to $V^{1-\epsilon}$. It follows now that V is homeomorphic to the discs $V^{1-\epsilon}$ and $f^{-1}([1-\epsilon, 1])$ glued together on their boundaries. $\square$

A Morse function determines the CW complex structure of a manifold V. The next theorem makes this precise:

Theorem 4.2. *Let $f\colon V \to \mathbb{R}$ be a function. Let a be a nondegenerate critical point of index k of f and let $\alpha = f(a)$. We suppose that for some sufficiently small $\epsilon > 0$, the set $f^{-1}([\alpha - \epsilon, \alpha + \epsilon])$ is compact and does not contain any critical point of f other than a. Then for every sufficiently small $\epsilon > 0$, the homotopy type of the space $V^{\alpha+\epsilon}$ is that of $V^{\alpha-\epsilon}$ with a cell of dimension k attached (the unstable manifold of a).*

Proof. We start by giving the main ideas of the proof. It consists of 3 steps:

1. We construct a function F, which coincides with f outside of a neighborhood U of a and for which $F < f$ inside the neighborhood U of a. For this function, $F^{-1}(]-\infty, \alpha-\epsilon]) = V^{\alpha-\epsilon} \cup H$, where H is a small neighborhood of a.

2. We show that $F^{-1}(]-\infty, \alpha-\epsilon]) = V^{\alpha-\epsilon} \cup H$ is a deformation retract of $V^{\alpha+\epsilon}$.

3. We show that $V^{\alpha-\epsilon} \cup D^k$, where D^k is a piece of the unstable manifold of a, is a deformation retract of $F^{-1}(]-\infty, \alpha-\epsilon]) = V^{\alpha-\epsilon} \cup H$.

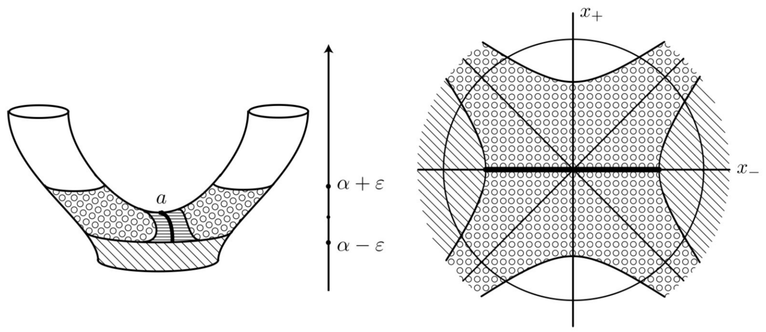


Figure 4.5: Morse chart used in the construction of F. [ADE13]

Step 1: Construction of F
We choose a Morse chart (U, h) in the neighborhood of a and an $\epsilon > 0$ small enough such that $f^{-1}([\alpha-\epsilon, \alpha+\epsilon])$ is compact and such that U contains the ball of radius $\sqrt{2\epsilon}$ with center 0. The disc D^k is the subset of U consisting of (x_-, x_+) such that $|x_-|^2 < \epsilon$ and $x_+ = 0$. In Figure 4.5, the sublevel set $V^{\alpha-\epsilon}$ is indicated with oblique hatching while $f^{-1}([\alpha-\epsilon, \alpha+\epsilon])$ is dotted. The cell, the disc D^k, is a tick line segment. We construct F by using a C^∞ function $\nu\colon [0, +\infty] \to [0, +\infty[$ with the following properties:

1. $\nu(0) > \epsilon$;

2. $\nu(s) = 0$ for $s \geq 2\epsilon$;

3. $-1 < \nu'(s) \leq 0$ for every s.

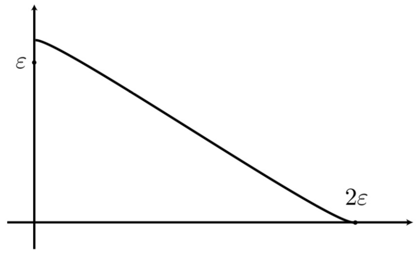


Figure 4.6: The function ν. [ADE13]

We the define F by setting:

$$F(x) = \begin{cases} f(x) & x \notin \Omega(a) \\ f(x) - \nu(|x_-|^2 + 2|x_+|^2) & x = h(x_-, x_+). \end{cases}$$

Define then $H\colon = \overline{F^{-1}(]-\infty, \alpha - \epsilon]) \setminus V^{\alpha-\epsilon}}$. We have then $F^{-1}(]-\infty, \alpha\epsilon]) = V^{\alpha-\epsilon} \cup H$ and H is the horizontally hatched part in the left image in 4.6.
Step 2: $F^{-1}(]-\infty, \alpha-\epsilon])$ is a retract of $V^{\alpha+\epsilon}$

Claim 4.1. *It holds that $F^{-1}(]-\infty, \alpha+\epsilon]) = V^{\alpha+\epsilon}$.*

Proof. Outside of the ellipsoid $|x_-|^2 + 2|x_+|^2 \leq 2\epsilon$, it holds that $F = f$. Inside of the ellipsoid it holds:

$$F(x) \leq f(x) = \alpha - |x_-|^2 + |x_+|^2 \leq \alpha + \frac{1}{2}|x_-|^2 + |x_+|^2 \leq \alpha + \epsilon$$

In particular, all the points inside the ellipsoid are contained in both sublevel sets. Thus, the claim follows. □

Claim 4.2. *F and f have the same critical points.*

Proof. By considering the coordinates inside the Morse chart U, we obtain:

$$dF = (-1 - \nu'(|x_-|^2 + 2|x_+|^2))2x_- \cdot dx_- + (1 - 2\nu'(|x_-|^2 + |x_+|^2))2x_+ \cdot dx_+$$

which vanishes only for $x_- = x_+ = 0$, so exactly for a. □

Claim 4.3. *$F^{-1}(]-\infty, \alpha-\epsilon])$ is a deformation retract of $V^{\alpha+\epsilon}$.*

Proof. By the first claim together with the fact that $F \leq f$, it follows that $F^{-1}([\alpha-\epsilon, \alpha+\epsilon]) \subset f^{-1}([\alpha-\epsilon, \alpha+\epsilon])$. In particular, this means that $F^{-1}([\alpha-\epsilon, \alpha+\epsilon])$ is compact. Thus, we may apply Theorem 4.1 to argue that $F^{-1}(]-\infty, \alpha-\epsilon])$ is diffeomorphic to $F^{-1}(]-\infty, \alpha+\epsilon]) = V^{\alpha+\epsilon}$. Indeed, $F^{-1}([\alpha-\epsilon, \alpha+\epsilon])$ does not contain any critical points of F: the only possible candidate would be a, but $F(a) = \alpha - \nu(0) < \alpha - \epsilon$. □

Step 3: $V^{\alpha-\epsilon} \cup D^k$ is a deformation retract of $V^{\alpha-\epsilon} \cup H$.
In this step, we construct a retraction from H onto D^k.

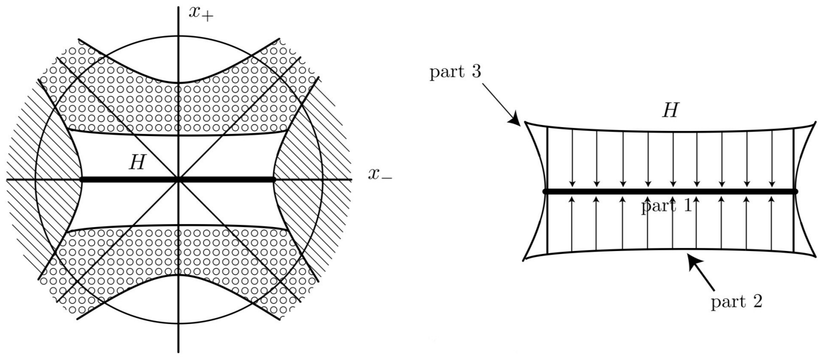


Figure 4.7: Deformation retract from H onto D^k. [ADE13]

We define the retraction r by following the arrows indicated in 4.7. Outside of $\Omega(a)$, r is the identity. We define r on U as follows:

1. On region 1 (Figure 4.7), so on $|x_-| \leq \epsilon$, as $r(t,(x_-,x_+)) = (x_-, tx_+)$.
2. On region 2, defined by $\epsilon \leq |x_-|^2 \leq |x_+|^2$, we set $r(t,(x_-,x_+)) = \left(x_-, (t + (1-t)\frac{\sqrt{|x_-|^2-\epsilon}}{|x_+|})x_+\right)$.
3. On region 3, which corresponds to $V^{\alpha-\epsilon}$ and where $|x_+|^2 + \epsilon \leq |x_-|^2$, we simply define r as the identity.

□

Example 4.2. *Let us now have look at the decomposition of the standing torus into a CW-complex:*
Step 1: *Point a is a critical point of index 0 with $f(a) = 0$. For a small $\epsilon > 0$, $V^{f(a)+\epsilon}$ is a simple 0-cell, which has the same homotopy type as a 2-cell:*

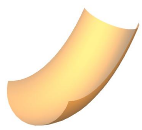
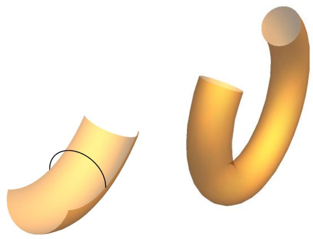


Figure 4.8: 2 cell.

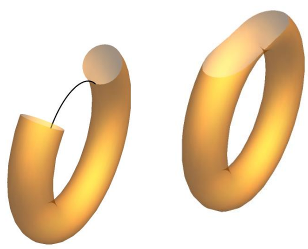
Step 2: *Point b is a critical point of index 1. For a small $\epsilon > 0$, $V^{f(b)+\epsilon}$ has the same homotopy type as a 2-cell (Figure 4.8) with a cell of dimension 1 attached:*



(a) (b)

Figure 4.9: (a) and (b) have the same homotopy type

Step 3: *Point c is a critical point of index 1. For a small* $\epsilon > 0$, $V^{f(c)+\epsilon}$ *has the same homotopy type as Figure 4.9 with a cell of dimension 1 attached:*



(a) (b)

Figure 4.10: (a) and (b) have the same homotopy type.

Step 4: *Point d is a critical point of index 2.* V *has the same homotopy type as 4.10 with a cell of dimension 2 attached:*

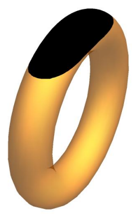


Figure 4.11: The standing torus.

4.5 The Smale Condition

In this section, we introduce the *Smale Condition.* Depending on whether all the stable and unstable manifolds meet *transversally*, we say that pseudo-gradient field adapted to a Morse function satisfies the *Smale Condition.* We start with an example.

Example 4.3. *Consider again the height function on the standing torus with pseudo-gradient field is simply the gradient for the metric induced by that on* $\mathbb{R}^3$. $W^s(a)$ *consists of the whole torus but the point* d *and the circle passing through* b *and* c *and* $W^u(a) = \{a\}$. $W^u(d)$ *consists of the whole torus but the circle passing trough* b *and* c *and* $W^s(d) = \{d\}$. *For the critical points* b *and* c, *consider Figure 4.12.*

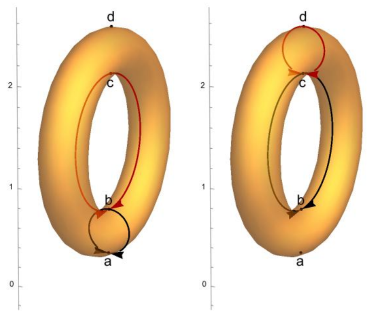


Figure 4.12: Stable manifolds (red) and unstable manifolds (black) of the critical points b and c of the height function on the standing torus.

In this example, the stable manifold of b *and the unstable manifold of* c *have two open intervals in common. As the torus can be parametrized over the set* $[0, 2\pi) \times [0, 2\pi)$, *we can also illustrate the flow lines in the square with side lenghts* 2π.

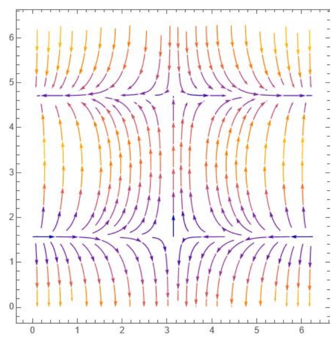


Figure 4.13: Different point of view the flow lines of the height function on the standing Torus

In contrast, look at the height function on the tilted torus and the corresponding flow lines.

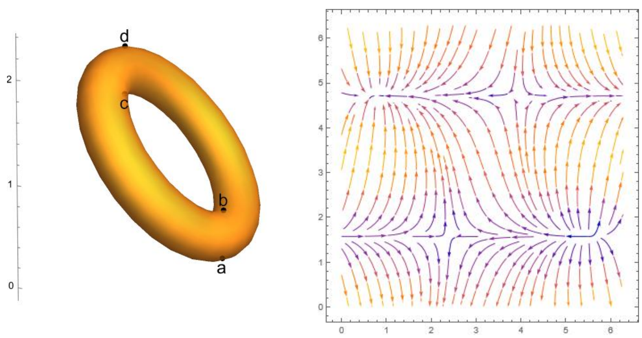


Figure 4.14: Flow lines of the height function on the tilted torus.

The unstable manifold of c and the stable manifold of b do not have two open intervals in common.

In general, the intersection of two submanifolds M and N of a manifold V is not a submanifold itself. However, if two submanifolds intersect in a certain way, their intersection is a submanifold.

Definition 4.8 (**Transversality**)**.** *Let V be a manifold. We say that two submanifolds M and N of V are* **transverse** *at the point $v \in V$ if*

$$\text{either } v \notin M \cap N \text{ or } v \in M \cap N \text{ and } T_vM + T_vN = T_vV.$$

We call M and N transverse*, denoted by $M \pitchfork N$, if they are transverse at every point.*

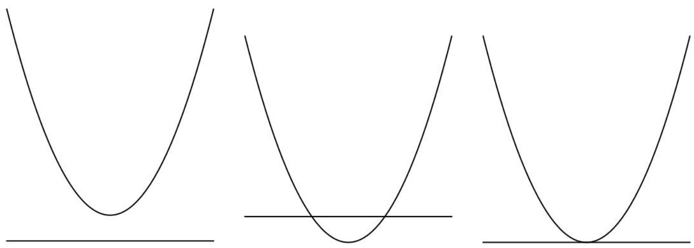


Figure 4.15: The curves are transverse in the left and middle plot. They are not transverse in the right plot.

Proposition 4.5. *Let M and N be two submanifolds of a manifold V. If $M \pitchfork N$, then $M \cap N$ is a submanifold of V whose codimension is equal to* $\operatorname{codim}(M) + \operatorname{codim}(N)$*.*

Definition 4.9 (**Smale Condition**)**.** *We say that a pseudo-gradient field X adapted to a Morse fuction $f\colon V \to \mathbb{R}$ satisfies the* **Smale condition** *if*

$$W^u(a) \pitchfork W^s(b) \quad \text{for all } a, b \in \operatorname{Crit}(f).$$

Example 4.4. *The height function on the standing torus with the pseudo-gradient field induced by the metric on $\mathbb{R}^3$ does not satisfy the Smale condition as the stable manifold of b and the unstable manifold of c do not intersect* *transversally. The height function on the tilted torus satisfies the Smale condition.*

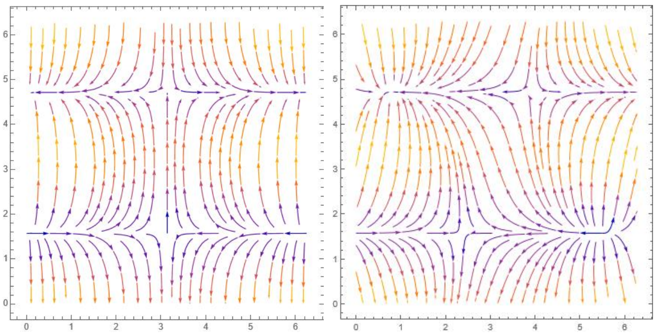


Figure 4.16: Left: flow lines of the height function on the standing torus, Right: flow lines of the height function on the tilted torus.

Some stable and unstable manifolds always meet transversally. It always holds:

1. $W^u(a) \pitchfork W^s(a)$, this can be easily seen in a Morse chart as in Section 4.2 where V_+ and V_- intersect transversally.

2. $W^u(a) \cap W^s(b) =$ if a and b are distinct and $f(a) \leq f(b)$. This implies in particular that $W^u(a) \pitchfork W^s(b)$.

Proposition 4.6. *Let $f\colon V \to \mathbb{R}$ be a Morse function and X a pseudo-gradient field adapted to it satisfying the Smale condition. Then*

$$\operatorname{codim}(W^u(a) \cap W^s(b)) = \operatorname{codim}(W^u(a)) + \operatorname{codim}(W^s(b)).$$

In particular, $\dim(W^u(a) \cap W^s(b)) = \operatorname{Ind}(a) - \operatorname{Ind}(b)$.

Proof. This follows immediately from Proposition 4.5. □

In particular, if the pseudo-gradient field satisfies the *Smale condition*, the corresponding flow lines never connect critical points $a, b \in V$ with the same index, as Proposition 4.6 would then imply that $\dim(W^u(a) \cap W^s(b)) - 1$. Assume now that the pseudo-gradient field satisfies the *Smale condition* and consider the set

$$W(a,b) = \{x \in V\colon \lim_{s \to -\infty} \varphi^s(x) = a \text{ and } \lim_{s \to +\infty} \varphi^s(x) = b\},$$

which consists of all points on the flow lines connecting a and b. Proposition 4.5 implies in that case that $W(a,b)$ is a manifold. The group $\mathbb{R}$ acts then on $W(a,b)$.

Proposition 4.7. *The group $\mathbb{R}$ of translations in time acts on $W(a,b)$ by $s \cdot x = \varphi^s(x)$. In addition, this action is free if $a \neq b$.*

Proof. It is clear that this is a group action. If $a \neq b$, there is no critical point in $W(a,b)$. Let $x \in W(a,b)$, since x is not a critical point, we know that $f(\varphi^s(x))$ is a strictly decreasing function of s, so that if $\varphi^s(x) = \varphi^{s'}(x)$, it follows that $s = s'$. Hence the action is free. □

We can consider now the quotient $\mathscr{L}(a,b) = W(a,b)/\mathbb{R}$, which is a manifold of dimension $\operatorname{Ind}(a) - \operatorname{Ind}(b) - 1$. In particular, it follows that $\mathscr{L}(a,b)$ is a 1-dimensional manifold if $\operatorname{Ind}(a) = \operatorname{Ind}(b) + 2$. This will play a role in Section 5.

4.6 Existence of Pseudo-Gradients

In this section, we will show that every pseudo-gradient field can be approximated by a pseudo-gradient satisfying the Smale condition.

Definition 4.10 (C^1**-proximity**). *Let V be a manifold and X a vector field. We say that a vector field X' is* **close** *to X with respect to $\epsilon > 0$ if for every cover V by charts $\varphi_i(U_i)$ and for every compact subset $K_i \subset U_i$, it holds that $||T\varphi_i^{-1}(X') - T\varphi_i^{-1}(X)||_{C^1} < \epsilon$ for the C^1-norm on K_i.*

Theorem 4.3 (**Smale Theorem**). *Let V be a manifold with boundary and let f be a Morse function on V with distinct critical values. We fix Morse charts in the neighborhood of each critical point of f. Let Ω be the union of these charts and let X be a pseudo-gradient field on V that is transversal to the boundary. Then for every $\epsilon > 0$ there exists a pseudo-gradient field X' that is close to X in the C^1-sense with respect to ϵ, equals X on Ω and for which we have*

$$W^s_{X'}(a) \pitchfork W^u_{X'}(b)$$

for all critical points a, b of f.

Proof. The proof relies on the following lemma.

Lemma 4.1. *Let $j \in \{1, \dots, q\}$ and let $\epsilon > 0$. There exists a vector field X' which is close to X with respect to ϵ such that:*

1. *The vector field X' coincides with X on the complement of $f^{-1}([\alpha_j + \epsilon, \alpha_j + 2\epsilon])$ in V.*

2. *The stable manifold of c_j (for X') is transversal to the unstable manifolds of all critical points, that is,*

$$W^s_{X'}(c_j) \pitchfork W^u_{X'}(c_i).$$

We use Lemma 4.1 to prove the theorem. We do a proof by induction over the number of critical points. Let $\mathscr{P}(r)$ denote the following property: Let $\epsilon > 0$, then there exists vector field X'_r which is close to X with respect to ϵ such that for every $p \leq r$ and every i, we have

$$W^s_{X'_r}(c_p) \pitchfork W^u_{X'_r}(c_i).$$

Then, we have that $\mathscr{P}(q)$ is exactly the theorem. We start with the **base case** $\mathbf{q = 1}$: $\mathscr{P}(1)$ is trivially true as c_1 is the maximum of f and $W^s(c_1) = \{c_1\}$. The **case** $\mathbf{q = 2}$ follows from the lemma with $j = 2$. We assume now that $\mathscr{P}(r-1)$ is true and we want to show that $\mathscr{P}(r)$ is also true. We apply the lemma to the vector field X'_{r-1} and $j = r$. Then, we obtain a vector field X'_r that coincides with X'_{r-1} outside of the narrow strip where $\alpha_r + \epsilon \leq f \leq \alpha_r + 2\epsilon$. In particular, since for every $p \leq r-1$, the stable manifold of c_p for X'_{r-1} lies above this strip, the stable manifold is the same for X'_{r-1} as it is for X'_r (see Figure 4.17).

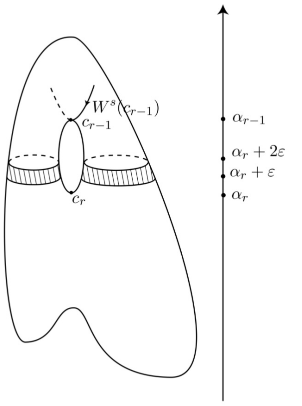


Figure 4.17

We therefore have

$$W^s_{X_r'}(c_p) \cap W^u_{X_r'}(c_i) = W^s_{X_r'}(c_p) \cap W^u_{X_r'}(c_i)$$

for $p \leq r-1$, which implies for every i

$$W^s_{X_r'}(c_p) \pitchfork W^u_{X_r'}(c_i).$$

For $p = r$, the lemma implies

$$W^s_{X_r'}(c_r) \pitchfork W^u_{X_r'}(c_i).$$

□

4.7 Classification of 1-dimensional compact Manifolds and Brouwer Fixed Point Theorem

In this section, we show that there are only two types of compact connected 1-dimensional manifolds. We use Morse functions to prove this. Additionally, we provide a proof for the *Brouwer Fixed Point Theorem*.

Theorem 4.4. *Let V be a compact connected manifold of dimension 1. Then V is diffeomorphic to S^1 if $\partial V = \emptyset$ and diffeomorphic to $[0,1]$ otherwise.*

Proof. Let X be a vector field that is incoming along the boundary and let f be a Morse function for which X is a adapted pseudo-gradient field. As V is 1-dimensional, all the critical points are local maxima or minima. So denote by $\{c_1, ..., c_k\}$ the minima of f. We know by a Proposition 4.3, that for every $1 \leq i \leq k$, the stable manifold $W^s(c_i)$ is diffeomorphic to an open interval. Let A_i be the closure of $W^s(c_i)$.
So, let us have a closer look at A_i for $1 \leq i \leq k$. A_i consists of $W^s(c_i)$ and the starting points of the two trajectories. These starting points:

1. either are both maxima, in which case they can either coincide or not

2. or at least one of them is a boundary point of V, in which case they are distinct.

The image above illustrates the two cases.

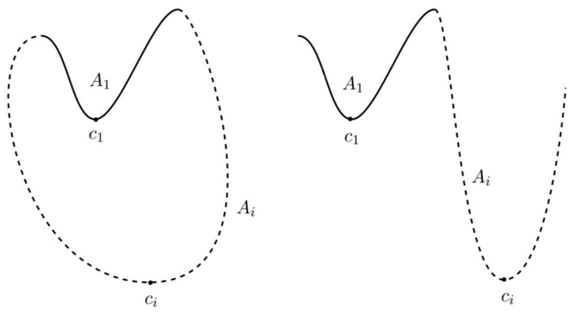


Figure 4.18: Left: Both starting points are maxima (they do not coincide), Right: One of the starting points is a boundary point

The proof is now based on the idea that every point $x \in V$ is contained in some A_i and that if $A_i \cap A_j \neq \emptyset$ for some $1 \leq i, j \leq k$, it holds that either $A_i \cup A_j$ is diffeomorphic to S^1 or diffeomorphic to $[0, 1]$. Indeed, if $x \in V$ is not a maximum, it is either a minimum or it holds that $df_x(X) < 0$ and x is thus contained in some stable manifold. Otherwise, x is a maximum and is contained in the closure of some stable manifold. If $k = 1$, then $A_1 = V$. Clearly, if A_1 consists of $W^s(c_1)$ and one additional point, the latter is a maximum and then A_1 is diffeomorphic to a S^1. Otherwise, A_1 consists of $W^s(c_1)$ and two additional points and A_1 is diffeomorphic to a closed interval. If $k \geq 2$, since V is connected, there exists $i \geq 2$, such that $A_1 \cap A_i \neq \emptyset$. Note that this intersections contains only local maxima, since these are the only points from which we can descend to two different minima. In $A_1 \cup A_i$, there are at most two points, which are either:

1. two maxima, hence $A_1 \cup A_i$ is diffeomorphic to S^1, which finishes the proof
2. only one points, then $A_1 \cup A_i$ is diffeomorphic to $[0, 1]$, if $A_1 \cup A_i = V$, we are done.

Otherwise, we continue adding $A_i's$ until they run out. □

Theorem 4.5 (Brouwer Fixed Point Theorem)**.** *Let $\varphi \colon D^n \to D^n$ be a continuous map, then it has a fixed point.*

Proof. We are going to do a proof by contradiction. It is possible to reduce to the case where φ is a C^∞ map. So, we may assume now that φ is a C^∞ map without any fixed points. Consider the retraction

$$r \colon D^n \to S^{n-1}$$

which sends $x \in D^n$ to the intersection point of the sphere S^{n-1} and the ray starting at $\varphi(x)$ and going through x. This is indeed a retraction as it it continuous and restricts to the identity on S^n. Theorem 3.1 (Sard's theorem) guarantees now that this map has regular values. As S^{n-1} has not measure 0, this means that there is a point $a \in S^{n-1}$, such that a is a regular value. Then, we know by regular value theorem, that $r^{-1}(a)$ is a submanifold of dimension 1 of D^n with boundary

$$\partial r^{-1}(a) = r^{-1}(a) \cap \partial D^n = \{a\}.$$

But a manifold of dimension 1 with boundary is diffeomorphic to a union of circles and closed intervals, (Note that we considered in Theorem 4.4 compact manifolds) such that its boundary consists of an even number of points. This is a contradiction, hence there is a fixed point. $\square$

Chapter 5

Morse Homology

In this chapter we consider a compact manifold V endowed with a Morse function f and a generic pseudo-gradient field X, where a *generic field* is such that is satisfies the *Smale condition.* The goal is to introduce *Morse homology* and we then have a look at some applications. This chapter is based on [ADE13].

5.1 Morse-Smale Chain Complex

The Morse chain complex is defined as follows:

Definition 5.1 (**Morse chain complex**). *1. $C_k(f)$ is defined as $C_k(f) = \sum_{c \in Crit_k(f)} a_c c \mid a_c \in \mathbb{Z}/2\mathbb{Z}\}$.*

2. The differential ∂_k is defined as follows:

$$\partial_k(a) = \sum_{b \in Crit_{k-1}(f)} n_X(a,b)b,$$

where $a \in Crit_k(f)$ and $n_X(a,b)$ is the number modulo 2 of trajectories of X going from a to b. Note that as ∂_k is a homomorphism, it is indeed enough to define it for critical points.

Proposition 5.1. *$(C_k(f), \partial_k)$ is a chain complex.*

Proof. We will give a rather heuristic proof. For details, consider [ADE13] p.57-63. Let a be a critical point of index $k+1$, then we have:

$$(\partial \circ \partial)(a) = \sum_{b \in Crit_{k-1}} \left(\sum_{c \in Crit_k} n_X(a,c) n_X(c,b) \right) b$$

So, we need to prove that given two critical points, a of index $k+1$ and b of index $k-1$, the sum

$$\sum_{c \in Crit_k} n_X(a,c) n_X(c,b)$$

is an even number. This number equals the cardinality of the disjoint union

$$\coprod_{c \in Crit_k(f)} \mathscr{L}(a,c) \times \mathscr{L}(c,b) =: \mathscr{L}$$

We want to argue now that $\mathscr{L}$ is the boundary of a 1-dimensional manifold, as those always have an even number of boundary points. Remember, that as a as index $k+1$ and b has index $k-1$, we have that $\dim(\mathscr{L}(a,b)) = 1$ and $\mathscr{L}(a,b)$ is a union of open intervals. We want to argue now that $\mathscr{L}$ can be identified with the boundary of $\overline{\mathscr{L}(a,b)}$. By considering Figure 5.1, we see that $\mathscr{L}(a,b)$ must be a union of open intervals, which can be compactified by adding the 'broken' trajectories, passing through the critical points c_1 and c_2.

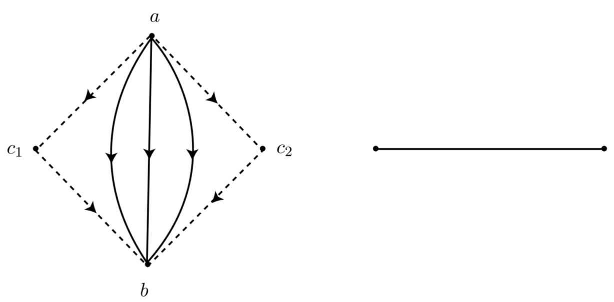


Figure 5.1: Left: Trajectories from a to b with added trajectories. Right: $\mathscr{L}(a,b) \cup \mathscr{L}$. [ADE13]

□

The height function on the standing torus does not satisfy the *Smale condition*, in contrary to the tilted torus, which we consider in the next example.

Example 5.1 (Tilted Torus). *Consider the height function f on the tilted torus.*

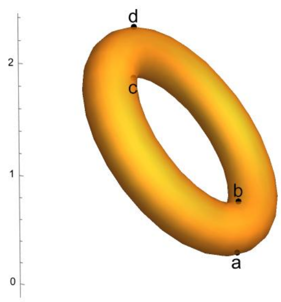


Figure 5.2: Height function on the tilted torus with 4 critical points.

As for the standing torus, we have 4 critical points. a is a critical point of index 0, b and c are critical points of index 1 and d is a critical point of index 2. Thus, we have that $C_0(f) = \langle a \rangle$, $C_1(f) = \langle b, c \rangle$ and $C_2(f) = \langle d \rangle$. By counting the flow lines connecting the critical points naively, we see that

$$\partial(d) = 2c + 2b \text{ and } \partial(c) = \partial(b) = 2a.$$

Hence, the homology is

$$\begin{cases} H_0 = \mathbb{Z}/2 \\ H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\ H_2 = \mathbb{Z}/2. \end{cases}$$

5.2 Applications of Morse Homology

In this section, we are going to state some fundamental theorems an applications of *Morse homology*, one of which is that the it does neither depend on the Morse function, nor on the pseudo-gradient field. We will then mainly focus on how Morse homology relates to the connectivity of the manifold.

Theorem 5.1. *Let V be a compact manifold. Let $f_0, f_1\colon \to \mathbb{R}$ be two Morse functions and let X_0, X_1 be pseudo-gradients adapted to f_0 and f_1, respectively, with the Smale property. Then there exists a morphism of complexes*

$$\Phi_*\colon (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

that induces an isomomorphism in the Morse homology.

Proof. [ADE13] P.69-71. □

From now on, we write $\mathrm{HM}_k(V, \mathbb{Z}/2)$ for the $k-$th homology group, which we are allowed to do by Theorem 5.1. We now use Theorem 5.1 to show that for a manifold V and two Morse functions f and f' that $|\mathrm{Crit}(f)| \equiv |\mathrm{Crit}(f')|$ modulo 2.

Lemma 5.1. *The number of critical points modulo 2 of a Morse function depends only on the manifold and not on the function.*

Proof. Let $f\colon V \to \mathbb{R}$ be an arbitrary Morse function of V. Consider the chain complex associated to it and a pseudo-gradient field:

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Then, we obtain the following equalities:

$$\begin{aligned} \#\mathrm{Crit}(f) &= \sum_{k=0}^{n} \dim\, C_K(f) \\ &\overset{(*)}{=} \sum_{k=0}^{n+1} (\dim\,\mathrm{Ker}\ \partial_k + \dim\,\mathrm{Im}\ \partial_k) \\ &\overset{(**)}{=} \sum_{k=0}^{n} (\dim\,\mathrm{Ker}\ \partial_k + \dim\,\mathrm{Im}\ \partial_{k+1}) \\ &= \sum_{k=0}^{n} (\dim\,\mathrm{Ker}\ \partial_k - \dim\,\mathrm{Im}\ \partial_{k+1}) \ \ mod\ 2 \\ &= \sum_{k=0}^{n} \dim\,\mathrm{HM}_k(V, \mathbb{Z}/2) \ \ \mathrm{mod}\ 2, \end{aligned}$$

where $(*)$ follows from rank-nullity and $(**)$ follows since $\dim\,\mathrm{Ker}\ \partial_{n+1} = \dim\,\mathrm{Im}\partial_0 = 0$. □

In a very similar manner, we can show the following result:

Proposition 5.2. *The number of critical points of a Morse function on a manifold V is greater than or equal the sum of the dimensions of the Morse (modulo 2) homology groups of this manifold.*

Proof. Again, as in Lemma 5.1, we get that:

$$\begin{aligned}\#\mathrm{Crit}(f) &= \sum_{k=0}^{n} (\dim \mathrm{Ker}\ \partial_k + \dim \mathrm{Im}\ \partial_{k+1}) \\ &\geq \sum_{k=0}^{n} (\dim \mathrm{Ker}\ \partial_k - \dim \mathrm{Im}\ \partial_{k+1}) \\ &= \sum_{k=0}^{n} \dim \mathrm{HM}_k(V, \mathbb{Z}/2).\end{aligned}$$

□

Example 5.2. *Consider again the height function on the tilted torus torus. We have calculated in Example* 5.1 *the homology groups, it follows now from Proposition* 5.2 *that every Morse function on the tilted torus must have at least 4 critical points.*

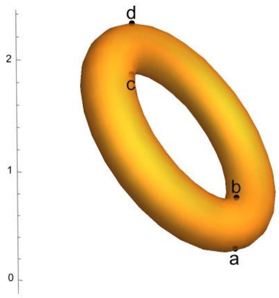


Figure 5.3: Height function on the tilted torus with 4 critical points.

It is easy to see that for a Morse function $f\colon V \to \mathbb{R}$, the critical points of index k are exactly the critical points of index $n-k$ of the Morse function $-f$. Additionally, if X is a pseudo-gradient field adapted to f, then $-X$ is a pseudo-gradient adapted to $-f$. As a consequence, we get the following result:

Proposition 5.3. *Let V be a compact manifold (without boundary) of dimension n, then* $\mathrm{HM}_{n-k}(V, \mathbb{Z}/2)$ *is isomorphic to* $\mathrm{HM}_k(V, \mathbb{Z}/2)$.

Proof. If $a \in \mathrm{Crit}_k(f)$, then let a^* denote the same point regarded as a critical point of the function $-f$. It holds then that $\{a\colon a \in \mathrm{Crit}_k(f)\}$ is a basis of $C_k(f)$ and $\{a^*\colon a \in \mathrm{Crit}_k(f)\}$ is a basis of $C_{n-k}(f)$. The vector space $C_{n-k}(-f) = C_k(f)^*$. Thus, the transpose of $\partial_X\colon C_k(f) \to C_{k-1}(f)$ is the differential of ∂_{-X} and the result follows. □

5.3 Homology and Connectedness

Morse homology also gives insight in the number of connected components of the associated manifold.

Proposition 5.4. *If V is a compact connected manifold, then*

$$\mathrm{HM}_0(V, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Proof. Let $\{a_1, \ldots, a_r\}$ be the critical points of f. We consider the $r-1$-dimensional vector subspace of $C_0(f)$ generated by the a_1+a_i for $2 \leq i \leq r$. If we can show that B is exactly the image of $\partial_X : C_1(f) \to C_0(f)$. We will only show $\partial_X(C_1(f)) \subset B$, ass the proof of the other inclusion relies on the theory of *broken trajectories*, which goes beyond the scope of this thesis. For details, we point the reader to [ADE13] p.88-89. We continue with showing $\partial_X(C_1(f)) \subset B$. Let c be a critical points of index 1. It follows from Proposition 4.3, that the unstable manifold of c is 1-dimensional. Therefore, there are only two trajectories of X starting from c. Both end up at two local minimia a_j and a_i. We obtain that

$$\partial_X c = a_j + a_i = (a_1 + a_j) + (a_1 + a_i) \in B,$$

thus $\partial_X(C_1(f)) \subset B$. □

By Proposition 5.4, we can deduce the following result.

Corollary 5.1. *Let V be a compact connected manifold of dimension n, then* $\mathrm{HM}_n \cong \mathbb{Z}/2$.

Proof. This follows immediately from Proposition 5.3. □

In Proposition 5.4 and Corollary 5.1, we have only considered connected manifolds. In general, the dimension of the n-th and 0-th Homology group coincides with the number of connected components of V.

Lemma 5.2. *Let V be a compact manifold of dimension n. Then* $\mathrm{HM}_0(V, \mathbb{Z}/2)$ *and* $\mathrm{HM}_n(V, \mathbb{Z}/2)$ *are $\mathbb{Z}/2$ spaces of dimension the number of connected components of V.*

Proof. The proof relies on the fact that flow lines run within the same connected component. We can write V as the disjoint union of its N connected component

$$V = \bigsqcup_{i=1}^{N} V_i.$$

We choose a Morse funcion f_i and a pseudo-gradient field X_i on every connected component. It holds that

$$C_k(\bigsqcup_{i=1}^{N} f_i) = \bigoplus_{i=1}^{N} C_k(f_i) \text{ and } \partial_{\sqcup X_i} = \oplus_{\partial_{X_i}}$$

In particular,

$$\mathrm{HM}_k(V, \mathbb{Z}/2) = \bigoplus_{i=1}^{N} \mathrm{HM}_k(V_i, \mathbb{Z}/2).$$

The result follows now from Proposition 5.4 and Corollary 5.1. □

Proposition 5.5. *If the manifold V admits a Morse function with no critical points of index 1, then it is simply connected.*

Proof. We may assume that V is pathwise connected. We choose a minimum a_0 of f as a base point. Let C be any differentiable loop through a_0 in V. Let b be a critical point of index k, it follows then from Proposition 4.3, that $\dim W^s(b) = n - k$. We may deform C in such a way that C meets none of the stable manifolds of the critical points of index greater than or equal to 2, in particular the loop C is then contained in the disjoint union of the stable manifolds of the local minima. As V is pathwise connected, C is contained in exactly one of these manifolds, namely that of a_0. As this is a disc, C is contractible. □

Morse homology also reflects additional features, like counting the number of holes of a space, much like Singular homology. Therefore, it is not surprising that Morse homology is the same as the *cellular homology* ([ADE13] P110-121), especially coinciding with the *Singular homology.*

Chapter 6

Spectral Surface Quadrangulation via Morse Theory

A *mesh* is a discretization of a geometric domain into small simple shapes, such as triangles or quadrilaterals in two dimensions and tetrahedra or hexahedra in three.

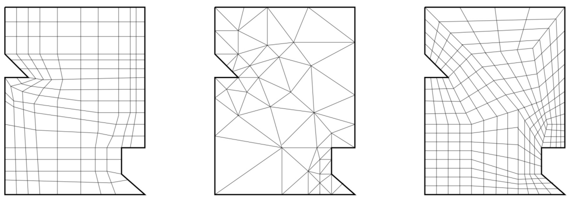


Figure 6.1: Three example meshes. [MB]

Raw surface input meshes are often generated from laser scanning, isosurface extraction, or other methods. However, they frequently suffer from irregularities and noise, which hinder the accuracy and efficiency of subsequent operations. One such subsequent operation is for example *surface reconstruction*, a common task in computer graphics and computer-aided design (CAD). Surface reconstruction aims to create a smooth and accurate representation of a surface from a set of points or a mesh. Much of the remeshing work in the graphics literature focuses on triangle meshes, tough many scientific applications benefit from good quadrilateral meshes. The particular remeshing process we will focus on is called *Spectral Surface Quadrangulation* ([Don+06]). This approach uses Laplacian eigenfunctions of a manifolds input meshes, which are the natural harmonics of the surface. They have the property that their extrema are distributed evenly accross a mesh, which connect via gradient flow into a quadrangular base mesh. The process of remeshing will consist of 3 steps:

1. Create a quadrangular base mesh, which is the *primal* or *dual* Morse-Smale complex according to a certain Laplacian eigenfunction.

2. Improve this quadrangular base mesh layout with an iterative relaxation algorithm which simultaneously produces a globally smooth parametrization of the complex.

3. Construct another well-shaped quadrilateral mesh with few extraordinary vertices from this new improved quadrilateral mesh. Extraordinary vertices are those that have more or fewer than four neighboring vertices.

This chapter is based on [Don+06].

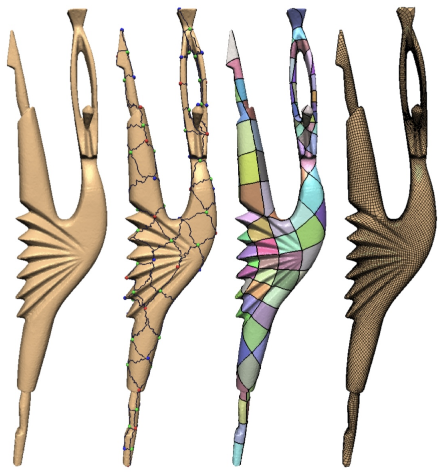


Figure 6.2: Visualization of full remeshing process of the surface of a dancer. [Don+06]

6.1 Laplacian Eigenfunctions

In this section we are going to give a brief overview of *Laplacian eigenfunctions* of a triangle shaped mesh M of a manifold. Those serve in the construction of a well-defined quadrangulation with well-shaped and evenly distributed cells. The key insight is that Laplacian eigenfunctions evenly distribute their extrema and so serve as ideal functions from which to generate a quadrangulated base domain.

6.1.1 Spectral Surface Analysis

The discrete Laplacian operator for every vertex i on piecewise linear function over a triangulated manifold is defined as

$$\Delta f_i = \sum_{j \in N_i} w_{ij}(f_j - f_i),$$

where N_i is the set of vertices adjacent to vertex i and $w_{ij} = \frac{1}{2}(\cot(\alpha_{ij}) + \cot(\beta_{ij}))$, where α_{ij} and β_{ij} are the angles opposite the edge (i, j).

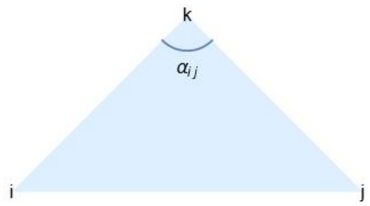


Figure 6.3: Visualization of the angle α_{ij} opposite to the edge (i, j).

We represent the function f by the column vector of its per-vertex values f, such that Laplace's equation will have the following form:

$$\Delta f = -Lf,$$

where

$$L_{ij} = \begin{cases} \sum_k w_{ik} & i = j \\ -w_{ij} & edge(i,j) \in M \\ 0 & otherwise \end{cases}.$$

Note, that as f is a picewise-linear function, f is indeed uniquely determined by its value on the vertices. The eigenvalues $\lambda_1 = 0 \leq \lambda_2 \leq ... \leq \lambda_n$ of L form the *spectrum* of the mesh M and the corresponding eigenvectors $e_1, ..., e_n$ of L define piecewise linear functions over M of progressively higher eigenvalues, which are called the *Laplacian eigenfunctions* of the mesh.

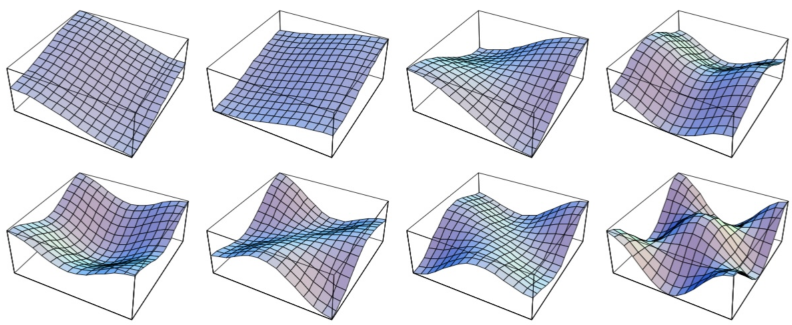


Figure 6.4: The first 8 non-constant eigenfunctions over a 15×15 planar grid, plotted as height functions. [Don+06]

Laplacian eigenfunctions have several useful properties, which is why they are well-suited for producing a well-shaped quadrangulation of M. On the one hand, their critical points are well-spaced over the surface and minima and maxima are interleaved in such a way that high valence nodes are extremly rare. In addition, multisaddles almost never arise, thus extraordinary points can only only occur at extrema. Those characteristics are also visible in Figure 6.4 and guarantee that the corresponding Morse-Smale complex produces a good quadrangulation of the surface. Usually, one finds that for most surfaces eigenfunctions with eigenvalues in the range $40 - 80$ will produce the most desirable complexes. In general, simple surfaces (sphere, torus) work well with lower eigenvalues whereas higher genus surfaces require higher eigenvalues.

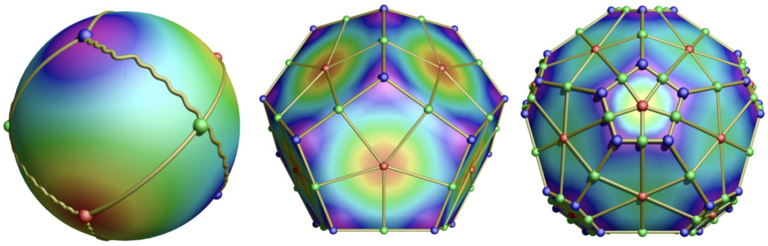


Figure 6.5: Surfaces whose 'ideal' complex contains more nodes require higher frequency eigenfunctions. Here we see the 10th, 46th, and 108th harmonic of a sphere, dodecahedron, and cornercut icosahedron, respectively. [Don+06]

6.1.2 Multiresolution Spectral Analysis

In this subsection, we are going to give a short explanation on how to determine which eigenfunction to choose. As solving *Laplace's equation* for a substantial number of eigenfunctions on a large mesh can be quite costly, *multiresolution techniques* can be quite useful to overcome this performance bottleneck. The idea is to carry out a small mesh change to get a coarser mesh, which preserves the topological type of the surface. *Multiresolution techniques* provide an easy way to find the small range of eigenfunctions that will produce complexes with a given number of critical points. Based on the number of critical points we want to have in our complex, we choose then an eigenvalue λ. We then compute a small number of eigenfunctions on the original mesh with eigenvalues close to λ.

6.2 Building a Quadrangular Base Complex

In this section, we describe how to construct the Morse-Smale complex to quadrangulate the surface.

6.2.1 The Morse-Smale Complex

Formally, the Morse-Smale complex is a cellular decomposition of a scalar function over a manifold, defined as the intersection of its ascending manifolds with its descending manifolds. In practice, given a function defined on the vertices of a triangulated manifold M it is computed by tracing lines of steepest ascent/descent. In general these lines segment M into foursided regions with two opposing saddles, a maximum and a minimum as corners.

6.2.2 Topological Noise removal

Often the eigenfunctions are smooth and we encounter increasing numbers of critical points are progressively higher frequencies. However, it can make sense to remove some critical points if the resulting complex is noisy:

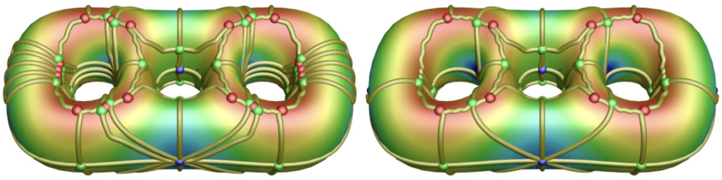


Figure 6.6: Left: Initial noisy complex, Right: Denoised complex. [Don+06]

To remove some potential 'noise' critical points, we can use *cancellations*, which can be seen ass a double edge contraction that removes a connected saddle-extremum pair. Basically, we can rank the saddle-extremum pairs by their *persistence*, which is the difference in their function value. We then cancel pairs in order of increasing persistence, up to a noise threshold.

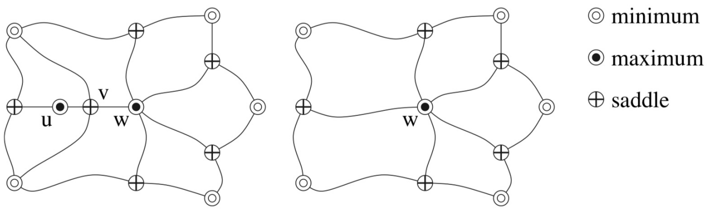


Figure 6.7: Morse-smale complex before and after caneling u and v. One saddle, one extremum, four paths, and two cells are removed. [Don+06]

6.2.3 Quasi-Dual Complexes

From each Morse-Smale complex, which we refer to here as the *primal complex*, it is possible to construct a *quasi-dual complex*. By computing the minimum-maximum diagonal within each Morse-Smale region, one can create another purely quadrangular complex. These quasi-dual complexes serve to expand the pool of possible base meshes.

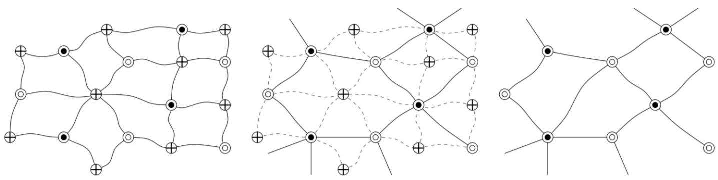


Figure 6.8: A primal complex (*a*) is replaced with min-max diagonals (*b*) to produce the quasi-dual complex (*c*). [Don+06]

One of the advantages of *quasi-dual complexes* is that they are in general more compact than their primal counterparts.

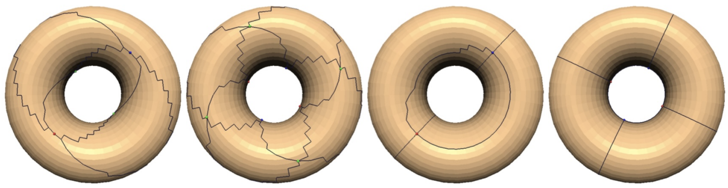


Figure 6.9: Complexes of torus eigenfunctions 8 and 16. [Don+06]

6.3 Remeshing

So far, we have constructed a quadrangular base complex over the surface. This base complex might be less then satisfactory, as the paths do not necessarily follow the surface shape in a natural way. Also, it might happen that multiple paths merge and follow the same edge chain. Those problems appear for example in the base complex in Figure 6.10.

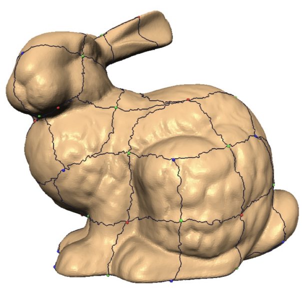


Figure 6.10: Base complex of bunny surface. [Don+06]

The idea is now to construct a globally smooth parametrization of the complex, which also corrections and optimizies the complex's embedding on the surface.

We are not going into detail on how construct this parametrization, for further details consider [Don+06] p.5-7.

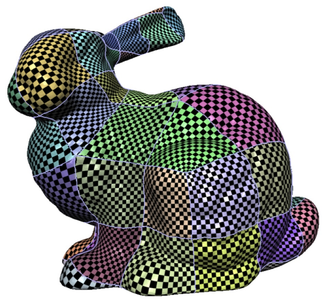


Figure 6.11: Complex of bunny surface after reparametrization. [Don+06]

After a valid globally parametrization has been found, we build a final semi-regular mesh. For each path in the complex, we must trace out the corresponding parametric boundary over the mesh. This gives us surface patches, each of which is equipped with a parametric mapping onto the unit square. Given a specific density d, we construct a regular $d \times d$ grid of quadrilaterals in this parametric domain and map their corners back onto the surface, thus producing the output mesh. Finally, coming back again to the question, which eigenfunction to choose. We have already briefly discussed in Subsection 6.2.2, how to select a certain range of eigenvalues based on the number of critical points, we want to have in the complex. Within this range, we usually select the eigenfunction for which the complex has the lowest parametric distortion.

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