DS-GA 3001.03: Homework Problem Set 3

Optimization and Computational Linear Algebra for Data Science

(Fall 2016)

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This homework problem set is due on October 4, before class at the homework dropbox in the CDS reception.

If you have any questions about the homework post them on the piazza page for the course, contact Vlad Kobzar at `vkobzar@cims.nyu.edu` **or myself at** `bandeira@cims.nyu.edu`**, or stop by our office hours.**

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

Problem 1.1 *Given* $x \in \mathbb{R}^n$, *and* $p \geq 1$ *we define*

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}.$$

Notice that $\|x\|_2$ *is the norm we defined before. Show that*

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Problem 1.2 *Given* $u \in \mathbb{R}^n$, *let* $P_{\langle u\rangle}$ *be the* $n \times n$ *matrix corresponding to the projection from* $\mathbb{R}^n$ *to the span of* u. *(A) What are the entries of* $P_{\langle u\rangle}$*? (B) For which pairs of vectors* $u, v \in \mathbb{R}^n$ *do we have* $P_{\langle u\rangle}P_{\langle v\rangle} = P_{\langle v\rangle}P_{\langle u\rangle}$.

Problem 1.3 *Let* U, V *be subspaces in* $\mathbb{R}^n$. *(A) is* $U \cup V$ *a subspace? (Prove or give a counter example). (B) is* $U \cap V$ *a subspace? (Prove or give a counter example).*

Problem 1.4 *Given a subspace* W, P_W *is the projection onto* W. *Let* U, V *be subspaces in* $\mathbb{R}^n$. *Give a necessary and sufficient condition for subspaces* U, V *to satisfy*

$$P_U P_V = P_{U\cap V}?$$

$(*)$ **Problem 1.5 (For Extra Credit)** *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix. Show that*

$$R(x) = \frac{x^T A x}{x^T x},$$

has a maximum $x^\natural$ *over* $\mathbb{R}^n \setminus \{0\}$, *and that this maximum satisfies:*

$$Ax^\natural = \lambda x^\natural,$$

where $\lambda = R(x^\natural)$.