A CHIANG-TYPE LAGRANGIAN IN $\mathbb{CP}^2$

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Abstract. We analyse a monotone lagrangian in $\mathbb{CP}^2$ that is hamiltonian isotopic to the standard lagrangian $\mathbb{RP}^2$, yet exhibits a distinguishing behaviour under reduction by one of the toric circle actions, namely it intersects transversally the reduction level set and it projects *one-to-one* onto a great circle in $\mathbb{CP}^1$. This lagrangian thus provides an example of *embedded composition* fitting work of Wehrheim-Woodward and Weinstein.

1. Introduction

Among the myriad of reasons that lagrangian submanifolds are so fundamental in symplectic mathematics is their role in the quest for a symplectic category. This quest was launched by Alan Weinstein in the early 80's with an eye towards quantization and was recently invigorated by results of Katrin Wehrheim and Chris Woodward on quilted Floer cohomology. We will address an instance of the latter's work within symplectic reduction.

Suppose we have a symplectic manifold (M,ω) with a hamiltonian action of a torus, where μ is the moment map and $\mu^{-1}(a)$ is a regular level. Assuming that the torus acts freely on this level set, then the corresponding orbit space is a new symplectic manifold called the reduced space, (M_{red},ω_{red}). We denote $\imath:\mu^{-1}(a)\hookrightarrow M$ the inclusion of the level and $\pi:\mu^{-1}(a)\twoheadrightarrow M_{red}$ the point-orbit projection map from the level.

Let L_1 and L_2 be lagrangian submanifolds of the original symplectic manifold (M,ω), and let ℓ_1 and ℓ_2 be lagrangian submanifolds of the reduced space (M_{red},ω_{red}) fulfilling the following relationships:

$$L_1 = \imath(\pi^{-1}(\ell_1)) \qquad (\star_1) \qquad\qquad \text{and} \qquad\qquad \ell_2 = \pi(\imath^{-1}(L_2)) \qquad (\star_2)\ ,$$

that means, L_1 is the preimage of ℓ_1 under π, whereas ℓ_2 is the image of $L_2\cap\mu^{-1}(a)$ under π.

In special circumstances (we have in mind the case when L_2 intersects $\mu^{-1}(a)$ transversally and the projection from this intersection onto ℓ_2 is *one-to-one*; see Section 2), there is a straightforward bijection of points in the following intersections via the point-orbit projection

$$L_1\cap L_2 \quad \overset{\pi}{\longleftrightarrow} \quad \ell_1\cap\ell_2\ .$$

Wehrheim and Woodward have shown in [8] that, under nice conditions, this geometric bijection of generators of the Floer complexes induces an isomorphism in Floer cohomology. Their paper generalizes Floer cohomology to sequences of *lagrangian correspondences* and establishes an isomorphism of the Floer cohomologies when such sequences are related by *composition* of lagrangian correspondences – the symplectic reduction result above being a special case.

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Whereas pairs of lagrangian submanifolds L_1 and ℓ_1 related by ($\star_1$) trivially abound, it has been challenging to identify interesting examples of pairs L_2 and ℓ_2 related by ($\star_2$), where the required special geometric circumstances are satisfied.

In this paper we exhibit compact lagrangian submanifolds $\mathcal{L}_2 \subset \mathbb{CP}^2$ and $\ell_2 \subset \mathbb{CP}^1$ of the latter type and with the desired intersection properties. Moreover, our lagrangians $\mathcal{L}_2$ and ℓ_2 are monotone (as required in the setting of [8]). A broader motivation for this example is the general goal of understanding lagrangian submanifolds – the special case where the ambient space is a toric manifold such as $\mathbb{CP}^2$ providing rich, user-friendly, and interesting exploration grounds. A broader exploration of such examples within toric manifolds is in the works.

In Section 2 we describe in detail our symplectic reduction set-up. In Section 3 we present the concrete example $\mathcal{L}_2$ as a non-standard lagrangian $\mathbb{RP}^2$ in $\mathbb{CP}^2$. In Section 4 we prove that $\mathcal{L}_2$ fulfills the desired properties culminating in Theorem 4.5. In Section 5 we rephrase relevant properties in terms of *embedded composition* required for a symplectic "category" [9].

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2. Lagrangians in a symplectic reduction scenario

Let (M, ω) be a $2n$-dimensional symplectic manifold equipped with a hamiltonian action of a k-dimensional real torus T^k where $k < n$ and corresponding moment map

$$\mu : M \longrightarrow \mathbb{R}^k \ .$$

We assume that $a \in \mathbb{R}^k$ is a regular value of μ such that the reduced space

$$M_{red} = \mu^{-1}(a)/T^k$$

is a manifold. The latter comes with the so-called *reduced* symplectic form ω_{red} satisfying the equality $\pi^*\omega_{red} = \iota^*\omega$, where π is the point-orbit projection from the level set and ι is the level set inclusion:

$$\begin{array}{ccc} \mu^{-1}(a) & \overset{\iota}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{red} & & \end{array}$$

Let $\ell_1 \subset M_{red}$ be a compact lagrangian submanifold of the reduced space. Then its preimage in M,

$$L_1 := \iota\left(\pi^{-1}(\ell_1)\right) \ ,$$

is always a compact lagrangian submanifold of (M,ω), which happens to lie entirely in the level set $\mu^{-1}(a)$ and be a T^k-bundle over M_{red}.

We are looking for a reverse picture where, starting from a lagrangian in (M,ω), we obtain a suitable lagrangian in (M_{red},ω_{red}). For that purpose, suppose now that we have a compact lagrangian submanifold $L_2 \subset M$ such that

- $L_2 \pitchfork \mu^{-1}(a)$, that is, L_2 and $\mu^{-1}(a)$ intersect *transversally* and
- $L_2 \cap \mu^{-1}(a) \overset{\pi}{\hookrightarrow} M_{red}$, that is, L_2 intersects each T^k-orbit in $\mu^{-1}(a)$ at most *once*,

so the intersection submanifold $L_2 \cap \mu^{-1}(a)$ injects into the reduced space M_{red} via the point-orbit projection π. In this case, we obtain an embedded lagrangian submanifold in the reduced space (M_{red},ω_{red}), namely

$$\ell_2 := \pi\left(L_2 \cap \mu^{-1}(a)\right) ,$$

and we call L_2 a *one-to-one transverse lifting* of ℓ_2.

In this note, we will concentrate on the case where the symplectic manifold is the complex projective plane, $M = \mathbb{CP}^2$, with a scaled Fubini-Study structure ω so that the total volume is $\frac{\pi^2}{2}$. We regard the circle action where a circle element $e^{i\theta} \in S^1$ $(0 \le \theta < 2\pi)$ acts by

$$[z_0 : z_1 : z_2] \longmapsto [z_0 : z_1 : e^{i\theta} z_2] .$$

This action has moment map

$$\mu_2 : \mathbb{CP}^2 \longrightarrow \mathbb{R} , \quad \mu_2[z_0 : z_1 : z_2] = -\frac{1}{2}\frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} .$$

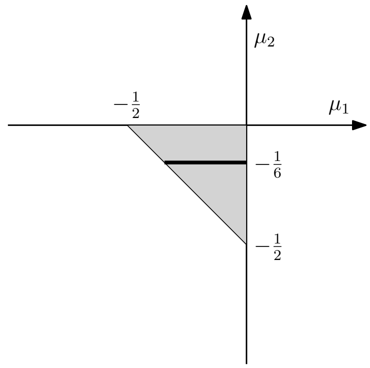
And we choose the level $a = -\frac{1}{6}$.[1] With these choices, we have $n = 2$, $k = 1$ and the reduced space (M_{red},ω_{red}) is a complex projective line $\mathbb{CP}^1$ with a scaled Fubini-Study form so that the total area is $\frac{2\pi}{3}$. The other standard hamiltonian circle action, namely

$$[z_0 : z_1 : z_2] \longmapsto [z_0 : e^{i\theta} z_1 : z_2] ,$$

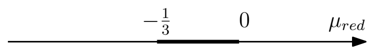
which has moment map $\mu_1[z_0 : z_1 : z_2] = -\frac{1}{2}\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}$, descends to the reduced space, where we denote the induced moment map $\mu_{red} : \mathbb{CP}^1 \to \mathbb{R}$.

Pictorially in terms of toric moment polytopes, we have for the original space $\mathbb{CP}^2$ a triangle as image by (μ_1,μ_2):

[1] All other levels in the range $-\frac{1}{4} < a < 0$ allow the *one-to-one transverse lifting property*, but this level (that of the so-called *Clifford torus* $\{[1 : e^{i\theta_1} : e^{i\theta_2}], \theta_1,\theta_2 \in \mathbb{R}\}$) ensures the monotonicity of ℓ_2 since it is then a great circle.



and for the reduced space $\mathbb{CP}^1$ (reduction carried out with respect to μ_2) the line segment $\mu_{red}(\mathbb{CP}^1) = [-\frac{1}{3}, 0]$:



Section 3 exhibits a non-standard lagrangian embedding of $\mathbb{RP}^2$ into $\mathbb{CP}^2$, which in Section 4 is shown to be a *one-to-one transverse lifting* of a great circle in $\mathbb{CP}^1$, so we get an example pair denoted $\mathcal{L}_2$, ℓ_2.

3. A non-standard lagrangian $\mathbb{RP}^2$ in $\mathbb{CP}^2$ from representation theory

The example at stake is a lagrangian submanifold $\mathcal{L}_2$ of $\mathbb{CP}^2$ which arises as an orbit for an SU(2)-action, similar to a lagrangian studied by River Chiang in [4]. We first define $\mathcal{L}_2$ as a subset of $\mathbb{CP}^2$ in terms of complex parameters α and β:

$$\boxed{\mathcal{L}_2 := \left\{ \left[\bar{\alpha}^2 + \bar{\beta}^2 : \sqrt{2}(\bar{\alpha}\beta - \alpha\bar{\beta}) : \alpha^2 + \beta^2 \right] \;\middle|\; \alpha, \beta \in \mathbb{C} \;, |\alpha|^2 + |\beta|^2 = 1 \right\} \;,}$$

and give the proofs in these terms, since it was in this perspective that this example first arose. Further down, we exhibit a convenient real parametrization of $\mathcal{L}_2$ and give the corresponding (simpler) real proofs.

Lemma 3.1. *The set $\mathcal{L}_2$ is a submanifold of $\mathbb{CP}^2$ diffeomorphic to $\mathbb{RP}^2$.*

Proof. We view $\mathbb{CP}^2$ as the space of homogeneous complex polynomials of degree 2 in two variables x and y up to scaling,

$$[z_0 : z_1 : z_2] \quad \longleftrightarrow \quad p_{[z_0:z_1:z_2]}(x,y) = z_0 y^2 + z_1 \sqrt{2} xy + z_2 x^2 \;,$$

and let the group SU(2) act as follows. An element

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2) \qquad \text{where} \qquad \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1 \;,$$

acts by the change-of-variables diffeomorphism

$$p(x,y) \longmapsto p\left((x,y)A\right) = p\left(\alpha x - \bar{\beta} y, \beta x + \bar{\alpha} y\right) . \tag{3.1}$$

Then the set $\mathcal{L}_2$ is simply the SU(2)-orbit of the polynomial $p_{[1:0:1]}(x,y) = y^2 + x^2$ or, equivalently, of the point $[1:0:1]$, because

$$(\beta x + \bar{\alpha} y)^2 + (\alpha x - \bar{\beta} y)^2 = \underbrace{(\bar{\alpha}^2 + \bar{\beta}^2)}_{z_0} y^2 + \underbrace{2(\bar{\alpha}\beta - \alpha\bar{\beta})}_{z_1\sqrt{2}} xy + \underbrace{(\alpha^2 + \beta^2)}_{z_2} x^2 .$$

So $\mathcal{L}_2$ is an SU(2)-homogeneous space. Since the stabilizer of $[1:0:1] \in \mathcal{L}_2$ is the subgroup

$$\mathcal{S} := \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & -i\cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

where the component of the identity is a circle subgroup of SU(2) (hence conjugate to the Hopf subgroup), it follows that $\mathcal{L}_2$ is diffeomorphic to

$$\mathrm{SU}(2) \,/\, \mathcal{S} \;\simeq\; \mathbb{RP}^2 .$$

□

Lemma 3.2. *The submanifold $\mathcal{L}_2$ is lagrangian.*

Proof. The fact that $\mathcal{L}_2$ is isotropic can be checked either directly (computing the vector fields generated by the SU(2)-action) or, as done here, analysing a hamiltonian action in the context of representation theory.

The standard hermitian metric on the vector space $V := \mathbb{C}^2$, namely

$$h(z,w) = \bar{z}^T w = \underbrace{\mathfrak{R}(\bar{z}^T w)}_{\substack{\text{euclidean} \\ \text{inner prod.}}} + i \underbrace{\mathfrak{I}(\bar{z}^T w)}_{\substack{\text{standard} \\ \text{sympl. str.}}} ,$$

induces a hermitian metric on the symmetric power $\mathrm{Sym}^2(V^*)$ for which

$$y^2 , \quad \sqrt{2}xy , \quad x^2 \tag{3.2}$$

is a unitary basis; here x and y (the linear maps extracting the first and second components of a vector in $\mathbb{C}^2$) form the standard unitary basis of V^*. We denote u_0, u_1, u_2 the corresponding coordinates in $\mathrm{Sym}^2(V^*)$ and write an element of $\mathrm{Sym}^2(V^*)$ as

$$p(x,y) = u_0 y^2 + u_1 \sqrt{2} xy + u_2 x^2 .$$

W.r.t. these coordinates, the symplectic structure on $\mathrm{Sym}^2(V^*)$ is the standard structure:

$$\Omega_0 = \tfrac{i}{2} \left(du_0 d\bar{u}_0 + du_1 d\bar{u}_1 + du_2 d\bar{u}_2 \right) .$$

The diagonal circle action on $\mathrm{Sym}^2(V^*)$ is hamiltonian with moment map linearly proportional to $|u_0|^2 + |u_1|^2 + |u_2|^2$. By symplectic reduction at the level of the unit sphere, we obtain the space of homogeneous complex polynomials of degree 2 in two variables

x and y up to scaling, $\mathbb{P}\big(\mathrm{Sym}^2(V^*)\big) \simeq S^5/S^1 \simeq \mathbb{CP}^2$, with the standard Fubini-Study structure (S^1 acts on S^5 by the Hopf action).

Now the action of SU(2) on $\mathrm{Sym}^2(V^*)$ by the change-of-variables (3.1) is a *unitary* representation (this is the 3-dimensional irreducible representation of SU(2)), hence hamiltonian with moment map[2]

$$\tilde{\mu} : \mathrm{Sym}^2(V^*) \longrightarrow \mathfrak{su}(2)^* \simeq \mathbb{R}^3$$

defined as follows w.r.t. the real basis

$$X_a := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad X_b := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad X_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

of the Lie algebra $\mathfrak{su}(2)$ and the unitary basis (3.2) of $\mathrm{Sym}^2(V^*)$:

$$\tilde{\mu}(u_0, u_1, u_2) = \begin{pmatrix} |u_0|^2 - |u_2|^2 \\ -\sqrt{2}\mathfrak{R}(u_0\bar{u}_1 + u_1\bar{u}_2) \\ -\sqrt{2}\mathfrak{I}(u_0\bar{u}_1 + u_1\bar{u}_2) \end{pmatrix} .$$

This SU(2)-action commutes with the diagonal S^1-action, hence descends to a hamiltonian action of SU(2) on $\mathbb{CP}^2$ with moment map

$$\tilde{\tilde{\mu}} : \mathbb{CP}^2 \longrightarrow \mathfrak{su}(2)^* \simeq \mathbb{R}^3 , \quad \tilde{\tilde{\mu}}[z_0 : z_1 : z_2] = \begin{pmatrix} |z_0|^2 - |z_2|^2 \\ -\sqrt{2}\mathfrak{R}(z_0\bar{z}_1 + z_1\bar{z}_2) \\ -\sqrt{2}\mathfrak{I}(z_0\bar{z}_1 + z_1\bar{z}_2) \end{pmatrix} ,$$

where now z_0, z_1, z_2 are homogeneous coordinates satisfying $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$.

[2]In general, a symplectic representation $\rho : G \to \mathrm{Sympl}(\mathbb{C}^N, \Omega_0)$ may be viewed as a hamiltonian action with moment map $\tilde{\mu} : \mathbb{C}^N \to \mathfrak{g}^*$, $\tilde{\mu}(z)(X) = \frac{i}{4}(z^*\widetilde{X}z - z^*\widetilde{X}^*z)$ where $z \in \mathbb{C}^N$ is a point in the symplectic manifold, $X \in \mathfrak{g}$ is an element in the Lie algebra of $Sympl(\mathbb{C}^N, \Omega_0)$, and $\widetilde{X}$ is the corresponding matrix representative. When the representation is *unitary*, the formula for the moment map reduces to $\tilde{\mu}(z)(X) = \frac{i}{2}z^*\widetilde{X}z$.

In the present case, we determine the matrix $\widetilde{X}$ for each of the Lie algebra basis elements $X_a, X_b, X_c \in \mathfrak{su}(2)$: We take their 1-parameter subgroups in SU(2),

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

respectively, and the corresponding curves in $\mathrm{Sym}^2(V^*)$ of the form

$$\begin{pmatrix} \bar{\alpha}^2 & -\sqrt{2}\bar{\alpha}\bar{\beta} & \bar{\beta}^2 \\ \sqrt{2}\bar{\alpha}\beta & |\alpha|^2 - |\alpha|^2 & -\sqrt{2}\alpha\bar{\beta} \\ \beta^2 & \sqrt{2}\alpha\beta & \alpha^2 \end{pmatrix}$$

with respect to the basis (3.2), to find the matrix representatives by differentiating at $t = 0$,

$$\widetilde{X}_a = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}, \quad \widetilde{X}_b = \begin{pmatrix} 0 & \sqrt{2}i & 0 \\ \sqrt{2}i & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}, \quad \widetilde{X}_c = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} .$$

The submanifold $\mathcal{L}_2$ is isotropic because it is an SU(2)-orbit and sits in the zero-level of the moment map $\tilde{\tilde{\mu}}$ – indeed we have $|z_0|^2 = |z_2|^2$ and $z_0\bar{z}_1 + z_1\bar{z}_2 = 0$ when $z_0 = \bar{z}_2$ and z_1 is imaginary. It follows that $\mathcal{L}_2$ is lagrangian because it is half-dimensional. □

There is a convenient **real description** of $\mathcal{L}_2$ by using real ($\mathfrak{R}$) and imaginary ($\mathfrak{I}$) parts of complex numbers and defining

$$X := \mathfrak{R}(\alpha^2 + \beta^2) \ , \quad Y := \mathfrak{I}(\alpha^2 + \beta^2) \quad \text{and} \quad Z := 2\mathfrak{I}(\bar{\alpha}\beta) \ .$$

In these terms, we have

$$\boxed{\mathcal{L}_2 = \left\{ \left[X - iY : \sqrt{2}iZ : X + iY \right] \mid X, Y, Z \in \mathbb{R} \ , \ X^2 + Y^2 + Z^2 = 1 \right\}} \ ,$$

which also shows that $\mathcal{L}_2 \simeq S^2/\pm 1 \simeq \mathbb{RP}^2$, thus reproving Lemma 3.1.

In order to recheck from this perspective that $\mathcal{L}_2$ is lagrangian, we use *action-angle coordinates* $(\theta_1, \mu_1, \theta_2, \mu_2)$, with $\theta_k \in \mathbb{R} \mod 2\pi$, $\mu_k < 0$ and $\mu_1 + \mu_2 > -\frac{1}{2}$, valid in the open dense subset of $\mathbb{CP}^2$ where all homogeneous coordinates are nonzero, which we informally refer to as the **interior of the moment polytope**.

We first observe that the action of the circle subgroup $(e^{i\theta}, e^{2i\theta})$ preserves $\mathcal{L}_2$,

$$\left[X - iY : e^{i\theta}\sqrt{2}iZ : e^{2i\theta}(X + iY) \right] = \Big[\underbrace{e^{-i\theta}(X - iY)}_{\widetilde{X} - i\widetilde{Y}} : \sqrt{2}iZ : \underbrace{e^{i\theta}(X + iY)}_{\widetilde{X} + i\widetilde{Y}} \Big] \ ;$$

thus the corresponding vector field $\frac{\partial}{\partial\theta_1} + 2\frac{\partial}{\partial\theta_2}$ is tangent to $\mathcal{L}_2$.

Now, w.r.t. to these action-angle coordinates, our multiple of the Fubini-Study form is $\omega = d\theta_1 \wedge d\mu_1 + d\theta_2 \wedge d\mu_2$. Since on a point $\left[X - iY : \sqrt{2}iZ : X + iY \right]$ of $\mathcal{L}_2$ we have

$$\mu_1 = -\tfrac{1}{2}Z^2 \qquad \text{and} \qquad \mu_2 = -\tfrac{1}{4}(1 - Z^2) \ ,$$

so $d\mu_1 = -ZdZ$ and $d\mu_2 = \frac{1}{2}ZdZ$, we can write the pullback of ω to $\mathcal{L}_2$ as

$$-Z\left(d\theta_1 - \tfrac{1}{2}d\theta_2\right) \wedge dZ \ .$$

But this pullback form vanishes identically, since we can choose a trivialization of $T\mathcal{L}_2$ (over an open dense subset) where the first element is given by the vector field $\frac{\partial}{\partial\theta_1} + 2\frac{\partial}{\partial\theta_2}$ above, thus concluding the alternative proof of Lemma 3.2.

4. Main theorem about $\mathcal{L}_2$

Lemma 4.1. *The submanifold $\mathcal{L}_2$ intersects transversally the moment map level set $\mu_2^{-1}(a)$.*

Proof. We claim that *in the interior of the moment polytope* the restriction of $d\mu_2$ to $\mathcal{L}_2$ never vanishes. This implies the lemma because the level set $\mu_2^{-1}(a)$ is a codimension one submanifold whose tangent bundle is the kernel of $d\mu_2$.

Using the real description of $\mathcal{L}_2$ framed above, we parametrize it (in the interior of the moment polytope) via real coordinates X and Y, choosing

$$Z = \sqrt{1 - X^2 - Y^2} \ , \qquad X^2 + Y^2 < 1 \ ,$$

so that, in the interior of the moment polytope $\mathcal{L}_2$ is the set of points

$$\left[\frac{X - iY}{\sqrt{2}i\sqrt{1 - X^2 - Y^2}} : 1 : \frac{X + iY}{\sqrt{2}i\sqrt{1 - X^2 - Y^2}} \right] .$$

We evaluate μ_2 in points of the above form:

$$\mu_2 \left[\tfrac{X-iY}{\sqrt{2}i\sqrt{1-X^2-Y^2}} : 1 : \tfrac{X+iY}{\sqrt{2}i\sqrt{1-X^2-Y^2}} \right] = -\tfrac{1}{4}(X^2 + Y^2) \ .$$

This shows that the differential of μ_2 only vanishes when μ_2 itself vanishes, which never happens in the interior of the moment polytope. □

Lemma 4.2. *In the interior of the moment polytope, the manifold $\mathcal{L}_2$ intersects at most once each orbit of the second circle action in $\mathbb{CP}^2$.*

Proof. Using the real description, we compare two points in $\mathcal{L}_2$ which map to the interior of the moment polytope, say

$$P = \left[X - iY : \sqrt{2}iZ : X + iY \right] \qquad \text{and} \qquad p = \left[x - iy : \sqrt{2}iz : x + iy \right]$$

where $X, Y, Z, x, y, z \in \mathbb{R} \smallsetminus \{0\}$ are such that $X^2 + Y^2 + Z^2 = 1$ and $x^2 + y^2 + z^2 = 1$.

The goal is to check that such points can never be nontrivially related by the second circle action, that is, to check that an equality

$$\left[X - iY : \sqrt{2}iZ : e^{i\theta}(X + iY) \right] = \left[x - iy : \sqrt{2}iz : x + iy \right]$$

can only occur when $e^{i\theta} = 1$. For that purpose, we compare the ratios of homogeneous coordinates

$$\underbrace{\frac{X - iY}{x - iy}}_{\overline{w}} = \underbrace{\frac{Z}{z}}_{\text{real}} = e^{i\theta} \cdot \underbrace{\frac{X + iY}{x + iy}}_{w} \ .$$

Since the middle term is real, it must be $w = \overline{w}$ and thus $e^{i\theta} = 1$. □

Remark 4.3. *A different proof of Lemma 4.2 was first found by Radivoje Bankovic in* [1]*. There he analysed $\mathcal{L}_2$ as a singular fibration over the moment map image tilted segment, within the 2-torus fibers, using our earlier lemmas.*

Remark 4.4. *The lagrangian $\mathcal{L}_2$ has double intersections with orbits of the first circle action: $[X - iY : \sqrt{2}iZ : X + iY]$ and $[X - iY : -\sqrt{2}iZ : X + iY]$. In general, the lagrangian $\mathcal{L}_2$ has single intersections with orbits of subgroups of the form $(e^{ik\theta}, e^{i(2k\pm1)\theta})$ with $k = 0, 1, 2, \ldots$. This can be checked as in the proof of Lemma 4.2.*

Theorem 4.5. *The lagrangian submanifold $\mathcal{L}_2$ is a one-to-one transverse lifting of a great circle ℓ_2 in $\mathbb{CP}^1$.*

Proof. Lemmas 4.1 and 4.2 show that $\mathcal{L}_2$ is a one-to-one transverse lifting of a compact lagrangian submanifold ℓ_2 in $\mathbb{CP}^1$. In order to see that ℓ_2 is a great circle it is enough to note that it lies in the middle level set of μ_{red}. □

Note that the lagrangian $\mathcal{L}_2$ is monotone (see the proof of theorem A in section 6 of Biran [2]), has minimal Maslov index 3 and has $HF_k(\mathcal{L}_2, \mathcal{L}_2) \simeq \mathbb{Z}_2$ for every $k = 0, 1, 2$ (see also Corollary 1.2.11 of Biran and Cornea [3]). Having chosen the level $a = -\frac{1}{6}$, we obtain that the corresponding lagrangian ℓ_2 in the reduced space is a great circle, hence also monotone.

The result in Lemma 4.2 is interesting in itself since it shows a contrast between the lagrangian $\mathcal{L}_2$ and the standard embedding of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ as the fixed locus of complex conjugation: the standard embedding intersects each orbit of the second circle action twice (in the interior of the moment polytope), namely, for each point $[y_0 : y_1 : y_2]$ with $y_0, y_1, y_2 \in \mathbb{R}$, there is also the point $[y_0 : y_1 : -y_2]$ related by the second circle action, and similarly for the first circle action.

However, $\mathcal{L}_2$ is hamiltonian isotopic to the standard $\mathbb{RP}^2$ according to Theorem 6.9 of Li and Wu [6] going back to work of Hind [5]. We can even explicitly write such a hamiltonian isotopy.

5. Embedded composition

We recall the relevant notions of *lagrangian correspondence* and *lagrangian composition* introduced by Weinstein [9] as, respectively, "morphism" between symplectic manifolds and "morphism composition," and the notion of *embedded composition* introduced by Wehrheim and Woodward [8] in order to promote Weinstein's "category" to an actual 2-category.

Let (M_0, ω_0), (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. For convenience we refer to $(M_i, -\omega_i)$ simply as M_i^- and let M_i^+ mean (M_i, ω_i), so these spaces are the same manifold equipped with opposite symplectic forms. Then $M_i^- \times M_j^+$ denotes the product manifold $M_i \times M_j$ equipped with $-\omega_i + \omega_j$ where we mean to add here the standard pullbacks to a product manifold.

A *lagrangian correspondence* L is simply a lagrangian submanifold of such a product $M_i^- \times M_j^+$. One writes L^T for the same submanifold regarded as a lagrangian in the reverse product $M_j^- \times M_i^+$, related by the *transposition* map $M_i \times M_j \to M_j \times M_i$, $(p, q) \mapsto (q, p)$.

The notion of *lagrangian correspondence* generalizes that of *symplectomorphism*, since the graph of a symplectomorphism $\varphi : M_i \to M_j$ may be viewed as a lagrangian correspondence in $M_i^- \times M_j^+$. But now the symplectic manifolds may even have different dimensions. A lagrangian L in a symplectic manifold (M, ω) may be viewed as a lagrangian in the product $pt^- \times M^+$ where the point manifold is a trivial symplectic manifold. More interestingly, the level set $\mu^{-1}(a)$ in a reduction scenario as in Section 2 may be viewed as a lagrangian correspondence in $M_{red}^- \times M^+$; this widely known fact is explained below.

Two lagrangian correspondences may be *composed* as follows, generalizing the composition of symplectomorphisms. Let L_{01} be a lagrangian submanifold of $M_0^- \times M_1^+$ and let L_{12} be a lagrangian submanifold of $M_1^- \times M_2^+$. The *composition* of L_{01} and L_{12} is the following subset of $M_0 \times M_2$:

$$L_{01} \circ L_{12} := \{(p_0, p_2) \mid \exists p_1 \in M_1 \text{ with } (p_0, p_1) \in L_{01} \text{ and } (p_1, p_2) \in L_{12}\} .$$

In terms of the factor projections

$$\begin{array}{ccccc} & & M_0 \times M_1 \times M_1 \times M_2 & & \\ & \pi_0 \swarrow & & \searrow \pi_2 & \\ M_0 & & & & M_2 \end{array}$$

the composition is the set

$$L_{01} \circ L_{12} = (\pi_0, \pi_2)(L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2) ,$$

where Δ_{M_1} denotes the preimage of the middle diagonal in $M_0 \times M_1 \times M_1 \times M_2$ via the natural projection onto $M_1 \times M_1$; hence, it is the subset of all points of the form (p_0, p_1, p_1, p_2).

Of course, this composition might be a singular set. Yet, in the best scenario, we have that:

- the intersection $L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2$ is *transverse* , so this is a submanifold with dimension $\frac{1}{2} \dim(M_0 \times M_2)$ and
- the projection map (π_0, π_2) from the above intersection to $M_0 \times M_2$ is an *injective immersion*,

so, in this case, the lagrangian composition is an embedded submanifold of $M_0 \times M_2$. Furthermore, this composition is indeed lagrangian with respect to the symplectic form $-\omega_0 + \omega_2$. In this best scenario, we say that the lagrangian submanifold $L_{01} \circ L_{12}$ of $M_0^- \times M_2^+$ is an **embedded composition** (a.k.a. *congenial composition* [7]).

In the reduction framework described in Section 2 we can view the level set $\mu^{-1}(a)$ as a lagrangian in two different ways and use it to compose with other lagrangians. Concretely, we consider the lagrangian submanifolds defined by

$$L_\mu \;:=\; (\pi, \iota)(\mu^{-1}(a)) \;\subset\; M_{red}^- \times M^+ \; .$$

and

$$L_\mu^T \;:=\; (\iota, \pi)(\mu^{-1}(a)) \;\subset\; M^- \times M_{red}^+$$

These submanifolds are isotropic because $\pi^* \omega_{red} = \iota^* \omega$. Although these submanifolds depend on the specific level set, we omit a in their notation.

In terms of L_μ, of L_μ^T and of lagrangian composition, lagrangians ℓ_1, L_1, ℓ_2 and L_2 as in Section 2 (now viewed in $pt^- \times M_{red}^+$ or $pt^- \times M^+$) may be described as

$$L_1 = \ell_1 \circ L_\mu = \iota(\pi^{-1}(\ell_1))$$

and

$$\ell_2 = L_2 \circ L_\mu^T = \pi(L_2 \cap \mu^{-1}(a)) \ .$$

The first composition is always a straightforward embedded composition, whereas the second is an embedded composition exactly when the two earlier conditions are satisfied:

- L_2 and $\mu^{-1}(a)$ *intersect transversally* and
- L_2 intersects each T^k-orbit in $\mu^{-1}(a)$ *at most once*.

Therefore, $\ell_2 = L_2 \circ L_\mu^T$ is an embedded composition exactly when L_2 is a one-to-one transverse lifting of ℓ_2.

Since $\ell \circ L_\mu = \iota(\pi^{-1}(\ell))$ is always an embedded composition, informally we have a map

$$\begin{array}{ccc} \{\text{lagrangians in } M_{red}\} & \xrightarrow{\circ L_\mu} & \{\text{lagrangians in } M\} \\ \ell & \longmapsto & \iota(\pi^{-1}(\ell)) \end{array}$$

where we view M_{red} and M as $pt^- \times M_{red}^+$ and $pt^- \times M^+$, respectively.

However, in the other direction there is no well-defined map:

$$\{\text{lagrangians in } M\} \quad \overset{\circ L_\mu^T}{\dashrightarrow} \quad \{\text{lagrangians in } M_{red}\}$$

In these terms, the lagrangian $\mathcal{L}_2 \subset \mathbb{CP}^2$ from Sections 3 and 4 is such that $\ell_2 = \mathcal{L}_2 \circ L_\mu^T \subset \mathbb{CP}^1$ is an *embedded composition*; hence, the above “map” is well defined for $\mathcal{L}_2$.

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