Invariant measures of Hamiltonian systems with prescribed asymptotic Maslov index

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Abstract

We study the properties of the asymptotic Maslov index of invariant measures for time-periodic Hamiltonian systems on the cotangent bundle of a compact manifold M. We show that if M has finite fundamental group and the Hamiltonian satisfies some general growth assumptions on the momenta, the asymptotic Maslov indices of periodic orbits are dense in the half line $[0, +\infty)$. Furthermore, if the Hamiltonian is the Fenchel dual of an electromagnetic Lagrangian, every non-negative number r is the limit of the asymptotic Maslov indices of a sequence of periodic orbits which converges narrowly to an invariant measure with asymptotic Maslov index r. We discuss the existence of minimal ergodic invariant measures with prescribed asymptotic Maslov index by the analogue of Mather's theory of the beta function, the asymptotic Maslov index playing the role of the rotation vector.

Introduction

The Lagrangian Grassmannian $\mathscr{L}(n)$ is the space of n-dimensional linear subspaces of $\mathbb{R}^{2n}$ on which the symplectic form $\omega_0 = \sum_{j=1}^n dp^j \wedge dq^j$ vanishes. The fundamental group of $\mathscr{L}(n)$ is infinite cyclic. The Maslov index of a closed loop of Lagrangian subspaces is a measure of its winding number in $\mathscr{L}(n)$. It was introduced by Maslov in his book on perturbation methods in quantum mechanics [19], and its geometric interpretation both as a topological winding number and as an algebraic intersection number was discussed by Arnold in an appendix of the same book and in [5]. Here we will use the definition by Robbin and Salamon [23] of the Maslov index $\mu(\lambda, \lambda_0)$ of a path λ in $\mathscr{L}(n)$ with arbitrary end-points with respect to some $\lambda_0 \in \mathscr{L}(n)$.

If $x(t) = (q(t), p(t))$ is the orbit of a point (q_0, p_0) by a Hamiltonian flow on the phase space $\mathbb{R}^{2n}$, we can define $\mu_T(q_0, p_0)$ as the Maslov index of the evolution of the vertical space $(0) \times \mathbb{R}^n$ by the differential of the flow at (q_0, p_0), with respect to the vertical space itself, over the time interval $[0, T]$. When the Hamiltonian is strictly convex in the momentum variables p, this Maslov index is non-negative, and it coincides - up to a suitable additive constant - with the Morse index of the curve $q(t)$, seen as an extremal curve of the Lagrangian action functional with fixed end-points at $t = 0$ and $t = T$, as proved by Duistermaat in [10]. For a general Hamiltonian, this interpretation as a Morse index is recovered by Floer's approach to the study of the Hamiltonian action functional, as shown by Salamon and Zehnder in [25]. These facts remain true if we replace the phase space $\mathbb{R}^{2n}$ by the cotangent bundle T^*M of an arbitrary manifold, and the vertical Lagrangian space $(0) \times \mathbb{R}^n$ by the vertical subbundle $T^vT^*M = \ker D\pi$, where $\pi : T^*M \to M$ denotes the projection.

In this paper we are interested in the *asymptotic Maslov index*, that is in the limit

$$\hat{\mu}(q_0, p_0) := \lim_{T \to +\infty} \frac{\mu_T(q_0, p_0)}{T}.$$

Ergodic theory implies that the above limit exists for almost every initial condition (q_0, p_0), with respect to any invariant probability measure η on T^*M, and that the *asymptotic Maslov index* of an invariant probability measure η is well-defined:

$$\hat{\mu}(\eta) := \int \hat{\mu}\, d\eta = \lim_{T \to +\infty} \int \frac{\mu_T}{T}\, d\eta,$$

provided that the functions μ_T are in $L^1(T^*M, \eta)$. This invariant was introduced by Ruelle [24], who also showed that the asymptotic Maslov index of an invariant measure is continuous with respect to the narrow topology of measures and with respect to perturbations of the Hamiltonian system, a fact which fails to be true for other ergodic invariants such as the Lyapunov exponents. Under different names, this invariant appears in the context of systems defined by a Hamiltonian which is convex in the (q, p) variables in the book of Ekeland [11], and in the context of Lagrangian systems in [1] and [7]. Contreras, Gambaudo, Iturriaga and Paternain [8] have extended this invariant to symplectic manifolds with a fixed Lagrangian bundle, and have studied the properties of the asymptotic Maslov index of the Liouville measure on the energy level of optical Hamiltonians.

Here we deal mainly with time-periodic Hamiltonian systems on the cotangent bundle of a compact manifold M. We start by recalling the definition and the continuity properties of the asymptotic Maslov index in this context, paying special attention to invariant measures which are not compactly supported (as often happens when there is no conservation of energy). Let $I(H) \subset \mathbb{R}$ be the set - actually an interval - consisting of the asymptotic Maslov indices of the invariant measures. Our first result is that when M is compact and has finite fundamental group and the Hamiltonian satisfies some very general growth assumptions, the interval $I(H)$ contains the whole half-line $[0, +\infty)$. Actually, the asymptotic Maslov indices of the contractible closed orbits with integer period are dense in $[0, +\infty)$. The proof is based on Floer's approach by J-holomorphic curves, together with a result from [2] which allows to deal with general growth conditions on the Hamiltonian. A large interval $I(H)$ is in sharp contrast with what may happen with Hamiltonian systems on configuration spaces having infinite fundamental group: all the invariant measures for the geodesic flow on a compact Riemannian manifold with non-positive sectional curvature - which necessarily has infinite fundamental group - have vanishing asymptotic Maslov index, so $I(H) = \{0\}$ in this case.

If a positive number r is not the asymptotic Maslov index of any contractible periodic orbit - generically we expect a countable number of periodic orbits - it seems natural to consider a sequence of periodic orbits with asymptotic Maslov index converging to r and try to take a limit. Our second and main result is that this is indeed possible for a Hamiltonian which is the Fenchel transform of an electro-magnetic Lagrangian

$$L(t, q, v) = \frac{1}{2}\langle v, v\rangle + \langle A(t, q), v\rangle - V(t, q),$$

provided that the manifold M is compact and has finite fundamental group. Under these assumptions, we prove that for any $r \geq 0$ at least one of the following statements must hold:

(i) There is a contractible closed orbit with integer period and asymptotic Maslov index r.

(ii) There is a sequence of contractible closed orbits with integer minimal periods $k_j \to +\infty$ and asymptotic Maslov indices $\hat{\mu}_j \to r$. The corresponding sequence of probability measures narrowly converges to an invariant probability measure η with finite second moment and asymptotic Maslov index $\hat{\mu}(\eta) = r$.

In the last section we outline the analogue of Mather's theory of the beta function, the asymptotic Maslov index $\hat{\mu} \in I(H)$ playing the role of the rotation class $\rho \in H_1(M, \mathbb{R})$. For sake of simplicity, we deal with electromagnetic Lagrangians on compact manifolds with finite fundamental group, so that $I(H) = [0, +\infty)$ and for any $r \geq 0$ there is an invariant measure with asymptotic Maslov index r and finite action. In this case, the function

$$\beta : [0, +\infty) \to \mathbb{R},$$

which associates to r the minimal Lagrangian action over all the invariant measures with asymptotic Maslov index r is convex and has quadratic growth. In particular, β must have infinitely many strict extremal points, and if r is such a point the minimum $\beta(r)$ is achieved by an ergodic invariant measure. Notice that the topological assumption on M implies that $H_1(M,\mathbb{R}) = (0)$, so Mather theory would produce just one class of minimal invariant measures, while the fact that $I(H) = [0,+\infty)$ allows to find a diverging sequence of numbers which are asymptotic Maslov indices of ergodic action-minimizing measures. Of course, in this situation there are also infinitely many periodic orbits, which produce ergodic invariant measures, but these measures need not minimize the action among invariant measures with prescribed asymptotic Maslov index.

Other related problems remain open. Mather [20] has proved that measures minimizing the action with given rotation class are supported in the image of a Lipschitz section of TM. What is the structure of a minimizing measure with given asymptotic Maslov index? Mañé [18] has proved that a measure which minimizes the action among *closed* measures is invariant. Can one extend the notion of asymptotic Maslov index to some class of closed measures, and prove that a measure which minimizes the action among closed ones with given asymptotic Maslov index is invariant?

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1 The Maslov index

In this section we review the basic properties of the Maslov index, following [23].

We consider coordinates $(q^1,\dots,q^n,p^1,\dots,p^n)$ on the Euclidean space $\mathbb{R}^{2n}$, endowed with its standard symplectic structure, given by the alternating 2-form

$$\omega_0 = \sum_{j=1}^{n} dp^j \wedge dq^j.$$

The symplectic group - that is the group of linear isomorphisms of $\mathbb{R}^{2n}$ preserving ω_0 - is denoted by $\mathrm{Sp}(2n)$. Let $\mathscr{L}(n)$ be the manifold of Lagrangian subspaces of $(\mathbb{R}^{2n},\omega_0)$, that is n-dimensional linear subspaces of $\mathbb{R}^{2n}$ on which ω_0 vanishes identically.

Let $\lambda : [a,b] \to \mathscr{L}(n)$ be a continuous path of Lagrangian subspaces, and let λ_0 be a fixed Lagrangian subspace. The *Maslov index* $\mu(\lambda,\lambda_0)$ is a half-integer counting the intersections of $\lambda(t)$ with λ_0, with suitable orientation signs and multiplicity. Rather than defining it, we prefer to list its characterizing properties:

Naturality. If $\Phi \in \mathrm{Sp}(2n)$, then $\mu(\Phi\lambda,\Phi\lambda_0) = \mu(\lambda,\lambda_0)$.

Juxtaposition. If $a < c < b$, then $\mu(\lambda,\lambda_0) = \mu(\lambda|_{[a,c]},\lambda_0) + \mu(\lambda|_{[c,b]},\lambda_0)$.

Product. If $n' + n'' = n$ and $\mathscr{L}(n') \times \mathscr{L}(n'')$ is identified with a submanifold of $\mathscr{L}(n)$ in the obvious way, then $\mu(\lambda' \oplus \lambda'', \lambda_0' \oplus \lambda_0'') = \mu(\lambda',\lambda_0') + \mu(\lambda'',\lambda_0'')$.

Homotopy. If $\lambda,\lambda' : [a,b] \to \mathscr{L}(n)$ have the same end-points and are homotopic with fixed end-points, then $\mu(\lambda,\lambda_0) = \mu(\lambda',\lambda_0)$.

Localization. If $\lambda(t) = \mathrm{graph}\, S(t)$, the graph of a path of symmetric endomorphisms of $\mathbb{R}^n$, then $\mu(\lambda,\mathbb{R}^n \times (0)) = (\mathrm{sign}\, S(b) - \mathrm{sign}\, S(a))/2$, where sign denotes the signature.

Zero. For every path $\lambda : [a,b] \to \mathscr{L}(n)$ such that the dimension of $\lambda(t) \cap \lambda_0$ does not depend on t, the Maslov index $\mu(\lambda,\lambda_0)$ is zero.

A related invariant is the *Conley-Zehnder index* $\mu_{CZ}(\Phi)$ of a continuous path $\Phi : [a,b] \to \mathrm{Sp}(2n)$ in the symplectic group. It can be defined as

$$\mu_{CZ}(\Phi) = \mu(\mathrm{graph}\,\Phi, \Delta),$$

the Maslov index of the graph of Φ with respect to the diagonal Δ of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Here $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is endowed with the symplectic form $\omega \oplus (-\omega)$, so that the diagonal, and more generally the graph of every symplectic isomorphism, is a Lagrangian subspace. When $\Phi(a) = I$ and $\Phi(b) - I$ is invertible, $\mu_{CZ}(\Phi)$ is an integer. The Conley-Zehnder index is significant when $\Phi : \mathbb{R} \to \mathrm{Sp}(2n)$ is the fundamental solution of a linear k-periodic Hamiltonian system. In this situation, the limit

$$\hat{\mu}_{CZ}(\Phi) := \lim_{h \to +\infty} \frac{\mu_{CZ}(\Phi|_{[0,hk]})}{hk}$$

exists, and

$$|\mu_{CZ}(\Phi|_{[0,hk]}) - hk\, \hat{\mu}_{CZ}(\Phi)| \leq 2n, \tag{1}$$

as it is proved in [25]. See also [16] for finer estimates on the behavior of the Conley-Zehnder index under iteration. The quantity $\hat{\mu}_{CZ}(\Phi)$ is called the *Bott index* of the linear Hamiltonian system solved by Φ.

Let $\Phi : [a, b] \to \mathrm{Sp}(2n)$ be a continuous path in the symplectic group, and let $\lambda_0 \in \mathscr{L}(n)$. Then

$$|\mu_{CZ}(\Phi) - \mu(\Phi\lambda_0, \lambda_0)| \leq 2n. \tag{2}$$

In fact, by Theorem 3.2 in [23],

$$\mu(\Phi\lambda_0, \lambda_0) = \mu(\mathrm{graph}\, \Phi, \lambda_0 \times \lambda_0),$$

so

$$\mu_{CZ}(\Phi) - \mu(\Phi\lambda_0, \lambda_0) = \mu(\mathrm{graph}\, \Phi, \Delta) - \mu(\mathrm{graph}\, \Phi, \lambda_0 \times \lambda_0)$$

is the *Hörmander index* of the 4-uple of Lagrangian subspaces of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$

$$(\lambda_0 \times \lambda_0, \Delta; \mathrm{graph}\, \Phi(a), \mathrm{graph}\, \Phi(b)),$$

a half-integer which does not depend on the path Φ and whose absolute value does not exceed $2n$ (see [15], [10], or [23, Theorem 3.5]).

2 The asymptotic Maslov index on cotangent bundles

Let M be a compact n-dimensional smooth manifold without boundary. The manifold T^*M is endowed with the symplectic structure

$$\omega = \sum_{j=1}^{n} dp^j \wedge dq^j,$$

where $(q^1, \dots, q^n, p^1, \dots, p^n)$ are cotangent local coordinates on T^*M. The fibers of the vertical subbundle of TT^*M,

$$T_x^v T^*M = \ker D\pi(x),$$

where $\pi : T^*M \to M$ is the projection, are Lagrangian subspaces of the symplectic vector spaces $T_x T^*M$.

Let $H \in C^\infty(\mathbb{T} \times T^*M)$ be a time-periodic Hamiltonian on the cotangent bundle of M. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ indicates the circle. The induced Hamiltonian vector field X_H is the time-periodic tangent vector field on T^*M defined by

$$\omega(X_H(t, x), \xi) = -D_x H(t, x)[\xi], \quad \forall \xi \in T_x T^*M.$$

At the moment, we only assume that X_H generates a global flow on the extended phase space $\mathbb{T} \times T^*M$, that is

$$\phi : \mathbb{R} \times \mathbb{T} \times T^*M \to \mathbb{T} \times T^*M, \quad \phi_t(s, x_0) = (s + t, \varphi_{s+t}(s, x_0)),$$

where $\varphi : \mathbb{R} \times \mathbb{T} \times T^*M \to T^*M$ solves the Cauchy problem

$$\partial_t \varphi_t(s, x_0) = X_H(t, \varphi_t(s, x_0)), \quad \varphi_s(s, x_0) = x_0.$$

Since M is compact, this is true for instance if H is coercive and it satisfies the estimate

$$\partial_t H(t, x) \leq c(1 + H(t, x)), \quad \forall (t, x) \in \mathbb{T} \times T^*M,$$

for some positive constant c.

Consider the orbit $x(t) := \varphi_t(s, x_0)$ such that $x(s) = x_0 \in T^*M$. The isomorphism

$$D_x \varphi_t(s, x_0) : T_{x_0} T^*M \to T_{x(t)} T^*M$$

is symplectic, so

$$\lambda_t(s, x_0) := D_x \varphi_t(s, x_0) T^v_{x_0} T^*M$$

is a Lagrangian subspace of $T_{x(t)} T^*M$. By use of a symplectic trivialization of $x^*(TT^*M)$,

$$U(t) : (\mathbb{R}^{2n}, \omega_0) \longrightarrow (T_{x(t)} T^*M, \omega_{x(t)})$$

mapping the Lagrangian subspace $\lambda_0 := (0) \times \mathbb{R}^n$ into the vertical subspace $T^v_{x(t)} T^*M$, we can transform the path $t \mapsto \lambda_t(s, x_0)$ into a path

$$\lambda^U(t) := U(t)^{-1} \lambda_t(s, x_0)$$

of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. By the Juxtaposition, Homotopy, and Zero properties of the Maslov index, the number

$$\mu_t(s, x_0) := \mu(\lambda^U|_{[s, s+t]}, \lambda_0) - \frac{n}{2}$$

does not depend on the choice of the trivialization U (see for instance [21, 3]). The normalization constant $-n/2$ appears because when the Hamiltonian is strictly convex on the fibers and t is positive but smaller than the first conjugate instant, we would like $\mu_t(s, x_0)$ to be zero. The function μ_t is neither lower nor upper semicontinuous, but if we denote by $\underline{\mu}_t$ and $\overline{\mu}_t$ its lower and upper semi-continuous envelopes, we have the bounds

$$\underline{\mu}_t \leq \mu_t \leq \overline{\mu}_t, \quad \overline{\mu}_t - \underline{\mu}_t \leq 2n. \tag{3}$$

Now we would like to consider the limit for $t \to +\infty$ of the quotient μ_t/t. This is possible thanks to the following:

THEOREM 2.1 *Let η be a Borel probability measure on $\mathbb{T} \times T^*M$ such that:*

(i) η is invariant for the flow ϕ;

(ii) for every $t \geq 0$ the function μ_t is in $L^1(\mathbb{T} \times T^*M, \eta)$.

Then for η-almost every $(s, x_0) \in \mathbb{T} \times T^*M$ the limit

$$\hat{\mu}(s, x_0) := \lim_{t \to +\infty} \frac{\mu_t(s, x_0)}{t}$$

exists, and it defines a ϕ-invariant function in $L^1(\mathbb{T} \times T^*M, \eta)$. Furthermore,

$$\int_A \hat{\mu}(s, x_0)\, d\eta(s, x_0) = \lim_{t \to +\infty} \frac{1}{t} \int_A \mu_t(s, x_0)\, d\eta(s, x_0),$$

for every ϕ-invariant Borel set $A \subset \mathbb{T} \times T^*M$.

The quantity $\hat{\mu}(s, x_0)$ is the *asymptotic Maslov index* of the point (s, x_0), and the quantity

$$\hat{\mu}(\eta) := \int_{\mathbb{T}\times T^*M} \hat{\mu}(s, x_0)\, d\eta(s, x_0) = \lim_{t\to+\infty} \frac{1}{t} \int_{\mathbb{T}\times T^*M} \mu_t(s, x_0)\, d\eta(s, x_0)$$

is the *asymptotic Maslov index* of the invariant measure η. The function μ is bounded on compact subsets of $\mathbb{R} \times \mathbb{T} \times T^*M$, so the summability assumption (ii) is trivially satisfied when η has compact support.

Theorem 2.1 is an easy consequence of Kingman subadditive ergodic theorem. In Derriennic's formulation [9], this theorem says that if ϕ_t is a measure preserving flow on the probability space $(\Omega, \mathcal{F}, \eta)$ and f_t is one-parameter family of real valued functions in $L^1(\Omega, \mathcal{F}, \eta)$ satisfying

$$\int_\Omega f_t(\omega)\, d\eta(\omega) \geq -c(t+1), \quad f_{s+t} \leq f_s + f_t \circ \phi_s, \quad \forall t, s \in [0, +\infty), \tag{4}$$

for some positive constant c, then the limit of f_t/t exists η-almost everywhere, and the integral of such a limit over any ϕ-invariant $A \in \mathcal{F}$ coincides with the limit of the integral of f_t/t over A. The Naturality and the Juxtaposition properties of the Maslov index imply that

$$|\mu_{t+t'}(s, x) - \mu_t(s, x) - \mu_{t'}(\phi_t(s, x))| \leq 2n, \tag{5}$$

It easily follows that, if $\Omega := \mathbb{T} \times T^*M$ and $\mathcal{F}$ is its Borel sigma-algebra, the one-parameter family of functions

$$f_t(s, x) := \mu_t(s, x) + 2n$$

satisfies the hypotheses (4) of the subadditive ergodic theorem. It is also possible to derive Theorem 2.1 from the standard Birkhoff ergodic theorem, by lifting the dynamical system ϕ to the Lagrangian bundle of T^*M, as in [8]. By means of such a lift, the asymptotic Maslov index can be seen as a special case of Schwartzman asymptotic cycles, see [26].

In the case of an autonomous Hamiltonian, one can work with the phase space T^*M instead of $\mathbb{T} \times T^*M$, and it is natural to apply the analogue of Theorem 2.1 to the invariant measure induced by the Liouville measure on each energy surface.

In the case of a time-periodic Hamiltonian, it is better to think of an invariant measure as a generalized orbit. Let us examine the case of an invariant measure associated to a k-periodic orbit, with k a positive integer. We can identify a k-periodic orbit $x : \mathbb{R}/k\mathbb{Z} \to T^*M$ of X_H with the invariant probability measure η on $\mathbb{T} \times T^*M$ defined by

$$\eta(A) = \frac{1}{k} |\, \{t \in [0, k] \mid (t, x(t)) \in A\}\, |,$$

for every Borel subset of $\mathbb{T} \times T^*M$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. We recall that the k-periodic orbit x has a well defined Conley-Zehnder index

$$\mu^k_{CZ}(x) := \mu_{CZ}(\Phi),$$

where $\Phi : [0, k] \to \mathrm{Sp}(2n)$ is the path in the symplectic group $\mathrm{Sp}(2n)$ obtained by conjugating $t \mapsto D_x\varphi_t(0, x(0))$ by a symplectic trivialization preserving the vertical subbundle (see for instance [25], [31], or [3]). By (2),

$$|\mu^k_{CZ}(x) - \mu_k(0, x(0))| = |\mu_{CZ}(\Phi) - \mu(\Phi\lambda_0, \lambda_0)| \leq 2n.$$

From this estimate we deduce that the asymptotic Maslov index of the invariant probability measure η coincides with the *Bott index* of the k-periodic orbit x,

$$\hat{\mu}(\eta) = \lim_{h\to+\infty} \frac{\mu^{hk}_{CZ}(x)}{hk}.$$

Then the inequality (1) with $h = 1$ yields the estimate

$$|\mu^k_{CZ}(x) - k\, \hat{\mu}(\eta)| \leq 2n. \tag{6}$$

Other interpretations of the asymptotic Maslov index are possible. For instance, if one fixes a metric on M, one can consider the polar decomposition for the differential of the flow. The orthogonal part in this decomposition is actually unitary with respect to the almost complex structure associated to the metric, and one can recover the asymptotic Maslov index by taking the asymptotic time average of the logarithm of the complex determinant of these unitary isomorphisms. This is the original approach adopted by Ruelle in [24] when the phase space is $\mathbb{R}^{2n}$.

3 The continuity of the asymptotic Maslov index

We denote by $\mathscr{P}(\mathbb{T}\times T^*M)$ the space of Borel probability measures on $\mathbb{T}\times T^*M$. We recall that a sequence $(\eta_j)\in\mathscr{P}(\mathbb{T}\times T^*M)$ *narrowly* converges to a measure $\eta\in\mathscr{P}(\mathbb{T}\times T^*M)$ if it converges to η in the weak-* topology of the dual space of $C_b(\mathbb{T}\times T^*M)$, that is if

$$\lim_{j\to+\infty}\int_{\mathbb{T}\times T^*M} f\,d\eta_j=\int_{\mathbb{T}\times T^*M} f\,d\eta,$$

for every bounded continuous function f on $\mathbb{T}\times T^*M$. The narrow convergence is induced by a metrizable topology, so we do not have to distinguish between continuity and sequential continuity. The narrow limit of an invariant measure is invariant.

As observed by Ruelle [24], the asymptotic Maslov index is continuous with respect to the narrow topology. We recall that a Borel function f is said to be uniformly integrable with respect to the sequence of measures (η_j) if

$$\lim_{c\to+\infty}\sup_{j\in\mathbb{N}}\int_{\{|f|\geq c\}}|f|\,d\eta_j=0.$$

THEOREM 3.1 *Assume that $(\eta_j)\subset\mathscr{P}(\mathbb{T}\times T^*M)$ is a sequence of ϕ-invariant probability measures which narrowly converges to η, and that for every $t\geq 0$ the function μ_t is uniformly integrable with respect to the sequence of measures (η_j). Then $(\hat{\mu}(\eta_j))$ converges to $\hat{\mu}(\eta)$.*

The following proof is adapted from the appendix in [24]. See [8] for a proof using the already mentioned lift to the Lagrangian bundle of T^*M.

Proof. Set $M_t(\eta):=\int\mu_t\,d\eta$, so that

$$\lim_{t\to+\infty}\frac{M_t(\eta)}{t}=\hat{\mu}(\eta). \tag{7}$$

Integrating the inequality (5) with respect to a ϕ-invariant probability measure η we obtain

$$|M_{t+t'}(\eta)-M_t(\eta)-M_{t'}(\eta)|\leq 2n,\quad \forall t,t'\in\mathbb{R}.$$

It easily follows that

$$|M_{ht}(\eta)-hM_t(\eta)|\leq 2(h-1)n,\quad \forall h\in\mathbb{N},\ h\geq 1,\ \forall t\in\mathbb{R}.$$

Then, if $t>0$,

$$\left|\frac{M_{ht}(\eta)}{ht}-\frac{M_t(\eta)}{t}\right|\leq\frac{2(h-1)n}{ht},$$

and taking the limit for $h\to+\infty$, we obtain the estimate

$$\left|\hat{\mu}(\eta)-\frac{M_t(\eta)}{t}\right|\leq\frac{2n}{t}, \tag{8}$$

which shows that the limit (7) is uniform over the space of all ϕ-invariant probability measures. Fix a number $t\geq 0$. By (3), the uniform integrability of μ_t implies that of $\underline{\mu}_t$. Since $\underline{\mu}_t$ is also lower-semicontinuous, from the semicontinuity property of narrow limits (see Lemma 5.1.7 in [4]) we find

$$\liminf_{j\to+\infty} M_t(\eta_j) \geq \liminf_{j\to+\infty} \int_{\mathbb{T}\times T^*M} \underline{\mu}_t \, d\eta_j \geq \int_{\mathbb{T}\times T^*M} \underline{\mu}_t \, d\eta \geq M_t(\eta) - 2n.$$

Similarly,

$$\limsup_{j\to+\infty} M_t(\eta_j) \leq \limsup_{j\to+\infty} \int_{\mathbb{T}\times T^*M} \overline{\mu}_t \, d\eta_j \leq \int_{\mathbb{T}\times T^*M} \overline{\mu}_t \, d\eta \leq M_t(\eta) + 2n.$$

The above inequalities together with the uniform estimate (8) imply that $(\hat{\mu}(\eta_j))$ converges to $\hat{\mu}(\eta)$. □

4 The range of the asymptotic Maslov index

We denote by $I(H) \subset \mathbb{R}$ the interval consisting of the asymptotic Maslov indices assumed by all the invariant probability Borel measures η for which μ_t is integrable for every $t \geq 0$. In general, a Hamiltonian vector field may have no invariant probability measures, so $I(H)$ could be empty.

Choose as H the energy of the geodesic flow on a compact Riemannian manifold M with non-positive sectional curvature. Since every geodesic on such a manifold has no conjugate points, the Maslov index of every orbit of the corresponding Hamiltonian system is zero. Therefore, in this case the interval $I(H)$ is just the singleton $\{0\}$. Another interesting example is the pendulum, with configuration space $M = S^1$. The unstable equilibrium and the homoclinic points have asymptotic Maslov index zero, as well as the non-contractible periodic orbits. On the other hand, the contractible periodic orbits have positive asymptotic Maslov index, growing from zero - for the contractible periodic orbits which are close to the homoclinic orbits - to a positive maximum - achieved at the stable equilibrium point. This follows from the fact that for such orbits the vertical subspace makes exactly one complete turn in one period, and from the fact that the period increases with the amplitude. Therefore, in this case $I(H)$ is the bounded interval $[0, a]$, where $a > 0$ is the asymptotic Maslov index of the stable equilibrium.

In both these examples the fundamental group of M is infinite. Manifolds with finite fundamental group exhibit a completely different behavior: for a fairly general class of Hamiltonians H the interval $I(H)$ contains the half-line $[0, +\infty)$.

We denote by $Y : T^*M \to TT^*M$ the Liouville vector field on T^*M, whose expression in local coordinates is

$$Y(q, p) = \sum_{j=1}^{n} p^j \partial_{p^j}.$$

THEOREM 4.1 *Assume that the compact manifold M has finite fundamental group, and that the Hamiltonian $H \in C^\infty(\mathbb{T} \times T^*M)$ satisfies:*

(i) The action integrand $DH(t, x)[Y(x)] - H(t, x)$ is coercive on $\mathbb{T} \times T^*M$.

(ii) The Hamiltonian $H(t, x)$ is superlinear on the fibers of T^*M, uniformly in $t \in \mathbb{T}$.

(iii) The Hamiltonian vector field X_H generates a global flow.

Then $I(H)$ contains the half-line $[0, +\infty)$. More precisely, the set of asymptotic Maslov indices assumed by the contractible closed orbits with integer period is dense in $[0, +\infty)$.

The function $DH[Y] - H$ is called action integrand because it coincides with the integrand of the Hamiltonian action along orbits. Condition (i) is a sort of radial convexity assumption for large values of $|p|$, and it is satisfied for instance by a Hamiltonian which is the Fenchel transform of a Tonelli Lagrangian (see the next section). Condition (i), together with (iii), implies a priori estimates for orbits with bounded action. The superlinearity assumption (ii) guarantees that the periodic orbit problem "sees" the whole topology of the free loop space of M. A Hamiltonian H with linear growth would typically have periodic orbits with bounded Conley-Zehnder index, and it would produce a bounded interval $I(H)$.

Proof. Assume first that M is simply connected. Then the free loop space of M has infinitely many non-vanishing Betti numbers, as proved by Sullivan in [28]. So there is a diverging sequence of natural numbers (h_j) such that the h_j-th singular homology group of the free loop space of M does not vanish.

Let $r \in [0, +\infty)$ and fix some $\epsilon > 0$. The fact that the sequence (h_j) diverges easily implies that for every $k_0 > 0$ the set

$$\left\{ \frac{h_j}{k} \mid j \in \mathbb{N}, \ k \in \mathbb{N}, \ k \geq k_0 \right\}$$

is dense in $[0, +\infty)$. Therefore, we can find $j \in \mathbb{N}$ and $k \in \mathbb{N}$, $k > 0$, such that

$$\left| r - \frac{h_j}{k} \right| < \frac{\epsilon}{3}, \quad \frac{2n}{k} < \frac{\epsilon}{3}.$$

Since the h_j-th singular homology group of the free loop space of M does not vanish, and the Hamiltonian H satisfies the above conditions (i), (ii), and (iii), the Hamiltonian vector field X_H has at least one k-periodic orbit x with Conley-Zehnder index $\mu_{CZ}(x)$ satisfying

$$|\mu^k_{CZ}(x) - h_j| \leq 2n.$$

This existence statement is proved by minimax arguments on the space of finite energy solutions of the Floer equation on the cylinder, see [2, Theorem 6.3]. Let η be the probability measure on $\mathbb{T} \times T^*M$ associated to the k-periodic orbit x. By (6),

$$|\mu^k_{CZ}(x) - k\hat{\mu}(\eta)| \leq 2n,$$

hence we conclude that

$$\begin{aligned} |\hat{\mu}(\eta) - r| &\leq \frac{1}{k}|k\hat{\mu}(\eta) - \mu^k_{CZ}(x)| + \frac{1}{k}|\mu^k_{CZ}(x) - h_j| + \left| \frac{h_j}{k} - r \right| \\ &\leq \frac{2n}{k} + \frac{2n}{k} + \frac{\epsilon}{3} < \epsilon, \end{aligned}$$

proving the density statement for simply connected configuration spaces. If the fundamental group of M is finite, the universal covering of M is compact. Therefore, the asymptotic Maslov indices of the orbits with integer period of the Hamiltonian vector field lifted to such a compact manifold are dense in $[0, +\infty)$. Projecting a periodic orbit of the lifted system onto M produces a contractible periodic orbit with the same asymptotic Maslov index. □

5 Convex Hamiltonians

Assume that the Hamiltonian H satisfies the Tonelli conditions: it is C^2-strictly convex and superlinear on the fibers of T^*M, meaning that

$$\partial_{pp} H(t, q, p) > 0, \quad \lim_{|p| \to +\infty} \frac{H(t, q, p)}{|p|} = +\infty,$$

where the norm $|\cdot|$ comes from some Riemannian metric on M (by the compactness of $\mathbb{T} \times M$, the superlinearity condition does not depend on the choice of this metric). Then the Fenchel transform defines a smooth Lagrangian on $\mathbb{T} \times TM$,

$$L(t, q, v) := \max_{p \in T^*_q M} \left(p[v] - H(t, q, p) \right),$$

which satisfies the Tonelli conditions, that is it is C^2-strictly convex and superlinear on the fibers of TM. Conversely, a Tonelli Lagrangian defines by the dual Fenchel transform a Tonelli Hamiltonian.

The Legendre duality defines a loop of diffeomorphisms from the tangent bundle to the cotangent bundle,

$$\mathcal{L} : \mathbb{T} \times TM \to \mathbb{T} \times T^*M, \quad (t, q, v) \mapsto (t, q, D_v L(t, q, v)),$$

such that

$$L(t, q, v) = p[v] - H(t, q, p) \quad \Longleftrightarrow \quad (t, q, p) = \mathcal{L}(t, q, v).$$

A curve $x(t) = (q(t), p(t))$ in the cotangent bundle is an orbit of the Hamiltonian vector field X_H if and only if the curve $q : \mathbb{R} \times M$ is a solution of the Euler-Lagrange equation

$$\frac{d}{dt} \partial_v L(t, q(t), \dot{q}(t)) = \partial_q L(t, q(t), \dot{q}(t)), \tag{9}$$

if and only if for every $s < t$ the curve $q|_{[s,t]}$ is an extremal of the Lagrangian action

$$\mathbb{A}_s^t(\gamma) = \int_s^t L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau,$$

among all absolutely continuous curves γ with end-points $\gamma(s) = q(s)$ and $\gamma(t) = q(t)$. The second variation of $\mathbb{A}_s^t$ at the extremal q is a continuous symmetric bilinear form on the Hilbert space W consisting of the $W^{1,2}$ sections ξ of $q^*(TM)$ such that $\xi(s) = 0$, $\xi(t) = 0$. The C^2-strict convexity of L guarantees that this bilinear form is a compact perturbation of a coercive form. Therefore, its Morse index $i_s^t(q)$ and its nullity $\nu_s^t(q)$ are finite. By the classical relationship between the Morse index and the Maslov index, we have the identities

$$\underline{\mu}_t(s, x_0) = i_s^{s+t}(q), \quad \overline{\mu}_t(s, x_0) = i_s^{s+t}(q) + \nu_s^{s+t}(q), \quad \forall s \in [0, 1),\ t \in [0, +\infty),$$

where x_0 is the point on T^*M whose orbit starting at time s corresponds to the curve $t \mapsto (\gamma(t), \dot{\gamma}(t))$ via the Legendre transform. See for instance [10]. Therefore, in the case of convex Hamiltonians the asymptotic Maslov index can be interpreted as an asymptotic Morse index,

$$\hat{\mu}(s, x_0) = \lim_{t \to +\infty} \frac{i_s^{s+t}(q)}{t} = \lim_{t \to +\infty} \frac{i_s^{s+t}(q) + \nu_s^{s+t}(q)}{t},$$

and this number is always non-negative. In this context, a corollary of Theorem 4.1 is that if L is a Tonelli Lagrangian generating a global flow on the tangent bundle of a compact manifold with finite fundamental group, the asymptotic Morse indices of closed orbits with integer period are dense in $[0, +\infty)$. In this case, the range of the asymptotic Morse index is exactly $[0, +\infty)$.

Now we restrict our attention to electro-magnetic systems, produced by Lagrangians of the form

$$L(t, q, v) = \frac{1}{2} \langle v, v \rangle + \langle A(t, q), v \rangle - V(t, q), \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is a Riemannian structure on M - the kinetic energy - A is a smooth vector field - the magnetic potential - and V is a smooth real function - the scalar potential. Both A and V are 1-periodic in time. In this case, the Legendre transform is

$$\mathcal{L}(t, q, v) = (t, q, p), \quad \text{where } p[w] = \langle v + A(t, q), w \rangle, \quad \forall w \in T_q M, \tag{11}$$

and the Hamiltonian is

$$H(t, q, p) = \frac{1}{2} \langle p - \alpha(t, q), p - \alpha(t, q) \rangle + V(t, q),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product induced on T^*M and α is the 1-form given by

$$\alpha(t, q)[v] = \langle A(t, q), v \rangle, \quad \forall v \in T_q M.$$

The corresponding Euler-Lagrange equation can be written in the form

$$\nabla_t \dot{\gamma} + \partial_t A(t,\gamma) - (\nabla A(t,\gamma)^* - \nabla A(t,\gamma))\dot{\gamma} + \operatorname{grad} V(t,\gamma) = 0,$$

where ∇ is the covariant derivative associated to the metric of M, ∇_t denotes the covariant derivative along γ, the superscript $*$ denotes the metric adjoint, and grad denotes the gradient. Therefore, if γ is a solution of the Euler-Lagrange equation, we have the estimate

$$\frac{d}{dt}|\dot{\gamma}(t)|^2 = 2\langle \nabla_t \dot{\gamma}(t), \dot{\gamma}(t)\rangle \leq c_1(1+|\dot{\gamma}(t)|^2)$$

for some constant c_1. It follows that the Euler-Lagrange flow on $\mathbb{T} \times TM$ - hence also the Hamiltonian flow on $\mathbb{T} \times T^*M$ - is globally defined, and

$$1 + |\dot{\gamma}(t)|^2 \leq (1+|\dot{\gamma}(s)|^2)e^{c_1|t-s|}, \quad \forall s,t \in \mathbb{R}. \tag{12}$$

The second variation of the Lagrangian action functional at an extremal curve γ on the time interval $[s, s+t]$ is

$$d^2\mathbb{A}_s^{s+t}(\gamma)[\xi,\xi] = \int_s^{s+t} \Big(\langle \nabla_\tau \xi, \nabla_\tau \xi\rangle + 2\langle \nabla_\xi A, \nabla_\tau \xi\rangle$$
$$+ \langle \operatorname{Hess} A[\xi,\xi], \dot{\gamma}\rangle - \operatorname{Hess} V[\xi,\xi] - \langle R(\xi,\dot{\gamma})\xi, \dot{\gamma} + A\rangle \Big)\, d\tau,$$

where Hess denotes the Riemannian Hessian and R is the Riemann tensor. The above formula shows that there exists a number c_2 such that

$$d^2\mathbb{A}_s^{s+t}(\gamma)[\xi,\xi] \geq \int_s^{s+t} \left(\frac{1}{2}|\nabla_\tau \xi|^2 - c_2(1+|\dot{\gamma}|^2)|\xi|^2 \right) d\tau.$$

Together with (12), this estimate implies that

$$d^2\mathbb{A}_s^{s+t}(\gamma)[\xi,\xi] \geq \frac{1}{2}\int_s^{s+t} |\nabla_\tau \xi|^2\, d\tau - c_2(1+|\dot{\gamma}(s)|^2)e^{c_1|t|} \int_s^{s+t} |\xi|^2\, d\tau.$$

Writing ξ as $\xi = \sum_{j=1}^n \varphi_j \xi_j$, where $\xi_1, \dots, \xi_n$ is an orthonormal frame along γ built by parallel transport, and $\varphi_1, \dots, \varphi_n$ are real-valued functions, the quadratic form on the right-hand side equals

$$\sum_{j=1}^n \left[\frac{1}{2}\int_s^{s+t} |\dot{\varphi}_j|^2\, d\tau - c_2(1+|\dot{\gamma}(s)|^2)e^{c_1|t|}\int_s^{s+t} |\varphi_j|^2\, d\tau \right].$$

The Morse index of the above quadratic form on the Sobolev space $W_0^{1,2}([s,s+t],\mathbb{R}^n)$ is

$$n \left\lfloor \frac{|t|\sqrt{2c_2(1+|\dot{\gamma}(s)|^2)}e^{c_1|t|/2}}{\pi} \right\rfloor.$$

Therefore, the Morse index of a solution on a fixed time interval grows at most linearly with the initial velocity: there exists a positive function $t \mapsto c_3(t)$ such that

$$i_s^{s+t}(\gamma) \leq c_3(t)(1+|\dot{\gamma}(s)|).$$

By the form (11) of the Legendre transform, we deduce a similar estimate for the function μ_t:

$$\mu_t(s, q_0, p_0) \leq c_4(t)(1+|p_0|), \tag{13}$$

for a suitable function $t \mapsto c_4(t)$. In particular, if a ϕ-invariant probability measure η on $\mathbb{T} \times T^*M$ has finite first moment, then the function μ_t is in $L^1(\mathbb{T} \times T^*M, \eta)$, and the asymptotic Maslov index of η is well-defined.

6 Electro-magnetic Lagrangians

The aim of this section is to prove the following theorem:

THEOREM 6.1 *Assume that the manifold M is compact and has finite fundamental group. Let $L \in C^\infty(\mathbb{T} \times TM)$ be a Lagrangian of the form (10). Let r be a non-negative number. Then at least one of the following statements holds:*

(i) There exists a contractible closed orbit with integer period and asymptotic Maslov index r.

(ii) There exists a sequence of contractible closed orbits with integer minimal periods $k_j \to +\infty$ and asymptotic Maslov indices $\hat{\mu}_j \to r$. The corresponding sequence of probability measures on $\mathbb{T} \times TM$ narrowly converges to an invariant probability measure $\eta \in \mathscr{P}(\mathbb{T} \times TM)$ with finite second moment and asymptotic Maslov index $\hat{\mu}(\eta) = r$.

In both cases, there is an invariant probability measure η on $\mathbb{T} \times TM$ with finite second moment, asymptotic Maslov index r, and action bound

$$\mathbb{A}(\eta) := \int L(t, q, v)\, d\eta(t, q, v) \leq C(1 + r^2), \tag{14}$$

where C is a positive number.

When the Lagrangian L does not depend on time, the above statement can be made more precise: either there is a contractible closed orbit with asymptotic Maslov index r, or there is a sequence of contractible closed orbits with minimal periods $T_j \to +\infty$ such that the corresponding probability measures on TM narrowly converge to an invariant probability measure which is supported on a (compact) energy level and has asymptotic Maslov index r. See Remark 6.4 after the end of the proof.

Let $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ be the space of k-periodic curves in M of Sobolev class $W^{1,2}$. This is an infinite dimensional Hilbert manifold, and it has the homotopy type of $\Lambda(M)$, the free loop space of M. By the form (10) of the Lagrangian, the (average) action functional

$$\mathbb{A}_k(\gamma) := \frac{1}{k} \int_0^k L(t, \gamma(t), \dot{\gamma}(t))\, dt$$

is twice continuously differentiable on $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$. Its critical points are precisely the k-periodic solutions of the Euler-Lagrange equation (9), and the second differential of $\mathbb{A}_k$ at a critical point γ is a compact perturbation of a coercive symmetric bilinear form on $T_\gamma W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$. In particular, the Morse index $m_k(\gamma)$ is finite. We denote by $m_k^*(\gamma)$ the large Morse index of the critical point γ, that is the Morse index plus the nullity. Since the elements of the kernel of the second differential of $\mathbb{A}_k$ solve a second order ODE, there holds

$$m_k(\gamma) \leq m_k^*(\gamma) \leq m_k(\gamma) + 2n.$$

Furthermore, if x is the k-periodic orbit on T^*M corresponding to γ by the Legendre transform,

$$m_k(\gamma) \leq \mu_{CZ}^k(x) \leq m_k^*(\gamma),$$

see [10] and [31]. Therefore, the asymptotic Maslov index of the invariant probability measure η associated to the k-periodic solution γ can be expressed in terms of the Morse indices of the Lagrangian action functional by

$$\hat{\mu}(\eta) = \lim_{h \to +\infty} \frac{m_{hk}(\gamma)}{hk} = \lim_{h \to +\infty} \frac{m_{hk}^*(\gamma)}{hk}.$$

If we endow $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ with the complete Riemannian metric

$$\langle \xi, \eta \rangle_{W^{1,2}} = \int_0^k \left(\langle \xi, \eta \rangle + \langle \nabla_t \xi, \nabla_t \eta \rangle \right) dt, \quad \forall \xi, \eta \in T_\gamma W^{1,2}(\mathbb{R}/k\mathbb{Z}, M),$$

the functional $\mathbb{A}_k$ satisfies the Palais-Smale condition on $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ (see [6], or the appendix in [2]). This fact allows to find k-periodic orbits by minimax arguments. Indeed, let h be a natural number such that the h-th singular homology group of $\Lambda(M)$ - hence also of $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ - does not vanish. Let $\Gamma_k(h)$ be the class of compact subsets K of $W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ such that the inclusion mapping $i : K \hookrightarrow W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ induces a non-zero homomorphism between the h-th homology groups. Then

$$a_k(h) := \inf_{K \in \Gamma_k(h)} \max_{\gamma \in K} \mathbb{A}_k(\gamma) \tag{15}$$

is finite, and there exists a critical point $\gamma \in W^{1,2}(\mathbb{R}/k\mathbb{Z}, M)$ such that

$$\mathbb{A}_k(\gamma) = a_k(h), \quad m_k(\gamma) \leq h \leq m_k^*(\gamma).$$

See for instance [30]. In the following lemma we prove an estimate on the growth rate of $a_k(h)$ in terms of k.

LEMMA 6.2 *There exists a constant c_1 such that for every $h \in \mathbb{N}$ for which $H_h(\Lambda(M)) \neq 0$, and for every integer $k \geq 1$, there holds*

$$a_k(h) \leq \frac{2}{k^2} a_1(h) + c_1.$$

Proof. The reparametrization mapping

$$\varphi_k : W^{1,2}(\mathbb{R}/\mathbb{Z}, M) \to W^{1,2}(\mathbb{R}/k\mathbb{Z}, M), \quad \varphi_k(\gamma)(t) := \gamma(t/k),$$

is a homeomorphism. Therefore,

$$a_k(h) = \inf_{K \in \Gamma_k(h)} \max_{\gamma \in K} \mathbb{A}_k(\gamma) = \inf_{K \in \Gamma_1(h)} \max_{\gamma \in K} \mathbb{A}_k(\varphi_k(\gamma)). \tag{16}$$

Let $\gamma \in W^{1,2}(\mathbb{R}/\mathbb{Z}, M)$. By a change of variable and by simple manipulations

$$\begin{aligned}
\mathbb{A}_k(\varphi_k(\gamma)) &= \frac{1}{k} \int_0^k \Big[\frac{1}{2k^2} \langle \dot\gamma(t/k), \dot\gamma(t/k) \rangle + \frac{1}{k} \langle A(t, \gamma(t/k)), \dot\gamma(t/k) \rangle - V(t, \gamma(t/k)) \Big] \, dt \\
&= \frac{1}{k} \int_0^1 \Big[\frac{1}{2k} \langle \dot\gamma(t), \dot\gamma(t) \rangle + \langle A(kt, \gamma(t)), \dot\gamma(t) \rangle - kV(kt, \gamma(t)) \Big] \, dt \\
&= \frac{2}{k^2} \mathbb{A}_1(\gamma) + \int_0^1 \Big[\frac{1}{k} \langle A(kt, \gamma(t)), \dot\gamma(t) \rangle - \frac{2}{k^2} \langle A(t, \gamma(t)), \dot\gamma(t) \rangle \\
&\quad - V(kt, \gamma(t)) + \frac{2}{k^2} V(t, \gamma(t)) - \frac{1}{2k^2} \langle \dot\gamma(t), \dot\gamma(t) \rangle \Big] \, dt.
\end{aligned}$$

Together with the Cauchy-Schwarz inequality this yields the estimate

$$\mathbb{A}_k(\varphi_k(\gamma)) \leq \frac{2}{k^2} \mathbb{A}_1(\gamma) + \left(\frac{1}{k} + \frac{2}{k^2} \right) \|A\|_\infty \|\dot\gamma\|_{L^2} + \left(1 + \frac{2}{k^2} \right) \|V\|_\infty - \frac{1}{2k^2} \|\dot\gamma\|_{L^2}^2. \tag{17}$$

Since $k \geq 1$,

$$\begin{aligned}
\left(\frac{1}{k} + \frac{2}{k^2} \right) \|A\|_\infty \|\dot\gamma\|_{L^2} - \frac{1}{2k^2} \|\dot\gamma\|_{L^2}^2 &\leq \frac{3}{k} \|A\|_\infty \|\dot\gamma\|_{L^2} - \frac{1}{2k^2} \|\dot\gamma\|_{L^2}^2 \\
&\leq \max_{r \geq 0} \left(3\|A\|_\infty r - \frac{1}{2} r^2 \right) = \frac{9}{2} \|A\|_\infty^2.
\end{aligned}$$

Therefore, (17) shows that for every $\gamma \in W^{1,2}(\mathbb{R}/\mathbb{Z}, M)$ there holds

$$\mathbb{A}_k(\varphi_k(\gamma)) \leq \frac{2}{k^2} \mathbb{A}_1(\gamma) + c_1,$$

where

$$c_1 := 3\|V\|_\infty + \frac{9}{2}\|A\|_\infty^2.$$

The conclusion follows by (16). □

In the next lemma we prove an estimate on the growth rate of $a_1(h)$ in terms of h.

LEMMA 6.3 *If M is compact and simply connected, then there exist positive constants c_2 and c_3 such that*

$$a_1(h) \leq c_2 h^2 + c_3,$$

for every $h \in \mathbb{N}$ for which $H_h(\Lambda(M)) \neq 0$.

Proof. Let $L(\gamma)$ be the length of a loop γ with respect to the metric $\langle\cdot,\cdot\rangle$,

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt.$$

Gromov has proved that the fact that the compact Riemannian manifold M is simply connected has the following consequence: there exists a number C_0 such that every singular homology class $\alpha \in H_h(W^{1,2}(\mathbb{R}/\mathbb{Z}, M))$ is represented by a singular cycle whose support is contained in the length sublevel

$$\left\{\gamma \in W^{1,2}(\mathbb{R}/\mathbb{Z}, M) \mid L(\gamma) \leq C_0 h\right\},$$

for every $h \in \mathbb{N}$ (see [13] and [14], Theorem 7.3). By a reparametrization argument, we can transform this cycle into a cycle in the same homology class α consisting of loops γ which satisfy the estimate

$$\|\dot{\gamma}\|_\infty \leq Ch,$$

for a suitable constant C, not depending on h. See Appendix A for details. Therefore, for every $h \in \mathbb{N}$ for which the h-th homology group of the free loop space of M does not vanish there exists K in $\Gamma_1(h)$ such that

$$\max_{\gamma \in K} \|\dot{\gamma}\|_\infty \leq Ch.$$

Then, if γ belongs to K,

$$\mathbb{A}_1(\gamma) \leq \int_0^1 \left(\frac{1}{2}|\dot{\gamma}|^2 + \|A\|_\infty|\dot{\gamma}| + \|V\|_\infty\right) dt$$
$$\leq \frac{1}{2}C^2h^2 + \|A\|_\infty Ch + \|V\|_\infty \leq C^2h^2 + \frac{1}{2}\|A\|_\infty^2 + \|V\|_\infty.$$

We conclude that the desired estimate holds with $c_1 = C^2$ and $c_2 = \|A\|_\infty^2/2 + \|V\|_\infty$. □

Proof. [of Theorem 6.1] By lifting the system to the universal covering, we may assume that M is simply connected. Up to adding a positive constant, we may assume that the Lagrangian L is non-negative.

Assume that (i) does not hold: there are no contractible closed orbits with integer period and asymptotic Maslov index r. Arguing as in the proof of Theorem 4.1, we can find two diverging sequences of natural numbers (h_j) and (k_j) such that the h_j-th homology group of $\Lambda(M)$ does not vanish, $k_j \geq 1$, and (h_j/k_j) converges to r. By the minimax (15), we can find a k_j-periodic solution γ_j with

$$\mathbb{A}_{k_j}(\gamma_j) = a_{k_j}(h_j), \quad m_{k_j}(\gamma_j) \leq h_j \leq m^*_{k_j}(\gamma_j).$$

By Lemmas 6.2 and 6.3,

$$\mathbb{A}_{k_j}(\gamma_j) = a_{k_j}(h_j) \leq \frac{2}{k_j^2}a_1(h_j) + c_1 \leq 2c_2\frac{h_j^2}{k_j^2} + 2c_3\frac{1}{k_j^2} + c_1,$$

so the average action of γ_j has a uniform bound of the form,

$$\mathbb{A}_{k_j}(\gamma_j) \leq C(r^2 + 1), \quad \forall j \in \mathbb{N}. \tag{18}$$

Let $k'_j \in \mathbb{N}$ be the minimal period of the periodic solution γ_j. If by contradiction k'_j does not diverge, up to a subsequence we may assume that $k_j \equiv k$ is constant. Then (γ_j) is a sequence of k-periodic solutions with bounded action. It easily follows that (γ_j) is bounded in $C^2(\mathbb{R}/\mathbb{Z}, M)$ (see for instance [2, Lemma 1.1]), so a subsequence of (γ_j) converges in C^1 to a contractible k-periodic solution γ which has asymptotic Maslov index r (for instance because the associated invariant probability measures on $\mathbb{T} \times TM$ converge narrowly). This contradiction shows that the minimal periods diverge.

Let η_j be the invariant probability measure on $\mathbb{T} \times TM$ associated to γ_j, that is

$$\eta_j(A) = \frac{1}{k_j} | \{ t \in [0, k_j] \mid (t, \gamma_j(t), \dot{\gamma}_j(t)) \in A \} |,$$

for every Borel subset A of $\mathbb{T} \times TM$. Fix some number $M > 0$. By the Tchebichev inequality - recall that $L \geq 0$ - and by (18),

$$\begin{aligned} \eta_j(\{(t,q,v) \in \mathbb{T} \times TM \mid L(t,q,v) \geq M\}) &\leq \frac{1}{M} \int_{\mathbb{T} \times TM} L(t,q,v)\, d\eta_j(t,q,v) \\ &= \frac{1}{M} \frac{1}{k_j} \int_0^{k_j} L(t, \gamma_j(t), \dot{\gamma}_j(t))\, dt = \frac{1}{M} \mathbb{A}_{k_j}(\gamma_j) \leq \frac{c}{M}. \end{aligned} \tag{19}$$

Since the function L is coercive on $\mathbb{T} \times TM$, we conclude that the sequence (η_j) is tight, so by the Prokhorov theorem up to a subsequence it narrowly converges to some invariant probability measure η on $\mathbb{T} \times TM$. Actually, since L grows quadratically in v, the bound (19) shows that the sequence (η_j) has uniformly integrable second moments, so η has finite second moment (see for instance [4, section 5.1.1]). In particular, by (13) for every $t \geq 0$ the function μ_t is uniformly integrable with respect to the sequence (η_j), so by Theorem 3.1 the invariant probability measure η has asymptotic Maslov index r. This concludes the proof of (ii). Since the action is lower semi-continuous with respect to the narrow topology of measures, the estimate (14) follows from (18) (in case (ii), case (i) is similar). □

Remark 6.4 *As mentioned before, if the Lagrangian L does not depend on time then either there is a contractible closed orbit with asymptotic Maslov index r, or there is a sequence of contractible closed orbits with diverging minimal periods such that the corresponding probability measures on TM narrowly converge to an invariant probability measure which is supported on an energy level and has asymptotic Maslov index r.*

Indeed, the above proof produces a sequence of contractible closed orbits (γ_j) with asymptotic Maslov index converging to r and bounded average action. The energy H - read on TM by the Legendre transform - is an integral of the Euler-Lagrange flow, so the bound on the average action implies that each curve $(\gamma_j, \dot{\gamma}_j)$ is contained in an energy level $\{H = H_j\}$, with (H_j) a bounded sequence of real numbers. If the minimal periods of γ_j are bounded, by an easy limiting argument we obtain a contractible closed orbit with asymptotic Maslov index r. Another consequence is that the corresponding probability measures on TM narrowly converge - up to a subsequence - to a probability measure which is also supported on an energy level.

7 Minimizing measures and Aubry-Mather theory

In this section we develop an analogue of the ergodicity analysis in Aubry-Mather theory (see for instance [17, Chapter III] or [27] for an introduction to the subject) replacing the rotation vector of a closed measure with the asymptotic Maslov index of the measure.

For sake of simplicity, we deal with an electromagnetic Lagrangian of the form (10) on a manifold M which is assumed to be compact and to have finite fundamental group. See Remark 7.4 below for possible generalizations.

By Theorem 6.1, for any $r \in [0,+\infty)$ there is an invariant probability measure η on $\mathbb{T} \times TM$ with asymptotic Maslov index $\hat{\mu}(\eta) = r$ and finite action

$$\mathbb{A}(\eta) := \int_{\mathbb{T}\times TM} L(t,q,v)\, d\eta(t,q,v).$$

Therefore, we can define a real function β on $[0,+\infty)$ as

$$\beta(r) := \inf_{\hat{\mu}(\eta)=r} \mathbb{A}(\eta),$$

where the infimum is taken over all the invariant measures on $\mathbb{T} \times TM$ for which μ_t is integrable for every $t \geq 0$, and $\hat{\mu}(\eta) = r$.

Proposition 7.1 *For every $r \geq 0$, the infimum in the definition of $\beta(r)$ is attained. Moreover, the function β has the following properties:*

(i) β is convex;

(ii) β has quadratic growth at infinity, meaning that

$$a_1 r^2 - A_1 \leq \beta(r) \leq a_2 r^2 + A_2,$$

for some positive constants a_1, a_2, A_1, A_2.

Proof. In order to prove that the infimum defining $\beta(r)$ is attained, it suffices to show that for every $r \geq 0$ and $c \in \mathbb{R}$ the space of invariant measures with asymptotic Maslov index r and action not exceeding c is compact in the narrow topology. Let (η_j) be a sequence of invariant measures with $\hat{\mu}(\eta_j) = r$ and $\mathbb{A}(\eta_j) \leq c$. Since L grows quadratically in v at infinity, the upper bound on $\mathbb{A}(\eta_j)$ implies that the sequence (η_j) has uniformly integrable second moments. In particular (η_j) is tight, so by Prokhorov theorem up to a subsequence it narrowly converges to some probability measure η. The measure η is invariant, and since $\mathbb{A}$ is narrowly lower semi-continuous we get $\mathbb{A}(\eta) \leq c$. By the estimate (13), for every $t \geq 0$ the function μ_t is uniformly integrable with respect to the sequence of measures (η_j), so Theorem 3.1 implies that the invariant probability measure η has asymptotic Maslov index r. This concludes the proof of the narrow compactness.

(i) Since the asymptotic Maslov index is linear in η, that is

$$\hat{\mu}(s\eta_1 + (1-s)\eta_2) = s\hat{\mu}(\eta_1) + (1-s)\hat{\mu}(\eta_2) \qquad \text{for } 0 \leq s \leq 1,$$

the convexity of β follows easily from the definition.

(ii) The upper quadratic bound follows from the existence of an invariant measure satisfying the estimate (14) in Theorem 6.1. As for the lower quadratic bound, notice that by (8), by (13), and by the Cauchy-Schwarz inequality, any invariant probability measure η satisfies

$$\begin{aligned}\hat{\mu}(\eta) \leq 2n + M_1(\eta) = 2n + \int \mu_1(s,q,p)\, d\eta(s,q,p) \\ \leq 2n + c_4(1) \int (1+|p|)\, d\eta(s,q,p) \leq 2n + c_4(1) + c_4(1) \left(\int |p|^2\, d\eta(s,q,p) \right)^{1/2}.\end{aligned}$$

Since L has a lower quadratic bound, the above inequality implies an estimate of the form

$$\hat{\mu}(\eta) \leq b_1 + b_2 \left(\int L(s,q,p)\, d\eta(s,q,p) \right)^{1/2} = b_1 + b_2\, \mathbb{A}(\eta)^{1/2},$$

for suitable positive numbers b_1, b_2. The lower quadratic bound for β follows. □

Let us denote by $\mathcal{M}_r \subset \{\eta \mid \hat{\mu}(\eta) = r\}$ the set of invariant measures with minimal action among those with asymptotic Maslov index r. Thanks to the convexity and quadratic growth of β we can now prove that, although in general minimizing measures are not ergodic, there exists a diverging sequence of values $r \in [0, +\infty)$ for which $\mathcal{M}_r$ contains an ergodic measure.

We recall the concept of ergodic decomposition of an invariant measure. Let η be an invariant probability measure on $\mathbb{T} \times TM$, and define $\mathscr{E}(\eta)$ to be the space of ergodic invariant probability measures whose support is contained in the support of η. We endow this space with the narrow topology. Then there exists a unique probability measure λ on the Borel σ-algebra of $\mathscr{E}(\eta)$ such that

$$\eta(A) = \int_{\mathscr{E}(\eta)} \nu(A)\, d\lambda(\nu),$$

for every Borel set $A \subset \mathbb{T} \times TM$. The measure λ is called the *ergodic decomposition* of η and the ergodic measures $\nu \in \operatorname{supp}(\lambda)$ are called the *ergodic components* of η. Since η need not have compact support, this decomposition result does not follow at once from Choquet theorem, but it requires a more sophisticated proof. It is proved, in the more general setting of a locally compact group (in our case $\mathbb{R}$) acting in a measurable way on a complete separable metric space (in our case the support of η), by Varadarajan in [29, Theorem 4.4].

We also recall that a point $r \in [0, +\infty)$ is said to be an extremal point of the convex function β if $\beta(r) < s\beta(r_1) + (1-s)\beta(r_2)$ for all $r_1, r_2 \in [0, +\infty)$ and $0 < s < 1$ satisfying $sr_1 + (1-s)r_2 = r$. We can now prove the following:

Proposition 7.2 *If r is an extremal point of β, then $\mathcal{M}_r$ contains an ergodic measure.*

Combining this proposition with the fact that β has quadratic growth at infinity, we conclude that there exists a diverging sequence (r_j) such that $\mathcal{M}_{r_j}$ contains an ergodic measure.

Proof. Let us fix $\eta \in \mathcal{M}_r$. By the ergodic decomposition theorem stated above, we can write

$$\beta(r) = \int_{\mathbb{T}\times TM} L\, d\eta = \int_{\mathscr{E}(\eta)} \left(\int_{\mathbb{T}\times TM} L\, d\nu \right) d\lambda(\nu). \tag{20}$$

Moreover, by Theorem 2.1,

$$\hat{\mu}(\eta) = \int_{\mathbb{T}\times TM} \hat{\mu}(s,x)\, d\eta = \int_{\mathscr{E}(\eta)} \left(\int_{\mathbb{T}\times TM} \hat{\mu}(s,x)\, d\nu \right) d\lambda(\nu) = \int_{\mathscr{E}(\eta)} \hat{\mu}(\nu)\, d\lambda(\nu).$$

Since β is convex, by Jensen inequality we get

$$\beta(r) = \beta(\hat{\mu}(\eta)) = \beta\left(\int_{\mathscr{E}(\eta)} \hat{\mu}(\nu)\, d\lambda(\nu) \right) \leq \int_{\mathscr{E}(\eta)} \beta(\hat{\mu}(\nu))\, d\lambda(\nu),$$

which together with (20) gives

$$\int_{\mathscr{E}(\eta)} \left(\int_{\mathbb{T}\times TM} L\, d\nu \right) d\lambda(\nu) \leq \int_{\mathscr{E}(\eta)} \beta(\hat{\mu}(\nu))\, d\lambda(\nu).$$

Since, by the definition of β, $\int_{\mathbb{T}\times TM} L\, d\nu \geq \beta(\hat{\mu}(\nu))$, we conclude that

$$\int_{\mathbb{T}\times TM} L\, d\nu = \beta(\hat{\mu}(\nu)) \quad \text{and} \quad \beta(\hat{\mu}(\eta)) = \int_{\mathscr{E}(\eta)} \beta(\hat{\mu}(\nu))\, d\lambda(\nu) \qquad \text{for } \lambda-\text{a.e. } \nu,$$

The latter identity, combined with the fact that $r = \hat{\mu}(\eta)$ is an extremal point of β, implies that $\hat{\mu}(\nu) = r$ for λ-a.e. ν. We have shown that λ-a.e. ergodic component of an element of $\mathcal{M}_r$ belongs to $\mathcal{M}_r$. □

Remark 7.3 *We observe that, without the assumption that r is an extremal point of β, from the equality $\beta(\hat{\mu}(\eta)) = \int_{\mathscr{E}(\eta)} \beta(\hat{\mu}(\nu))\, d\lambda(\nu)$ one deduces that, for λ-a.e. ν, $\beta(\hat{\mu}(\nu))$ belongs to the so called supported domain of r, that is the interval given by*

$$\bigcap_{p \in \partial\beta(r)} \{u \in [0,+\infty) \mid \beta(u) = \beta(r) + p(u-r)\},$$

where $\partial\beta(r)$ denotes the subdifferential of β at r, that is

$$\partial\beta(r) := \{p \in \mathbb{R} \mid \beta(u) = \beta(r) + p(u-r) \ \forall u \in [0,+\infty)\}.$$

Remark 7.4 *The results of this section can be extended to an autonomous Tonelli Lagrangian (the assumption of non-dependence on t allows to recover compactness in a simple way), proving in this case a superlinear growth at infinity on the function β (see Proposition 7.1 (ii)). The configuration space M could be any compact manifold, provided that we replace the half-line $[0,+\infty)$ by the interval $I(H)$, the range of the asymptotic Maslov index.*

A Appendix: Singular cycles with bounded speed in the loop space

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, and let

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\, dt$$

denote the length of an absolutely continuous curve $\gamma : [0,1] \to M$. Gromov [13] has proved the following result:

Theorem A.1 *Assume that the manifold M is compact and simply connected. Then there is a number C_0 such that for every $h \in \mathbb{N}$ every singular homology class*

$$\alpha \in H_h(C^0(\mathbb{R}/\mathbb{Z}, M))$$

is represented by a singular cycle whose support is contained in the length sublevel

$$\left\{\gamma \in C^0(\mathbb{R}/\mathbb{Z}, M) \mid \gamma \text{ is absolutely continuous and } L(\gamma) \leq C_0 h\right\}.$$

See also Theorem 7.3 in [14], and Theorem 5.10 in the book of Paternain [22] for detailed proofs. The cycle produced by Gromov's argument is made of loops whose speed may have large variation, so the length estimate does not imply a good estimate on the uniform norm of the speed, and not even on the energy (as erroneously claimed in [22]). However, such an estimate can be easily obtained by a second homotopy, as shown by the following:

Corollary A.2 *Assume that the manifold M is compact and simply connected. Then there exists a number C such that for every $h \in \mathbb{N}$ every singular homology class*

$$\alpha \in H_h(C^\infty(\mathbb{R}/\mathbb{Z}, M))$$

is represented by a singular cycle whose support is contained in

$$\{\gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, M) \mid \|\dot{\gamma}\|_\infty \leq Ch\}.$$

Since the inclusions

$$C^\infty(\mathbb{R}/\mathbb{Z}, M) \hookrightarrow W^{1,2}(\mathbb{R}/\mathbb{Z}, M) \hookrightarrow C^0(\mathbb{R}/\mathbb{Z}, M)$$

are homotopy equivalences, the above result holds a fortiori when the space of smooth loops is replaced by the space of $W^{1,2}$ loops, or by the space of continuous loops. Corollary A.2 can be deduced from the proof of Lemma 2.12 in [12, Lemma 2.12]. For sake of completeness, we include a proof also here.

Proof. We may assume that $h \geq 1$, because the (unique) zero-homology class is realized by any constant loop. Let α be a homology class of degree h and let a_0 be the singular cycle given by Theorem A.1, so that

$$[a_0] = \alpha, \quad L(\gamma) \leq C_0 h, \quad \forall \gamma \in \operatorname{supp} a_0. \tag{A1}$$

By regularization and up to choosing a larger C_0, we may assume that a_0 is a singular cycle in $C^\infty(\mathbb{R}/\mathbb{Z}, M)$. For instance, such a regularization can be achieved by embedding M into some Euclidean space, by smoothing the loops in α by a convolution keeping them in a tubular neighborhood of M, and by projecting onto M.

In order to obtain a cycle a_1 which is homologous to a_0 and is supported in a suitable speed sublevel, the natural idea would be to build a homotopy changing the parametrization of the loops in a_0, so that they become parametrized by constant speed. However, such a construction presents technical difficulties due to the fact that the speed of some loops may vanish somewhere. We can overcome this difficulty by reparametrizing the loops in a_0 by slowly varying speed, as follows.

Given a loop $\gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, M)$, we define the real function σ_γ on $[0,1]$ by

$$\sigma_\gamma(t) := \frac{1}{\int_0^1 \sqrt{1+|\dot\gamma(s)|^2}\, ds} \int_0^t \sqrt{1+|\dot\gamma(s)|^2}\, ds.$$

Notice that $\sigma_\gamma(0) = 0$, $\sigma_\gamma(1) = 1$. Since the square root is subadditive, we have

$$\sigma'_\gamma(t) = \frac{\sqrt{1+|\dot\gamma(t)|^2}}{\int_0^1 \sqrt{1+|\dot\gamma(s)|^2}\, ds} \geq \frac{\sqrt{1+|\dot\gamma(t)|^2}}{1+L(\gamma)} > 0, \tag{A2}$$

so σ_γ is a smooth diffeomorphism of $[0,1]$ onto itself. Denote by τ_γ its inverse, and notice that the mapping $\gamma \mapsto \tau_\gamma$ is continuous in the C^∞ topology. Then the homotopy

$$F : [0,1] \times C^\infty(\mathbb{R}/\mathbb{Z}, M) \to C^\infty(\mathbb{R}/\mathbb{Z}, M), \quad F(\lambda, \gamma)(t) := \gamma(\lambda \tau_\gamma(t) + (1-\lambda)t),$$

is continuous, so the cycle $a_0 = F(0,\cdot)_* a_0$ is homologous to the cycle

$$a_1 := F(1,\cdot)_* a_0.$$

If $\gamma_1 = F(1,\gamma)$, γ in the support of a_0, is a loop in the support of a_1, differentiating the identity $\gamma_1(\sigma_\gamma(t)) = \gamma(t)$ we find by (A2)

$$|\dot\gamma_1(\sigma_\gamma(t))| = \frac{|\dot\gamma(t)|}{\sigma'_\gamma(t)} \leq \frac{|\dot\gamma(t)|}{\sqrt{1+|\dot\gamma(t)|^2}}(1+L(\gamma)) \leq 1 + L(\gamma).$$

Therefore, by (A1)

$$\|\dot\gamma_1\|_\infty \leq 1 + L(\gamma) \leq 1 + C_0 h \leq (1+C_0)h.$$

Therefore, a_1 is the desired cycle in $C^\infty(\mathbb{R}/\mathbb{Z}, M)$. □

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