ON STABLE SOLUTIONS FOR BOUNDARY REACTIONS: A DE GIORGI-TYPE RESULT IN DIMENSION 4+1

ALESSIO FIGALLI AND JOAQUIM SERRA

Abstract. We prove that every bounded stable solution of

$$(-\Delta)^{1/2}u + f(u) = 0 \qquad \text{in } \mathbb{R}^3$$

is a 1D profile, i.e., $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^2$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing bounded stable solution in dimension one. Equivalently, stable critical points of boundary reaction problems in $\mathbb{R}^{d+1}_+ = \mathbb{R}^{d+1} \cap \{x_{d+1} \geq 0\}$ of the form

$$\int_{\{x_{d+1}\geq 0\}} \frac{1}{2}|\nabla U|^2\,dx\,dx_{d+1} + \int_{\{x_{d+1}=0\}} F(U)\,dx$$

are 1D when $d = 3$.

These equations have been studied since the 1940's in crystal dislocations. Also, as it happens for the Allen-Cahn equation, the associated energies enjoy a Γ-convergence result to the perimeter functional. In particular, when $f(u) = u^3 - u$ (or equivalently when $F(U) = \frac{1}{4}(1-U^2)^2$), our result implies the analogue of the De Giorgi conjecture for the half-Laplacian in dimension 4, namely that monotone solutions are 1D.

Note that our result is a PDE version of the fact that stable embedded minimal surfaces in $\mathbb{R}^3$ are planes. It is interesting to observe that the corresponding statement about stable solutions to the Allen-Cahn equation (namely, when the half-Laplacian is replaced by the classical Laplacian) is still unknown for $d = 3$.

1. Introduction

1.1. **De Giorgi conjecture on the Allen-Cahn equation.** In 1978, De Giorgi stated the following famous conjecture [16]:

Conjecture 1.1. *Let $u \in C^2(\mathbb{R}^d)$ be a solution of the Allen-Cahn equation*

$$-\Delta u = u - u^3, \quad |u| \leq 1, \tag{1.1}$$

satisfying $\partial_{x_d} u > 0$. Then, if $d \leq 8$, all level sets $\{u = \lambda\}$ of u must be hyperplanes.

To motivate this conjecture, we need to explain its relation to minimal surfaces.

1.2. **Allen-Cahn vs. minimal surfaces.** It is well-known that (1.1) is the condition of vanishing first variation for the Ginzburg-Landau energy

$$\mathcal{E}_1(v) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1-v^2)^2 \right) dx.$$

By scaling, if u is a local minimizer of $\mathcal{E}_1$ (namely, a minimizer with respect to compactly supported variations), then $u_\varepsilon(x) := u(\varepsilon^{-1}x)$ is a local minimizer of the

Date: June 27, 2019.

ε-energy

$$\mathcal{E}_{1,\varepsilon}(v) := \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\left(\frac{\varepsilon^2}{2}|\nabla v|^2 + \frac{1}{4}(1-v^2)^2\right)dx. \tag{1.2}$$

In [28, 29], Modica and Mortola established the Γ-convergence of $\mathcal{E}_{1,\varepsilon}$ to the perimeter functional as $\varepsilon \downarrow 0$. As a consequence, the rescalings u_ε have a subsequence u_{ε_k} such that

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}},$$

and E is a local minimizer of the perimeter in $\mathbb{R}^d$. This result was later improved by Caffarelli and Cordoba [9], who showed a density estimate for minimizers of $\mathcal{E}_{1,\varepsilon}$, and proved that the super-level sets $\{u_{\varepsilon_k} \geq \lambda\}$ converge locally uniformly (in the sense of Hausdorff distance) to E for each fixed $\lambda \in (-1,1)$. Hence, at least heuristically, minimizers of $\mathcal{E}_{1,\varepsilon}$ for ε small should behave similarly to sets of minimal perimeter.

1.3. **Classifications of entire minimal surfaces and De Giorgi conjecture.** Here we recall some well-known facts on minimal surfaces:[1]

(i) If E is a local minimizer of the perimeter in $\mathbb{R}^d$ with $d \leq 7$, then E is a halfspace.

(ii) The Simons cone $\{x_1^2+x_2^2+x_3^2+x_4^2 < x_5^2+x_6^2+x_7^2+x_8^2\}$ is a local minimizer of perimeter in $\mathbb{R}^8$ which is not a halfspace.

Also, we recall that these results hold in one dimension higher if we restrict to minimal graphs:

(i') If $E = \{x_d > h(x_1,\ldots,x_{d-1}) \,:\, h:\mathbb{R}^{d-1}\to\mathbb{R}\}$, ∂E is a minimal surface, and $d \leq 8$, then h is affine (equivalently, E is a halfspace).

(ii') There is a non-affine entire minimal graph in dimension $d = 9$.

These assertions combine several classical results. The main contributions leading to (i)-(ii)-(i')-(ii') are the landmark papers of De Giorgi [14, 15] (improvement of flateness – Bernstein theorem for minimal graphs), Simons [39] (classification of stable minimal cones), and Bombieri, De Giorgi, and Giusti [5] (existence of a nontrivial minimal graph in dimension $d = 9$, and minimizing property of the Simons cone).

Note that, in the assumptions of Conjecture 1.1, the function u satisfies $\partial_{x_d} u > 0$, a condition that implies that the super-level sets $\{u \geq \lambda\}$ are epigraphs. Thus, if we assume that $d \leq 8$, it follows by (i') and the discussion in Section 1.2 that the level sets of $u_\varepsilon(x) = u(\varepsilon^{-1}x)$ should be close to a hyperplane for $\varepsilon \ll 1$. Since

$$\{u_\varepsilon = \lambda\} = \varepsilon\{u = \lambda\},$$

this means that all blow-downs of $\{u = \lambda\}$ (i.e., all possible limit points of $\varepsilon\{u = \lambda\}$ as $\varepsilon \downarrow 0$) are hyperplanes. Hence, the conjecture of De Giorgi asserts that, for this to be true, the level sets of u had to be already hyperplanes.

[1]Note that, here and in the sequel, the terminology "minimal surface" denotes a critical point of the area functional (in other words, a surface with zero mean curvature).

1.4. **Results on the De Giorgi conjecture.** Conjecture 1.1 was first proved, about twenty years after it was raised, in dimensions $d = 2$ and $d = 3$, by Ghoussoub and Gui [23] and Ambrosio and Cabré [3], respectively. Almost ten years later, in the celebrated paper [32], Savin attacked the conjecture in the dimensions $4 \le d \le 8$, and he succeeded in proving it under the additional assumption

$$\lim_{x_d \to \pm\infty} u = \pm 1. \tag{1.3}$$

Shortly after, Del Pino, Kowalczyk, and Wei [19] established the existence of a counterexample in dimensions $d \ge 9$.

It is worth mentioning that the extra assumption (1.3) in [32] is only used to guarantee that u is a local minimizer of $\mathcal{E}_1$. Indeed, while in the case of minimal surfaces epigraphs are automatically minimizers of the perimeter, the same holds for monotone solutions of (1.1) under the additional assumption (1.3). Note also that (1.3) guarantees that, for $\lambda \in (-1, 1)$, the sets $\{u > \lambda\}$ are epigraphs of the form $\big\{x_d > h_\lambda(x_1, \ldots, x_{d-1})\big\}$ with $h_\lambda : \mathbb{R}^{n-1} \to \mathbb{R}$, while under the original monotonicity assumption of De Giorgi the functions h_λ could take also the values $\pm\infty$ in some regions of $\mathbb{R}^{n-1}$.

1.5. **Monotone vs. stable solutions.** Before introducing the problem investigated in this paper, we make a connection between monotone and stable solutions.

It is well-known (see [2, Corollary 4.3]) that monotone solutions to (1.1) in $\mathbb{R}^d$ are stable solutions, i.e., the second variation of $\mathcal{E}_1$ is nonnegative. Actually, in the context of monotone solutions it is natural to consider the two limits

$$u^\pm := \lim_{x_d \to \pm\infty} u,$$

which are functions of the first $d - 1$ variables $x_1, \ldots, x_{d-1}$ only, and one can easily prove that $u^\pm$ are stable solutions of (1.1) in $\mathbb{R}^{d-1}$. If one could show that these functions are 1D, then the results of Savin [32] would imply that u was also 1D.

In other words, the following implication holds:

$$\text{stable solutions to (1.1) in } \mathbb{R}^{d-1} \text{ are 1D} \quad \Rightarrow \quad \text{De Giorgi conjecture holds in } \mathbb{R}^d.$$

1.6. **Boundary reaction and line tension effects.** A natural variant of the Ginzburg-Landau energy, fist introduced in the 1940's in the context of crystal dislocations by Peierls and Nabarro [31, 30], and later studied by Alberti, Bouchitté, and Seppecher [1] and Cabré and Solà-Morales [8], consists in studying a Dirichlet energy with boundary potential on a half space $\mathbb{R}^{d+1}_+ := \{x_{d+1} > 0\}$ (the choice of considering $d + 1$ dimensions will be clear by the discussion in the next sections). In other words, one considers the energy functional

$$\mathcal{J}(V) := \int_{\mathbb{R}^{d+1}_+} \frac{1}{2}|\nabla V|^2 \, dx \, dx_{d+1} + \int_{\{x_{d+1}=0\}} F(V) \, dx,$$

where $F : \mathbb{R} \to \mathbb{R}$ is some potential. Then, the Euler-Lagrange equation corresponding to $\mathcal{J}$ is given by

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^{d+1}_+, \\ \partial_\nu U = -f(U) & \text{on } \{x_{d+1} = 0\}, \end{cases} \tag{1.4}$$

where $f = F'$, and $\partial_\nu U = -\partial_{x_{d+1}} U$ is the exterior normal derivative. When $f(U) = \sin(cU)$, $c \in \mathbb{R}$, the above problem is called the Peierls-Navarro equation and appears in a model of dislocation of crystals [24, 40]. Also, the same equation is central for the analysis of boundary vortices for soft thin films in [26]. Other motivations, as well as constructions of oscillating solutions, can be found in [13].

1.7. **Non-local interactions.** To state the analogue of the De Giorgi conjecture in this context we first recall that, for a harmonic function V, the energy $\mathcal{J}$ can be rewritten in terms of its trace $v := V|_{x_{d+1}=0}$. More precisely, a classical computation shows that (up to a multiplicative dimensional constant) the Dirichlet energy of V is equal to the $H^{1/2}$ energy of v:

$$\int_{\mathbb{R}^{d+1}_+} |\nabla V|^2 \, dx_1 \cdots dx_d \, dx_{d+1} = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v(x) - v(y)|^2}{|x-y|^{d+1}} \, dx \, dy$$

(see for instance [10]). Hence, instead of $\mathcal{J}$, one can consider the energy functional

$$\mathcal{E}(v) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2} \frac{|v(x) - v(y)|^2}{|x-y|^{d+1}} \, dx \, dy + \int_{\mathbb{R}^d} F(v(x)) \, dx$$

and because harmonic functions minimize the Dirichlet energy, one can easily prove that

$$U \text{ is a local min. of } \mathcal{J} \text{ in } \mathbb{R}^{d+1}_+ \quad \Leftrightarrow \quad \begin{cases} u = U|_{x_{d+1}=0} \text{ is a local min. of } \mathcal{E} \text{ in } \mathbb{R}^d \\ \text{and } U \text{ is the harmonic extension of } u. \end{cases}$$

Hence, in terms of the function u, the Euler-Lagrange equation (1.4) corresponds to the first variation of $\mathcal{E}$, namely

$$(-\Delta)^{1/2} u + f(u) = 0 \quad \text{in } \mathbb{R}^d, \tag{1.5}$$

where

$$(-\Delta)^{1/2} u(x) = 2 \, \text{p.v.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+1}} \, dy.$$

1.8. **Γ-convergence of nonlocal energies to the classical perimeter, and the De Giorgi conjecture for the $1/2$-Laplacian.** Analogously to what happens with the classical Allen-Cahn equation, there is a connection between solutions of $(-\Delta)^{1/2} u = u - u^3$ and minimal surfaces. Namely, if u is a local minimizer of $\mathcal{E}$ in $\mathbb{R}^d$ with $F(u) = \frac{1}{4}(1-u^2)^2$, then the rescaled function $u_\varepsilon(x) = u(\varepsilon^{-1} x)$ is a local minimizer of the ε-energy

$$\mathcal{E}_\varepsilon(v) := \frac{1}{\varepsilon \log(1/\varepsilon)} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\varepsilon}{2} \frac{|v(x) - v(y)|^2}{|x-y|^{d+1}} \, dx \, dy + \int_{\mathbb{R}^d} \frac{1}{4} (1 - v^2)^2 \, dx \right).$$

As happened for the energies $\mathcal{E}_{1,\varepsilon}$ in (1.2), the papers [1, 27] established the Γ-convergence of $\mathcal{E}_\varepsilon$ to the perimeter functional as $\varepsilon \downarrow 0$, as well as the existence of a subsequence u_{ε_k} such that

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}},$$

where E is a local minimizer of the perimeter in $\mathbb{R}^d$. Moreover, Savin and Valdinoci [36] proved density estimates for minimizers of $\mathcal{E}_\varepsilon$, implying that $\{u_{\varepsilon_k} \geq \lambda\}$ converge locally uniformly to E for each fixed $\lambda \in (-1, 1)$.

Hence, the discussion in Section 1.3 motivates the validity of the De Giorgi conjecture when $-\Delta$ is replaced with $(-\Delta)^{1/2}$, namely:

Conjecture 1.2. *Let $u \in C^2(\mathbb{R}^d)$ be a solution of the fractional Allen-Cahn equation*

$$(-\Delta)^{1/2}u = u - u^3, \quad |u| \le 1, \tag{1.6}$$

satisfying $\partial_{x_d} u > 0$. Then, if $d \le 8$, all level sets $\{u = \lambda\}$ of u must be hyperplanes.

In this direction, Cabré and Solà-Morales proved the conjecture for $d = 2$ [8]. Later, Cabré and Cinti [6] established Conjecture 1.2 for $d = 3$. Very recently, under the additional assumption (1.3), Savin first announced in [33] and then gave in [34] a proof of Conjecture 1.2 in the remaining dimensions $4 \le d \le 8$. Thanks to the latter result, the relation between monotone and stable solutions explained in Section 1.5 holds also in this setting.

1.9. **Stable solutions vs. stable minimal surfaces.** Exactly as in the setting of Conjecture 1.1, given u as in Conjecture 1.2 it is natural to introduce the two limit functions $u^\pm := \lim_{x_d \to \pm\infty} u$. These functions depend only on the first $d-1$ variables $x_1, \dots, x_{d-1}$, and are stable solutions of (1.6) in $\mathbb{R}^{d-1}$.

As mentioned at the end of last section, the classification of stable solutions to (1.6) in $\mathbb{R}^{d-1}$, $3 \le d-1 \le 7$, together with the improvement of flatness for $(-\Delta)^{1/2}u = u - u^3$ proved in [34], would imply the full Conjecture 1.2 in $\mathbb{R}^d$.

The difficult problem of classifying stable solutions of (1.6) (or of (1.1)) is connected to the following well-known conjecture for minimal surfaces:

Conjecture 1.3. *Stable embedded minimal hypersurfaces in $\mathbb{R}^d$ are hyperplanes as long as $d \le 7$.*

A positive answer to this conjecture is only known to be true in dimension $d = 3$, a result of Fischer-Colbrie and Schoen [22] and Do Carmo and Peng [20]. Note that, for minimal cones, the conjecture is true (and the dimension 7 sharp) by the results of Simons [39] and Bombieri, De Giorgi, and Giusti [5].

Conjecture 1.3 above suggests a "stable De Giorgi conjecture":

Conjecture 1.4. *Let $u \in C^2(\mathbb{R}^d)$ be a stable solution of* (1.1) *or of* (1.6). *Then, if $d \le 7$, all level sets $\{u = \lambda\}$ of u must be hyperplanes.*

As explained before, the validity of this conjecture would imply both Conjectures 1.1 and 1.2.

1.10. **Results of the paper.** As of now, Conjecture 1.4 has been proved only for $d = 2$ (see [4, 23] for (1.1), and [8] for (1.6)). The main result of this paper establishes its validity to (1.6) for $d = 3$, a case that heuristically corresponds to the classification in $\mathbb{R}^3$ of stable minimal surfaces of [22]. Note that, for the classical case (1.1), Conjecture 1.4 in the case $d = 3$ is still open.

This is our main result:

Theorem 1.5. *Let u be a bounded stable solution of* (1.5) *with $d = 3$, and assume that $f \in C^{0,\alpha}$ for some $\alpha > 0$. Then u is 1D profile, namely, $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^2$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing bounded stable solution to* (1.5) *in dimension one.*

As explained before, as an application of Theorem 1.5 and the improvement of flatness for $(-\Delta)^{1/2}u = u - u^3$ in [34], we obtain the following:

Corollary 1.6. *Conjecture 1.2 holds true in dimension $d = 4$.*

A key ingredient behind the proof of Theorem 1.5 is the following general energy estimate which holds in every dimension $d \geq 2$:

Proposition 1.7. *Let $R \geq 1$, $M_o \geq 2$, and $\alpha \in (0,1)$. Let u be a stable solution of*

$$(-\Delta)^{1/2}u + f(u) = 0, \quad |u| \leq 1 \qquad \text{in } B_R \subset \mathbb{R}^d, \tag{1.7}$$

where $f : [-1,1] \to \mathbb{R}$ satisfies $\|f\|_{C^{0,\alpha}([-1,1])} \leq M_o$. Then there exists a constant $C > 0$, depending only on d and α, such that

$$\int_{B_{R/2}} |\nabla u| \leq CR^{d-1}\log(M_oR)$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus B^c_{R/2}\times B^c_{R/2}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+1}}dxdy \leq CR^{d-1}\log^2(M_oR).$$

Because of recent results on the structure of stable solutions to fractional Allen-Cahn equations, it is likely that Proposition 1.7 is sharp: indeed, for $d \geq 3$ one may expect to build a stable solution in B_R that saturates the bounds above by taking $\log R$ catenoidal ends at mutual distance $(\log R)^{-1}R$ in B_R, and then construct a stable solution that has these catenoidal ends as 0-level set.

On the other hand, at least in low dimensions, for global stable solutions one would like to improve the bounds by a factor $\log(M_oR)$ (since that corresponds to the case when u is a 1D profile). This is indeed what we do in $\mathbb{R}^3$: in Section 4 we are able to bootstrap the estimates from Proposition 1.7 to obtain the sharp energy bound, from which Theorem 1.5 follows easily.

1.11. **Comments on the results.** As we have explained above, the classification of global stable critical points to boundary reactions is the natural boundary analogue of the similar problem for Allen-Cahn. While originally this boundary problem is of purely local nature (see (1.4)) and indeed it was studied as a local problem in [8], by looking at it as a nonlocal equation we are able to use some of the recent techniques developed in these areas. In particular we can exploit some arguments developed in [12] in the context of the so-called nonlocal minimal surfaces.

However, while in [12] uniform area bounds are a rather easy consequence of stability, in our setting this approach leads to non-sharp bounds (see Proposition 1.7). Such bounds turn out to be insufficient to classify entire solutions in $\mathbb{R}^3$ by the "standard" approach (in minimal surfaces, Allen-Cahn, etc.) based on testing stability with a logarithmic cutoff function. It is well-known (see e.g. [37]) that this standard approach works when one has an energy growth of the type $CR^2G(R)$ with $\sum_{k=1}^\infty \frac{1}{G(2^k)} = +\infty$, but even being sharp in every step we can only get $CR^2\log^2 R$, which does not satisfy the previous condition since $\sum_{k=1}^\infty \frac{1}{k^2} < +\infty$. This is actually not surprising: a purely nonlocal method as the one in [12] cannot provide a sharp energy growth control, because our energy is nonlocal only at small scales (recall that it approaches the local perimeter at large scales, see Section 1.8) and thus estimates based on nonlocal interactions must degenerate at large scales.

A cornerstone of our paper, which makes possible the classification result in $\mathbb{R}^3$, is the recurrence relation (4.5), that relates the natural renormalized energies across different scales. We believe it is quite remarkable that such a closed recursive relation holds true and it is interesting to point out that, in order to get it, it is absolutely essential to use the sharp interpolation inequality in Lemma 3.1. The interested reader may note that, if any of the steps of the paper was made in a slightly non-sharp way, then the recurrence relation obtained instead of (4.5) would not be closed (as it would involve some constant depending on R) and then whole proof would break down completely.

1.12. **Further directions.** A series of recent papers [17, 18, 11] used the Allen-Cahn equation as a tool to construct, via min-max procedures, minimal hypersurfaces with prescribed Morse index on compact Riemannian manifolds. In particular, in [11], Chodosh and Mantoulidis have used this approach to construct multiplicity-one minimal surfaces in compact 3-manifolds with bounded energy and prescribed Morse index, giving a new proof of the Yau's conjecture recently proved by Irie, Marques, and Neves [25].

The main ingredients needed for the construction in [11] are:

- curvature estimates for the 0-level set of stable solutions of $-\Delta u = \varepsilon^{-2}(u - u^3)$ that are robust as $\varepsilon \downarrow 0$;
- sharp lower bounds in terms of ε for the "sheet distance" (i.e., the distance between two consecutive connected components of the 0-level set, whenever more than one component exists).

It turns out that, for stable embedded minimal surfaces on 3-manifolds, the flatness result for complete surfaces in $\mathbb{R}^3$ (i.e. the analogue of Theorem 1.5) implies a universal curvature estimate for minimal discs through a blow-up argument (see for instance [42]). Similarly, for Allen-Cahn, a classification result for stable solutions with quadratic energy growth (a very strong extra assumption with respect to the result in our Theorem 1.5, that does not require any energy bound) is used in [11] to obtain curvature estimates for the 0-level set of stable solutions of $-\Delta u = \varepsilon^{-2}(u-u^3)$ on 3-manifolds. In this case, though, the obtained curvature estimates are not universal but depend on energy bounds for the solutions (since so does the available classification result). In the case of Allen-Cahn the analysis of clustering sheets in [41], which leads to a striking regularity result for stable configurations, is also an essential tool to obtain these curvature estimates.

One outcome of our paper is that, in the case of the half-Laplacian, the classification result can be proven without assuming any energy bound. Also, the methods introduced here seem to lead to "sheet distance" lower bounds for stable solutions that are much stronger than the ones available for Allen-Cahn. Hence, a natural further development is to exploit these techniques (combining them with the natural extensions of the results in [41]) to provide universal curvature and energy estimates for stable solutions of $(-\Delta)^{1/2}u = \varepsilon^{-1}(u-u^3)$. Among other applications, they could then potentially be used to construct minimal surfaces on manifolds (as an alternative to the Allen-Cahn equation for the min-max constructions mentioned above).

1.13. **Structure of the paper.** In the next section we collect all the basic estimates needed for the proof of Proposition 1.7. Then, in Section 3 we prove Proposition 1.7. Finally, in Sections 4 we prove Theorem 1.5.

Acknowledgments: This project has received funding from the European Research Council under the Grant Agreement No. 721675 "Regularity and Stability in Partial Differential Equations (RSPDE)".

2. Ingredients of the proofs

We begin by introducing some notation.

Given $R > 0$, we define the energy of a function inside $B_R \subset \mathbb{R}^d$ as

$$\mathcal{E}(v; B_R) := \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus B_R^c\times B_R^c} \frac{1}{2}\frac{|v(x)-v(y)|^2}{|x-y|^{d+1}}\,dx\,dy + \int_{B_R} F(v(x))\,dx,$$

where $B_R^c = \mathbb{R}^d \setminus B_R$, and F is a primitive of f. Note that equation (1.7) is the condition of vanishing first variation for the energy functional $\mathcal{E}(\,\cdot\,; B_R)$.

We say that a solution u of (1.7) is *stable* if the second variation at u of $\mathcal{E}$ is nonnegative, that is

$$\int_{B_R} \big((-\Delta)^{1/2}\xi + f'(u)\xi\big)\xi\,dx \ge 0 \quad \text{for all } \xi \in C^2_c(B_R). \tag{2.1}$$

Also, we say that u is stable in $\mathbb{R}^d$ if it is stable in B_R for all $R \ge 1$.

An important ingredient in our proof consists in considering variations of a stable solution u via a suitable smooth 1-parameter family of "translation like" deformations. This kind of idea has been first used by Savin and Valdinoci in [35, 37], and then in [12, 7]. More precisely, given $R \ge 3$, consider the cut-off functions

$$\varphi^0(x) := \begin{cases} 1 & \text{for } |x| \le \frac{1}{2} \\ 2-2|x| & \text{for } \frac{1}{2} \le |x| \le 1 \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

$$\varphi^1(x) := \begin{cases} 1 & \text{for } |x| \le \sqrt{R} \\ 2-2\frac{\log|x|}{\log R} & \text{for } \sqrt{R} \le |x| \le R \\ 0 & \text{for } |x| \ge R, \end{cases}$$

$$\varphi^2(x) := \begin{cases} 1 & \text{for } |x| \le R_* \\ 2-2\dfrac{\log\log|x|}{\log\log R} & \text{for } R_* \le |x| \le R \\ 0 & \text{for } |x| \ge R, \end{cases}$$

where $R^* := \exp(\sqrt{\log R})$.

For a fixed unit vector $\boldsymbol{v} \in \mathbb{S}^{n-1}$ define

$$\Psi^i_{t,\boldsymbol{v}}(z) := z + t\varphi^i(z)\boldsymbol{v}, \qquad t \in \mathbb{R}, \quad z \in \mathbb{R}^d, \quad i = 0,1,2. \tag{2.2}$$

Then, given a function $v : \mathbb{R}^d \to \mathbb{R}$ and $t \in (-1,1)$ with $|t|$ small enough (so that $\Psi^i_{t,\boldsymbol{v}}$ is invertible), we define the operator

$$\mathcal{P}^i_{t,\boldsymbol{v}} v(x) := v\big((\Psi^i_{t,\boldsymbol{v}})^{-1}(x)\big). \tag{2.3}$$

Also, we use $\mathcal{E}^{\mathrm{Sob}}$ and $\mathcal{E}^{\mathrm{Pot}}$ to denote respectively the fractional Sobolev term and the Potential term appearing in the definition of $\mathcal{E}$:[2]

$$\mathcal{E}^{\mathrm{Sob}}(u;B_R) := \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus B_R^c\times B_R^c} \frac{|u(x)-u(y)|^2}{|x-y|^{d+1}}\,dx\,dy,$$

$$\mathcal{E}^{\mathrm{Pot}}(u;B_R) := \int_{B_R} F(u(x))\,dx.$$

We shall use the following bounds:

Lemma 2.1. *There exists a dimensional constant C such that the following hold for all $v : \mathbb{R}^d \to \mathbb{R}$, $|t|$ small, and $\boldsymbol{v} \in \mathbb{S}^{d-1}$:*

(1) We have

$$\mathcal{E}(\mathcal{P}^0_{t,\boldsymbol{v}} v;B_1) + \mathcal{E}(\mathcal{P}^0_{-t,\boldsymbol{v}} v;B_1) - 2\mathcal{E}(v;B_1) \le Ct^2\mathcal{E}^{\mathrm{Sob}}(v;B_2). \tag{2.4}$$

(2) For $R = 2^{2k}$, $k \ge 1$, we have

$$\mathcal{E}(\mathcal{P}^1_{t,\boldsymbol{v}} v;B_R) + \mathcal{E}(\mathcal{P}^1_{-t,\boldsymbol{v}} v;B_R) - 2\mathcal{E}(v;B_R) \le C\frac{t^2}{k^2}\sum_{j=1}^{k}\frac{\mathcal{E}^{\mathrm{Sob}}(v;B_{2^{k+j}})}{2^{2(k+j)}}. \tag{2.5}$$

(3) For $R \ge 4$,

$$\mathcal{E}(\mathcal{P}^2_{t,\boldsymbol{v}} v;B_R) + \mathcal{E}(\mathcal{P}^2_{-t,\boldsymbol{v}} v;B_R) - 2\mathcal{E}(v;B_R) \le C\frac{t^2}{\log\log R}\sup_{\rho\ge 2}\frac{\mathcal{E}^{\mathrm{Sob}}(v;B_\rho)}{\rho^2\log\rho}. \tag{2.6}$$

Proof. The lemma follows as in [12, Lemma 2.1] and [7, Lemma 2.3]. However, since we do not have a precise reference for the estimates that we need, we give a sketch of proof. Note that, by approximation, it suffices to consider the case when $v \in C^2_c(\mathbb{R}^d)$.

First observe that, since $\boldsymbol{v}$ has unit norm, the Jacobian of the change of variables $z \mapsto \Psi^i_{t,\boldsymbol{v}}(z)$ is given by

$$J^i_t(z) := |\det(D\Psi^i_{t,\boldsymbol{v}}(z))| = |\det(\mathrm{Id} + t\nabla\varphi^i(z)\otimes\boldsymbol{v})| = 1 + t\partial_{\boldsymbol{v}}\varphi^i(z). \tag{2.7}$$

Set

$$R^i := \begin{cases} 1 & \text{if } i = 0 \\ R & \text{if } i = 1,2. \end{cases} \tag{2.8}$$

Then, performing the change of variables $x := \Psi^i_{t,\boldsymbol{v}}(z)$, we get

$$\mathcal{E}^{\mathrm{Pot}}(\mathcal{P}^i_{t,\boldsymbol{v}} v;B_{R^i}) = \int_{B_{R^i}} F\big(v\big((\Psi^i_{t,\boldsymbol{v}})^{-1}(x)\big)\big)\,dx = \int_{B_{R^i}} F(v(z))\big(1 + t\partial_{\boldsymbol{v}}\varphi^i(z)\big)\,dz,$$

[2]To simplify the notation we define $\mathcal{E}^{\mathrm{Sob}}$ without the coefficient 1/2, so that

$$\mathcal{E} = \frac{1}{2}\mathcal{E}^{\mathrm{Sob}} + \mathcal{E}^{\mathrm{Pot}}.$$

thus

$$\mathcal{E}^{\rm Pot}(\mathcal{P}^i_{t,\boldsymbol{v}}v;B_{R^i})+\mathcal{E}^{\rm Pot}(\mathcal{P}^i_{-t,\boldsymbol{v}}v;B_{R^i})-2\mathcal{E}^{\rm Pot}(v;B_{R^i})=0.$$

Hence, we only need to estimate the second order incremental quotient of $\mathcal{E}^{\rm Sob}$. To this aim, using the same change of variable and setting

$$A_r:=\mathbb{R}^n\times\mathbb{R}^n\setminus B_r^c\times B_r^c\qquad (r>0)$$

and $K(z):=|z|^{-(d+1)}$, we have (note that $\Psi^i_{t,\boldsymbol{v}}$ preserves B_{R^i})

(2.9)

$$\mathcal{E}^{\rm Sob}(\mathcal{P}^i_{t,\boldsymbol{v}}v;B_{R^i})=\iint_{A_{R^i}}|v(y)-v(\overline{y})|^2K\big(\Psi^i_{t,\boldsymbol{v}}(y)-\Psi^i_{t,\boldsymbol{v}}(\overline{y})\big)\,J^i_t(y)\,dy\,J^i_t(\overline{y})\,d\overline{y}\,.$$

Recalling that $\Psi^i_{t,\boldsymbol{v}}(y)-\Psi^i_{t,\boldsymbol{v}}(\overline{y})=y-\overline{y}+t\big(\varphi^i(y)-\varphi^i(\overline{y})\big)\boldsymbol{v}$ and defining

$$\varepsilon^i(y,\overline{y}):=\frac{\varphi^i(y)-\varphi^i(\overline{y})}{|y-\overline{y}|},$$

as in the proof of [12, Lemma 2.1] we have, for $|t|$ small,

$$\big|K\big(z\pm t\varepsilon|z|\boldsymbol{v}\big)-K(z)\mp t\partial_{\boldsymbol{v}}K(z)\varepsilon|z|\big|\le Ct^2\varepsilon^2K(z) \tag{2.10}$$

and

(2.11)

$$|\varepsilon^i(y,\overline{y})|+\frac{|J^i_t(y)-1|}{t}+\frac{|J^i_t(\overline{y})-1|}{t}\le$$
$$\le\begin{cases} C & \text{if } i=0,\\[4pt] \dfrac{C}{\log R\,\max\{\sqrt{R},\min(|y|,|\overline{y}|)\}} & \text{if } i=1,\\[4pt] \dfrac{C}{\log\log R\,\log\rho\,\rho}\quad \text{for } \rho\ge R_*,\ |y|\ge\rho,\ |\overline{y}|\ge\rho & \text{if } i=2.\end{cases}$$

Then, using (2.9), (2.7), (2.10), and (2.11), and decomposing $A_R=A_{2^{2k}}=A_{2^k}\cup\big(\cup_{j=1}^kA_{2^{k+j}}\setminus A_{2^{k+j-1}}\big)$ when $i=1$, an easy computation yields

$$\mathcal{E}^{\rm Sob}(\mathcal{P}^i_{t,\boldsymbol{v}}v;B_{R^i})+\mathcal{E}^{\rm Sob}(\mathcal{P}^i_{-t,\boldsymbol{v}}v;B_{R^i})-2\mathcal{E}^{\rm Sob}(v;B_{R^i})\le$$
$$\le\begin{cases} Ct^2\displaystyle\iint_{A_1}|v(y)-v(\overline{y})|^2K(y-\overline{y})\,dy\,d\overline{y} & \text{if } i=0,\\[4pt] C\dfrac{t^2}{k^2}\displaystyle\sum_{j=1}^k\iint_{A_{2^{k+j}}}\frac{1}{2^{2(k+j)}}|v(y)-v(\overline{y})|^2K(y-\overline{y})\,dy\,d\overline{y} & \text{if } i=1\end{cases}$$

(see the proof of [12, Lemma 2.1] for more details). Therefore (2.4) and (2.5) follow.

The proof of (2.6) needs a more careful estimate. For $\rho>0$, we denote

$$e(\rho):=\mathcal{E}^{\rm Sob}(v;B_\rho)=\iint_{A_\rho}|v(y)-v(\overline{y})|^2K(y-\overline{y})\,dy\,d\overline{y}.$$

Note that

$$e'(\rho)=\lim_{h\downarrow0}\frac{1}{h}\iint_{A_{\rho+h}\setminus A_\rho}|v(y)-v(\overline{y})|^2K(y-\overline{y})\,dy\,d\overline{y}.$$

Observing that in the complement of A_ρ we have $|y| \geq \rho$ and $|\overline{y}| \geq \rho$, and using (2.9), (2.7), (2.10), and (2.11), we obtain

$$\begin{aligned}
&\mathcal{E}^{\mathrm{Sob}}(\mathcal{P}^2_{t,\boldsymbol{v}}v; B_R) + \mathcal{E}^{\mathrm{Sob}}(\mathcal{P}^2_{-t,\boldsymbol{v}}v; B_R) - 2\mathcal{E}^{\mathrm{Sob}}(v; B_R) \leq \\
&\qquad \leq \frac{Ct^2}{(\log\log R)^2}\left(\frac{e(R_*)}{(\log R_*\, R_*)^2} + \int_{R_*}^R \frac{e'(\rho)}{(\log\rho\,\rho)^2}d\rho\right) \\
&\qquad \leq \frac{Ct^2}{(\log\log R)^2}\left(S + C\int_{R_*}^R \frac{e(\rho)}{(\log\rho)^2\,\rho^3}d\rho\right) \\
&\qquad \leq \frac{Ct^2}{(\log\log R)^2}\left(S + \int_{R_*}^R \frac{S}{\log\rho\,\rho}d\rho\right) \\
&\qquad \leq \frac{Ct^2}{\log\log R}\, S,
\end{aligned}$$

where

$$S := \sup_{\rho\geq 2} \frac{e(\rho)}{\rho^2\log\rho} = \sup_{\rho\geq 2}\frac{\mathcal{E}^{\mathrm{Sob}}(v; B_\rho)}{\rho^2\log\rho},$$

so (2.6) follows. □

The following is a basic BV estimate in $B_{1/2}$ for stable solutions in a ball.

Lemma 2.2. *Let $i \in \{0, 1, 2\}$, R^i as in (2.8), and let $u \in C^{1,\alpha}(\overline{B_{R^i}})$ be a stable solution to $(-\Delta)^{1/2}u + f(u) = 0$ in B_{R^i} with $|u| \leq 1$. Assume there exists $\eta > 0$ such that, for $|t|$ small enough, we have*

$$\mathcal{E}(\mathcal{P}^i_{t,\boldsymbol{v}}u; B_{R^i}) + \mathcal{E}(\mathcal{P}^i_{-t,\boldsymbol{v}}u; B_{R^i}) - 2\mathcal{E}(u; B_{R^i}) \leq \eta t^2 \qquad \forall\, \boldsymbol{v} \in \mathbb{S}^{d-1}. \tag{2.12}$$

Then

$$\left(\int_{B_{1/2}} \big(\partial_{\boldsymbol{v}}u(x)\big)_+\, dx\right)\left(\int_{B_{1/2}} \big(\partial_{\boldsymbol{v}}u(y)\big)_-\, dy\right) \leq 2\eta \tag{2.13}$$

and

$$\int_{B_{1/2}} |\nabla u| \leq C(1 + \sqrt{\eta}), \tag{2.14}$$

for some dimensional constant C.

Proof. The proof is similar to the ones of [12, Lemmas 2.4 and 2.5] or [7, Lemma 2.5 and 2.6]. The key point is to note that, since u is stable,

$$\mathcal{E}(\mathcal{P}^i_{t,\boldsymbol{v}}u; B_{R^i}) - \mathcal{E}(u; B_{R^i}) \geq -o(t^2),$$

hence (2.12) implies

$$\mathcal{E}(\mathcal{P}^i_{t,\boldsymbol{v}}u; B_{R^i}) - \mathcal{E}(u; B_{R^i}) \leq 2\eta t^2$$

for $|t|$ small enough.

On the other hand, still by stability, the two functions

$$\overline{u} := \max\{u, P^i_{t,\boldsymbol{v}}u\} \quad \text{and} \quad \underline{u} := \min\{u, P^i_{t,\boldsymbol{v}}u\}$$

satisfy

$$\mathcal{E}(\overline{u}; B_{R^i}) - \mathcal{E}(u; B_{R^i}) \geq -o(t^2), \qquad \mathcal{E}(\underline{u}; B_{R^i}) - \mathcal{E}(u; B_{R^i}) \geq -o(t^2).$$

Hence, combining these inequalities with the identity

$$\mathcal{E}(\overline{u}; B_{R^i}) + \mathcal{E}(\underline{u}; B_{R^i}) + 2\iint_{B_{R^i}\times B_{R^i}} \frac{(\mathcal{P}^i_{t,\boldsymbol{v}}u-u)_+(x)(\mathcal{P}^i_{t,\boldsymbol{v}}u-u)_-(y)}{|x-y|^{d+1}}\,dx\,dy \le$$
$$\le \mathcal{E}(\mathcal{P}^i_{t,\boldsymbol{v}}u; B_{R^i}) + \mathcal{E}(u; B_{R^i})$$

we obtain

$$2\iint_{B_{R^i}\times B_{R^i}} \frac{(\mathcal{P}^i_{t,\boldsymbol{v}}u-u)_+(x)(\mathcal{P}^i_{t,\boldsymbol{v}}u-u)_-(y)}{|x-y|^{d+1}}\,dx\,dy \le 4\eta t^2$$

Noticing that $\mathcal{P}^i_{t,\boldsymbol{v}}u(x) = u(x-t\boldsymbol{v})$ for $x \in B_{1/2}$ and that $|x-y|^{-d-1} \ge 1$ for $x, y \in B_{1/2}$ we obtain the bound

$$\iint_{B_{1/2}\times B_{1/2}} \left(\frac{u(x-t\boldsymbol{v})-u(x)}{t}\right)_+ \left(\frac{u(y-t\boldsymbol{v})-u(y)}{t}\right)_- dx\,dy \le 2\eta$$

for all $|t|$ small enough, so (2.13) follows by letting $t \to 0$.

In other words, if we define

$$A^\pm_{\boldsymbol{v}} := \int_{B_{1/2}} \big(\partial_{\boldsymbol{v}} u(x)\big)_\pm\,dx,$$

we have proved that $\min\{A^+_{\boldsymbol{v}}, A^-_{\boldsymbol{v}}\}^2 \le A^+_{\boldsymbol{v}} A^-_{\boldsymbol{v}} \le 2\eta$. In addition, since $|u| \le 1$, by the divergence theorem

$$|A^+_{\boldsymbol{v}} - A^-_{\boldsymbol{v}}| = \left|\int_{B_{1/2}} \partial_{\boldsymbol{v}} u(x)\,dx\right| \le \int_{\partial B_{1/2}} |\boldsymbol{v}\cdot\nu_{\partial B_{1/2}}| \le C.$$

Combining these bounds, this proves that

$$\int_{B_{1/2}} |\partial_{\boldsymbol{v}} u(x)|\,dx = A^+_{\boldsymbol{v}} + A^-_{\boldsymbol{v}} = |A^+_{\boldsymbol{v}} - A^-_{\boldsymbol{v}}| + 2\min\{A^+_{\boldsymbol{v}}, A^-_{\boldsymbol{v}}\} \le C + 2\sqrt{2\eta},$$

from which (2.14) follows immediately. □

We now recall the following general lemma due to Simon [38] (see also [12, Lemma 3.1]):

Lemma 2.3. *Let $\beta \in \mathbb{R}$ and $C_0 > 0$. Let $\mathcal{S} : \mathcal{B} \to [0, +\infty]$ be a nonnegative function defined on the class $\mathcal{B}$ of open balls $B \subset \mathbb{R}^n$ and satisfying the following subadditivity property:*

$$B \subset \bigcup_{j=1}^N B_j \quad \Longrightarrow \quad \mathcal{S}(B) \le \sum_{j=1}^N \mathcal{S}(B_j).$$

Also, assume that $\mathcal{S}(B_1) < \infty$. Then there exists $\delta = \delta(n, \beta) > 0$ such that if

$$\rho^\beta \mathcal{S}\big(B_{\rho/4}(z)\big) \le \delta\rho^\beta \mathcal{S}\big(B_\rho(z)\big) + C_0 \quad \textit{whenever } B_\rho(z) \subset B_1$$

then

$$\mathcal{S}(B_{1/2}) \le CC_0,$$

where C depends only on d and β.

Finally, we state an optimal bound on the $H^{1/2}$ norm of the mollification of a bounded function with the standard heat kernel, in terms of the BV norm and the parameter of mollification (see [21, Lemma 2.1] for a proof):

Lemma 2.4. *Let $H_{d,t}(x) := (4\pi t)^{-d/2} e^{-|x|^2/4t}$ denote the heat kernel in $\mathbb{R}^d$. Given $u \in BV(\mathbb{R}^d)$ with $|u| \le 1$, set $u_\varepsilon := u * H_{d,\varepsilon^2}$. Then, for $\varepsilon \in (0, 1/2)$, we have*

$$[u_\varepsilon]^2_{H^{1/2}(\mathbb{R}^d)} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{d+1}}\, dx\, dy \le C \log\frac{1}{\varepsilon}\, \|u\|_{BV(\mathbb{R}^d)},$$

where C is a dimensional constant.

3. Proof of Proposition 1.7

As a preliminary result we need the following (sharp) interpolation estimate.

Lemma 3.1. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a bounded function, with $|u| \le 1$. Assume that u is Lipschitz in B_2, with $\|\nabla u\|_{L^\infty(B_2)} \le L_o$ for some $L_o \ge 2$. Then*

$$\mathcal{E}^{\mathrm{Sob}}(u; B_1) \le C \log L_o \left(1 + \int_{B_2} |\nabla u|\, dx\right)$$

where C depends only on d.

Proof. Let $\eta \in C^\infty_c(B_2)$, $0 \le \eta \le 1$, be a radial cutoff function such that $\eta = 1$ in $B_{3/2}$ and $\|\nabla \eta\|_{L^\infty(\mathbb{R}^d)} \le 3$, and set $\widetilde{u} := \eta u$. Observe that, since $|u| \le 1$, $0 \le \eta \le 1$, $\|\nabla u\|_{L^\infty(B_2)} \le L_o$, and η is supported inside B_2, we have (recall that $L_0 \ge 2$)

$$\|\nabla \widetilde{u}\|_{L^\infty(\mathbb{R}^d)} \le \|\nabla u\|_{L^\infty(B_2)} + \|\nabla \eta\|_{L^\infty(\mathbb{R}^d)} \le L_0 + 3 \le 3L_0. \tag{3.1}$$

Now, since $|u| \le 1$, we have

$$\mathcal{E}^{\mathrm{Sob}}(u; B_1) \le \mathcal{E}^{\mathrm{Sob}}(\widetilde{u}; B_1) + C \tag{3.2}$$

where C depends only on d. On the other hand, it follows by Lemma 2.4 that

$$\|\widetilde{u}_\varepsilon\|^2_{H^{1/2}(\mathbb{R}^d)} \le C \log\frac{1}{\varepsilon}\, \|\widetilde{u}\|_{BV(\mathbb{R}^d)}. \tag{3.3}$$

We also observe that, because of (3.1),[3]

$$[\widetilde{u}_\varepsilon - \widetilde{u}]^2_{H^{1/2}(\mathbb{R}^d)} \le C\varepsilon \|\nabla \widetilde{u}\|^2_{L^2(\mathbb{R}^d)} \le C\varepsilon L_o \int_{\mathbb{R}^d} |\nabla \widetilde{u}|\, dx. \tag{3.4}$$

[3]The first inequality in (3.4) can be proven using Fourier transform, noticing that

$$\widehat{(\widetilde{u}_\epsilon - \widetilde{u})}(\xi) = (e^{-\varepsilon^2|\xi|^2} - 1)\widehat{\widetilde{u}}(\xi),$$

and that $\frac{|e^{-\varepsilon^2|\xi|^2}-1|^2}{\varepsilon|\xi|}$ is universally bounded. Indeed,

$$\begin{aligned}[\widetilde{u}_\varepsilon - \widetilde{u}]^2_{H^{1/2}(\mathbb{R}^d)} &= \int |\xi| \left|\widehat{(\widetilde{u}_\epsilon - \widetilde{u})}(\xi)\right|^2 d\xi = \int |\xi| \left|e^{-\varepsilon^2|\xi|^2} - 1\right|^2 |\widehat{\widetilde{u}}(\xi)|^2\, d\xi \\ &\le C\varepsilon \int |\xi|^2 |\widehat{\widetilde{u}}(\xi)|^2\, d\xi = C\varepsilon \|\nabla \widetilde{u}\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, choosing $\varepsilon = (L_o)^{-1}$ in (3.3) and (3.4), and using a triangle inequality, we get (recall that $L_0 \geq 2$)

$$[\widetilde{u}]^2_{H^{1/2}(\mathbb{R}^d)} \leq 2[\widetilde{u}_\varepsilon]^2_{H^{1/2}(\mathbb{R}^d)} + 2[\widetilde{u}_\varepsilon - \widetilde{u}]^2_{H^{1/2}(\mathbb{R}^d)} \leq C \log L_o \int_{\mathbb{R}^d} |\nabla \widetilde{u}| \, dx.$$

Finally, we note that

$$\mathcal{E}^{\rm Sob}(\widetilde{u}; B_1) \leq \mathcal{E}^{\rm Sob}(\widetilde{u}; B_2) = [\widetilde{u}]^2_{H^{1/2}(\mathbb{R}^d)}$$

and that (cp. (3.1))

$$\int_{\mathbb{R}^d} |\nabla \widetilde{u}| \, dx \leq C + \int_{B_2} |\nabla u| \, dx.$$

Hence, recalling (3.2), we obtain

$$\mathcal{E}^{\rm Sob}(u; B_1) \leq C + C \log L_o \left(1 + \int_{B_2} |\nabla u| \, dx \right),$$

and the lemma follows. □

We can now prove Proposition 1.7.

Proof of Proposition 1.7. This proof is similar to the proof of Theorem 2.1 in [7] (see also the proof Theorem 1.7 in [12]). Here we need to use, as a new ingredient, the estimate from Lemma 3.1. Throughout the proof, C denotes a generic dimensional constant.

- *Step 1.* Let v be a stable solution $(-\Delta)^{1/2} v + g(v) = 0$ in B_3 satisfying $|v| \leq 1$ in all of $\mathbb{R}^d$.

First, using (2.4) in Lemma 2.1 and then Lemma 2.2 with $i = 0$ and $R^i = 1$, we obtain

$$\int_{B_{1/2}} |\nabla v| \, dx \leq C_d \left(1 + \sqrt{\mathcal{E}^{\rm Sob}(v; B_1)} \right) \tag{3.5}$$

for some dimensional constant $C_d > 0$. Note that this estimate is valid for every stable solution v, independently of the nonlinearity g.

On the other hand, note that if $\|g\|_{C^{0,\alpha}([-1,1])} \leq M_o$ for some $M_o \geq 2$, then by the interior regularity estimates for $(-\Delta)^{1/2}$ we have

$$\|\nabla v\|_{L^\infty(B_2)} \leq L_o := C M_o, \tag{3.6}$$

where C depends only on d and α. Therefore, combining (3.5) with Lemma 3.1, we obtain

$$\begin{aligned} \int_{B_{1/2}} |\nabla v| \, dx &\leq C \left(1 + \sqrt{C \log M_0 \left(1 + \int_{B_2} |\nabla v| \, dx \right)} \right) \\ &\leq \frac{C \log M_0}{\delta} + \delta \int_{B_2} |\nabla v| \, dx \qquad \forall\, \delta \in (0,1), \end{aligned} \tag{3.7}$$

where we used the inequality $2\sqrt{ab} \leq \delta a + b/\delta$ for $a, b \in \mathbb{R}_+$.

- *Step 2.* For v as in Step 1 and $B_\rho(z) \subset B_1$ we note that the function

$$\widetilde{v}(x) := v(z + 2\rho x)$$

satisfies $(-\Delta)^{1/2}\widetilde{v} + \widetilde{g}(\widetilde{v}) = 0$ with $\widetilde{g}(s) := 2\rho g(s)$. In particular $\|\widetilde{g}\|_{C^{0,\alpha}([-1,1])} \le 2\rho M_o \le 2M_o$, so estimate (3.7) applied to $\widetilde{v}$ yields

$$\int_{B_{1/2}} |\nabla \widetilde{v}|\, dx \le \frac{C\log(2M_0)}{\delta} + \delta \int_{B_2} |\nabla \widetilde{v}|\, dx,$$

or equivalently

$$\rho^{1-d}\int_{B_{\rho/4}(z)} |\nabla v|\, dx \le \frac{C\log(2M_0)}{\delta} + \delta\rho^{1-d}\int_{B_{\rho}(z)} |\nabla v|\, dx. \tag{3.8}$$

Hence taking δ small enough and using Lemma 2.3 with $\mathcal{S}(B) := \int_B |\nabla v|\, dx$ and $\beta := 1-d$, we obtain

$$\int_{B_1} |\nabla v|\, dx \le C\log M_0, \tag{3.9}$$

where C depends only on d and α. Also, it follows by Lemma 3.1 and (3.6) that

$$\mathcal{E}^{\mathrm{Sob}}(v, B_{1/2}) \le C\log^2 M_0. \tag{3.10}$$

- *Step 3.* If u is a stable solution u of $(-\Delta)^{1/2}u = f(u)$ in B_R, given $x_o \in B_{R/2}$ we consider the function $v(x) := u\big(x_o + \frac{R}{6}x\big)$. Note that this function satisfies (3.9) and (3.10) with M_0 replaced by M_0R, hence the desired estimates follow easily by scaling and a covering argument. □

4. Proof of Theorem 1.5

We are given u a bounded stable solution of $(-\Delta)^{1/2}u + f(u) = 0$ in $\mathbb{R}^3$ with $f \in C^{0,\alpha}$. Up to replacing u by $u/\|u\|_{L^\infty(\mathbb{R}^3)}$ and $f(s)$ by $f(\|u\|_{L^\infty(\mathbb{R}^3)}s)/\|u\|_{L^\infty(\mathbb{R}^3)}$, we can assume that $|u| \le 1$ and we want to show that u is 1D. We split the proof in three steps. Throughout the proof, C_f will denote a positive constant depending only on f.

- *Step 1.* By Proposition 1.7 we have

$$\mathcal{E}^{\mathrm{Sob}}(u; B_R) \le C_f R^2 \log^2 R \tag{4.1}$$

for all $R \ge 2$. Take $k \ge 1$, $R = 2^{2(k+1)}$, and $v(x) := u(Rx)$. Note that, by elliptic regularity, $\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le C_f$, thus $\|\nabla v\|_{L^\infty(\mathbb{R}^3)} \le C_f R$. Also, v is still a stable solution of a semilinear equation in all of $\mathbb{R}^3$. Hence, using (2.5) in Lemma 2.1 and then Lemma 2.2 with $i = 1$ and $R^i = R$, we obtain

$$\int_{B_{1/2}} |\nabla v|\, dx \le C\left(1 + \sqrt{\frac{1}{k}\sum_{j=1}^{k} \frac{\mathcal{E}^{\mathrm{Sob}}(v; B_{2^{k+j}})}{k2^{2(k+j)}}}\right) \tag{4.2}$$

for some universal constant C. On the other hand, using Lemma 3.1 and the bound $\|\nabla v\|_{L^\infty(\mathbb{R}^3)} \le C_f R$, we have

$$\frac{\mathcal{E}^{\mathrm{Sob}}(v; B_{1/4})}{\log R} \le C_f\left(1 + \int_{B_{1/2}} |\nabla v|\, dx\right). \tag{4.3}$$

Thus, recalling that $R = 2^{2(k+1)} = 4 \cdot 2^{2k}$, $v(x) = u(Rx)$, rewriting (4.2) and (4.3) in terms of u we get (here we use that $d = 3$)

$$
\begin{aligned}
\frac{\mathcal{E}^{\rm Sob}(u; B_{2^{2k}})}{k 2^{4k}} &\le C_f \left(1 + \sqrt{\frac{1}{k}\sum_{j=1}^{k} \frac{\mathcal{E}^{\rm Sob}(u; B_{2^{3k+j+2}})}{k 2^{2(k+j)} 2^{4(k+1)}}}\right) \\
&\le C_f \left(1 + \sqrt{\frac{1}{2k}\sum_{\ell=1}^{2k} \frac{\mathcal{E}^{\rm Sob}(u; B_{2^{2(k+\ell)}})}{(k+\ell) 2^{4(k+\ell)}}}\right) \qquad \forall\, k \ge 1.
\end{aligned}
\tag{4.4}
$$

- *Step 2.* Given $j \ge 1$ set

$$
\mathcal{A}(j) := \frac{\mathcal{E}^{\rm Sob}(u; B_{2^{2j}})}{j 2^{4j}},
$$

so that (4.4) can be rewritten as

$$
\mathcal{A}(k) \le C_f \left(1 + \sqrt{\frac{1}{2k}\sum_{\ell=1}^{2k} \mathcal{A}(k+\ell)}\right). \tag{4.5}
$$

We claim that

$$
\mathcal{A}(k) \le M \quad \text{for all } k \ge 1 \tag{4.6}
$$

for some constant M depending only on f.

Indeed, assume by contradiction that $\mathcal{A}(k_0) \ge M$ for some large constant M to be chosen later. Rewriting (4.5) as

$$
2c_f \mathcal{A}(k_0) - 1 \le \sqrt{\frac{1}{2k_0}\sum_{\ell=1}^{2k_0} \mathcal{A}(k_0+\ell)}, \qquad c_f := \frac{1}{2C_f},
$$

then, provided $M \ge \frac{1}{c_f}$, if $\mathcal{A}(k_0) \ge M$ we find

$$
c_f M \le \sqrt{\frac{1}{2k_0}\sum_{\ell=1}^{2k_0} \mathcal{A}(k_0+\ell)}.
$$

This implies that there exists $k_1 \in \{k_0+1, \dots, 3k_0\}$ such that

$$
c_f M \le \sqrt{\mathcal{A}(k_1)},
$$

that is

$$
(c_f M)^2 \le \mathcal{A}(k_1).
$$

Hence, choosing M large enough so that $\widetilde{M} := (c_f M)^2 \ge \frac{1}{c_f}$, we can repeat exactly the same argument as above with M replaced by $\widetilde{M}$ and k_0 replaced by k_1 in order to find $k_2 \in \{k_1+1, \dots, 3k_1\}$ such that

$$
(c_f \widetilde{M})^2 = c_f^6 M^4 \le \mathcal{A}(k_2).
$$

Iterating further we find $k_1 < k_2, < k_3 < \ldots < k_m < \ldots$ such that $k_{m+1} \le 3k_m$ and

$$
c_f^{2^{m+1}-2} M^{2^m} \le \mathcal{A}(k_m).
$$

Now, ensuring that M is large enough so that $\theta := c_f^2 M > 1$, we obtain

$$c_f^{-2}\theta^{2^m} \le \mathcal{A}(k_m). \tag{4.7}$$

On the other hand, recalling (4.1) and using that $k_m \le 3^m k_0$, we have

$$\mathcal{A}(k_m) \le C\,3^m k_0. \tag{4.8}$$

The exponential bound from (4.8) clearly contradicts the super-exponential growth in (4.7) for m large enough. Hence, this provides the desired contradiction and proves (4.6)

- *Step 3.* Rephrasing (4.6), we proved that

$$\mathcal{E}^{\mathrm{Sob}}(u; B_R) \le C_f R^2 \log R \tag{4.9}$$

for all $R \ge 2$. In other words, we have obtained an optimal energy estimate in large balls B_R (note that 1D profiles saturates (4.9)). Having improved the energy bound of Proposition 1.7 from $R^2 \log^2 R$ to (4.9), we now conclude that u is a 1D profile as follows.

Given $\boldsymbol{v} \in \mathbb{S}^{d-1}$ and using the perturbation $\mathcal{P}^2_{t,\boldsymbol{v}}$ as in (2.2)-(2.3), it follows by (2.6), (2.12), and (2.13) that

$$\iint_{B_{1/2}\times B_{1/2}} \big(\partial_{\boldsymbol{v}} u(x)\big)_+ \big(\partial_{\boldsymbol{v}} u(y)\big)_- \,dx\,dy \le \frac{C_f}{\log\log R}.$$

Hence, taking the limit as $R \to \infty$ we find that

$$\iint_{B_{1/2}\times B_{1/2}} \big(\partial_{\boldsymbol{v}} u(x)\big)_+ \big(\partial_{\boldsymbol{v}} u(y)\big)_- \,dx\,dy = 0,$$

thus

$$\text{either} \qquad \partial_{\boldsymbol{v}} u \ge 0 \quad \text{in } B_{1/2} \qquad \text{or} \qquad \partial_{\boldsymbol{v}} u \le 0 \quad \text{in } B_{1/2} \qquad \forall\, \boldsymbol{v} \in \mathbb{S}^{d-1}.$$

Since this argument can be repeated changing the center of the ball $B_{1/2}$ with any other point, by a continuity argument we obtain that

$$\text{either} \qquad \partial_{\boldsymbol{v}} u \ge 0 \quad \text{in } \mathbb{R}^3 \qquad \text{or} \qquad \partial_{\boldsymbol{v}} u \le 0 \quad \text{in } \mathbb{R}^3 \qquad \forall\, \boldsymbol{v} \in \mathbb{S}^{d-1}.$$

Thanks to this fact, we easily conclude that u is a 1D monotone function, as desired. □

References

[1] G. Alberti, G. Bouchitté, and S. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. 144 (1998), 1–46.

[2] G. Alberti, L. Ambrosio, and X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday, Acta Appl. Math. 65 (2001), 9–33.

[3] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. 13 (2000), 725–739.

[4] H. Berestycki, L. Caffarelli, and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, dedicated to Ennio De Giorgi, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 25 (1997), 69–94.

[5] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969), 243–268.

[6] X. Cabré and E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. 28 (2010), 1179–1206.

[7] X. Cabré, E. Cinti, and J. Serra, *Stable nonlocal phase transitions*, preprint.

[8] X. Cabré and J. Solá Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. 58 (2005), 1678–1732.

[9] L. Caffarelli and A. Cordoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. 48 (1995), 1–12.

[10] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), 1245–1260.

[11] O. Chodosh and C. Mantoulidis, *Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates*, preprint, `https://arxiv.org/abs/1803.02716`.

[12] E. Cinti, J. Serra, and E. Valdinoci, *Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces*, J. Diff. Geom., to appear.

[13] J. Dávila, M. del Pino, M. Musso, *Bistable boundary reactions in two dimensions.* Arch. Ration. Mech. Anal. 200 (2011), no. 1, 89-140.

[14] E. de Giorgi, *Frontiere orientate di misura minima* (Italian), Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61 Editrice Tecnico Scientifica, Pisa 1961.

[15] E. de Giorgi, *Una estensione del teorema di Bernstein* (Italian), Ann. Scuola Norm. Sup. Pisa (3) 19 (1965), 79–85.

[16] E. de Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188.

[17] M. A. M. Guaraco, Min–max for phase transitions and the existence of embedded minimal hypersurfaces, J. Differential Geom. 108 (2018), no. 1, 91–133.

[18] M. A. M. Guaraco and P. Gaspar, *The Weyl Law for the phase transition spectrum and density of limit interfaces*, Geom. Funct. Anal. 29 (2019), no. 2, 382–410.

[19] M. del Pino, M. Kowalczyk, and J. Wei, *A conjecture by de Giorgi in large dimensions*, Ann. of Math. 174 (2011), 1485–1569.

[20] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), 903–906.

[21] A. Figalli and D. Jerison, *How to recognize convexity of a set from its marginals*, J. Funct. Anal. 266 (2014), no. 3, 1685–1701.

[22] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Maths. 33 (1980), 199–211.

[23] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311 (1998), 481–491.

[24] A. Garroni and S. Müller, *Γ-limit of a phase-field model of dislocation*, SIAM J. Math. Anal. 36 (2005), no. 6, 1943–1964.

[25] K. Irie, F. C. Marques, and A. Neves, *Density of minimal hypersurfaces for generic metrics*, Ann. of Math. (2), 187 (2018), no. 3, 963–972.

[26] M. Kurzke, *Boundary vortices in thin magnetic films.* Calc. Var. Partial Differential Equations 26 (2006), no. 1, 1–28.

[27] M.d.M. González, *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Part. Diff. Eq. 36 (2009), 173–210.

[28] L. Modica, *Γ-convergence to minimal surfaces problems and global solutions of $\Delta u = 2(u^3 - u)$*, Proc. of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 223—244, Pitagora, Bologna, 1979.

[29] L. Modica and S. Mortola, *Un esempio di Γ-convergenza* (Italian), Boll. Un. Mat. Ital. B (5) 14 (1977), 285–299.

[30] F.R.N. Nabarro, *Dislocations in a simple cubic lattice* Proc. Phys. Soc., 59 (1947), 256–272.

[31] R. Peierls, *The size of a dislocation*, Proc. Phys. Soc., 52 (1940), 34–37.

[32] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. (2) 169 (2009), 41–78.

[33] O. Savin, *Rigidity of minimizers in nonlocal phase transitions*, Anal. PDE 11 (2018), no. 8, 1881–1900.

[34] O. Savin, *Rigidity of minimizers in nonlocal phase transitions II*, Anal. Theory Appl. 35 (2019), no. 1, 1–27.

[35] O. Savin and E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations 48 (2013), 33–39.

[36] O. Savin and E. Valdinoci, *Density estimates for a variational model driven by the Gagliardo norm*, J. Math. Pures Appl. 101 (2014), 1–26.

[37] O. Savin and E. Valdinoci, *Some monotonicity results for minimizers in the calculus of variations*, J. Funct. Anal. 264 (2013), 2469–2496.

[38] L. Simon, *Schauder estimates by scaling*, Calc. Var. Partial Differential Equations 5 (1997), 391–407.

[39] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968), 62–105.

[40] J. F. Toland, *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. 145 (1997), 136–150.

[41] K. Wang and J. Wei, *Finite Morse index implies finite ends*, Comm. Pure Appl. Math. 72 (2019), no. 5, 1044–1119.

[42] B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. 88 (1987), pp. 24–256.

ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland.

Email address: alessio.figalli@math.ethz.ch

ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland.

Email address: joaquim.serra@math.ethz.ch