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On the Kostant Multiplicity Formula for Group Actions with Non-isolated Fixed Points

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1. INTRODUCTION

Let M be a compact, connected, $2d$-dimensional manifold equipped with a symplectic form, ω, and with a Hermitian line bundle, $\mathbf{L}$, and an almost-complex structure, J, which are compatible with ω. (By *compatible* we mean that

$$c(\mathbf{L}) = [\omega] \tag{1.1}$$

and that the bilinear form

$$g_p(v, w) = \omega_p(J_p v, w);\ v, w \in T_p M, \tag{1.2}$$

is symmetric and positive definite.) From J_p one gets a Dolbeault structure on the exterior algebra of T_p:

$$\bigwedge^i (T_p) \otimes \mathbf{C} = \sum_{i=j+k} \bigwedge_p^{j,k}.$$

For $\xi \in T_p^*$, let $\xi_{0,1}$ be the $\bigwedge^{0,1}$-component of ξ, and let

$$\gamma_p(\xi) \colon \mathbf{L}_p \otimes \bigwedge_p^{0,i} \to \mathbf{L}_p \otimes \bigwedge_p^{0,i+1}$$

be the map,

$$\gamma_p(\xi)\, w = \xi_{0,1} \wedge w.$$

The compatibility condition (1.2) implies that g_p and ω_p are the real and imaginary parts of a Hermitian inner product; and from this inner product

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and the inner product on $\mathbf{L}_p$ one gets inner products on the domain and range of $\gamma_p(\xi)$. Let $\gamma_p(\xi)^*$ be the transpose of $\gamma_p(\xi)$, and let

$$\sigma_p(\xi)\colon \mathbf{L}_p \otimes \textstyle\bigwedge_p^{0,\,\text{even}} \to \mathbf{L}_p \otimes \textstyle\bigwedge_p^{0,\,\text{odd}} \tag{1.3}$$

be the sum of $\gamma_p(\xi)$ and $\gamma_p(\xi)^*$. For $\xi \neq 0$ this map is bijective; so there exists a first order elliptic differential operator, D, whose symbol is (1.3). We will denote by $\operatorname{Ind}(D)$ the virtual vector space

$$\operatorname{kernel}(D) - \operatorname{cokernel}(D). \tag{1.4}$$

Now let G be a compact connected Lie group, and let τ be an effective action of G on M which preserves ω and J. We will assume that there is an action, τ_1, of G on $\mathbf{L}$ which is compatible with τ and hence, in particular (see [GS]) that τ is a Hamiltonian action with moment map, $\phi\colon M \to \mathfrak{g}^*$. From τ_1 one gets an induced action of G on the sections of $\mathbf{L} \otimes \bigwedge^{0,\,*}$ and, by averaging, one can make D commute with this action. Thus one gets a representation of G on $\operatorname{Ind}(D)$ which, up to isomorphism, is a *Hamiltonian* invariant of M, i.e., depends on (τ, ϕ) but doesn't depend on J or D. To compute this invariant, one can, without loss of generality, assume that G is abelian (see appendix A) in which case this representation is completely determined by its weight multiplicities. If M^G is finite, these are given by the Kostant multiplicity formula:

$$\#(\alpha, \operatorname{Ind}(D)) = \sum (-1)^{\sigma_i} N_i(\alpha) \tag{1.5}$$

the left hand side being the multiplicity of the weight, α, and N_i being the "Kostant partition function" associated with the isotropy representation of G at the ith fixed point.[1, 2]

In this article we will show that a formula of this type is true when M^G isn't finite. Let's denote the connected components of M^G by F_i, $i = 1, \ldots N$, and let $\mathbf{N}F_i$ be the normal bundle of F_i. $\mathbf{N}F_i$ splits into a direct sum of vector subbundles

$$\mathbf{E}_{i,1} \oplus \cdots \oplus \mathbf{E}_{i,m}, \tag{1.6}$$

m depending on i, such that the isotropy representation of $\mathfrak{g}$ on $\mathbf{E}_{i,j}$ is multiplication by a fixed weight, $\alpha_{i,j}$ (where $\alpha_{i,j} \neq \alpha_{i,k}$ for $j \neq k$). We will

[1] For the definition of N_i and σ_i, see below.

[2] This formula was discovered by Kostant [Ko] in the middle fifties in the setting of coadjoint orbits and, in the late eighties, extended by Guillemin, Lerman and Sternberg [GLS] to the setting above.

polarize these weights as in [GLS] by choosing an element, v, of $\mathfrak{g}$ such that $\alpha_{i,j}(v) \neq 0$ for all i, j, and setting

$$\alpha^{\#}_{i,j} = \varepsilon_{i,j} \alpha_{i,j} \tag{1.7}$$

where

$$\varepsilon_{i,j} = \operatorname{sign} \alpha_{i,j}(v). \tag{1.8}$$

(These polarized weights have the property that they all lie in the half-space $0 < (\xi, v)$.) Let $n_{i,j}$ be the rank of the vector bundle, $\mathbf{E}_{i,j}$, and let

$$\delta_i = \sum_{\varepsilon_{i,j} = -1} n_{i,j} \alpha^{\#}_{i,j} \qquad \text{and} \qquad \sigma_i = \sum_{\varepsilon_{i,j} = -1} n_{i,j}. \tag{1.9}$$

For every m-tuple of non-negative integers, $k = (k_1, ..., k_m)$, let $\mathbf{E}_i(k)$ be the tensor product

$$\left(\bigotimes_{j=1}^{m} \mathscr{S}^{k_j}(\mathbf{E}^{\#}_{i,j}) \right) \otimes \left(\bigotimes_{\varepsilon_{i,j} = -1} \bigwedge^{n_{i,j}} (\mathbf{E}^{\#}_{i,j}) \right) \tag{1.10}$$

where $\mathbf{E}^{\#}_{i,j} = \mathbf{E}_{i,j}$ or $\mathbf{E}^{*}_{i,j}$ depending on whether $\varepsilon_{i,j}$ is 1 or -1. Finally let $\Delta_i(\alpha)$ be the convex polytope in $\mathbf{R}^m$ consisting of all m-tuples, $(s_1, ..., s_m)$, $s_i \geqslant 0$, for which

$$\sum_j s_j \alpha^{\#}_{i,j} + \phi_i = \alpha \tag{1.11}$$

where ϕ_i is the value of ϕ on F_i. (The fact that the $\alpha^{\#}_{i,j}$'s are polarized implies that $\Delta_i(\alpha)$ is compact.) Our generalization of the Kostant formula is the following:

THEOREM 1. *The multiplicity with which* α *occurs as a weight of the representation of* G *on* Ind(D) *is equal to the sum* (1.5) *where*

$$N_i(\alpha) = \sum_{k \in \Delta_i(\alpha - \delta_i)} \int_{F_i} \operatorname{Ch}(\mathbf{E}_i(k) \otimes \mathbf{L}) \operatorname{Todd}(F_i) \tag{1.12}$$

Todd(F_i) *being the Todd class of* F_i (*with the almost-complex structure induced on* F_i *by* J) *and* Ch$(\mathbf{E}_i(k) \otimes \mathbf{L})$ *being the Chern character of* $\mathbf{E}_i(k) \otimes \mathbf{L}$.

Remark. If M^G is finite (1.12) reduces to

$$N_i(\alpha) = \sum_{k \in \Delta_i(\alpha - \delta_i)} \binom{k_1 + n_{i,1} - 1}{n_{i,1} - 1} \cdots \binom{k_m + n_{i,m} - 1}{n_{i,m} - 1}. \tag{1.13}$$

The formula (1.12) has an interesting "semi-classical" limit. Replacing the line bundle, $\mathbf{L}$, by its nth tensor power, one gets, in analogy with (1.3), an elliptic symbol

$$\sigma_p^{(n)}(\xi)\colon \mathbf{L}_p^n \otimes \bigwedge\nolimits_p^{0,\,\text{even}} \to \mathbf{L}_p^n \otimes \bigwedge\nolimits_p^{0,\,\text{odd}}.$$

Let D_n be a G-invariant elliptic operator with this as its symbol and let $\gamma = \dim G$.

Theorem 2. *As n tends to infinity, the quantity* $n^{-(d-\gamma)}\,\#(n\alpha, \operatorname{Ind}(D_n))$ *tends to*

$$\sum (-1)^{\sigma_i} \int_{\Delta_i(\alpha)} \operatorname{Res}_i(s)\, ds \tag{1.14}$$

where $\operatorname{Res}_i(s)$ *is the residue at* $z = 0$ *of*

$$\exp\left(\sum s_j z_j\right) \int_{F_i} \frac{\exp[\omega]}{c_{i,1}(z_1) \cdots c_{i,m}(z_m)} \tag{1.15}$$

and $c_{i,j}(z)$ *is the Chern polynomial of* $\mathbf{E}^{\#}_{i,j}$.

For the case of isolated fixed points (1.14) reduces to:

$$\sum (-1)^{\sigma_i} \int_{\Delta_i(\alpha)} \frac{s_1^{n_{i,1}-1} \cdots s_m^{n_{i,m}-1}}{(n_{i,1}-1)! \cdots (n_{i,m}-1)!}. \tag{1.16}$$

In [GLS] it was proved that the function of α defined by (1.16) is the Radon–Nikodym derivative

$$\frac{d\mu_{DH}}{d\mu_{Leb}} \tag{1.17}$$

where μ_{DH} is the Duistermaat–Heckman measure and μ_{Leb} is the standard Lebesgue measure on $\mathfrak{g}^*$ (suitably normalized). It turns out that the same is true for (1.14):

Theorem 3. *The piece-wise polynomial function of* α *defined by* (1.14) *is the Radon–Nikodym derivative*, (1.17).

The results above are true, with small modifications, for orbifolds. In particular in Theorem 1, the integral

$$\int_{F_i} \operatorname{Ch}(\mathbf{E}_i(k) \otimes \mathbf{L}) \operatorname{Todd}(F_i)$$

is the "Riemann–Roch" number of the vector bundle $\mathbf{E}_i(k) \otimes \mathbf{L}$, and the orbifold version of Theorem 1 is true if one replaces this by the *Kawasaki* Riemann–Roch number (see [Ka]).

We will conclude this summary of our results by saying a few words about the proofs: It was shown by Cartier in [Ca] that the Kostant multiplicity formula can be derived from the Weyl character formula by expanding the Weyl denominator into a trigonometric series and computing the coefficient of $e^{i\alpha}$.[3] In this article we will show that (1.12) can be derived, by essentially the same argument, from the equivariant index theorem of Atiyah–Segal–Singer for spin^c-Dirac (see [AS]). Duistermaat [Du] has recently proved that the orbifold analogue of this theorem is true[4]; and, as a consequence, the proof which we give of Theorem 1 in Section 2 can easily be adapted to the orbifold setting.[5]

2. THE PROOF OF THEOREM 1

The equivariant index theorem says that for $x \in \sqrt{-1}\,\mathfrak{g}$, x close to zero, the trace of $\exp \sqrt{-1}\, x$ on the virtual vector space (1.4) is equal to the sum over the fixed point components, F_i, of local traces, $\chi_{F_i}(x)$, where

$$\chi_{F_i}(x) = e^{\phi_i(x)} \int_{F_i} \frac{e^{[\omega]}\,\mathrm{Todd}(F_i)}{\prod_j \det(I - \exp(\alpha_{i,j}(x)\, I + \Omega(\mathbf{E}_{i,j})))} \tag{2.1}$$

$\Omega(\mathbf{E}_{i,j})$ being the curvature form associated with a connection on $\mathbf{E}_{i,j}$. To simplify notation in the paragraph below we omit the subscript i's in (2.1) and set $F_i = F$, $\mathbf{E}_{i,j} = \mathbf{E}_j$, $\alpha_{i,j} = \alpha_j$, $\phi_i = \phi_F$, $\varepsilon_{i,j} = \varepsilon_j$, etc. If $\varepsilon_j = -1$, the jth term in the denominator can be rewritten:

$$(-1)^{n_j}\, e^{n_j \alpha_j(x)} \det \exp \Omega(\mathbf{E}_j) \det(I - e^{-\alpha_j(x)} \exp(-\Omega(\mathbf{E}_j)))$$

and if we substitute this into (2.1) and let $\mathbf{D}$ be the line bundle

$$\bigotimes_{\varepsilon_j = -1} \bigwedge^{n_j} (\mathbf{E}_j^{\#})$$

[3] This is less trivial than it sounds: There are several ways of expanding the Weyl denominator into a trigonometric series, and for some of these expansions the coefficient of $e^{i\alpha}$ will be given by a divergent infinite sum.

[4] He has, in fact, proved a somewhat deeper result of which this is a consequence: that the "local" version of Atiyah–Segal–Singer is true for spin^c-Dirac.

[5] For a more detailed account of the orbifold versions of theorems 1 to 3, see [CG].

we can rewrite (2.1) in "polarized" form

$$(-1)^{\sigma}\, e^{(\delta+\phi_F)(x)} \int_F \frac{\exp[\omega] \exp \Omega(\mathbf{D}) \operatorname{Todd}(F)}{\prod_j \det(I - e^{\alpha_j^{\#}(x)} \exp \Omega(\mathbf{E}_j^{\#}))}.$$

By Theorem 1 of appendix B this can be expanded into an infinite trigonometric series.

$$(-1)^{\sigma} \sum_k c_k e^{k_1 \alpha_1^{\#} + \cdots + k_m \alpha_m^{\#} + \delta + \phi_F} \tag{2.2}$$

summed over all non-negative integer m-tuples, k, where c_k is equal to

$$\int_F \operatorname{trace} \tau_{k_1}(\exp \Omega(\mathbf{E}_1^{\#})) \cdots \operatorname{trace} \tau_{k_m}(\exp \Omega(\mathbf{E}_m^{\#})) \exp(\omega + \Omega(\mathbf{D})) \operatorname{Todd}(F) \tag{2.3}$$

or

$$\int_F \operatorname{Ch}(\mathbf{E}(k) \otimes \mathbf{L}) \operatorname{Todd}(F). \tag{2.4}$$

(Notice that since the $\alpha_i^{\#}$ are polarized, the quantity

$$k_1 \alpha_1^{\#}(v) + \cdots + k_m \alpha_m^{\#}(v) + \delta(v) + \phi_F(v)$$

tends to $+\infty$ as $k_1 + \cdots + k_m$ tends to $+\infty$. Thus for any constant, C, there are only a finite number of k's for which this quantity is less than C.) On the other hand, for $x \in \sqrt{-1}\,\mathfrak{g}$ the trace of $exp\sqrt{-1}\, x$ on the vector space (1.4) is equal to

$$\sum \#(\alpha, \operatorname{Ind}(D))\, e^{\alpha(x)} \tag{2.5}$$

and by comparing (2.2) with (2.5) one gets the identity (1.12).

3. THE PROOF OF THEOREM 2

By Theorem 1, $\#(n\alpha, \operatorname{Ind}(D_n))$ is equal to the sum

$$\sum (-1)^{\sigma_i} N_i^{(n)}(n\alpha) \tag{3.1}$$

where

$$N_i^{(n)}(n\alpha) = \sum_k \int_{F_i} \operatorname{Ch}(\mathbf{E}_i(k) \otimes \mathbf{L}^n) \operatorname{Todd}(F_i) \tag{3.2}$$

summed over all non-negative integral solutions, k, of the equation

$$k_1 \alpha^{\#}_{i,1} + \cdots + k_m \alpha^{\#}_{i,m} + \delta_i + n\phi_i = n\alpha. \tag{3.3}$$

(Notice that if we replace $\mathbf{L}$ by $\mathbf{L}^n$ we must replace ω by $n\omega$ and ϕ by $n\phi$.) As in Section 2 we will omit all subscript i's from now on and let $F_i = F$, $\alpha_{i,j} = \alpha_j$, etc. Let $2p = \dim F$ and $q = \dim \Delta(\alpha)$. By (3.2)

$$n^{-(d-\gamma)} N^{(n)}(n\alpha) = n^{-(d-\gamma-p)} \sum_k \int_F n^{-p} \mathrm{Ch}(\mathbf{E}(k) \otimes \mathbf{L}^n)\ \mathrm{Todd}(F)$$

which is equal to

$$n^{-(d-\gamma-p)} \sum_k \int_F \exp[\omega]\ \mathrm{trace}\ \tau_{k_1}(\exp \Omega(\mathbf{E}^{\#}_1)/n) \cdots \mathrm{trace}\ \tau_{k_m}(\exp \Omega(\mathbf{E}^{\#}_m)/n) \tag{3.4}$$

up to an error of order $O(1/n)$. (Proof: With ω replaced by $n\omega$ in (2.3), the integrand in this expression can be expanded into a sum of terms of the form

$$n^{-p} (n\omega)^r \wedge \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \wedge \Omega(\mathbf{D})^l \wedge T_\mu$$

where Ω_{i_a} is a coefficient of the curvature form, $\Omega(\mathbf{E}^{\#}_{i_a})$, and T_μ is the component of degree 2μ of $\mathrm{Todd}(F)$. However this term can only contribute to the integral if $r + s + l + \mu = p$ in which case it can be rewritten as

$$\omega^r \wedge (\Omega_{i_1}/n) \wedge \cdots \wedge (\Omega_{i_s}/n) \wedge (\Omega(\mathbf{D})/n)^l \wedge T_\mu/n^\mu.$$

Moreover, the terms in this sum for which l or μ is positive can be discarded since they contribute errors of order $O(1/n)$.)

By Theorem 2 of Appendix B, (3.4) is equal, up to an error of order $O(1/n)$, to

$$n^{-q}\ \mathrm{Res}_{z=0} \sum_k e^{(k_1/n) z_1 + \cdots + (k_m/n) z_m} \int_F \frac{\exp[\omega]}{\det(zI - \Omega(\mathbf{E}^{\#}_1)) \cdots \det(zI - \Omega(\mathbf{E}^{\#}_m))}$$

summed over all k satisfying

$$\frac{k_1}{n} \alpha_1 + \cdots + \frac{k_m}{n} \alpha_m + \phi_F + \frac{\delta}{n} = \alpha$$

and as n tends to infinity this tends to the integral

$$\int_{\Delta(\alpha)} \mathrm{Res}_{z=0}\, e^{sz} \left(\int_F \frac{\exp[\omega]}{c_{\mathbf{E}_1^\#}(z_1) \cdots c_{\mathbf{E}_m^\#}(z_m)} \right) ds.$$

4. THE PROOF OF THEOREM 3

By definition the Duistermaat–Heckman measure is the "push-forward" by the moment map of the symplectic measure on M, *i.e.*, for a Borel subset, B, of $\mathfrak{g}^*$

$$\mu_{DH}(B) = \int_{\phi^{-1}(B)} \frac{\omega^d}{d!}.$$

The inverse Fourier transform of μ_{DH} is the function

$$\check{\mu}_{DH}(x) = \int_M e^{\sqrt{-1}\,(\phi,\, x)} \frac{\omega^d}{d!}$$

and by the "exact stationary phase" formula [DH] this is equal to the sum over fixed point components

$$\sum_i e^{\sqrt{-1}\,\phi_i(x)} \int_{F_i} \frac{\exp[\omega]}{\prod_j \det(\sqrt{-1}\,\alpha_{ij}(x)\, I + \Omega(\mathbf{E}_{ij}))} \tag{4.1}$$

providing $\alpha_{ij}(x) \neq 0$ for all i and j. Dropping the subscript i's and setting $y = \sqrt{-1}\, x$, the ith summand becomes

$$e^{\phi_F(y)} \int_F \frac{\exp[\omega]}{\prod_j \det(\alpha_j(y)\, I + \Omega(\mathbf{E}_j))} \tag{4.2}$$

or

$$(-1)^\sigma\, e^{\phi_F(y)} \int_F \frac{\exp[\omega]}{\prod_j \det(\alpha_j^\#(y) + \Omega(\mathbf{E}_j^\#))}. \tag{4.3}$$

By Theorem 1 of Appendix B this is equal to

$$\frac{(-1)^\sigma\, e^{\phi_F(y)}}{e^{[\omega]} \prod_j \alpha_j^\#(y)^{n_j}} \sum_{k=0}^{\infty} \frac{1}{\prod_j \alpha_j^\#(y)^{k_j}} \int_F \prod_j \mathrm{trace}\, \tau_{k_j}(-\Omega(\mathbf{E}_j^\#)). \tag{4.4}$$

(Note that this sum is finite. The terms on the right are zero if $2 \sum n_j k_j >$ dim F.) By the Fourier inversion formula the Radon–Nikodym derivative

(1.17) is the Fourier transform of (4.1), and we can compute this by computing the Fourier transforms of the summands in (4.4) and summing over k and the fixed point components. By formula C9 of Appendix C, the Fourier transform of

$$\frac{e^{\phi_F(y)}}{\prod_j \alpha_j^{\#}(y)^{k_j+n_j}}, \qquad y=\sqrt{-1}\,x,$$

is the function

$$f_k(\alpha)=\int_{\Delta(\alpha)} \frac{s_1^{k_1+n_1-1}}{(k_1+n_1-1)!} \cdots \frac{s_m^{k_m+n_m-1}}{(k_m+n_m-1)!}\, ds. \tag{4.5}$$

Substituting this into (4.4) one gets

$$(-1)^{\sigma}\int_{\Delta(\alpha)} ds\left(\int_F e^{[\omega]}\prod_j \frac{s_j^{k_j+n_j-1}}{(k_j+n_j-1)!}\,\text{trace}\,\tau_{k_j}(-\Omega(\mathbf{E}_j^{\#}))\right). \tag{4.6}$$

However, by formula B3 of Appendix B,

$$\text{trace}\,\tau_{k_j}(-\Omega(\mathbf{E}_j^{\#}))=\frac{1}{2\pi i}\int_{\Gamma_j} \frac{z^{n_j+k_j-1}}{\det(zI+\Omega(\mathbf{E}_j^{\#}))} \tag{4.7}$$

Γ_j being a small contour about the origin in the z_j plane. If $k_j<0$ the integral on the right is zero, so by substituting (4.7) into (4.6) and summing over all $k_j\geqslant 0$ (or, equivalently, over all $k_j+n_j-1\geqslant 0$) one gets for the Fourier transform of (4.4):

$$(-1)^{\sigma}\int_{\Delta(\alpha)} ds\left(\text{Res}_{z=0}\, e^{sz}\int_F \frac{\exp[\omega]}{\prod_j c_{\mathbf{E}_j^{\#}}(z_j)}\right). \tag{4.8}$$

APPENDIX A

By the "shifting trick" (see [GS], Section 6) it suffices to compute the multiplicity with which the trivial representation occurs in the representation of G on the space (1.4) and (as was pointed out to us by Michèle Vergne) this can easily be computed from the weight multiplicities of the representation of the Cartan subgroup, T, of G on the space (1.4). More explicitly the following result is true: Let G be a compact semi-simple Lie group and $\rho\colon G\to U(Q)$ a representation of G on a finite dimensional Hilbert space, Q. Restricting ρ to T, Q breaks up into weight spaces

$$Q_{\xi}, \qquad \xi\in\mathbf{Z}_T$$

($\mathbf{Z}_T$ being the weight lattice of T). Then

$$\dim Q^G = \frac{1}{|W|} \sum C_\xi \dim Q_\xi, \tag{A1}$$

the C_ξ's being the Fourier coefficients of the function, $\prod_{\alpha \in \Delta} (1 - e^{i\alpha})$. In other words,

$$\prod_{\alpha \in \Delta} (1 - e^{i\alpha(x)}) = \sum_\xi C_\xi e^{i\xi(x)}, \qquad x \in \mathfrak{t}. \tag{A2}$$

(Here Δ is the set of roots of G.)

Proof. (A1) can be extracted from the following result of Weyl (see [He], page 194, Corollary 5.16).

THEOREM. *Let $\chi \in C^\infty(G)$ be a class function (i.e., $\chi(aga^{-1}) = \chi(g)$ for all a and g.) Then*

$$\int_G \chi(g)\, dg = \frac{1}{|W|} \int_T \theta(x)\, \chi(x)\, dx \tag{A3}$$

dg and dx being Haar measures on G and T, $\theta(x)$ being the function (A2) *and $|W|$ being the cardinality of the Weyl group.*

Comments. 1. $\theta(x)$ is real and non-negative, as one can see by writing it as the product of the function

$$\prod_{\alpha \in \Delta_+} (1 - e^{i\alpha}) \tag{A4}$$

times its conjugate. In particular, $\theta = \bar{\theta}$, i.e., $C_\xi = C_{-\xi}$.

2. Let δ be half the sum of the positive roots. It is clear from (A4) that $C_\xi \neq 0 \Rightarrow \xi/2$ lies in the convex hull of $\{\omega\delta, w \in W\}$.

Let's apply (A3) to the character, χ, of representation ρ. Noting that for $x \in \mathfrak{t}$:

$$\chi(\exp x) = \sum e^{i\xi(x)} \dim Q_\xi \tag{A5}$$

one gets, by Schur's lemma

$$\dim Q^G = \int \chi(g)\, dg = \langle \chi, 1 \rangle_{L^2} \tag{A6}$$

(1 being the character of the trivial representation), and hence, by (A3) and (A4)

$$\dim Q^G = \frac{1}{|W|} \int \chi(\exp x)\, \theta(x)\, dx$$

$$= \frac{1}{|W|} \int \left(\sum_{\xi} e^{i\xi(x)} \dim Q_\xi \right) \left(\sum_{\xi} C_\xi e^{-i\xi(x)} \right) dx$$

$$= \frac{1}{|W|} \sum C_\xi \dim Q_\xi .$$

APPENDIX B

Let V be a d-dimensional vector space over the complex numbers and let τ_k be the standard representation of $GL(V)$ on the kth symmetric product, $\mathscr{S}^k(V)$.

THEOREM (B1). *For $z \in \mathbf{C}$, z large, and $B \in GL(V)$,*

$$\det(z - B)^{-1} = z^{-d} \sum_{k=0}^{\infty} s^{-k} \operatorname{trace} \tau_k(B) \tag{B1}$$

Proof. Without loss of generality we can assume that B is diagonalizable with eigenvalues, $\lambda_1, ..., \lambda_d$; in which case the left hand side of (B1) becomes

$$z^{-d} \prod_{j=1}^{d} (1 - \lambda_j z^{-1})^{-1}. \tag{B2}$$

Expanding each of the factors $(1 - \lambda_j z^{-1})^{-1}$ into a geometric series one can rewrite (B2) in the form

$$z^{-d} \left(\sum z^{-k} t_k \right)$$

where

$$t_k = \sum_{|I| = k} \lambda_1^{i_1} \cdots \lambda_d^{i_d},$$

and the right hand side of this expression is trace $\tau_k(B)$. Q.E.D

COROLLARY. *Let Γ be a contour about the origin containing the zeroes of* $\det(z-B)$. *Then*

$$\frac{1}{2\pi i}\int_{\Gamma} z^{d+k-1}\det(z-B)^{-1}\,dz=\operatorname{trace}\,\tau_k(B). \tag{B3}$$

Remark. By analyticity, (B1) and (B3) are valid for any endomorphism, $B\colon V\to V$; *i.e.*, B doesn't necessarily have to be in $GL(V)$.

From (B3) we will deduce the following useful estimate:

THEOREM (B2). *Let A be an endomorphism of V. Then*

$$n^{-(d-1)}\operatorname{trace}\,\tau_k(\exp A/n)$$
$$=\frac{1}{2\pi i}\left(\int_{\Gamma} e^{((d+k-1)/n)\,z}\det(z-A)^{-1}\,dz\right)\left(1+O\left(\frac{1}{n}\right)\right) \tag{B4}$$

the $O(1/n)$ being uniform in k.

Proof. Without loss of generality we can assume that A is diagonalizable with eigenvalues $\mu_1, ..., \mu_d$ and that $e^{\mu_1}, ..., e^{\mu_d}$ are distinct. Then by (B3) trace $\tau_k(\exp A/n)$ is equal to the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma} z^{d+k-1}(z-e^{\mu_1/n})^{-1}\cdots(z-e^{\mu_d/n})^{-1}\,dz$$

which, by the residue formula, is equal to

$$\sum_{i=1}^{d} e^{(d+k-1)\,\mu_i/n}\prod_{j\neq i}(e^{\mu_i/n}-e^{\mu_j/n})^{-1}$$

or

$$n^{d-1}\left(\sum_{i=1}^{d} e^{(d+k-1)\,\mu_i/n}\prod_{j\neq i}(\mu_i-\mu_j)^{-1}\right)\left(1+O\left(\frac{1}{n}\right)\right)$$

and, again by the residue formula, this is equal to:

$$n^{d-1}\left(\frac{1}{2\pi i}\int_{\Gamma} e^{((d+k-1)/n)\,z}\prod_{i=1}^{d}(z-\mu_i)^{-1}\right)\left(1+O\left(\frac{1}{n}\right)\right).$$

Dividing by n^{d-1} and replacing $\prod(z-\mu_i)$ by $\det(z-A)$ we obtain (B4).

APPENDIX C

Let $\alpha_1, ..., \alpha_d$ be vectors in $\mathbf{R}^n$ which are "polarized" in the sense that, for some $v \in \mathbf{R}^n$, the inner products, (α_i, v), are all positive. Given $\phi \in \mathbf{R}^n$ consider the function

$$e^{i(\phi, x)} \prod_{j=1}^{d} (\alpha_j, x)^{-1}. \tag{C1}$$

Since this function isn't well-defined on all of $\mathbf{R}^n$, its Fourier transform is also not well-defined. However, there is a unique measure, μ, on $\mathbf{R}^n$, with the following two properties:

1. The inverse Fourier transform of μ is equal to (C1) on the set

$$(\alpha_j, v) \neq 0, \qquad j = 1, ..., d.$$

2. μ is supported in the half space

$$(\xi, v) \geqslant 0.$$

Proof. One can take for μ the measure

$$H_{\alpha_1} * \cdots * H_{\alpha_d} * \delta_\phi \tag{C2}$$

where δ_ϕ is the delta-measure at ϕ and

$$H_{\alpha_i}(f) = \int_0^\infty f(t\alpha_i)\, dt \tag{C3}$$

for continuous functions of compact support, f. Q.E.D

Another description of this measure is the following: Let

$$\mathbf{R}^d_+ = \{(s_1, ..., s_d), s_i \geqslant 0\}$$

be the positive orthant in $\mathbf{R}^d$ and let $L: \mathbf{R}^d_+ \to \mathbf{R}^n$ be the map

$$L(s_1, ..., s_d) = \sum s_i \alpha_i + \phi.$$

The assumption that the α_i's are polarized implies that this is a proper mapping so the measure

$$L_* ds_1 \cdots ds_d \tag{C4}$$

is well defined.

THEOREM (C1). *The measures* (C2) *and* (C4) *are equal.*

Proof. In the special case of $\mathbf{R}^n = \mathbf{R}^d$ and $\alpha_i = e_i$ (the ith standard basis vector) this is just the Fubini theorem. Thus one can write Lebesgue measure on $\mathbf{R}^d_+$ as the convolution product

$$H_{e_1} * \cdots * H_{e_d}.$$

The theorem follows from the fact that $L_*(H_{e_i}) = H_{\alpha_i}$, and the fact that, for any pair of measures, μ_1 and μ_2, with support in $\mathbf{R}^d_+$,

$$L_*(\mu_1 * \mu_2) = L_*(\mu_1) * L_*(\mu_2). \qquad \text{Q.E.D}$$

COROLLARY (C2). *If the vectors,* $\alpha_1, ..., \alpha_d$, *span* $\mathbf{R}^n$ *the measure* (C2) *is absolutely continuous with respect to Lebesgue measure.*

Proof. If suffices to prove that the set of critical points of the map, L, is of measure zero, which will be the case if and only if $\alpha_1, ..., \alpha_d$ span $\mathbf{R}^n$. Q.E.D

Thus, if these hypotheses are satisfied, one can write the measure (C2) in the form

$$f(\xi)\, d\xi_1 \cdots d\xi_n \tag{C5}$$

the function f being in L^1_{loc}. In fact it is easy to see that, up to a scalar multiple,[6]

$$f(\xi) = \text{volume } \Delta(\xi), \tag{C6}$$

$\Delta(\xi)$ being the convex polytope:

$$\left\{ s \in \mathbf{R}^d_+, \sum s_i \alpha_i + \phi = \xi \right\}. \tag{C7}$$

By abuse of notation we will refer to (C6) as the *Fourier transform* of the function (C1). Let us compute, in the same spirit, the Fourier transform, g, of the function

$$e^{i(\phi, x)} \prod_{j=1}^{d} (\alpha_j, x)^{-N_j}. \tag{C8}$$

[6] By an appropriate normalization of Lebesgue measure in the space, $\sum s_i \alpha_i = 0$, one can make this scalar equal to one.

Letting $N=N_1+\cdots+N_d$, it follows from what we've just proved that $g(\xi)$ is the volume of the polytope consisting of all N-tuples

$$t=(t_{1,1},\ldots,t_{1,N_1},\ldots,t_{d,1},\ldots,t_{d,N_d})$$

in $\mathbf{R}^N_+$ satisfying

$$\sum_{i=1}^{d}\left(\sum_{j=1}^{N_i} t_{i,j}\right)\alpha_i+\phi=\xi.$$

Let's denote this polytope by $\tilde{\Delta}(\xi)$. From the mapping

$$\mathbf{R}^N_+\to\mathbf{R}^d_+,\qquad s_i=\sum_{j=1}^{N_i} t_{i,j},$$

one gets a fibration of $\tilde{\Delta}(\xi)$ over $\Delta(\xi)$, the volume of the fiber over s being

$$\frac{s_1^{N_i-1}}{(N_1-1)!}\cdots\frac{s_d^{N_d-1}}{(N_d-1)!}.$$

Hence

$$g(\xi)=\text{volume }\tilde{\Delta}(\xi)=\int_{\Delta(\xi)}\frac{s_1^{N_1-1}}{(N_1-1)!}\cdots\frac{s_d^{N_d-1}}{(N_d-1)!}\,ds. \qquad \text{(C9)}$$

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