DS-GA 1014: Homework Problem Set 7

Optimization and Computational Linear Algebra for Data Science (Fall 2018)

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This homework problem set is due on October 26 on NYU Classes.

If you have questions about the homework feel free to contact Brett Bernstein (`brett.bernstein@nyu.edu`) or myself, or stop by our office hours.

Unless otherwise stated all answers must be mathematically justified, even questions that ask for a computation.

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to ask on Piazza.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count, from Friday to Monday counts only 1 day.

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

In class we saw:

- (Spectral Decomposition) For every symmetric matrix $A \in \mathbb{R}^{n\times n}$, there exists $V \in \mathbb{R}^{n\times n}$ satisfying $V^T V = I$, and $\Lambda \in \mathbb{R}^{n\times n}$ a diagonal matrix, with diagonal elements $\lambda_1, \ldots, \lambda_n$ (called the eigenvalues of A) such that
$$A = V\Lambda V^T.$$
- (Singular Value Decomposition) For every matrix $B \in \mathbb{R}^{m\times n}$, there exist matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ satisfying $U^T U = I$ and $V^T V = I$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with non-negative diagonal entries $\sigma_1, \ldots, \sigma_{\min\{n,m\}} \geq 0$ called the singular values of B (for a rectangular matrix, diagonal means that for all $i \neq j$, $\Sigma_{ij} = 0$), such that
$$B = U\Sigma V^T.$$

You are welcome (and encouraged) to use these facts in the homework.

Problem 7.1 *Prove that a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is orthogonal if and only if its eigenvalues all have absolute value* 1*.*

Problem 7.2 *The trace of a matrix $A \in \mathbb{R}^{n\times n}$,* $\mathrm{Tr}(A)$ *is defined as*
$$\mathrm{Tr}(A) = \sum_{k=1}^{n} A_{kk}.$$

1. *Show that for any matrices $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{n\times m}$ we have $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$.*
2. *Use the previous part to show that for any symmetric matrix $A \in \mathbb{R}^{n\times n}$, its trace is equal to the sum of its eigenvalues.*

Problem 7.3 *For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we say that A is positive semidefinite (also written $A \succeq 0$) if all of its eigenvalues are non-negative ($\lambda_k \geq 0$, for $k = 1, \ldots, n$).*

1. *Show that for every symmetric matrix A satisfying $A \succeq 0$, there exists a matrix U such that*
$$A = UU^T.$$
2. *Is the converse true? Either prove that every A of the form $A = UU^T$ (for some matrix U) must satisfy $A \succeq 0$ or give a counterexample.*

Problem 7.4 *Let $B \in \mathbb{R}^{m\times n}$ with $m \leq n$ and let $B = U\Sigma V^T$ be its singular value decomposition.*

1. *Show that the singular values of B are the square roots of the eigenvalues of BB^T.*
2. *Prove that the columns of V are eigenvectors for $B^T B$ and the columns of U are eigenvectors for BB^T.*

3. *Suppose $m = n$ and B is invertible. Give a singular value decomposition for B^{-1}.*

(∗) **Problem 7.5 (For Extra Credit)** *For $B \in \mathbb{R}^{m\times n}$ we define*

$$\|B\|_2 = \max_{x\neq 0} \frac{\|Bx\|_2}{\|x\|_2},$$

and

$$\gamma(B) = \min_{x\neq 0} \frac{\|Bx\|_2}{\|x\|_2}.$$

Can you relate $\|B\|_2$ and $\gamma(B)$ with the singular values of B?