DS-GA 3001.03: Homework Problem Set 2

Optimization and Computational Linear Algebra for Data Science

(Fall 2016)

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This homework problem set is due on September 27, before class at the homework dropbox in the CDS reception.

If you have any questions about the homework feel free to contact Vlad Kobzar at `vkobzar@cims.nyu.edu` **or myself at** `bandeira@cims.nyu.edu`**, or stop by our office hours.**

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

Problem 1.1 *Show that for any $n \geq 2$, any set of $n+1$ vectors in $\mathbb{R}^n$ needs to be linearly dependent. (Vlad will use this fact to prove that dimension is well-defined in the next Section)*

Problem 1.2 *Given a matrix $L \in \mathbb{R}^{n \times m}$ (meaning that L is a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$) show that*

$$\ker(L) = \ker(L^T L).$$

Problem 1.3 *Use the result from the proof above (and the fundamental theorem of linear algebra) to show that*

$$\mathrm{Rank}(L) = \mathrm{Rank}(L^T)$$

Hint: Start by showing that for any two matrices A, B we have $\mathrm{Rank}(A) \geq \mathrm{Rank}(AB)$.

Problem 1.4 *(Essentially in Problem 45 on Section 3.5. of Strang's book) Let V and W be two subspaces in $\mathbb{R}^n$. Show that if*

$$\dim(V) + \dim(W) > n,$$

then there must exist a non-zero vector in the intersection of them, i.e.: $V \cap W \neq \{0\}$.

($*$) **Problem 1.5 (For Extra Credit)** *(Essentially in Problem 46 on Section 3.5. of Strang's book) Given a matrix $A \in \mathbb{R}^{n \times n}$ satisfying $A^2 = 0$ show that* $\mathrm{Rank}(A) \leq \lfloor n/2 \rfloor$.