ON THE ISOPERIMETRIC PROBLEM FOR RADIAL LOG-CONVEX DENSITIES

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ABSTRACT. Given a smooth, radial, uniformly log-convex density e^V on $\mathbb{R}^n$, $n \ge 2$, we characterize isoperimetric sets E with respect to weighted perimeter $\int_{\partial E} e^V d\mathcal{H}^{n-1}$ and weighted volume $m = \int_E e^V$ as balls centered at the origin, provided $m \in [0, m_0)$ for some (potentially computable) $m_0 > 0$; this affirmatively answers conjecture [RCBM, Conjecture 3.12] for such values of the weighted volume parameter. We also prove that the set of weighted volumes such that this characterization holds true is open, thus reducing the proof of the full conjecture to excluding the possibility of bifurcation values of the weighted volume parameter. Finally, we show the validity of the conjecture when V belongs to a C^2-neighborhood of $c|x|^2$ ($c > 0$).

1. Introduction

1.1. **Background.** Isoperimetric problems in a space with density, a natural generalization of the classical Gaussian isoperimetric problem [Bo1, SC, Eh, CK, CFMP], have received an increasing attention in recent years; see [BBMP, Bo2, CJQW, CMV, DDNT, DHHT, FuMP2, KZ, RCBM, MS, MM, MP]. We refer the reader to [Mo1] for a quick excursion into the theory of manifolds with density.

As to now, very little is known about the isoperimetric problem with general densities. We consider here a quite basic open question, which can been introduced through an elementary analysis of first and second variation formulae. Precisely, denoting by E an open set with smooth boundary in $\mathbb{R}^n$, let us consider the isoperimetric-type problem

$$\phi_V(m) = \inf\left\{\int_{\partial E} e^V \, d\mathcal{H}^{n-1} : \int_E e^V = m\,, \quad E \subset \mathbb{R}^n\right\}, \qquad m > 0\,, n \ge 2\,, \tag{1.1}$$

associated to a positive density e^V on $\mathbb{R}^n$, with $V \colon \mathbb{R}^n \to \mathbb{R}$ radially increasing, that is

$$V(x) = v(|x|)\,, \quad \text{for } v\colon (0,\infty) \to \mathbb{R} \text{ increasing}\,. \tag{1.2}$$

The naive intuition that balls centered at the origin should be the only isoperimetric sets (minimizers) in (1.1) is not correct. Indeed, as shown in [DDNT], if $n = 2$ and $e^V = |x|^p$ (i.e. $v(r) = p\log(r)$, $p > 0$), then isoperimetric sets are Euclidean disks whose boundaries pass through the origin. By computing first and second variations in (1.1), one sees that every isoperimetric set E with boundary of class C^2 satisfies the stationarity condition

$$H_E^V = H_E + \nabla V \cdot \nu_E = \text{constant} \quad \text{on } \partial E\,, \tag{1.3}$$

and, for every $u \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\partial E} u\, e^V \, d\mathcal{H}^{n-1} = 0$, the stability inequality

$$\int_{\partial E} \left(|\nabla_E u|^2 - |\mathrm{II}_E|^2 u^2\right) e^V \, d\mathcal{H}^{n-1} + \int_{\partial E} \nabla^2 V(\nu_E, \nu_E)\, u^2 \, e^V \, d\mathcal{H}^{n-1} \ge 0 \tag{1.4}$$

holds. (Here, H_E denotes the mean curvature of ∂E computed with respect to the outer unit normal ν_E to E, $\nabla_E u = \nabla u - (\nabla u \cdot \nu_E)\nu_E$ is the tangential gradient of u with respect to ∂E, and II_E is the second fundamental form of ∂E. Our convention is that H_E is the trace of II_E, so that $H_B = n - 1$ if B is the Euclidean unit ball of $\mathbb{R}^n$.)

In particular, balls $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ centered at the origin always satisfy the stationarity condition (1.3), with

$$H^V_{B_r} = \frac{n-1}{r} + v'(r) \quad \text{on } \partial B_r\,. \tag{1.5}$$

On the other hand, B_r satisfies the stability inequality (1.4) if and only if $v''(r) \geq 0$. Indeed, B_r satisfies (1.4) if and only if

$$e^{v(r)} \int_{\partial B_r} \Big(|\nabla_{B_r} u|^2 - \frac{n-1}{r^2}\, u^2 \Big)\, d\mathcal{H}^{n-1} + v''(r) e^{v(r)} \int_{\partial B_r} u^2\, d\mathcal{H}^{n-1} \geq 0\,, \tag{1.6}$$

for every $u \in C^\infty_c(\mathbb{R}^n)$ such that $0 = e^{v(r)} \int_{\partial B_r} u\, d\mathcal{H}^{n-1}$, i.e., with $\int_{\partial B_r} u\, d\mathcal{H}^{n-1} = 0$. Then the stability of balls in the Euclidean isoperimetric problem implies of course that

$$\int_{\partial B_r} \Big(|\nabla_{B_r} u|^2 - \frac{n-1}{r^2}\, u^2 \Big)\, d\mathcal{H}^{n-1} \geq 0\,, \tag{1.7}$$

whenever $u \in C^\infty_c(B_r)$ and $\int_{\partial B_r} u\, d\mathcal{H}^{n-1} = 0$, with equality if and only if $u = x \cdot e$ for some $e \in \mathbb{R}^n$ (this corresponds to an infinitesimal translation in the direction e). Taking this into account, one easily sees that (1.6) holds true on B_r if and only if $v''(r) \geq 0$.

Hence, in the spirit of [RCBM, Conjecture 3.12], we are naturally led to formulate the following *isoperimetric log-convex density conjecture*: for radially *increasing* log-convex densities e^V on $\mathbb{R}^n$, balls centered at the origin are isoperimetric sets. We shall refer to the *strong* form of the conjecture as to the claim that balls centered at the origin are the *unique* isoperimetric sets. Note that the assumption of v being increasing is somehow necessary for the validity of this conjecture; for example, as noticed by Morgan [Mo2], if $v(r) = (r-1)^2$, then isoperimetric sets with sufficiently small weighted volume have to be uniformly close to a point on $\mathbb{S}^{n-1}$ (a rigorous justification of this assertion can be derived, for example, using the arguments from sections 5 and 6 below). For the sake of clarity, let us also recall that the existence of isoperimetric sets was proved under the assumption of the conjecture in [RCBM, Theorem 2.1], and, more generally, whenever v is increasing with $v(r) \to +\infty$ as $r \to \infty$ in [MP, Theorem 3.3].

The validity of the isoperimetric log-convex density conjecture has been supported, up to date, by the following results. For $v(r) = c\,r^2$, $c > 0$, the conjecture was proved (before its formulation) by Borell [Bo2, Theorem 4.12] through a symmetrization argument. In this case, balls centered at the origin are actually the only isoperimetric sets [RCBM, Theorem 5.2]. This case is somehow special because the densities $a\, e^{c\,|x|^2}$, $a, c > 0$, can be characterized by the property that the natural notion of Schwartz symmetrization preserving the *weighted volume* $\mathrm{Vol}(E)$ decreases the *weighted perimeter* $\mathrm{Per}(E)$, where

$$\mathrm{Vol}(E) = \int_E e^V\,, \qquad \mathrm{Per}(E) = \int_{\partial E} e^V\, d\mathcal{H}^{n-1}\,;$$

see [BBMP, Theorem 3.8]. In [MM], the conjecture has been proved for $n = 2$ and $v(r) = c\,r^p$, $p \geq 2$, $c > 0$. A stronger evidence in favor of the conjecture has been provided by Kolesnikov and Zhdanov [KZ] through an enlightening argument based on the divergence theorem. They show that if v is increasing with $v'' \geq \alpha > 0$ on $(0, \infty)$, then there exists $\bar m > 0$ such that, for any $m > \bar m$, balls centered at the origin are the only isoperimetric sets with weighted volume m.

1.2. **Main results.** Before stating our main results, we introduce some useful notation: given $m > 0$, we denote by $r(m) > 0$ the radius such that the ball $B(m) = B_{r(m)}$ satisfies

$$\mathrm{Vol}(B(m)) = m\,; \tag{1.8}$$

clearly $r(m)$ is uniquely determined as soon as, for example, v is bounded from below on $[0,\infty)$. Moreover, we denote by $M(v)$ the set of those $m>0$ such that $B(m)$ is the unique isoperimetric set with weighted volume m.

Theorem 1.1. *If $v\in C^2([0,\infty);[0,\infty))$ is an increasing convex function with*

$$\inf_{[0,r]} v''>0 \qquad \forall\, r>0\,, \tag{1.9}$$

then the following two assertions hold true:

(i) *$M(v)$ is open;*

(ii) *there exists $m_0>0$, depending on n and v only, such that $(0,m_0)\subset M(v)$ (the value of m_0 is potentially computable; see Remark 1.6).*

Remark 1.1 (Bifurcation and proof of the complete conjecture)**.** By Theorem 1.1 (resp. by the above mentioned result of Kolesnikov and Zhdanov, if $\inf_{(0,\infty)} v''>0$) we may reduce the proof of the conjecture to showing that no bifurcation phenomena can occur. More precisely, to prove the conjecture one should show the validity of the following statement:

If $\tilde m>0$ and $(0,\tilde m)\subset M(v)$ (resp. $(\tilde m,\infty)\subset M(v)$), then $\tilde m\in M(v)$.

In other words, we would need to exclude the existence of $\tilde m>0$ such that both $B(\tilde m)$ and $E\neq B(\tilde m)$ are isoperimetric sets with weighted volume $\tilde m$, but $B(m)$ is the only isoperimetric set with weighted volume m for every $m<\tilde m$ (resp. $m>\tilde m$).

Combining Theorem 1.1-(ii), the validity of the conjecture for $e^{c\,|x|^2}$ ($c>0$) [Bo2, RCBM], Kolesnikov-Zhdanov's Theorem [KZ, Proposition 6.7], and a variant of the argument used in the proof of Theorem 1.1-(i), we shall prove our second main result, namely, the validity of the conjecture on every density e^V lying in a sufficiently small C^2-neighborhood of $e^{c\,|x|^2}$ (see Theorem 1.2 below). An interesting consequence of this result is that it shows that the validity of the conjecture for all the weighted volumes is not a completely exceptional feature related to the special tensorial structure of $e^{c\,|x|^2}$.

We now state our theorem, introducing the following notation: given $w\in C^0([0,\infty))$ and $R>0$, we shall denote by $\Omega[w](R,\cdot)$ the modulus of continuity of w over $[0,R]$, defined as

$$\Omega[w](R,\sigma)=\sup\Big\{|w(r)-w(s)|: r,s\in[0,R]\,,|r-s|<\sigma\Big\}\,,\qquad \sigma>0\,. \tag{1.10}$$

Observe that, by the local uniform continuity of w, $\Omega[w](R,\sigma)\to 0$ as $\sigma\to 0^+$.

Theorem 1.2. *Given $n\ge 2$, $c>0$, and a function $\Omega_0:[0,\infty)\times[0,\infty)\to[0,\infty)$ with $\Omega_0(R,0^+)=0$ for every $R>0$, there exists a positive constant δ, depending on n, c, and Ω_0 only, with the following property: if $v\in C^2([0,\infty);[0,\infty))$ is an increasing convex function with*

$$\|v-c\,r^2\|_{C^2([0,\infty))}<\delta\,,\qquad \Omega[v''](R,\sigma)\le\Omega_0(R,\sigma)\,,\qquad \forall\, R,\sigma>0\,, \tag{1.11}$$

then for every $m>0$ the ball $B(m)$ is the unique isoperimetric set in (1.1), that is, $M(v)=(0,\infty)$.

Before entering into a closer description of the strategy of proof of Theorem 1.1 we make the following three remarks.

Remark 1.2 (Densities and Euclidean geometry)**.** Theorem 1.1-(ii) may resemble for certain aspects the main results appearing in our previous paper [FM]. Indeed, in both cases, the starting point of our arguments is exploiting the smallness of the "mass parameter" in combination with the quantitative isoperimetric inequality [FuMP, FiMP, CL] to deduce the L^1-proximity of minimizers to balls. However, apart from this similarity, the two problems (and, consequently, the remaining parts of the proofs of our theorems) are completely different. In particular, what makes the study of this problem extremely delicate is the elusive interaction between geometric quantities such as the mean curvature of E, or the integrand $|\nabla_E u|^2 - |\mathrm{II}_E|^2 u^2$ in the second variation of the Euclidean perimeter of E (see (1.6) and (1.7)), with the density e^V. This point is understood, for example, by noticing that the stationarity condition (1.3) does not possess any scaling property; or realizing that the natural notion of Schwarz symmetrization which preserves weighted volume does not decrease weighted perimeter, unless $v(r) = c\,r^2$, $c > 0$. If these features of the problem make unlikely its solution by symmetrization techniques, it should also be noted that, at present, no characterization results for isoperimetric sets in problems with density have been obtained via mass transportation techniques; and this is true even in the case of the well-studied Gaussian isoperimetric problem, corresponding to $v(r) = -c\,r^2$, $c > 0$. The proof of the characterization result of isoperimetric sets stated in Theorem 1.1-(ii) and Theorem 1.2 is thus quite atypical in its genre: indeed, we will manage to prove a global minimality property by combining tools such as the (global) quantitative Euclidean isoperimetric inequality, strict stability properties of candidate minimizers in the problem with density (obtained in section 2), and improved convergence theorems for sequences of almost-minimizers of Euclidean perimeter (almost-minimizers are defined in (3.6), and related results are discussed in section 3).

Remark 1.3 (Global stability inequalities)**.** As mentioned above, in proving Theorems 1.1 and 1.2 we shall establish several local stability results, including in particular a stability result for the ball $B(m)$ with respect to its small C^1-perturbations; see Theorem 2.3. Therefore, by using a selection principle in the spirit of [CL, AFM, DM], it may be possible to deduce from our results global stability inequalities for radial uniformly log-convex densities e^V. We will not investigate further this direction since it does not seem to cast further light on the isoperimetric log-convex density conjecture, whose understanding is our primary interest here.

Remark 1.4 (Perturbation principle)**.** The perturbation argument behind Theorem 1.2 can be suitably adapted to show that, loosely speaking, if v is an increasing, uniformly convex function for which the strong conjecture holds at weighted volume m (i.e., $m \in M(v)$), then the validity of the conjecture "propagates" to any w close in C^2 to v (again with a uniform bound of the modulus of continuity of w''), for any value $\tilde{m}$ sufficiently close to m. However, since the main explicit example of the validity of the conjecture is obtained by setting $v(r) = c\,r^2$, we have decided to focus on Theorem 1.2 rather than stating a more abstract result.

Notation 1. Although the isoperimetric problem (1.1) can be directly formulated on open sets with smooth boundary, the discussion of the existence and regularity properties of isoperimetric sets requires passing through a generalized formulation of the problem. Referring readers to [AFP, Ma] for the technical details (which will play a very marginal role in our arguments), we shall work here in the framework of the theory of sets of finite perimeter. In particular, given a set of locally finite perimeter $E \subset \mathbb{R}^n$, we shall denote by $|E|$ its Lebesgue measure, by $\partial^* E$ its reduced boundary, by $\partial^{1/2} E$ the set of points of density one-half of E (recall that $\partial^* E \subset \partial^{1/2} E$ and $\mathcal{H}^{n-1}(\partial^{1/2} E \setminus \partial^* E) = 0$, so one can interchangeably use the two sets when integrating with respect to $d\mathcal{H}^{n-1}$), by $P(E;F) = \mathcal{H}^{n-1}(F \cap \partial^* E)$ the distributional perimeter of E relative to the Borel set $F \subset \mathbb{R}^n$, and we shall set for brevity

$$\mathrm{Per}(E;F) = \int_{F \cap \partial^* E} e^V \, d\mathcal{H}^{n-1} \,,$$

for the weighted perimeter of E relative to F as well. By standard approximation theorems, the value of $\phi_V(m)$ in (1.1) is unaffected if we minimize among sets of locally finite perimeter, or among open sets with smooth or Lipschitz or polyhedral boundary.

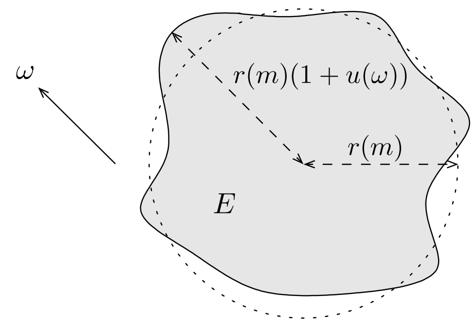

FIGURE 1. A set E defined by $u \in C^1(\mathbb{S}^{n-1};[-1,\infty))$ as in (1.14).

1.3. **Strategy of proof, Theorem 1.1-(i).** We now pass to describe the strategy of proof of Theorem 1.1, starting from statement (i), and introducing the family of isoperimetric sets of weighted volume m, defined as

$$\mathcal{M}_V(m) = \Big\{ E \subset \mathbb{R}^n : \mathrm{Per}(E) = \phi_V(m) \Big\} . \tag{1.12}$$

To prove Theorem 1.1-(i), we first show that for every $m_2 > m_1 > 0$ there exists a positive constant ε (depending on n, m_1, m_2, and v only) such that, if $\gamma = \inf_{[0,r(m_2)]} v'' > 0$, then

$$\mathrm{Per}(E) \ge \mathrm{Per}(B(m)) \Big\{ 1 + \frac{\gamma\, r(m_1)^2}{4} \fint_{\mathbb{S}^{n-1}} u^2 \Big\} , \tag{1.13}$$

whenever

$$E = \Big\{ t\,\Big(1 + u(\omega)\Big)\,\omega : \omega \in \mathbb{S}^{n-1}\,, 0 \le t < r(m) \Big\} , \qquad m = \mathrm{Vol}(E) \in (m_1, m_2)\,, \tag{1.14}$$

for some

$$u \in C^1(\mathbb{S}^{n-1};[-1,\infty))\,, \qquad \|u\|_{C^1(\mathbb{S}^{n-1})} < \varepsilon\,;$$

see Theorem 2.3 in section 2, and Figure 1. This implies in particular that balls centered at the origin are the unique isoperimetric sets in the restricted competition class of their small C^1-perturbations (a result which seems to be new in itself).

We next argue by contradiction; that is, we assume the existence of $m > 0$ such that $\mathcal{M}_V(m) = \{B(m)\}$, and of sequences $\{m_h\}_{h\in\mathbb{N}}$ and $\{E_h\}_{h\in\mathbb{N}}$ with $m_h \to m$ as $h \to \infty$, $E_h \in \mathcal{M}_V(m_h)$, and $|E_h \Delta B(m_h)| > 0$ for every $h \in \mathbb{N}$. Exploiting the minimality of E_h we show that, up to extracting a subsequence, $|E_h \Delta B(m)| \to 0$ as $h \to \infty$. At the same time, the minimality of the E_h in the global isoperimetric problem with density (1.1) implies in turn their (uniform) local *almost-minimality* with respect to the Euclidean perimeter: precisely, there exist positive constants C and r_0 such that

$$P(E_h; B(x,r)) \le P(F; B(x,r)) + C\, r^n\,,$$

whenever $h \in \mathbb{N}$ and $E_h \Delta F \subset\subset B(x,r)$ for some $x \in \mathbb{R}^n$ and $r \le r_0$. In particular, $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of uniform almost-minimizers of the perimeter in $\mathbb{R}^n$ which converges in L^1 to a smooth limit set, namely, $B(m)$. Then, the regularity theory for almost-minimizers of the perimeter (see section 3) implies the existence of a sequence $\{\hat{u}_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{S}^{n-1};[-1,\infty))$ such that

$$E_h = \Big\{ t\,\Big(1 + \hat{u}_h(\omega)\Big)\omega : \omega \in \mathbb{S}^{n-1}\,, 0 \le t < r(m) \Big\} , \qquad \lim_{h\to\infty} \|\hat{u}_h\|_{C^1} = 0\,.$$

(Here, $\|\cdot\|_{C^1} = \|\cdot\|_{C^1(\mathbb{S}^{n-1})}$.) Equivalently, since $m_h \to m$ there exists $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{S}^{n-1};[-1,\infty))$ such that

$$E_h = \Big\{ t\,\Big(1 + u_h(\omega)\Big)\omega : \omega \in \mathbb{S}^{n-1}\,, 0 \le t < r(m_h) \Big\} , \qquad \lim_{h\to\infty} \|u_h\|_{C^1} = 0\,.$$

Hence, for h sufficiently large we can apply (1.13) to conclude

$$\mathrm{Per}(E_h) \geq \mathrm{Per}(B(m_h))\left\{1 + \frac{\gamma\, r(m_1)^2}{4} \fint_{\mathbb{S}^{n-1}} u_h^2\right\}.$$

Since none of the E_h's is a ball, we have $u_h \not\equiv 0$ for every $h \in \mathbb{N}$; hence $\mathrm{Per}(E_h) > \mathrm{Per}(B(m_h))$, against $E_h \in \mathcal{M}_V(m_h)$.

Remark 1.5. Notice that, having used a compactness argument (together with the assumption $m \in M(v)$) to deduce that $|E_h \Delta B(m)| \to 0$ as $h \to \infty$, we have no information on the size of the neighborhood of m contained in $M(v)$.

1.4. **Strategy of proof, Theorem 1.1-(ii).** The strategy of proof is somehow similar to that of statement (i), although quite subtler in several aspects. Consider $E \in \mathcal{M}_V(m)$ for m small. By the quantitative isoperimetric inequality [FiMP], by the above mentioned regularity theory for almost-minimizers of the perimeter, and thanks to some uniform decay estimates for the diameter of E that we shall prove in section 5, we deduce that, if m is small enough, then there exist $\hat{u} \in C^1(\mathbb{S}^{n-1}; [-1, \infty))$ and a ball $B(\hat{x}_0, \hat{r})$ such that

$$E = \hat{x}_0 + \left\{t\left(1 + \hat{u}(\omega)\right)\omega : \omega \in \mathbb{S}^{n-1}, 0 \leq t < \hat{r}\right\}, \qquad \lim_{m \to 0} |\hat{x}_0| + \|\hat{u}\|_{C^1} + \hat{r} = 0.$$

Two major difficulties arise at this point. First, even in the case that $\hat{x}_0 = 0$, we cannot derive a contradiction directly from (1.13), as the constant ε defining the range of applicability of (1.13) depends on m, and may be smaller than $\|\hat{u}\|_{C^1}$. Second, it may actually be that $\hat{x}_0 \neq 0$, and having no information on the relative sizes of $|\hat{x}_0|$ and $\hat{r}$, we do not know if it is possible to parameterize E over $B(m)$ through C^1-small functions (actually, it may even be possible that E does not contain the origin).

The key idea here is that of parameterizing the sets E with respect to the ball $B(x_0, r)$ having the same weighted volume and "weighted barycenter"; precisely, we prove the existence of $x_0 \in \mathbb{R}^n$, $r > 0$, and $u \in C^1(\mathbb{S}^{n-1}; [-1, \infty))$, with

$$\mathrm{Vol}(B(x_0, r)) = m, \qquad \int_{B(x_0, r)} x\, e^{V(x)}\, dx = \int_E x\, e^{V(x)}\, dx, \tag{1.15}$$

and

$$E = x_0 + \left\{t\left(1 + u(\omega)\right)\omega : \omega \in \mathbb{S}^{n-1}, 0 \leq t < r\right\}, \qquad \lim_{m \to 0} |x_0| + \|u\|_{C^1} + r = 0.$$

Exploiting the matching of weighted barycenters in (1.15), we are able to take advantage of the *Euclidean* stability of $B(x_0, r)$ to show that if $\|u\|_{C^1} + |x_0| < \varepsilon_0$, then

$$\mathrm{Per}(E) \geq \mathrm{Per}(B(x_0, r))\left\{1 - C\, r\, |x_0| \fint_{\mathbb{S}^{n-1}} |u| + \frac{1}{C} \fint_{\mathbb{S}^{n-1}} u^2\right\}, \tag{1.16}$$

where C and ε_0 are *independent of* m; see Theorem 2.1, inequality (2.6), and the proof of Theorem 2.5; notice also the presence of a negative term of order one in (1.16), which reflects the non-stationarity of balls not centered at the origin in (1.1) in the isoperimetric problem with radially symmetric density. By a further analysis of the behavior of weighted perimeter on balls, we see that if $|x_0| \leq \varepsilon_1$, then

$$\mathrm{Per}(B(x_0, r)) \geq \mathrm{Per}(B(m))\left\{1 + \frac{|x_0|^2}{C}\right\}, \tag{1.17}$$

where ε_1 and C are, once again, independent of m; see Theorem 2.4. In conclusion, for m small enough, combining (1.16) and (1.17) with the elementary inequality

$$r\, |x_0| \fint_{\mathbb{S}^{n-1}} |u| \leq \frac{r}{2}\, |x_0|^2 + \frac{r}{2} \fint_{\mathbb{S}^{n-1}} |u|^2, \tag{1.18}$$

we deduce that

$$\mathrm{Per}(E) \geq \mathrm{Per}(B(m))\Big\{1 + \frac{1}{2C}\Big(|x_0|^2 + \fint_{\mathbb{S}^{n-1}} u^2\Big)\Big\}.$$

Since $E \in \mathcal{M}_V(m)$, this implies $x_0 = 0$ and $u = 0$, thus $E = B(m)$, as desired.

Remark 1.6. The value m_0 appearing in Theorem 1.1-(ii) is explicitly computable. Indeed, no compactness argument is ever used in the proof, the constant from the quantitative isoperimetric inequality in [FiMP] is explicit, and all constants appearing in the theory of almost-minimizers of perimeter from [T2, T1], as well as those appearing in the various other steps of the proof outlined above, are (in principle) computable.

1.5. **Organization of the paper.** In section 2 we gather the various stability estimates needed in the proof of Theorem 1.1 and Theorem 1.2, and show in particular that for every $m > 0$, $B(m)$ is the unique isoperimetric set among its small C^1-perturbations. In section 3 we prove several regularity, symmetry, boundedness, and almost-minimality properties of isoperimetric sets which are needed to apply the results proved in section 2 to our problem; this is done without needing the convexity of v. In section 4 we prove Theorem 1.1-(i). Then, after showing a quantitative decay estimate on the diameters of isoperimetric sets in the small weighted volume regime (see section 5), in section 6 we complete the proof of Theorem 1.1-(ii). Finally, in section 7, we prove Theorem 1.2.

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2. Some quantitative stability properties of balls centered at the origin

This section is devoted to the proof of various inequalities expressing in a quantitative way the stability of balls centered at the origin among special families of comparison sets; these inequalities are obtained, of course, under suitable uniform convexity assumptions on v. The key results are: Theorem 2.3, showing in particular that balls centered at the origin are isoperimetric sets among their C^1-small radial perturbations; Theorem 2.4, where we prove a stability (uniform with respect to the weighted volume parameter) of balls centered at the origin among balls whose centers are sufficiently close to the origin; and Theorem 2.5, where the stability of balls centered at the origin among C^1-small radial perturbations of balls whose weighted barycenters are sufficiently close to the origin, is again proved to be uniform in the small weighted volume parameter.

2.1. **Two basic lower-bounds.** We start our analysis of local stability properties by proving the two basic lower bounds on the weighted perimeter of a C^1-small perturbation of a ball. The first result, Theorem 2.1, is concerned with C^1-small perturbations of balls centered at the origin; the second result, Theorem 2.2, deals with balls not centered at the origin. Note that, in Theorem 2.1, v is not required to be increasing.

Theorem 2.1. *Given $n \geq 2$, non-negative constants α, β, and γ, and $r_2 \geq r_1 > 0$, there exist*

$$\varepsilon_0 = \varepsilon_0(n, \alpha, \beta, \gamma) \in \Big(0, \frac{1}{2}\Big), \qquad C_0 = C_0(n, \alpha, \beta, \gamma, r_1, r_2) < \infty,$$

with the following property: If $v : (0, \infty) \to (0, \infty)$ is twice differentiable with

$$\alpha \geq |v'|, \qquad \beta \geq v'' \geq \gamma, \qquad \text{on} \quad \big[(1-\varepsilon_0)\, r_1, (1+\varepsilon_0)\, r_2\big], \tag{2.1}$$

and if $r>0$, $u\in C^1(\mathbb{S}^{n-1};[-1,\infty))$, *and*

$$E=\Big\{t\Big(1+u(\omega)\Big)\omega:\omega\in\mathbb{S}^{n-1}\,,0\le t<r\Big\}\,,$$

are such that

$$r\in[r_1,r_2]\,,\qquad \|u\|_{C^1(\mathbb{S}^{n-1})}\le\varepsilon_0\,,\qquad \mathrm{Vol}(E)=\mathrm{Vol}(B_r)\,,\tag{2.2}$$

then

$$\begin{aligned}\mathrm{Per}(E)\ge\mathrm{Per}(B_r)\Big\{1+\Big(1-C_0\|u\|_{C^1}\Big)\frac{\gamma\,r^2}{2}\fint_{\mathbb{S}^{n-1}}u^2\\ +\fint_{\mathbb{S}^{n-1}}\Big(1-C_0\|u\|_{C^1}\Big)\frac{|\nabla u|^2}{2}-\Big(n-1+C_0\|u\|_{C^1}\Big)\frac{u^2}{2}\Big\}\,.\end{aligned}\tag{2.3}$$

Here $\|u\|_{C^1}=\|u\|_{C^1(\mathbb{S}^{n-1})}$, ∇u *is the tangential gradient of* u *on* $\mathbb{S}^{n-1}$, *and integration over* $\mathbb{S}^{n-1}$ *is with respect to* $\mathcal{H}^{n-1}$.

Theorem 2.2. *Given* $n\ge2$, *non-negative constants* α, β, *and* γ, *and* $r_0>0$, *there exist*

$$\varepsilon_0=\varepsilon_0(n,\alpha,\beta,\gamma)\in\Big(0,\frac12\Big)\,,\qquad C_0=C_0(n,\alpha,\beta,\gamma,r_0)<\infty\,,$$

with the following property: If $v:(0,\infty)\to(0,\infty)$ *is twice differentiable with*

$$\alpha\ge v'\ge0\,,\qquad\beta\ge v''\ge\gamma\,,\qquad\text{on}\quad[0,2\,r_0]\,,\tag{2.4}$$

and if $x_0\in\mathbb{R}^n$, $r>0$, $u\in C^1(\mathbb{S}^{n-1};[-1,\infty))$, *and*

$$E=x_0+\Big\{t\Big(1+u(\omega)\Big)\omega:\omega\in\mathbb{S}^{n-1}\,,0\le t<r\Big\}\,,$$

are such that

$$|x_0|\le\frac{r_0}{2}\,,\qquad r\le r_0\,,\qquad\|u\|_{C^1(\mathbb{S}^{n-1})}\le\varepsilon_0\,,\qquad\mathrm{Vol}(E)=\mathrm{Vol}(B(x_0,r))\,,\tag{2.5}$$

then

$$\begin{aligned}\mathrm{Per}(E)\ge\mathrm{Per}(B(x_0,r))\Big\{1-C_0\,r\,|x_0|\fint_{\mathbb{S}^{n-1}}|u|\\ +\fint_{\mathbb{S}^{n-1}}\Big(1-C_0\Big(\|u\|_{C^1}+r_0\Big)\Big)\frac{|\nabla u|^2}{2}-\Big(n-1+C_0\Big(\|u\|_{C^1}+r_0\Big)\Big)\frac{u^2}{2}\Big\}\,.\end{aligned}\tag{2.6}$$

Beginning of proof of Theorems 2.1 and 2.2. The first part of the proof of the two theorems is common. In the following, we shall denote by C a positive constant depending on n, α, β, γ, and either r_1,r_2 or r_0 depending on which of the two statements we are considering; the value of C may change at each appearance of the constant.

Given $x_0\in\mathbb{R}^n$ and $\omega\in\mathbb{S}^{n-1}$ we introduce the function $\phi_\omega:(0,\infty)\to(0,\infty)$ defined as

$$\phi_\omega(r)=\int_0^r e^{V(x_0+s\omega)}s^{n-1}\,ds\,,\qquad r>0\,.\tag{2.7}$$

We notice that, for every $r>0$,

$$\begin{aligned}\phi_\omega'(r)&=e^{V(x_0+r\omega)}r^{n-1}\,,\\ \phi_\omega''(r)&=e^{V(x_0+r\omega)}r^{n-1}\Big(\frac{n-1}{r}+\nabla V(x_0+r\omega)\cdot\omega\Big)\,,\\ \phi_\omega'''(r)&=e^{V(x_0+r\omega)}r^{n-1}\bigg(\Big(\frac{n-1}{r}+\nabla V(x_0+r\omega)\cdot\omega\Big)^2-\frac{n-1}{r^2}+\nabla^2V(x_0+r\omega)[\omega,\omega]\bigg)\,.\end{aligned}$$

• *Step one:* We show that, under the assumption of both theorems (and with $x_0 = 0$ in the case of Theorem 2.1), we have

$$\begin{aligned}\mathrm{Per}(E) - \mathrm{Per}(B(x_0,r)) \;\geq\; & \int_{\mathbb{S}^{n-1}} \Big(r\phi''_\omega(r)\,u + \frac{r^2\phi''_\omega(r)^2}{\phi'_\omega(r)}\,\frac{u^2}{2}\Big) \\ & + \Big(1 - C\,\|u\|_{C^0}\Big)\,\frac{\gamma\,r^2}{2}\int_{\mathbb{S}^{n-1}} \phi'_\omega(r)\,u^2 \\ & + \Big(1 - C\,\|u\|_{C^1}\Big)\int_{\mathbb{S}^{n-1}} \phi'_\omega(r)\,\frac{|\nabla u|^2}{2} \\ & - \Big(n-1 + C\,\|u\|_{C^0}\Big)\int_{\mathbb{S}^{n-1}} \phi'_\omega(r)\,\frac{u^2}{2}\,. \end{aligned} \tag{2.8}$$

We start by observing that

$$\partial E = x_0 + \Big\{ r\,\Big(1 + u(\omega)\Big)\omega : \omega \in \mathbb{S}^{n-1} \Big\}\,.$$

By applying the area formula on $\mathbb{S}^{n-1}$ to the Lipschitz function $f : \mathbb{S}^{n-1} \to \partial E$ defined as $f(\omega) = x_0 + r(1 + u(\omega))\omega$, $\omega \in \mathbb{S}^{n-1}$, we easily find that

$$\begin{aligned}\mathrm{Per}(E) &= \int_{\mathbb{S}^{n-1}} \Big(r(1+u)\Big)^{n-1} e^{V(x_0 + r(1+u)\omega)}\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} \\ &= \int_{\mathbb{S}^{n-1}} \phi'_\omega(r(1+u))\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}}\,. \end{aligned} \tag{2.9}$$

By the elementary inequalities

$$\frac{1}{(1+t)^2} \geq 1 - 2\,t\,, \qquad t > -1\,,$$

$$\sqrt{1+s} \geq 1 + \frac{s}{2} - \frac{s^2}{8}\,, \qquad s \geq 0\,,$$

we get

$$\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} \geq 1 + \Big(1 - C\,\|u\|_{C^1}\Big)\frac{|\nabla u|^2}{2} \quad \text{on } \mathbb{S}^{n-1}\,. \tag{2.10}$$

Then, noticing that

$$\partial E \cup \partial B_r \subset \overline{B_{(1+\varepsilon_0)\,r_2}} \setminus B_{(1-\varepsilon_0)\,r_1}\,, \qquad \text{in the case of Theorem 2.1}\,, \tag{2.11}$$

$$E \cup B(x_0, r) \subset B_{2\,r_0}\,, \qquad \text{in the case of Theorem 2.2}\,, \tag{2.12}$$

in both cases we find

$$\Big| V(x_0 + r(1+u)\omega) - V(x_0 + r\omega) \Big| \leq \alpha\, r\,\|u\|_{C^0}\,, \qquad \forall\,\omega \in \mathbb{S}^{n-1}\,,$$

and since $(1+t)^{n-1} \geq 1 + (n-1)t > 0$ for $|t| < 1/(n-1)$, we obtain

$$\Big(r(1+u)\Big)^{n-1} e^{V(x_0 + r(1+u)\omega)} \geq r^{n-1} e^{V(x_0 + r\omega)}\Big(1 - C\,\|u\|_{C^0}\Big) \quad \text{on } \mathbb{S}^{n-1}\,. \tag{2.13}$$

Thus, combining (2.10) and (2.13) with (2.9) we get

$$\mathrm{Per}(E) \geq \int_{\mathbb{S}^{n-1}} \phi'_\omega(r(1+u)) + \Big(1 - C\,\|u\|_{C^1}\Big)\int_{\mathbb{S}^{n-1}} \phi'_\omega(r)\frac{|\nabla u|^2}{2}\,. \tag{2.14}$$

We now notice that, by Taylor's formula, we can find a function $\bar\theta : \mathbb{S}^{n-1} \to [0,1]$ such that, if we set $\bar u = \bar\theta\,u$, then

$$\phi'_\omega(r(1+u)) = \phi'_\omega(r) + r\,\phi''_\omega(r)u + r^2\,\phi'''_\omega(r(1+\bar u))\,\frac{u^2}{2} \quad \text{on } \mathbb{S}^{n-1}\,. \tag{2.15}$$

Moreover, since

$$\nabla^2 V(x) = v''(|x|)\,\frac{x}{|x|}\otimes\frac{x}{|x|} + \frac{v'(|x|)}{|x|}\left(\mathrm{Id} - \frac{x}{|x|}\otimes\frac{x}{|x|}\right), \qquad \forall\, x\in\mathbb{R}^n\,, \tag{2.16}$$

we get

$$\nabla^2 V(x_0 + r(1+\bar u)\omega)[\omega,\omega] \ge \gamma\,,$$

where in the case of Theorem 2.1 we used that ω is parallel to $x_0+r(1+\bar u)\omega$ (since $x_0=0$), while in the case of Theorem 2.2 we used that $v'(r)\ge\gamma r$ (since $v'(0)\ge 0$ and $v''\ge\gamma$ on $[0,2r_0]$). Thus, by the formula for ϕ_ω''' and by (2.13) we obtain

$$\begin{aligned} & r^2(1+\bar u)^2\,\phi_\omega'''(r(1+\bar u)) \\ \ge\;& \Big(1\pm C\,\|u\|_{C^0}\Big)\phi_\omega'(r)\bigg\{\Big(n-1+\nabla V\Big(x_0+r(1+\bar u)\omega\Big)\cdot(r(1+\bar u)\omega)\Big)^2 \\ & \qquad\qquad -(n-1)+\gamma\, r^2(1+\bar u)^2\bigg\}\,, \end{aligned} \tag{2.17}$$

where $\pm$ is equal to $+$ if the expression inside curly brackets is negative, while $\pm=-$ if it is positive. Moreover, in both cases we have

$$\left|\nabla V\Big(x_0+r(1+\bar u)\omega\Big)\cdot(r(1+\bar u)\omega) - \nabla V(x_0+r\omega)\cdot r\omega\right| \le C\,\|u\|_{C^0}\,,$$

which combined with (2.17) gives (with the same convention for $\pm$ as before)

$$\begin{aligned} & r^2(1+\bar u)^2\,\phi_\omega'''(r(1+\bar u)) \\ \ge\;& \Big(1\pm C\,\|u\|_{C^0}\Big)\phi_\omega'(r)\bigg\{\gamma\, r^2(1+\bar u)^2-(n-1)+ \\ & \qquad\qquad \Big(n-1+\nabla V(x_0+r\omega)\cdot r\omega\Big)^2 - C\|u\|_{C^0}\bigg\} \\ =\;& \Big(1\pm C\,\|u\|_{C^0}\Big)\phi_\omega'(r)\bigg\{\gamma\, r^2(1+\bar u)^2-(n-1)+r^2\frac{\phi_\omega''(r)^2}{\phi_\omega'(r)^2} - C\|u\|_{C^0}\bigg\}\,. \end{aligned}$$

Multiplying by $(1+\bar u)^{-2}\ge(1-2\,\|u\|_{C^0})$ we find

$$\begin{aligned} r^2\,\phi_\omega'''(r(1+\bar u)) \;\ge\;& \Big(1\pm C\,\|u\|_{C^0}\Big)\,\phi_\omega'(r)\bigg\{r^2\frac{\phi_\omega''(r)^2}{\phi_\omega'(r)^2}+\gamma\, r^2-(n-1)-C\|u\|_{C^0}\bigg\} \\ \ge\;& r^2\frac{\phi_\omega''(r)^2}{\phi_\omega'(r)} + \Big(1\pm C\,\|u\|_{C^0}\Big)\,\phi_\omega'(r)\bigg\{\gamma\, r^2-(n-1)-C\|u\|_{C^0}\bigg\}\,, \end{aligned} \tag{2.18}$$

where in the last inequality we used that $r^2\phi_\omega''(r)^2\le C\,\phi_\omega'(r)^2$. We finally combine (2.14), (2.15), and (2.18), to obtain (2.8).

• *Step two:* We notice that the weighted volume constraint $\mathrm{Vol}(B(x_0,r))=\mathrm{Vol}(E)$ implies

$$\int_{\mathbb{S}^{n-1}}\int_0^r e^{V(x_0+s\omega)}s^{n-1}\,ds = \int_{\mathbb{S}^{n-1}}\int_0^{r(1+u)} e^{V(x_0+s\omega)}s^{n-1}\,ds\,,$$

that is,

$$\int_{\mathbb{S}^{n-1}}\Big(\phi_\omega(r(1+u))-\phi_\omega(r)\Big)=0\,. \tag{2.19}$$

By Taylor's formula, we may define $\tilde\theta:\mathbb{S}^{n-1}\to[0,1]$ so that, if $\tilde u=\tilde\theta u$, then

$$\phi_\omega(r(1+u)) = \phi_\omega(r)+r\,\phi_\omega'(r)u+r^2\,\phi_\omega''(r)\frac{u^2}{2}+r^3\,\phi_\omega''(r(1+\tilde u))\frac{u^3}{6}\quad\text{on }\mathbb{S}^{n-1}\,.$$

Inserting this expansion in (2.19) and noticing that, by (2.11) and (2.12), $r^2|\phi'''_\omega(r(1+\bar u))| \le C\,\phi'_\omega(r)$, we get the useful estimate

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}}\left(r\phi'_\omega(r)u+r^2\phi''_\omega(r)\frac{u^2}{2}\right)\right| &\le \int_{\mathbb{S}^{n-1}}|\phi'''_\omega(r(1+\tilde u))|\,\frac{(ru)^3}{6}\\ &\le C\,\|u\|_{C^0}\int_{\mathbb{S}^{n-1}} r\phi'_\omega(r)\,u^2\,,\end{aligned}\tag{2.20}$$

which, combined with $r\,|\phi''_\omega(r)|\le C\,\phi'_\omega(r)$, gives in particular

$$\left|\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)u\right|\le C\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,u^2\,.\tag{2.21}$$

We now conclude the proof of Theorem 2.1 and Theorem 2.2 by two separate arguments. □

Conclusion of proof of Theorem 2.1. We want to estimate the integral on the first line of (2.8). Since we are assuming that $x_0=0$, $\phi'_\omega(r)=r^{n-1}e^{v(r)}$ is constant with respect to $\omega\in\mathbb{S}^{n-1}$, so (2.20) gives

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}}\left(r\phi''_\omega(r)\,u+\frac{r^2\phi''_\omega(r)^2}{\phi'_\omega(r)}\,\frac{u^2}{2}\right)\right| &= \left|\frac{\phi''_\omega(r)}{\phi'_\omega(r)}\int_{\mathbb{S}^{n-1}}\left(r\phi'_\omega(r)\,u+r^2\phi''_\omega(r)\,\frac{u^2}{2}\right)\right|\\ &= \left|\left(\frac{n-1}{r}+v'(r)\right)\int_{\mathbb{S}^{n-1}}\left(r\phi'_\omega(r)\,u+r^2\phi''_\omega(r)\,\frac{u^2}{2}\right)\right|\\ &\le \frac{n-1+\alpha}{r}\,C\,\|u\|_{C^0}\int_{\mathbb{S}^{n-1}} r\phi'_\omega(r)\,u^2\\ &= C\|u\|_{C^0}\phi'_\omega(r)\fint_{\mathbb{S}^{n-1}}u^2\,.\end{aligned}$$

As $\phi'_\omega(r)=\mathrm{Per}(B_r)/n\omega_n$, inserting the above estimate in (2.8) and recalling that $\gamma\ge 0$, we easily get (2.3). □

Conclusion of proof of Theorem 2.2. We have to prove (2.6). Let us first show that

$$\begin{aligned}\mathrm{Per}(E)-\mathrm{Per}(B(x_0,r)) \;\ge\; & -Cr|x_0|\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)|u|\\ &+\Big(1-C\,\|u\|_{C^1}\Big)\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,\frac{|\nabla u|^2}{2}\\ &-\Big(n-1+C\Big(\|u\|_{C^0}+r\Big)\Big)\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,\frac{u^2}{2}\,.\end{aligned}\tag{2.22}$$

To this end, we notice that by the formulas for ϕ'_ω and ϕ''_ω, and by (2.20),

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\left(r\phi''_\omega(r)\,u+\frac{r^2\phi''_\omega(r)^2}{\phi'_\omega(r)}\,\frac{u^2}{2}\right)=\int_{\mathbb{S}^{n-1}}\frac{\phi''_\omega(r)}{\phi'_\omega(r)}\left(r\phi'_\omega(r)u+r^2\phi''_\omega(r)\frac{u^2}{2}\right)\\ &=\int_{\mathbb{S}^{n-1}}\frac{n-1}{r}\left(r\phi'_\omega(r)u+r^2\phi''_\omega(r)\frac{u^2}{2}\right)\\ &\qquad+\int_{\mathbb{S}^{n-1}}\nabla V(x_0+r\omega)\cdot\omega\left(r\phi'_\omega(r)u+r^2\phi''_\omega(r)\frac{u^2}{2}\right)\\ &\ge -C\,\|u\|_{C^0}\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,u^2+r\int_{\mathbb{S}^{n-1}}\nabla V(x_0+r\omega)\cdot\omega\left(\phi'_\omega(r)u+r\phi''_\omega(r)\frac{u^2}{2}\right)\\ &\ge -C\Big(\|u\|_{C^0}+r\Big)\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)u^2+r\int_{\mathbb{S}^{n-1}}\Big(\nabla V(x_0+r\omega)\cdot\omega\Big)\phi'_\omega(r)u\,,\end{aligned}$$

where in the last step we have used again that $r\,|\phi''_\omega(r)|\le C\,\phi'_\omega(r)$ for every $r\le r_0$. We now notice that, since $|\nabla V(x_0+r\omega)-\nabla V(r\omega)|\le\beta|x_0|$ for every $r\le r_0$ and $\omega\in\mathbb{S}^{n-1}$,

$$
\begin{aligned}
& r\int_{\mathbb{S}^{n-1}}\nabla V(x_0+r\omega)\cdot\omega\,\phi'_\omega(r)\,u && (2.23)\\
&= r\int_{\mathbb{S}^{n-1}}\nabla V(r\omega)\cdot\omega\,\phi'_\omega(r)\,u+r\int_{\mathbb{S}^{n-1}}\Big(\nabla V(x_0+r\omega)-\nabla V(r\omega)\Big)\cdot\omega\,\phi'_\omega(r)\,u\\
&= r\,v'(r)\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,u+r\int_{\mathbb{S}^{n-1}}\Big(\nabla V(x_0+r\omega)-\nabla V(r\omega)\Big)\cdot\omega\,\phi'_\omega(r)\,u\\
&\ge -C\left\{r\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,u^2+r|x_0|\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,|u|\right\},
\end{aligned}
$$

where in the last inequality (2.21) was also taken into account. Recalling (2.8) and neglecting the term with $\gamma\,r^2$, we obtain (2.22). Finally, to conclude the proof of (2.6) it suffices to observe that

$$
\Big|\phi'_\omega(r)-r^{n-1}e^{v(r)}\Big|\le\alpha\;|x_0|\,r^{n-1}\,e^{v(r)}\le\alpha\;r_0\,r^{n-1}\,e^{v(r)}\,, \tag{2.24}
$$

and

$$
\Big|\mathrm{Per}(B(x_0,r))-n\omega_n\,r^{n-1}e^{V(x_0)}\Big|\le\alpha\,|x_0|\,\mathrm{Per}(B(x_0,r))\le\alpha\,r_0\,\mathrm{Per}(B(x_0,r))\,. \tag{2.25}
$$

In particular

$$
\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)|u|\le n\omega_n r^{n-1}e^{v(r)}(1+C\,r_0)⨍_{\mathbb{S}^{n-1}}|u|\le C\,\mathrm{Per}(B(x_0,r))⨍_{\mathbb{S}^{n-1}}|u|\,,
$$

as well as,

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,\frac{|\nabla u|^2}{2}&\ge\mathrm{Per}(B(x_0,r))(1-\alpha\,r_0)^2⨍_{\mathbb{S}^{n-1}}\frac{|\nabla u|^2}{2}\,,\\
\int_{\mathbb{S}^{n-1}}\phi'_\omega(r)\,\frac{u^2}{2}&\le\mathrm{Per}(B(x_0,r))(1+\alpha\,r_0)^2⨍_{\mathbb{S}^{n-1}}\frac{u^2}{2}\,.
\end{aligned}
$$

By plugging these last three inequalities into (2.22), we obtain (2.6). □

2.2. **Stability of balls centered at the origin.** Following a technique developed by Fuglede to study the stability of the Euclidean isoperimetric inequality on nearly spherical domains, see [Fu1, Fu2], we now combine the lower bound (2.3) in Theorem 2.1 with an expansion in spherical harmonics to prove the minimality of balls centered at the origin with respect to C^1-small radial perturbations. Let us recall that, given $m>0$, we denote by $B(m)=B_{r(m)}$ the ball of radius $r(m)>0$ centered at the origin such that $\mathrm{Vol}(B(m))=\mathrm{Vol}(B_{r(m)})=m$. Notice that in this theorem v *is not required to be increasing*, but just locally uniformly convex.

Theorem 2.3. *Given $n\ge2$, positive constants α, β, and γ, and $m_2\ge m_1>0$, there exists*

$$
\varepsilon_1=\varepsilon_1(n,\alpha,\beta,\gamma,m_1,m_2)\in\left(0,\frac12\right),
$$

with the following property: If $v:[0,\infty)\to[0,\infty)$ is twice differentiable with

$$
\alpha\ge|v'|\,,\qquad\beta\ge v''\ge\gamma\,,\qquad\textit{on}\quad\Big[(1-\varepsilon_1)\,r(m_1),(1+\varepsilon_1)\,r(m_2)\Big]\,,
$$

and if $u\in C^1(\mathbb{S}^{n-1})$, $m\in[m_1,m_2]$, and

$$
E=\Big\{t\Big(1+u(\omega)\Big)\omega:\omega\in\mathbb{S}^{n-1}\,,0\le t<r(m)\Big\}
$$

are such that

$$
\|u\|_{C^1}\le\varepsilon_1\min\Big\{r(m_1)^2,1\Big\}\,,\qquad\mathrm{Vol}(E)=m\,,
$$

then

$$\mathrm{Per}(E) \geq \mathrm{Per}(B(m))\Big\{1+\frac{\gamma\, r(m_1)^2}{4}\fint_{\mathbb{S}^{n-1}} u^2\Big\}\,. \tag{2.26}$$

In particular, if E is an isoperimetric set then $E=B(m)$.

Remark 2.1. As a consequence of Theorem 2.3, $B(m)$ is the unique isoperimetric set among its C^1-small perturbations. In fact, it is a strictly stable isoperimetric set in this restricted competition class, and (2.26) quantitatively shows that, due to the uniform convexity of v, this minimality property becomes increasingly stronger as we increase the weighted volume parameter. At the same time we should notice that this theorem is not particularly useful in the small weighted volume regime: indeed, both the lower bound on $\mathrm{Per}(E)-\mathrm{Per}(B(m))$ in (2.26) and the range of applicability of (2.26) in terms of the size of $\|u\|_{C^1(\mathbb{S}^{n-1})}$ degenerate as $m_1\to 0^+$.

Proof of Theorem 2.3. Let ε_0 be the constant determined by Theorem 2.1 in correspondence with n, α, β, γ, $r_1=r(m_1)$, and $r_2=r(m_2)$, and let C denote a generic constant depending on n, α, β, γ, r_1, and r_2 only. Applying (2.3) with $r=r(m)$ (note that $r\in(r_1,r_2)$ since $v\geq 0$ and thus $\mathrm{Vol}(B_r)$ is strictly increasing as a function of r), we have

$$\begin{aligned}\mathrm{Per}(E) \quad \geq \quad & \mathrm{Per}(B_r)\Big\{1+\Big(1-C\,\|u\|_{C^1}\Big)\frac{\gamma\, r^2}{2}\fint_{\mathbb{S}^{n-1}} u^2\Big\} \\ & +\mathrm{Per}(B_r)\fint_{\mathbb{S}^{n-1}}\Big(1-C\,\|u\|_{C^1}\Big)\frac{|\nabla u|^2}{2}-\Big(n-1+C\,\|u\|_{C^1}\Big)\frac{u^2}{2}\,,\end{aligned} \tag{2.27}$$

provided $\|u\|_{C^1}\leq\varepsilon_0$. Our goal is now to estimate from below that term in the second line. To this end, let us consider the orthonormal basis of $L^2(\mathbb{S}^{n-1})$ given by the spherical harmonics $\{Y_{j,k}\,:\,1\leq k\leq n_j\}_{j=0}^\infty$, that is

$$\fint_{\mathbb{S}^{n-1}} Y_{j,k}Y_{\ell,q}=\delta_{j,\ell}\delta_{k,q}\,,\qquad \fint_{\mathbb{S}^{n-1}}|\nabla Y_{j,k}|^2=\lambda_j=j(n-2+j)\,,$$

and denote the coefficients of u with respect to this basis as

$$c_{j,k}=\fint_{\mathbb{S}^{n-1}} Y_{j,k}\,u\,.$$

Since $\lambda_1=n-1$ and $\lambda_j\geq 2n$ for every $j\geq 2$, we find

$$\begin{aligned}\Big(1-C\|u\|_{C^1}\Big)&\fint_{\mathbb{S}^{n-1}}|\nabla u|^2-\Big(n-1+C\|u\|_{C^1}\Big)\fint_{\mathbb{S}^{n-1}}u^2 \\ &=\Big(1-C\|u\|_{C^1}\Big)\sum_{j,k}\lambda_j c_{j,k}^2-\Big(n-1+C\|u\|_{C^1}\Big)\sum_{j,k}c_{j,k}^2 \\ &\geq\Big(n+1-C\,\|u\|_{C^1}\Big)\sum_{j\geq 2}\sum_k c_{j,k}^2-C\,\|u\|_{C^1}\sum_{k=1}^{n_1}c_{1,k}^2-n\,c_0^2 \\ &\geq -C\,\|u\|_{C^1}\sum_{k=1}^{n_1}c_{1,k}^2-n\,c_0^2\,.\end{aligned} \tag{2.28}$$

As $x_0=0$, $\phi_\omega'(r)=r^{n-1}e^{v(r)}$ is constant on $\mathbb{S}^{n-1}$; hence, by (2.21) we find that

$$\Big|\int_{\mathbb{S}^{n-1}}u\Big|\leq C\int_{\mathbb{S}^{n-1}}u^2\leq C\,\|u\|_{C^0}\int_{\mathbb{S}^{n-1}}|u|\,,$$

so that, by Hölder inequality and recalling that $Y_0=1$,

$$c_0^2=\Big|\fint_{\mathbb{S}^{n-1}}u\Big|^2\leq C\,\|u\|_{C^0}^2\fint_{\mathbb{S}^{n-1}}u^2\,. \tag{2.29}$$

Since

$$\fint_{\mathbb{S}^{n-1}} u^2 = \sum_{j\ge 0}\sum_k c_{j,k}^2 \ge \sum_{k=1}^{n_1} c_{1,k}^2\,, \tag{2.30}$$

combining (2.27), (2.28), (2.29), and (2.30) we conclude

$$\begin{aligned}\mathrm{Per}(E) &\ge \mathrm{Per}(B_r)\Big\{1+\Big(\Big(1-C\,\|u\|_{C^1}\Big)\frac{\gamma\,r^2}{2}-C\,\|u\|_{C^1}\Big)\fint_{\mathbb{S}^{n-1}} u^2\Big\}\\ &\ge \mathrm{Per}(B_r)\Big\{1+\Big(\Big(1-C\,\|u\|_{C^1}\Big)\frac{\gamma\,r(m_1)^2}{2}-C\,\|u\|_{C^1}\Big)\fint_{\mathbb{S}^{n-1}} u^2\Big\}\,,\end{aligned}$$

where in the last inequality we have use the fact that $r = r(m) \ge r(m_1)$. Then (2.26) follows immediately provided $\|u\|_{C^1} \le \varepsilon_1 \min\{r(m_1)^2, 1\}$ for a suitable $\varepsilon_1 \le \varepsilon_0$. □

2.3. **Perturbation of balls not centered at the origin.** We shall now explain how to exploit (2.6) under the assumption that $B(x_0, r)$ and E not only have the same weighted volume, but also share the same weighted barycenter; as noticed in the introduction, this analysis will be needed to tackle the conjecture in the small weighted volume regime, although we shall not use a small weighted volume assumption in the following discussion.

We start by showing that balls centered at the origin are the unique isoperimetric sets among balls centered sufficiently close to the origin, uniformly with respect to the weighted volume parameter; see Theorem 2.4. In Theorem 2.5, starting from (2.6) in Theorem 2.1, we extend this uniform stability property among all C^1-small radial perturbations of balls having weighted barycenter sufficiently close to the origin. We recall the notation $\Omega[w](R,\cdot)$ for the modulus of continuity of a continuous function w over the interval $[0, R]$; see (1.10).

Theorem 2.4. *Given $n \ge 2$, positive constants α, β, γ, and m_0, and $v \in C^2([0,\infty);[0,\infty))$ a convex increasing function with*

$$\alpha \ge v' \ge 0\,, \qquad \beta \ge v'' \ge \gamma\,, \qquad \text{on } [0, 2\,r(m_0)]\,,$$

there exists

$$t_0 = t_0\Big(n, \alpha, \beta, \Omega[v''](2\,r(m_0),\cdot)\Big) > 0\,,$$

with the following property: if $|x_0| \le t_0$ and $r > 0$ is such that $\mathrm{Vol}(B(x_0, r)) = m \le m_0$, then

$$\mathrm{Per}(B(x_0,r)) \ge \mathrm{Per}(B(m))\Big\{1+\frac{\gamma}{8n}\,|x_0|^2\Big\}\,. \tag{2.31}$$

Proof. Fix $m < m_0$, and denote by $s(t)$ the radius of the ball centered at te_1 satisfying

$$m = \mathrm{Vol}(B(t\,e_1, s(t))) = \int_{B_{s(t)}} e^{V(x+te_1)}\,dx\,, \qquad t>0\,;$$

notice that $s \in C^2([0,\infty);[0, r(m_0)])$ and $s(0) = r(m)$. Correspondingly, let us consider the function $f \in C^2([0,\infty);(0,\infty))$ defined as

$$f(t) = \mathrm{Per}(B(te_1, s(t))) = \int_{\mathbb{S}^{n-1}} s(t)^{n-1}\, e^{V(s(t)\omega + t\,e_1)}\, d\mathcal{H}^{n-1}(\omega)\,, \qquad t>0\,.$$

We claim that

$$f''(0) \ge \frac{\gamma}{n}\,f(0)\,, \tag{2.32}$$

$$|f'| \le C\,f\,, \quad \text{on } [0, r(m_0)]\,, \tag{2.33}$$

$$\frac{f''}{f} \in C^0([0, r(m_0)])\,, \tag{2.34}$$

where C depends only on n, α, and β, and where the modulus of continuity of f''/f over $[0, r(m_0)]$ depends only on the moduli of continuity of v, v', and v'' over $[0, 2r(m_0)]$. Before proving these claims, let us explain how they lead to conclude the proof of the theorem.

By (2.33) the function $t \mapsto f(t)\, e^{C\,t}$ is increasing over $[0, r(m_0)]$, hence there exists $t_0 < r(m_0)$ such that

$$f(t) \geq \frac{f(0)}{2}\,, \qquad \forall\, t \in [0, t_0]\,; \tag{2.35}$$

moreover, by (2.34) and by (2.32), up to decrease the value of t_0,

$$\frac{f''(t)}{f(t)} \geq \frac{\gamma}{2n} \qquad \forall\, t \in [0, t_0]\,. \tag{2.36}$$

We may thus combine (2.35) and (2.36) to find that

$$f''(t) \geq \frac{\gamma}{4n}\, f(0)\,, \qquad \forall\, t \in [0, t_0]\,,$$

which by Taylor's formula gives

$$f(t) \geq f(0)\left(1 + \frac{\gamma}{8n} t^2\right), \qquad \forall\, t \in [0, t_0]\,. \tag{2.37}$$

If now $x_0 \in \mathbb{R}^n$ with $|x_0| < t_0$, and $r > 0$ is such that $\mathrm{Vol}(B(x_0, r)) = m$, then $r = s(t)$ and $\mathrm{Per}(B(x_0, r)) = \mathrm{Per}(B(t\, e_1, s(t)))$ for $t = |x_0|$ so that (2.37) gives exactly (2.31). We are thus left to prove the validity of (2.32), (2.33), and (2.34).

To this end, we start differentiating the identity defining $s(t)$ to find that

$$\begin{aligned} s'(t) &= -\frac{1}{f(t)} \int_{B_{s(t)}} e^{V(x+te_1)} \partial_1 V(x + te_1)\, dx\,, \\ s''(t) &= -\frac{1}{f(t)} \int_{B_{s(t)}} e^{V(x+te_1)} \Big(\partial_1 V(x + te_1)^2 + \partial_{11} V(x + te_1)\Big)\, dx \\ &\quad -\frac{s'(t)}{f(t)} \int_{\partial B_{s(t)}} e^{V(\omega+te_1)} \partial_1 V(\omega + te_1)\, d\mathcal{H}^{n-1}(\omega) \\ &\quad +\frac{f'(t)}{f(t)^2} \int_{B_{s(t)}} e^{V(x+te_1)} \partial_1 V(x + te_1)\, dx. \end{aligned} \tag{2.38}$$

Observing that

$$\begin{aligned} e^{V(x+te_1)} \partial_1 V(x + te_1) &= \partial_1 \left(e^{V(x+te_1)}\right), \\ e^{V(x+te_1)} \Big(\partial_1 V(x + te_1)^2 + \partial_{11} V(x + te_1)\Big) &= \partial_1 \left(e^{V(x+te_1)} \partial_1 V(x + te_1)\right), \end{aligned}$$

and setting $\nu_1 = \nu_{B_{s(t)}} \cdot e_1$, we can rewrite $s''(t)$ as

$$\begin{aligned} s''(t) &= -\frac{1}{f(t)} \int_{\partial B_{s(t)}} e^{V(\omega+te_1)} \partial_1 V(\omega + te_1) \nu_1(x)\, d\mathcal{H}^{n-1}(\omega) \\ &\quad -\frac{s'(t)}{f(t)} \int_{\partial B_{s(t)}} e^{V(\omega+te_1)} \partial_1 V(\omega + te_1)\, d\mathcal{H}^{n-1}(\omega) \\ &\quad +\frac{f'(t)}{f(t)^2} \int_{\partial B_{s(t)}} e^{V(\omega+te_1)} \nu_1(\omega)\, d\mathcal{H}^{n-1}(\omega)\,. \end{aligned} \tag{2.39}$$

We finally compute

$$
\begin{aligned}
f'(t) &= (n-1)\frac{s'(t)}{s(t)}f(t)+\int_{\mathbb{S}^{n-1}} s(t)^{n-1}e^{V(s(t)\omega+te_1)}\Big(s'(t)\omega\cdot\nabla V+\partial_1 V\Big)\,d\mathcal{H}^{n-1}(\omega)\,, \qquad (2.40)\\
f''(t) &= (n-1)\Big(\frac{s''(t)}{s(t)}-\frac{s'(t)^2}{s(t)^2}\Big)f(t)+(n-1)\frac{s'(t)}{s(t)}f'(t)\\
&\quad +(n-1)\frac{s'(t)}{s(t)}\int_{\mathbb{S}^{n-1}} s(t)^{n-1}e^{V(s(t)\omega+te_1)}\Big(s'(t)\omega\cdot\nabla V+\partial_1 V\Big)\,d\mathcal{H}^{n-1}(\omega)\\
&\quad +\int_{\mathbb{S}^{n-1}} s(t)^{n-1}e^{V(s(t)\omega+te_1)}\Big(s'(t)\omega\cdot\nabla V+\partial_1 V\Big)^2\,d\mathcal{H}^{n-1}(\omega)\\
&\quad +s''(t)\int_{\mathbb{S}^{n-1}} s(t)^{n-1}e^{V(s(t)\omega+te_1)}\Big(\omega\cdot\nabla V\Big)\,d\mathcal{H}^{n-1}(\omega)\\
&\quad +\int_{\mathbb{S}^{n-1}} s(t)^{n-1}e^{V(s(t)\omega+te_1)}\nabla^2 V\Big(s'(t)\omega+e_1,s'(t)x+e_1\Big)\,d\mathcal{H}^{n-1}(\omega)\,,
\end{aligned}
$$

where $\partial_1 V$, ∇V, and $\nabla^2 V$ are all evaluated at $te_1+s(t)x$. We now notice that, by (2.38), (2.40), and by symmetry,

$$
\begin{aligned}
s'(0) &= \frac{1}{f(0)}\int_{B_{s(0)}} e^{v(|x|)}\frac{v'(|x|)}{|x|}\,(x\cdot e_1)\,dx=0\,,\\
f'(0) &= s(0)^{n-1}e^{v(s(0))}\frac{v'(s(0))}{s(0)}\int_{\mathbb{S}^{n-1}}(\omega\cdot e_1)\,d\mathcal{H}^{n-1}(\omega)=0\,.
\end{aligned}
$$

In particular, since

$$
1=\fint_{\mathbb{S}^{n-1}}|\omega|^2 d\mathcal{H}^{n-1}(\omega)=n\fint_{\mathbb{S}^{n-1}}(\omega\cdot e_1)^2\,d\mathcal{H}^{n-1}(\omega) \qquad (2.41)
$$

using (2.39) we may compute $s''(0)$ as

$$
s''(0)=-v'(s(0))\fint_{\mathbb{S}^{n-1}}(e_1\cdot\omega)^2\,d\mathcal{H}^{n-1}(\omega)=-\frac{v'(s(0))}{n}\,.
$$

Finally, taking into account that $f(0)=n\omega_n s(0)^{n-1}e^{v(s(0))}$, we find

$$
\begin{aligned}
\frac{f''(0)}{f(0)} &= (n-1)\frac{s''(0)}{s(0)}+v'(s(0))^2\fint_{\mathbb{S}^{n-1}}(\omega\cdot e_1)^2\,d\mathcal{H}^{n-1}(\omega)\\
&\quad +s''(0)\,v'(s(0))+\fint_{\mathbb{S}^{n-1}}\partial^2_{11}V(s(0)\omega)\,d\mathcal{H}^{n-1}(\omega)\\
&= -\frac{(n-1)}{n}\frac{v'(s(0))}{s(0)}+\fint_{\mathbb{S}^{n-1}}\partial^2_{11}V(s(0)\omega)\,d\mathcal{H}^{n-1}(\omega)\,.
\end{aligned}
$$

Recalling (2.16), we find that

$$
\partial_{11}V(r\,\omega)=v''(r)\,(\omega\cdot e_1)^2+\frac{v'(r)}{r}\Big(1-(\omega\cdot e_1)^2\Big)\,,\qquad \forall\, r>0\,,\omega\in\mathbb{S}^{n-1}\,,
$$

and thus, again by (2.41) and recalling that v is uniformly convex on $[0,r(m_0)]$,

$$
\frac{f''(0)}{f(0)}=-\frac{(n-1)}{n}\frac{v'(s(0))}{s(0)}+\frac{v''(s(0))}{n}+\frac{v'(s(0))}{s(0)}\Big(1-\frac{1}{n}\Big)=\frac{v''(s(0))}{n}\,. \qquad (2.42)
$$

This proves the validity of (2.32), while (2.33) and (2.34) follow easily by examining the formulas for f, f', f'', s', and s'' derived above. □

We now extend the uniform stability property of Theorem 2.4 to C^1-small radial perturbations of balls with small weighted barycenter.

Theorem 2.5. *Given $n\ge 2$, positive constants α, β, γ, and m_0, $v\in C^2([0,\infty);[0,\infty))$ a convex increasing function with*
$$\alpha\ge v'\ge 0\,,\qquad \beta\ge v''\ge\gamma\,,\qquad\text{on }[0,2\,r(m_0)]\,,$$
there exist
$$\varepsilon_2=\varepsilon_2(n,\alpha,\beta,\gamma,m_0)>0\,,\qquad C_1=C_1(n,\alpha,\beta,\gamma,m_0)<\infty\,,$$
$$r_1=r_1\Big(n,\alpha,\beta,\gamma,m_0,\Omega[v''](2\,r(m_0),\cdot)\Big)>0\,,$$
with the following property: if $x_0\in\mathbb{R}^n$, $r>0$, $u\in C^1(\mathbb{S}^{n-1};[-1,\infty))$ and
$$E=x_0+\Big\{t\,(1+u(\omega))\omega:\omega\in\mathbb{S}^{n-1}\,,0\le t<r\Big\}\,,$$
are such that
$$|x_0|\le r_1\,,\qquad r\le r_1\,,\qquad \|u\|_{C^1(\mathbb{S}^{n-1})}\le\varepsilon_2\,,$$
and satisfy the weighted volume and weighted barycenter constraints
$$\mathrm{Vol}(E)=\mathrm{Vol}(B(x_0,r))=m\le m_0\,,\qquad \int_{B(x_0,r)}x\,e^{V(x)}\,dx=\int_E x\,e^{V(x)}dx\,,\tag{2.43}$$
then
$$\mathrm{Per}(E)\ge\mathrm{Per}(B(m))\Big\{1+\frac{1}{C_1}\Big(|x_0|^2+\fint_{\mathbb{S}^{n-1}}u^2\Big)\Big\}\,.\tag{2.44}$$
In particular, if E is an isoperimetric set, then $E=B(m)$.

Proof. Let ε_0 be the positive constant defined by Theorem 2.1 in correspondence with n, α, β, γ, and $r_0=r(m_0)$, and let t_0 be the positive constant defined by Theorem 2.4 starting from n, α, β, and the moduli of continuity of v'' on $[0,2\,r(m_0)]$. We denote by C a generic constant depending on n, α, β, γ, and m_0 only. Provided
$$|x_0|\le\frac{r_0}{2}\,,\qquad r\le\min\{r_0,t_0\}\,,\qquad \|u\|_{C^1}\le\varepsilon_0\,,$$
as we can certainly assume by requiring $r_1\le\min\{t_0,r_0/2\}$ and $\varepsilon_2\le\varepsilon_0$, by (2.6) in Theorem 2.1 we have that
$$\begin{aligned}\mathrm{Per}(E)\ge\mathrm{Per}(B(x_0,r))\Big\{&1-C\,r\,|x_0|\fint_{\mathbb{S}^{n-1}}|u|\\&+\fint_{\mathbb{S}^{n-1}}\Big(1-C\,\Big(\|u\|_{C^1}+r_1\Big)\Big)\frac{|\nabla u|^2}{2}-\Big(n-1+C\Big(\|u\|_{C^1}+r_1\Big)\Big)\frac{u^2}{2}\Big\}\,,\end{aligned}\tag{2.45}$$
while (2.31) in Theorem 2.4 gives
$$\mathrm{Per}(B(x_0,r))\ge\mathrm{Per}(B(m))\Big\{1+\frac{\gamma}{8n}\,|x_0|^2\Big\}\,.\tag{2.46}$$
Expanding u in spherical harmonics as in the proof of Theorem 2.3, we see that
$$\begin{aligned}&\fint_{\mathbb{S}^{n-1}}\Big(1-C\Big(\|u\|_{C^1}+r_1\Big)\Big)\frac{|\nabla u|^2}{2}-\Big(n-1+C\Big(\|u\|_{C^1}+r_1\Big)\Big)\frac{u^2}{2}\\ \ge\;&\Big(n+1-C\,\Big(\|u\|_{C^1}+r_1\Big)\Big)\sum_{j\ge2}\sum_k c_{j,k}^2-C\,\Big(\|u\|_{C^1}+r_1\Big)\sum_{k=1}^{n_1}c_{1,k}^2-n\,c_0^2\,,\end{aligned}\tag{2.47}$$
where
$$c_0=\fint_{\mathbb{S}^{n-1}}u\,,\qquad c_{1,k}=\fint_{\mathbb{S}^{n-1}}(\omega\cdot e_k)\,u\,,\qquad k=1,\dots,n\,,\,(n_1=n)\,.$$
By (2.21), (2.24), and (2.25), we find
$$\Big|\fint_{\mathbb{S}^{n-1}}u\Big|\le C\,\Big(|x_0|\fint_{\mathbb{S}^{n-1}}|u|+\fint_{\mathbb{S}^{n-1}}u^2\Big)\,,\tag{2.48}$$

therefore

$$c_0^2 = \left(\fint_{\mathbb{S}^{n-1}} u \right)^2 \le C \, \left(|x_0|^2 + \|u\|_{C^0}^2 \right) \fint_{\mathbb{S}^{n-1}} u^2 \,. \tag{2.49}$$

By the barycenter constraint (2.43) we now see that, if we define

$$\psi_\omega(t) = \int_0^t s^n \, e^{V(x_0 + s\,\omega)} \, ds \,, \qquad t > 0 \,,$$

then

$$0 = \int_E x \, e^{V(x)} dx - \int_{B(x_0,r)} x \, e^{V(x)} \, dx = \int_{\mathbb{S}^{n-1}} \omega \, \Big(\psi_\omega(r(1+u)) - \psi_\omega(r) \Big) \,. \tag{2.50}$$

Thus, recalling the definition (2.24) of ϕ_ω, we have

$$\begin{aligned} \psi_\omega'(t) &= t^n \, e^{V(x_0 + t\,\omega)} = t \, \phi_\omega'(t) \,, \\ \psi_\omega''(t) &= t^n \, e^{V(x_0 + t\,\omega)} \Big(\frac{n}{t} + \nabla V(x_0 + t\,\omega) \cdot \omega \Big) \,. \end{aligned}$$

By Taylor's formula, this implies that

$$\psi_\omega(r(1+u)) - \psi_\omega(r) = r \phi_\omega'(r) \, r u + \psi_\omega''\Big(r(1 + \theta u) \Big) \frac{(ru)^2}{2}$$

for a suitable function $\theta : \mathbb{S}^{n-1} \to [0,1]$, and since $|\psi_\omega''(r(1+\theta u))| \le C \, \phi_\omega'(r)$, (2.50) gives

$$r^2 \left| \int_{\mathbb{S}^{n-1}} \omega \, u \, \phi_\omega'(r) \right| \;=\; \left| \int_{\mathbb{S}^{n-1}} \omega \, \psi_\omega'' \Big(r(1+\theta u) \Big) \frac{(ru)^2}{2} \right| \le C \, r^2 \int_{\mathbb{S}^{n-1}} \phi_\omega'(r) \, u^2 \,. \tag{2.51}$$

Again by (2.24), and since $\phi_\omega'(r) \le C \, r^{n-1} \, e^{v(r)}$, we deduce from (2.51) that

$$\left| \fint_{\mathbb{S}^{n-1}} \omega \, u \right| \le C \Big(|x_0| \fint_{\mathbb{S}^{n-1}} |u| + \fint_{\mathbb{S}^{n-1}} u^2 \Big) \le C \, \left(\|u\|_{C^0} + r_1 \right) \left(\fint_{\mathbb{S}^{n-1}} u^2 \right)^{1/2} ,$$

Therefore,

$$\sum_{k=1}^n c_{1,k}^2 = \left| \fint_{\mathbb{S}^{n-1}} \omega \, u \right|^2 \le C \, \left(\|u\|_{C^0} + r_1 \right)^2 \fint_{\mathbb{S}^{n-1}} u^2 \,. \tag{2.52}$$

By (2.49) and (2.52), since

$$\fint_{\mathbb{S}^{n-1}} u^2 = \sum_{j \ge 0} \sum_k c_{j,k}^2 \,,$$

we see that

$$\sum_{j \ge 2} \sum_k c_{j,k}^2 > \frac{1}{2} \fint_{\mathbb{S}^{n-1}} u^2 \,,$$

so that (2.45) and (2.47) give

$$\mathrm{Per}(E) \ge \mathrm{Per}(B(x_0,r)) \left\{ 1 + \frac{n}{2} \fint_{S^{n-1}} u^2 - C \, r \, |x_0| \fint_{\mathbb{S}^{n-1}} |u| \right\} . \tag{2.53}$$

Finally, using (2.46) we conclude that

$$\mathrm{Per}(E) \ge \mathrm{Per}(B(m)) \left\{ 1 + \frac{1}{C} \left(|x_0|^2 + \fint_{\mathbb{S}^{n-1}} u^2 \right) - C \, r |x_0| \fint_{\mathbb{S}^{n-1}} |u| \right\} ,$$

and (2.44) follows by Young's inequality (see (1.18)) provided r_1 is small enough. □

3. Some properties of isoperimetric sets

In this section we gather several properties of isoperimetric sets which shall be used in proving Theorems 1.1 and 1.2. We begin with a lemma which was first obtained by Morgan and Pratelli in [MP, Proof of Theorem 3.3] while proving the existence of isoperimetric sets when v is increasing and diverges as $r \to \infty$. As we shall use their argument to prove various uniform estimates, we include it in the following lemma for the sake of clarity. Given $n \geq 2$, let $\kappa(n)$ denote the isoperimetric constant on the sphere $\mathbb{S}^{n-1}$, that is,

$$\kappa(n) = \inf \left\{ \frac{\mathcal{H}^{n-2}(\partial G)}{\mathcal{H}^{n-1}(G)^{(n-2)/(n-1)}} : G \subset \mathbb{S}^{n-1}\,, \partial G \text{ smooth}\,, \mathcal{H}^{n-1}(G) \leq \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2} \right\}. \tag{3.1}$$

We also recall that $P(E)$ denotes the Euclidean distributional perimeter of a Borel set E.

Lemma 3.1. *Given $n \geq 2$ and $M > 0$ set*

$$R_0(n, M) := \left(\frac{2\,M}{n\omega_n} \right)^{1/(n-1)} . \tag{3.2}$$

If $v : (0, \infty) \to (0, \infty)$ is increasing and $P(E) \leq M$, then

$$\mathrm{Per}(E; B_r^c) \geq \kappa(n)^{(n-1)/n}\, e^{v(r)/n}\, \mathrm{Vol}(E \setminus B_r)^{(n-1)/n}\,, \qquad \forall\, r \geq R_0\,. \tag{3.3}$$

Remark 3.1. If v is convex, then the following "Euclidean" lower bound was proved by Kolesnikov and Zhdanov [KZ, Proposition 6.5]:

$$\mathrm{Per}(E) \geq n\omega_n^{1/n} e^{v(0)/n} \mathrm{Vol}(E)^{(n-1)/n}\,, \qquad \forall\, E \subset \mathbb{R}^n\,.$$

Proof of Lemma 3.1. Since perimeter is decreased by taking intersections with convex sets [Ma, Exercise 15.14], we have $P(E) \geq P(E \cap B_s)$, where, for a.e. $s > 0$, $P(E \cap B_s) = P(E; B_s) + \mathcal{H}^{n-1}(E \cap \partial B_s)$, see [Ma, Lemma 15.1]; hence,

$$P(E; B_s^c) \geq \mathcal{H}^{n-1}(E \cap \partial B_s)\,, \qquad \text{for a.e. } s > 0\,.$$

Since v is increasing and $P(E; B_s^c)$ is decreasing in s, this implies

$$\mathrm{Per}(E; B_r^c) \geq e^{v(r)} \operatorname*{ess\,sup}_{s>r} P(E; B_s^c) \geq e^{v(r)} \operatorname*{ess\,sup}_{s>r} \mathcal{H}^{n-1}(E \cap \partial B_s) \tag{3.4}$$

for every $r > 0$. Moreover, we also get

$$\frac{\mathcal{H}^{n-1}(E \cap \partial B_s)}{s^{n-1}} \leq \frac{P(E; B_s^c)}{s^{n-1}} \leq \frac{M}{s^{n-1}} \qquad \text{for a.e. } s > 0\,.$$

Since

$$\frac{M}{s^{n-1}} \leq \frac{n\omega_n}{2}\,, \qquad \forall\, s \geq R_0\,,$$

by a scaling and approximation argument and by definition of $\kappa(n)$, we get

$$\mathcal{H}^{n-2}(\partial^* E \cap \partial B_s) \geq \kappa(n) \mathcal{H}^{n-1}(E \cap \partial B_s)^{(n-2)/(n-1)}\,, \qquad \forall\, s \geq R_0\,.$$

Let us now recall that, being $\partial^* E$ a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$, then the coarea formula [Ma, Theorem 18.8]

$$\int_{\partial^* E} g\, \sqrt{|\nabla u|^2 - (\nu_E \cdot \nabla u)^2}\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}} dt \int_{\partial^* E \cap \{u = t\}} g\, d\mathcal{H}^{n-2}$$

holds for every Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ and Borel function $g : \mathbb{R}^n \to [0, \infty]$. In particular, if we set

$$u(x) = |x|\,, \qquad g(x) = 1_{\partial^* E \setminus B_r}(x)\, e^{V(x)}\,,$$

then the coarea formula gives

$$
\begin{aligned}
\mathrm{Per}(E;B_r^c) &\geq \int_{\partial^* E\setminus B_r} e^{V(x)}\sqrt{1-(\nu_E(x)\cdot x/|x|)^2}\,d\mathcal{H}^{n-1}\\
&= \int_r^\infty e^{v(s)}\mathcal{H}^{n-2}(\partial^*E\cap\partial B_s)\,ds\\
&\geq \kappa(n)\int_r^\infty e^{v(s)}\mathcal{H}^{n-1}(E\cap\partial B_s)^{(n-2)/(n-1)}\,ds\\
&\geq \frac{\kappa(n)\int_r^\infty e^{v(s)}\mathcal{H}^{n-1}(E\cap\partial B_s)\,dr}{\mathrm{esssup}_{s>r}\mathcal{H}^{n-1}(E\cap\partial B_s)^{1/(n-1)}}\\
&= \frac{\kappa(n)\mathrm{Vol}(E\setminus B_r)}{\mathrm{esssup}_{s>r}\mathcal{H}^{n-1}(E\cap\partial B_s)^{1/(n-1)}}\,,
\end{aligned}
$$

which combined with (3.4) immediately gives (3.3). □

An immediate corollary is the following theorem, see [MP, Theorem 3.3].

Theorem 3.1 (Existence theorem). *If $v:[0,\infty)\to\mathbb{R}$ is increasing, with*

$$\lim_{r\to\infty} v(r)=+\infty\,, \tag{3.5}$$

and $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of sets with uniformly bounded weighted perimeters and volumes, then there exists $E\subset\mathbb{R}^n$ such that, up to extracting a subsequence, $\mathrm{Vol}(E_h\Delta E)\to 0$ *as $h\to\infty$. In particular $\mathcal{M}_V(m)$ (the family of isoperimetric sets with weighted volume m, see* (1.12)*) is non-empty for every $m>0$.*

Proof. This follows easily from standard lower semicontinuity and compactness theorems for sets of finite perimeter [Ma, Chapter 12] thanks to condition (3.3), which prevents minimizing sequences to concentrate mass at infinity; see [MP, Section 3] for more details. □

We now gather the basic symmetry and regularity properties of isoperimetric sets. Concerning regularity properties, the key notion here is that of almost-minimality for the perimeter, introduced by Almgren in [Al] in a much more general context, and later rephrased and developed by Bombieri [B] and Tamanini [T1, T2] on integer rectifiable currents and sets of finite perimeter respectively. For the purposes of this paper, it suffices to say that a set of locally finite perimeter E in $\mathbb{R}^n$ is an *almost-minimizer* for the perimeter if there exist positive constants C and r_0 such that

$$P(E;B(x,r))\leq P(F;B(x,r))+C\,r^n\,, \tag{3.6}$$

whenever $E\Delta F\subset\subset B(x,r)$ and $r<r_0$. Referring to [T1, Chapter 1] or to [Ma, Chapter 21] for some heuristics, motivations, variants, and generalizations of this definition, we limit ourselves here to recall that if E satisfies (3.6), then ∂^*E is a $C^{1,1/2}$-hypersurface, and that, after modifying E on a set of Lebesgue measure zero, the singular set $\partial E\setminus\partial^*E$ has Hausdorff dimension at most $n-8$; see [T1, 1.9] or [T2, Theorem 1]. Moreover, if $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of almost-minimizers with uniform constants C and r_0, and E_h locally converges to E in L^1, then E is an almost-minimizer. In addition, given $x\in\partial^*E$ there exists $r_x>0$ and $h_x\in\mathbb{N}$ such that $B(x,r_x)\cap\partial E$ and $B(x,r_x)\cap\partial E_h$ $(h\geq h_x)$ are graphs (over a same $(n-1)$-dimensional disk) of functions u and u_h $(h\geq h_x)$ respectively, with $u_h\to u$ in $C^{1,1/2}$.

Theorem 3.2 (Qualitative properties of isoperimetric sets). *If $v:[0,\infty)\to[0,\infty)$ is smooth (resp. analytic) and $E\in\mathcal{M}_V(m)$, $m>0$, then (up to a modification of E on a set of measure zero) E is a bounded set and ∂E is a $C^{1,1/2}$ hypersurface on $\mathbb{R}^n$. Moreover*

$\partial E \setminus \{0\}$ *is smooth (resp. analytic), and there exists a line* ℓ *passing through the origin such that* E *is symmetric by rotation with respect to* ℓ*, with*

$$H_E^V = H_E + \nabla V \cdot \nu_E = \text{constant}\,, \qquad \text{on } \partial E \setminus \{0\}. \tag{3.7}$$

Moreover, if v *is increasing, then the isoperimetric function* ϕ_V *defined in* (1.1) *is strictly increasing and continuous on* $(0,\infty)$*, and*

$$\phi_V'(m^+) \le H_E^V \le \phi_V'(m^-)\,, \qquad \forall\, E \in \mathcal{M}_V(m)\,.$$

Remark 3.2. Notice that in the above result e^V does not need to be smooth at 0 (for instance, if $v(r) = r$ then $e^V = e^{|x|}$). However, if e^V is smooth also at the origin, then the proof below actually shows that ∂E is globally smooth (analytic).

Proof of Theorem 3.2. Some of the arguments in this proof are well known to specialists, but we include some details and references for convenience of the reader.

• *Step one:* First of all we observe that, up to change $E \in \mathcal{M}_V(m)$ on a set of Lebesgue measure zero, we can ensure that the reduced boundary $\partial^* E$ is dense in the topological boundary ∂E of E, with the characterization

$$\partial E = \Big\{ x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n\, r^n\,,\ \forall\, r > 0 \Big\}\,;$$

see for instance [Ma, Proposition 12.19]. The boundedness of E follows by a rather standard argument based on the isoperimetric inequality, that in the present context is detailed in [RCBM, Theorem 2.1]. As explained above, to show that $\partial^* E$ is a $C^{1,1/2}$-hypersurface which is relatively open (and dense) inside ∂E, and that the singular set $\Sigma(E) = \partial E \setminus \partial^* E$ has Hausdorff dimension at most $n-8$, it suffices to show that E is an almost-perimeter minimizer in $\mathbb{R}^n$; notice that, once $C^{1,1/2}$-regularity is proved, and since V is smooth outside the origin, then the (distributional form of the) Euler-Lagrange equation (3.7) considered in local coordinates will imply by standard elliptic regularity theory that $\partial^* E \setminus \{0\}$ is a smooth hypersurface in $\mathbb{R}^n$ - in fact, analytic, if v is so - see [Ma, Chapter 27] for more details. Since $E \subset B_{R-1}$ for some large radius $R > 0$, it suffices to prove that there exist positive constants C and $s_0 \in (0,1)$ (possibly depending on E, R, and v) such that

$$P(E; B(x,s)) \le P(F; B(x,s)) + C\, s^n \tag{3.8}$$

whenever $E \Delta F \subset\subset B(x,s) \subset B_R$ and $s < s_0$ (indeed, if $B(x,s) \not\subset B_R$ then $E \cap B(x,s) \subset B_{R-1} \cap B(x,s) = \emptyset$, so (3.8) is trivially satisfied).

Since e^V is uniformly positive and locally Lipschitz on $\mathbb{R}^n$, by a minor modification of [Ma, Lemma 17.21] we find two balls $B(x_1,r)$ and $B(x_2,r)$ lying at mutually positive distance, and two positive constants C_0 and σ_0 (depending on E, R, and v only), such that for every $\sigma \in (-\sigma_0, \sigma_0)$ there exist two sets of finite perimeter F_1 and F_2, with

$$E \Delta F_k \subset\subset B(x_k, r)\,, \qquad \mathrm{Vol}(F_k) = \mathrm{Vol}(E) + \sigma\,,$$
$$\Big| \mathrm{Per}(E; B(x,r)) - \mathrm{Per}(F_k; B(x,r)) \Big| \le C_0\, |\sigma|\,, \qquad k = 1,2\,.$$

(These sets are constructed using the flow of two smooth vector fields, respectively supported inside $B(x_1,r)$ and $B(x_2,r)$, which are almost pointing in the direction of the normal, see the proof of [Ma, Lemma 17.21] for more details.)

Let us now fix any ball $B(x,s) \subset B_R$ with $s < s_0$, where s_0 is chosen sufficiently small so that $\omega_n\, s_0^n < \sigma_0$ and

$$\text{either} \quad |x - x_1| > s_0 + r \quad \text{or} \quad |x - x_2| > s_0 + r\,.$$

If F is any set of finite perimeter with $E \Delta F \subset\subset B(x,s)$, and assuming without loss of generality that $B(x,s) \cap B(x_1,r) = \emptyset$, we define $\sigma := \mathrm{Vol}(E) - \mathrm{Vol}(F)$ and consider the

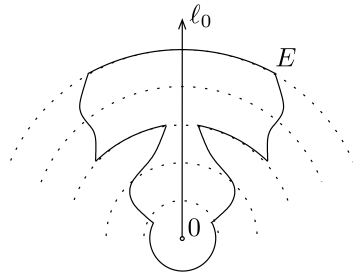

FIGURE 2. Symmetrization by spherical caps with respect to an half-line ℓ_0 preserves weighted volume and decreases weighted perimeter. In the picture, a set which is left invariant by spherical symmetrization (note that the orthogonal sections of E with respect to ℓ_0 need not to be $(n-1)$-dimensional balls).

set F_1 given by the construction described above. Then it is immediate to check that the set

$$G=\Big(F\cap B(x,s)\Big)\cup\Big(F_1\cap B(x_1,r)\Big)\cup\Big(E\cap\Big(B(x,s)\cup B(x_1,r)\Big)^c\Big)$$

satisfies $\mathrm{Vol}(G)=\mathrm{Vol}(E)$, so by minimality $\mathrm{Per}(E)\le\mathrm{Per}(G)$. We thus find

$$\mathrm{Per}(E;B(x,s)\cup B(x_1,r))\le\mathrm{Per}(F;B(x,s))+\mathrm{Per}(F_1;B(x_1,r))\,,$$

which in turn implies

$$\begin{aligned}\mathrm{Per}(E;B(x,s)) &\le \mathrm{Per}(F;B(x,s))+C_0\,|\mathrm{Vol}(E)-\mathrm{Vol}(F)|\\ &\le \mathrm{Per}(F;B(x,s))+C_0\,e^{v(R)}\,\omega_n\,s^n\,.\end{aligned}$$

Since e^V is a locally Lipschitz function on $\mathbb{R}^n$, there exists a constant C_1, depending only on R and v, such that

$$\begin{aligned}\mathrm{Per}(E;B(x,s)) &\ge e^{V(x)}(1-C_1\,s)\,P(E;B(x,s))\,,\\ \mathrm{Per}(F;B(x,s)) &\le e^{V(x)}\,(1+C_1\,s)\,P(F;B(x,s))\,.\end{aligned}$$

In conclusion, for a constant C_2 depending on E, v, and R only, we conclude that

$$P(E;B(x,s))\le(1+C_2\,s)\,P(F;B(x,s))+C_2\,s^n\,,\tag{3.9}$$

whenever $E\Delta F\subset\subset B(x,s)\subset B_R$ and $s<s_0$. Using (3.9) on $F=E\setminus B(x,s')$ for $s'<s<s_0$ such that $\mathcal{H}^{n-1}(\partial^*E\cap\partial B(x,s'))=0$, and noticing that a.e. $s'<s_0$ satisfies this last property, we conclude that

$$P(E;B(x,s))\le C_3\,s^{n-1}\,,\qquad\forall\,s<s_0\,,\tag{3.10}$$

where C_3 depends on E, R, and v only. We are thus in the position of proving (3.8): if $E\Delta F\subset\subset B(x,s)\subset B_R$ with $s<s_0$, and if $P(F;B(x,s))\ge C_3\,s^{n-1}$, then $P(E;B(x,s))\le P(F;B(x,s))$ by (3.10); if, instead, $P(F;B(x,s))\le C_3\,s^{n-1}$, then by (3.9) we find

$$P(E;B(x,s))\le P(F;B(x,s))+C_2(1+C_3)\,s^n\,;$$

in both cases (3.8) follows, as required.

• *Step two:* Given a half-line ℓ_0 through the origin and $E\subset\mathbb{R}^n$, we define the symmetrization by spherical caps E^* of E with respect to ℓ_0 by replacing the spherical slices $\{E\cap\partial B_r\}_{r>0}$ of E with spherical caps $\{K(E,r)\}_{r>0}$ such that $K(E,r)$ is centered on $\ell_0\cap\partial B_r$ and $\mathcal{H}^{n-1}(K(E,r))=\mathcal{H}^{n-1}(E\cap\partial B_r)$; see Figure 2. Using polar coordinates, it is immediate to check that $\mathrm{Vol}(E)=\mathrm{Vol}(E^*)$. Moreover, by the spherical isoperimetric inequality, the coarea formula

$$\int_{\partial^*E}g\,d\mathcal{H}^{n-1}=\int_{\{x\in\partial^*E:\nu_E(x)=\pm x/|x|\}}g\,d\mathcal{H}^{n-1}+\int_0^\infty ds\int_{\partial^*E\cap\partial B_s}\frac{g(x)\,d\mathcal{H}^{n-2}(x)}{\sqrt{1-(\nu_E(x)\cdot x/|x|)^2}}\,,$$

and Jensen's inequality, we see that $\mathrm{Per}(E) \ge \mathrm{Per}(E^*)$ for every $E \subset \mathbb{R}^n$, and that $\mathrm{Per}(E) = \mathrm{Per}(E^*)$ implies that almost every spherical slice of E is $\mathcal{H}^{n-1}$-equivalent to a spherical cap; see for example [CFMP, Section 4] for a detailed exposition of such a standard symmetrization argument in the case of the Gaussian (Ehrhard) symmetrization. This does not suffice yet to infer rotational symmetry with respect to ℓ_0, since the spherical caps defining the spherical slices of E may fail to be concentric.

• *Step three:* Let $E \in \mathcal{M}_V(m)$. By a continuity argument we can iteratively find $n-1$ mutually orthogonal hyperplanes H_i passing through the origin that define complementary half-spaces H_i^+ and H_i^- such that $\mathrm{Vol}(E \cap H_i^+) = \mathrm{Vol}(E \cap H_i^-) = \mathrm{Vol}(E)/2$. If $\{F_h\}_{h=1}^{2^{n-1}}$ is the family of sets obtained by first intersecting E with $\bigcap_{i=1}^{n-1} H_i^{\sigma(i)}$ for a fixed $\{\sigma(i)\}_{i=1}^{n-1} \subset \{+,-\}$, and then by iteratively reflecting the resulting set with respect to the H_i, then

$$\mathrm{Vol}(F_h) = m, \qquad \sum_{h=1}^{2^{n-1}} \mathrm{Per}(F_h) = 2^{n-1}\mathrm{Per}(E).$$

Combining this with the inequality $\mathrm{Per}(F_h) \ge \mathrm{Per}(E)$ for all $h = 1, \dots, 2^{n-1}$ (which follows by the minimility of E) we deduce that $\mathrm{Per}(F_h) = \mathrm{Per}(E)$, hence $\{F_h\}_{h=1}^{2^{n-1}} \subset \mathcal{M}_V(m)$. By step two, the spherical slices of each F_h are spherical caps which, by the reflection symmetries of F_h, are centered on the line $\ell = \bigcap_{i=1}^{n-1} H_i$. Since each F_h coincides with E on the region $\bigcap_{i=1}^{n-1} H_i^{\sigma(i)}$, we conclude that the spherical slices of E are spherical caps centered on ℓ. Exploiting the corresponding rotational symmetry we see that if the singular set $\Sigma(E)$ is not contained inside ℓ, i.e. $\Sigma(E) \setminus \ell \ne \emptyset$, then $\mathcal{H}^{n-2}(\Sigma(E)) > 0$, contrary to $\mathcal{H}^s(\Sigma(E)) = 0$ for every $s > n-8$. We thus conclude that $\Sigma(E) \subset \ell$.

• *Step four:* We now show that $\Sigma(E)$ is empty. Indeed, since E is symmetric by rotation with respect to ℓ, if $x \in \Sigma(E) \subset \ell$, then every tangent cone K to E at x is going to be symmetric by rotation with respect to ℓ, with $\partial K \setminus \{0\}$ analytic. Moreover, the fact that E is an almost minimizer for the perimeter implies that K is a global minimizer for the perimeter in $\mathbb{R}^n$. Since K is not an half-plane (otherwise x would belong to $\partial^* E$), we see that $\partial K \cap \mathbb{S}^{n-1}$ contains at least a non-equatorial $(n-2)$-dimensional sphere (without loss of generality we can assume that $n \ge 8$, otherwise $\Sigma(E)$ we already know to be empty): hence $|H_K| \ge c > 0$ on that non-equatorial sphere, so K cannot be stationary for the perimeter, a contradiction.

• *Step five:* We prove that ϕ_V is strictly increasing and continuous on $(0,\infty)$. Although this may be deduced from more general results on isoperimetric problems, in our situation the following elementary proof is possible. Fix $m > 0$ and $E \in \mathcal{M}_V(m)$. For every $\lambda > 0$, set $m(\lambda) = \mathrm{Vol}(\lambda\, E)$. Since $m(\lambda) = \lambda^n \int_E e^{V(\lambda\, x)}\, dx$ with v increasing, by differentiation we see that $m(\lambda)$ is of class C^1 and strictly increasing on $\lambda \in (0,\infty)$; similarly, $\mathrm{Per}(\lambda\, E) = \lambda^{n-1} \int_{\partial E} e^{V(\lambda\, x)}\, d\mathcal{H}^{n-1}(x)$ is of class C^1 and strictly increasing on $\lambda \in (0,\infty)$; we thus find that, if $\lambda \in (0,1)$, then

$$\phi_V(m) = \mathrm{Per}(E) > \mathrm{Per}(\lambda\, E) \ge \phi_V(m(\lambda))\,,$$

and by the arbitrariness of m and λ this proves that ϕ_V is strictly increasing on $(0,\infty)$. In addition, if $\lambda > 1$, then by dominated convergence (recall that E is bounded)

$$\mathrm{Per}(E) = \lim_{\lambda\to 1^+} \mathrm{Per}(\lambda\, E) \ge \limsup_{\lambda\to 1^+} \phi_V(m(\lambda)) \ge \phi_V(m) = \mathrm{Per}(E)\,,$$

so ϕ_V is continuous from the right on $(0,\infty)$. Finally, we fix a sequence $m_h \to m^-$ as $h \to \infty$, and prove that

$$\phi_V(m) \le \liminf_{h\to\infty} \phi_V(m_h)\,; \tag{3.11}$$

since ϕ_V is increasing, this will imply that ϕ_V is continuous from the left on $(0,\infty)$. To prove (3.11), without loss of generality and up to extracting a subsequence we may directly assume that

$$\liminf_{h\to\infty}\phi_V(m_h)=\lim_{h\to\infty}\phi_V(m_h)\,. \tag{3.12}$$

If we now consider $E_h\in\mathcal{M}_V(m_h)$, then $\mathrm{Per}(E_h)=\phi_V(m_h)\le\phi_V(m)$ for every $h\in\mathbb{N}$, and by Theorem 3.1 we deduce that, for some subsequence $h(k)\to\infty$, $E_{h(k)}$ converges to a set of finite perimeter E_*, with $\mathrm{Vol}(E_*)=m$; by lower-semicontinuity of perimeter,

$$\phi_V(m)\le\mathrm{Per}(E_*)\le\liminf_{k\to\infty}\mathrm{Per}(E_{h(k)})=\lim_{k\to\infty}\phi_V(m_{h(k)})=\lim_{h\to\infty}\phi_V(m_h)\,,$$

where we have used (3.12); this proves (3.11), thus the continuity of ϕ_V on $(0,\infty)$.

• *Step six:* Since ∂E is smooth, we can find $\varepsilon>0$ and a one-parameter family of diffeomorphisms $\{f_t\}_{|t|<\varepsilon}$, $f_t:\mathbb{R}^n\to\mathbb{R}^n$, satisfying

$$f_t(x)=x+t\,\nu_E(x)\,,\qquad\forall\,x\in\partial E\,.$$

By the area formula and a Taylor's expansion (recall that by stationarity H^V_E is constant on ∂E, see (1.3)),

$$\mathrm{Vol}(f_t(E))=\mathrm{Vol}(E)+t\int_{\partial E}e^V\,d\mathcal{H}^{n-1}+O(t^2)\,,$$
$$\mathrm{Per}(f_t(E))=\mathrm{Per}(E)+t\,H^V_E\int_{\partial E}e^V\,d\mathcal{H}^{n-1}+O(t^2)\,.$$

If we set $m(t)=\mathrm{Vol}(f_t(E))$, then

$$m'(0)=\int_{\partial E}e^V\,d\mathcal{H}^{n-1}>0\,,$$

so that, up to decrease the value of ε, we may safely assume that $m(t)$ is increasing on $(t-\varepsilon,t+\varepsilon)$. In particular, if $t\in(0,\varepsilon)$, then

$$\phi_V'(m^+)=\lim_{t\to0^+}\frac{\phi_V(m(t))-\phi_V(m)}{m(t)-m}\le\lim_{t\to0^+}\frac{\mathrm{Per}(f_t(E))-\mathrm{Per}(E)}{\mathrm{Vol}(f_t(E))-\mathrm{Vol}(E)}=H^V_E\,,$$

and analogously, if $t\in(-\varepsilon,0)$,

$$\phi_V'(m^-)=\lim_{t\to0^-}\frac{\phi_V(m(t))-\phi_V(m)}{m(t)-m}\ge\lim_{t\to0^-}\frac{\mathrm{Per}(f_t(E))-\mathrm{Per}(E)}{\mathrm{Vol}(f_t(E))-\mathrm{Vol}(E)}=H^V_E\,.$$

This concludes the proof. □

As seen in the proof above, upper density estimates (see (3.10)) and almost minimality properties (see (3.6)) for isoperimetric sets follow by rather standard arguments; and the same is true for boundedness estimates, as the one proved in [RCBM, Theorem 2.1]. However, to prove Theorems 1.1 and 1.2, we shall need these estimates to hold *uniformly* on isoperimetric sets in terms of their weighted volume, and of the growth at infinity and the local Lipschitz constants of v. Obtaining such uniform estimates requires a little extra care, as we detail in the next result.

Theorem 3.3 (Uniform estimates for isoperimetric sets)**.** *If $n\ge2$, $\alpha:(0,\infty)\to(0,\infty)$ and $\psi:[0,\infty)\to[0,\infty)$ are strictly increasing functions with*

$$\lim_{r\to\infty}\psi(r)=+\infty\,,$$

then for every $\bar m>0$ there exist positive constants R_1 (depending on n, α, ψ, and $\bar m$ only), and C_1, C_2, and C_3 (depending on n, α, R_1, and $\bar m$ only) with the following property: if $v:[0,\infty)\to[0,\infty)$ is locally Lipschitz, increasing, with $v(0)=0$, and such that

$$\operatorname*{ess\,sup}_{[0,r]}|v'|\le\alpha(r)\,,\qquad v(r)\ge\psi(r)\,,\qquad\forall\,r>0\,,$$

and if $E \in \mathcal{M}_V(m)$, $m < \bar m$, and $x \in \mathbb{R}^n$, then

$$
\begin{aligned}
E &\subset B_{R_1}\,, && (3.13)\\
P(E;B(x,r)) &\le C_1\, r^{n-1}\,, \qquad \forall\, r<1\,, && (3.14)\\
P(E;B(x,s)) &\le P(F;B(x,s)) + \frac{C_3}{m^{1/n}}\, s^n\,, \qquad \forall\, s<r_1\,, && (3.15)
\end{aligned}
$$

whenever $E\Delta F \subset\subset B(x,s)$, with $r_1 = \min\{1, m^{1/n}/C_2\}$.

Proof. Recalling (1.8) and since $v \ge 0$, we see that $m = \mathrm{Vol}(B(m)) \ge \omega_n\, r(m)^n$, that is $r(m) \le (m/\omega_n)^{1/n}$ for every $m>0$. Since $v(r) \le \alpha(r)\, r$ for every $r>0$, we deduce that

$$
\mathrm{Per}(B(m)) = n\omega_n r(m)^{n-1}\, e^{v(r(m))} \le K\, m^{(n-1)/n}\,, \qquad \forall m \le \bar m\,,
$$

where $K = K(n,\alpha,\bar m)$ is defined as

$$
K(n,\alpha,\bar m) = n\omega_n^{1/n}\, e^{(\bar m/\omega_n)^{1/n}\,\alpha((\bar m/\omega_n)^{1/n})}\,. \tag{3.16}
$$

Since $\phi_V(m) \le \mathrm{Per}(B(m))$, we thus conclude that, for every v as in the statement,

$$
\phi_V(m) \le K\, m^{(n-1)/n}\,, \qquad \forall\, m \le \bar m\,. \tag{3.17}
$$

We now divide the argument into various steps.

• *Step one:* Let $\varepsilon \in (0,1)$. We show that, if $E \in \mathcal{M}_V(m)$ with $m \le \bar m$, then

$$
\mathrm{Vol}(E\setminus B_r) \le \varepsilon\, m\,, \qquad \forall\, r \ge r_0(\varepsilon)\,, \tag{3.18}
$$

provided $r_0(\varepsilon) = r_0(n,\alpha,\psi,\bar m,\varepsilon)$ is defined as

$$
r_0(n,\alpha,\psi,\bar m,\varepsilon) = \max\left\{ R_0\left(n, K\bar m^{(n-1)/n}\right), \psi^{-1}\left(\log\left(\frac{K^n}{(\kappa(n)\varepsilon)^{n-1}}\right)\right)\right\}\,, \tag{3.19}
$$

with R_0, K, and $\kappa(n)$ as in (3.2), (3.16), and (3.1). Indeed, given $E \in \mathcal{M}_V(m)$, set

$$
r_E = \inf\Big\{ s>0 : \mathrm{Vol}(E\setminus B_r) \le \varepsilon\, m \quad \forall\, r \ge s\Big\}\,.
$$

Since $P(E) \le \mathrm{Per}(E)$, by (3.3) in Lemma 3.1 we have that

$$
\text{either} \quad r_E \le R_0(n,\phi_V(m))\,, \quad \text{or} \quad e^{v(r_E)} \le \frac{\phi_V(m)^n}{(\kappa(n)\varepsilon\, m)^{n-1}}\,.
$$

In the latter case we have

$$
\psi(r_E) \le v(r_E) \le \log\left(\frac{\phi_V(m)^n}{(\kappa(n)\varepsilon\, m)^{n-1}}\right)\,,
$$

which implies (recall that ψ is strictly increasing, thus invertible)

$$
r_E \le \psi^{-1}\left(\log\left(\frac{\phi_V(m)^n}{(\kappa(n)\varepsilon\, m)^{n-1}}\right)\right)\,.
$$

Hence (3.19) follows immediately from (3.17).

• *Step two:* Given $E \in \mathcal{M}_V(m)$ with $m < \bar m$, we define $m_E(r) = \mathrm{Vol}(E\setminus B_r)$, $r>0$. Then m_E is a decreasing function with

$$
\begin{aligned}
m_E'(r) &= -e^{v(r)}\,\mathcal{H}^{n-1}(E\cap\partial B_r)\,, && \text{for a.e. } r>0\,, && (3.20)\\
\mathrm{Per}(E\cap B_r) &= \mathrm{Per}(E;B_r) + |m_E'(r)|\,, && \text{for a.e. } r>0\,, && (3.21)\\
\mathrm{Per}(E\setminus B_r) &= \mathrm{Per}(E;B_r^c) + |m_E'(r)|\,, && \text{for a.e. } r>0\,, && (3.22)\\
m_E(r) &\le \frac{m}{4}\,, && \text{for every } r>s_0\,, && (3.23)
\end{aligned}
$$

provided $s_0 = s_0(n,\alpha,\psi,\bar m)$ is defined by

$$
s_0(n,\alpha,\psi,\bar m) = r_0\Big(n,\alpha,\psi,\bar m,\frac{1}{4}\Big)\,. \tag{3.24}
$$

Let now $\varphi : [0,\infty) \to [0,1]$ be such that

$$\varphi = 1 \quad \text{on } [0, s_0]\,, \qquad \varphi = 0 \quad \text{on } [2s_0, \infty)\,, \qquad \varphi' = -\frac{1}{s_0} \quad \text{on } [s_0, 2s_0]\,,$$

and define a one parameter family of Lipschitz maps $f_t : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$f_t(x) = \Big(1 + t\,\varphi(|x|)\Big)\,x\,, \qquad x \in \mathbb{R}^n\,.$$

We easily compute that

$$Jf_t(x) = (1 + t\,\varphi(|x|))^{n-1}\Big(1 + t\,\Big(\varphi(|x|) + \varphi'(|x|)|x|\Big)\Big)\,, \qquad \forall\, x \in \mathbb{R}^n\,.$$

In particular, since $\varphi(r) + r\,\varphi'(r) \ge -2$ on $(s_0, 2s_0)$ we have

$$\begin{array}{ll} Jf_t = (1+t)^n \ge 1 + n\,t & \text{on } B_{s_0}; \\ Jf_t \ge 1 - 2t & \text{on } B_{2s_0} \setminus B_{s_0}; \\ Jf_t = 1 & \text{on } B^c_{2s_0}. \end{array}$$

Combining this estimate with the fact that $e^{V(f_t)} \ge e^V$ (since v is increasing), and taking also (3.23) into account, we get

$$\begin{aligned} \mathrm{Vol}(f_t(E)) - \mathrm{Vol}(E) &= \int_E \Big(Jf_t\, e^{V(f_t)} - e^V\Big) \ge \int_{E\cap B_{2s_0}} (Jf_t - 1)\, e^V \\ &\ge n\mathrm{Vol}(E \cap B_{s_0})t - 2\mathrm{Vol}(E \cap (B_{2s_0} \setminus B_{s_0}))\,t \\ &\ge \Big(\frac{3n}{4} - \frac{1}{2}\Big) m\,t \ge m\,t\,. \end{aligned} \tag{3.25}$$

By (3.23) and (3.25), for every $r > s_0$ there exists $t(r) \in (0,1)$ such that

$$m_E(r) = \mathrm{Vol}(f_{t(r)}(E)) - \mathrm{Vol}(E)\,.$$

Let us define

$$F_r = f_{t(r)}(E \cap B_r)\,.$$

Since $f_t(E) \setminus B_{2\,s_0} = E \setminus B_{2\,s_0}$ for every $t < 1$, we see that

$$\begin{aligned} \mathrm{Vol}(F_r) &= \mathrm{Vol}(f_{t(r)}(E)) - \mathrm{Vol}(f_{t(r)}(E \setminus B_r)) \\ &= \mathrm{Vol}(E) + m_E(r) - \mathrm{Vol}(E \setminus B_r) = \mathrm{Vol}(E)\,. \end{aligned}$$

Hence $\mathrm{Per}(E) \le \mathrm{Per}(F_r)$, which in turn gives, for every $r > 2\,s_0$ such that $\mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0$ (that is, for a.e. $r > 2\,s_0$),

$$\mathrm{Per}(E; B_r^c) \le \mathrm{Per}(F_r; B_r) - \mathrm{Per}(E; B_r) + \mathrm{Per}(F_r; \partial B_r)\,.$$

Since $\mathrm{Per}(F_r; B_r) - \mathrm{Per}(E; B_r) = \mathrm{Per}(F_r; B_{2\,s_0}) - \mathrm{Per}(E; B_{2\,s_0})$ for $r > 2\,s_0$, and $\mathrm{Per}(F_r; \partial B_r) = e^{v(r)}\,\mathcal{H}^{n-1}(E \cap \partial B_r)$ for a.e. $r > 2\,s_0$, by (3.21) we deduce that the right hand side in the above formula is equal to

$$\mathrm{Per}(F_r; B_{2\,s_0}) - \mathrm{Per}(E; B_{2\,s_0}) + |m_E'(r)|\,,$$

and the latter is bounded by

$$\int_{B_{2\,s_0}\cap\partial E} \Big[\Big(1 + t(r)\Big)^{n-1} e^{V(x + t(r)x)} - e^{V(x)}\Big]\, d\mathcal{H}^{n-1} + |m_E'(r)|\,,$$

where we used that V is radially increasing, $0 \le \varphi \le 1$, and $\varphi' \le 0$ to infer that, for any M is locally $\mathcal{H}^{n-1}$-rectifiable in $\mathbb{R}^n$,

$$J^M f_t(x)\, e^{V(f_t(x))} \le (1 + t(r))^{n-1}\, e^{V(x + t(r)x)}\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in M,$$

where $J^M f_t$ denotes the tangential Jacobian of f_t.

We now observe that, since $e^{st}-1\le e^s t$ for all $s\ge 0$ and $t\in[0,1]$, for every $x\in B_{2\,s_0}$ we have

$$e^{V(x+t(r)x)}-e^{V(x)}\le e^{V(x)}\left(e^{2\,s_0\,\left(\sup_{[0,4\,s_0]}|v'|\right)\,t(r)}-1\right)\le e^{V(x)}\,e^{2\,s_0\,\alpha(4\,s_0)}\,t(r)\,.$$

Using that $(1+t)^{n-1}\le 1+2^{n-1}t$ for $t\in(0,1)$, and that $t(r)\le m_E(r)/m$ (see (3.25)), combining the estimates above we find

$$\begin{aligned}\mathrm{Per}(E;B_r^c) &\le \left(\left(1+2^{n-1}\,t(r)\right)\left(1+e^{2\,s_0\,\alpha(4\,s_0)}\,t(r)\right)-1\right)\mathrm{Per}(E;B_{2\,s_0})+|m_E'(r)|\\ &\le \left(2^{n-1}+e^{2\,s_0\,\alpha(4\,s_0)}+2^{n-1}e^{2\,s_0\,\alpha(4\,s_0)}\right)t(r)\phi_V(m)+|m_E'(r)|\\ &\le \frac{C_0}{m^{1/n}}\,m_E(r)+|m_E'(r)|\,,\end{aligned}$$

where in the last inequality we have used (3.17), and where $C_0=C_0(n,\alpha,\bar m)$ is defined as

$$C_0(n,\alpha,\bar m)=\left(2^{n-1}+e^{2\,s_0\,\alpha(4\,s_0)}+2^{n-1}e^{2\,s_0\,\alpha(4\,s_0)}\right)K\,. \tag{3.26}$$

Adding $|m_E'(r)|$ to both sides of the above inequality and using (3.22), we get

$$\mathrm{Per}(E\setminus B_r)\le\frac{C_0}{m^{1/n}}\,m_E(r)+2\,|m_E'(r)|\,,\qquad\text{for a.e. } r\in(2\,s_0,R_E)\,.$$

Setting $R_E=\sup\{r>0: m_E(r)>0\}$, since by (3.3) we have

$$\mathrm{Per}(E\setminus B_r)\ge\Big(\kappa(n)m_E(r)\Big)^{(n-1)/n}\,,$$

(recall that $s_0\ge R_0\left(n,K\bar m^{(n-1)/n}\right)$ by step one, and that $v\ge 0$), we conclude that

$$\Big(\kappa(n)\,m_E(r)\Big)^{(n-1)/n}\le\frac{C_0}{m^{1/n}}\,m_E(r)+2\,|m_E'(r)|\,,\qquad\text{for a.e. } r\in(2\,s_0,R_E)\,. \tag{3.27}$$

We now notice that

$$\frac{C_0}{m^{1/n}}\,m_E(r)\le\frac12\Big(\kappa(n)\,m_E(r)\Big)^{(n-1)/n}\,,\qquad\text{provided}\qquad m_E(r)\le\frac{\kappa(n)^{n-1}}{(2C_0)^n}\,m\,.$$

Thus, if we set $\varepsilon_0=\varepsilon_0(n,\alpha,\bar m)$ and $s_1=s_1(n,\alpha,\psi,\bar m)$ as

$$\varepsilon_0(n,\alpha,\bar m)=\min\left\{\frac{\kappa(n)^{n-1}}{(2C_0)^n},\frac14\right\},\qquad s_1(n,\alpha,\psi,\bar m)=\max\Big\{2\,s_0\,,r_0(\varepsilon_0)\Big\}\,, \tag{3.28}$$

then by step one

$$\frac12\Big(\kappa(n)\,m_E(r)\Big)^{(n-1)/n}\le 2\,|m_E'(r)|\,,\qquad\text{for a.e. } r\in(s_1,R_E)\,.$$

Since $m_E(r)>0$ if $r<R_E$, we may divide both sides by $m_E(r)^{(n-1)/n}$ and integrate the resulting inequality over (s_1,R_E) to conclude that

$$\frac{\kappa(n)^{(n-1)/n}}{4\,n}\,(R_E-s_1)\le m_E(s_1)^{1/n}\le(\varepsilon_0 m)^{1/n}\,.$$

By definition of R_E we finally deduce that, up to modification of E on a set of Lebesgue measure zero, we have $E\subset B_{R_1}$ for $R_1=R_1(n,\alpha,\psi,\bar m)$ defined as

$$R_1(n,\alpha,\psi,\bar m)=s_1+\frac{4\,n\varepsilon_0^{1/n}}{\kappa(n)^{(n-1)/n}}\,\bar m^{1/n}. \tag{3.29}$$

• *Step three:* We prove (3.14). If $B(x,r)\cap B_{R_1}=\emptyset$ then $P(E;B(x,r))=0$ by (3.13), and (3.14) follows trivially; if on the contrary $B(x,r)\cap B_{R_1}\neq\emptyset$, then by $r<1$ we have $|x|\le R_1+1$ and thus

$$\mathrm{Per}(B(x,r))\le n\omega_n\,r^{n-1}\,e^{v(R_1+1)}\le n\omega_n\,r^{n-1}\,e^{(R_1+1)\,\alpha(R_1+1)}\,.$$

At the same time, since ϕ_V is increasing and $\mathrm{Vol}(E\cup B(x,r))\geq\mathrm{Vol}(E)$, it must be $\mathrm{Per}(E)\leq\mathrm{Per}(E\cup B(x,r))$ for every $E\in\mathcal{M}_V(m)$, hence

$$\mathrm{Per}(E;B(x,r))\leq\mathrm{Per}(B(x,r))\leq C_1\,r^{n-1}\qquad\text{for a.e. } r\in(0,1),$$

where $C_1=C_1(n,\alpha,R_1,\bar m)=n\omega_n\,e^{(R_1+1)\,\alpha(R_1+1)}$. This inequality trivially extends to every $r\in(0,1)$, and (3.14) follows immediately from the fact that $P(E;B(x,r))\leq\mathrm{Per}(E;B(x,r))$ (recall that $v\geq 0$).

• *Step four:* We show the existence of $C_2=C_2(n,\alpha,R_1,\bar m)<\infty$ such that

$$\mathrm{Per}(E)\leq\left(1+C_2\left(1-\frac{\mathrm{Vol}(F)}{\mathrm{Vol}(E)}\right)^{+}\right)\mathrm{Per}(F)\,,\tag{3.30}$$

whenever $E\in\mathcal{M}_V(m)$, $m<\bar m$, $E\Delta F\subset\subset B(x,s)$, $x\in\mathbb{R}^n$, and $s<r_1$ for

$$r_1=\min\left\{1,\frac{m^{1/n}}{C_2}\right\}.\tag{3.31}$$

Since ϕ_V is increasing, we can directly assume that $\mathrm{Vol}(F)<\mathrm{Vol}(E)$; moreover, since $r_1\leq 1$ and $E\subset B_{R_1}$ we can also assume that $|x|\leq R_1+1$ (otherwise, we have necessarily $E=F$). Thus, $F\subset B_{R_1+2}$, and

$$0\leq\mathrm{Vol}(E)-\mathrm{Vol}(F)\leq e^{v(R_1+2)}\omega_n\,s^n\,,\tag{3.32}$$

so that

$$r_1<\min\left\{1,\Big(\frac{m}{2\omega_n\,e^{(R_1+2)\,\alpha(R_1+2)}}\Big)^{1/n}\right\}\qquad\text{implies}\qquad\mathrm{Vol}(F)\geq\frac{m}{2}\,.\tag{3.33}$$

Let us now consider the function $f(t)=\mathrm{Vol}((1+t)F)$, $t>0$; since $f(0)=\mathrm{Vol}(F)<m$ and $f(t)\to\infty$ as $t\to\infty$, there exists $t_F>0$ such that $f(t_F)=m$, and thus

$$\mathrm{Per}(E)\leq\mathrm{Per}((1+t_F)F)\leq(1+t_F)^{n-1}\int_{\partial^*F}e^{V(y+t_Fy)}\,d\mathcal{H}^{n-1}(y)\,.\tag{3.34}$$

In fact, since V is radially increasing,

$$\begin{aligned}\mathrm{Vol}(E)-\mathrm{Vol}(F)&=\mathrm{Vol}((1+t_F)F)-\mathrm{Vol}(F)\\&=(1+t_F)^n\int_F e^{V(x+t_Fx)}-\int_F e^{V(x)}\\&\geq[(1+t_F)^n-1]\,\mathrm{Vol}(F)\geq\frac{n\,m}{2}\,t_F\,,\end{aligned}\tag{3.35}$$

which gives

$$0<t_F\leq\frac{2}{n}\,\frac{\mathrm{Vol}(E)-\mathrm{Vol}(F)}{\mathrm{Vol}(E)}\,.\tag{3.36}$$

In particular $t_F\leq 2/n$, and since $F\subset B_{R_1+2}$ we find that, for every $y\in\partial^*F$,

$$e^{V(y+t_Fy)}\leq e^{t_F\,(R_1+3)\,\alpha(R_1+3)}\,e^{V(y)}\,,$$

so that, by (3.34),

$$\mathrm{Per}(E)\leq(1+C\,t_F)\,\mathrm{Per}(F)\,,\tag{3.37}$$

for some constant C depending on n, α, R_1, and $\bar m$ only. Plugging (3.36) into this last inequality we find a value of C_2 such that (3.30) for every $s<r_1$, with r_1 defined as in (3.31).

• *Step five:* We finally prove (3.15). If $E\in\mathcal{M}_V(m)$ with $m\leq\bar m$, then by (3.37), (3.36), and (3.32), we have

$$\mathrm{Per}(E)\leq\left(1+\frac{C_2}{m}\,\omega_n\,s^n\right)\mathrm{Per}(F)\,,$$

whenever $E\Delta F \subset\subset B(x,s)$, $x\in\mathbb{R}^n$, and $s<r_1$, with r_1 as in (3.31). Then, for a.e. $s<r_1$ (precisely, for those s such that $\mathcal{H}^{n-1}(\partial^*E\cap\partial B(x,s))=0$) this gives

$$\begin{aligned}\mathrm{Per}(E;B(x,s)) &\le \Big(1+\frac{C_2}{m}\,\omega_n\,s^n\Big)\,\mathrm{Per}(F;B(x,s))+\frac{C_2}{m}\,\omega_n\,s^n\,\mathrm{Per}(E;B(x,s)^c)\\ &\le \Big(1+\frac{C_2}{m}\,\omega_n\,s^n\Big)\,\mathrm{Per}(F;B(x,s))+\frac{C_2}{m}\,\phi_V(m)\,\omega_n\,s^n\,.\end{aligned}$$

On the one hand,

$$\mathrm{Per}(E;B(x,s))\ge e^{V(x)}e^{-\alpha(R_2)\,s}P(E;B(x,s))\ge e^{V(x)}(1-\alpha(R_1)\,s)\,P(E;B(x,s))\,;$$

on the other hand,

$$\mathrm{Per}(F;B(x,s))\le e^{V(x)}e^{\alpha(R_1+1)s}P(F;B(x,s))\,;$$

summarizing,

$$P(E;B(x,s))\le\Big(1+\frac{C_2}{m}\,\omega_n\,s^n\Big)(1+C\,s)\,P(F;B(x,s))+C_2\,\frac{\phi_V(m)}{m}\,\omega_n\,s^n\,,$$

where C denotes a generic constant depending on n, α, R_1, and $\bar m$ only. Since by the upper density estimate $P(E;B(x,s))\le C_1\,s^{n-1}$, if $P(F;B(x,s))\ge C_1\,s^{n-1}$ then (3.15) immediately follows. Otherwise we have

$$\begin{aligned}P(E;B(x,s)) &\le P(F;B(x,s))+\Big(\frac{C_2}{m}\,\omega_n\,s^n(1+C\,s)+C\,s\Big)C_1s^{n-1}+C_1\,\frac{\phi_V(m)}{m}\,\omega_n\,s^n\\ &\le P(F;B(x,s))+C\Big(1+\frac{\phi_V(m)}{m}+\frac{s^{n-1}}{m}\Big)s^n\,,\end{aligned}$$

and using that $s^{n-1}\le r_1^{n-1}\le m^{(n-1)/n}/C_2^{n-1}$ (by (3.31)) and $\phi_V(m)\le K\,m^{(n-1)/n}$ (by (3.17)), we finally obtain (3.15). □

4. Proof of Theorem 1.1-(i)

We are now in the position to combine Theorem 2.3 with the results of the previous sections to prove Theorem 1.1-(i).

Proof of Theorem 1.1-(i). Let $M(v)$ denote the sets of those $m>0$ such that $\mathcal{M}_V(m)=\{B(m)\}$, where $B(m)=B_{r(m)}$ is such that $\mathrm{Vol}(B(m))=m$. Arguing by contradiction, let us consider $m\in M(v)$ and a sequence $\{m_h\}_{h\in\mathbb{N}}$ with $m_h\to m$ such that for every $h\in\mathbb{N}$ there exists $E_h\in\mathcal{M}_V(m_h)$ with $|E_h\Delta B(m_h)|>0$. By Theorem 3.1 and the continuity of ϕ_V (see Theorem 3.2), up to a subsequence there exists $E\in\mathcal{M}_V(m)$ such that $\mathrm{Vol}(E_h\Delta E)\to 0$ as $h\to\infty$. But since $\mathcal{M}_V(m)=\{B(m)\}$, this implies $|E_h\Delta B(m)|\to 0$ as $h\to\infty$. Since m_h is bounded away from zero (as it converges to $m>0$), by Theorem 3.3 there exist positive constants C and r, independent of h, such that

$$P(E_h;B(x,s))\le P(F;B(x,s))+C\,s^n\,,$$

whenever $E_h\Delta F\subset\subset B(x,s)$, $s<r$; combining these two facts with [T1, Theorem 1.9], we find that $E_h\to B(m)=B_{r(m)}$ in C^1, meaning that for every $h\in\mathbb{N}$ large enough there exists $\bar u_h\in C^1(\mathbb{S}^{n-1};[-1,\infty))$ such that

$$E_h=\Big\{t\,(1+\bar u_h(x))\,x: x\in\mathbb{S}^{n-1}\,,0\le t<r(m)\Big\}\,,\qquad \lim_{h\to\infty}\|\bar u_h\|_{C^1}=0\,.$$

In turn, since $r(m_h)\to r(m)$, we find $u_h\in C^1(\mathbb{S}^{n-1};[-1,\infty))$ such that

$$E_h=\Big\{t\,(1+u_h(x))\,x: x\in\mathbb{S}^{n-1}\,,0\le t<r(m_h)\Big\}\,,\qquad \lim_{h\to\infty}\|u_h\|_{C^1}=0\,,$$

where, being $|E_h \Delta B(m_h)| > 0$, it must be $u_h \not\equiv 0$ on $\mathbb{S}^{n-1}$. Let us now consider an open interval (m_1, m_2) such that $m \in (m_1, m_2)$, and denote by $\varepsilon_1 = \varepsilon_1(n, \alpha, \beta, \gamma, m_1, m_2)$ the constant appearing in Theorem 2.3, with

$$\alpha = \sup_{[r(m_1)/2, 2\, r(m_2)]} |v'|\,, \qquad \beta = \sup_{[r(m_1)/2, 2\, r(m_2)]} v''\,, \qquad \gamma = \inf_{[r(m_1)/2, 2\, r(m_2)]} v''\,.$$

Hence, provided h is large enough we have

$$\|u_h\|_{C^1} \le \varepsilon_1 \min\Big\{1, r(m_1)^2\Big\}\,,$$

so that by (2.26)

$$\mathrm{Per}(E_h) \ge \mathrm{Per}(B(m_h))\Big\{1 + \frac{\gamma\, r(m_1)^2}{4} \fint_{\mathbb{S}^{n-1}} u^2\Big\} > \mathrm{Per}(B(m_h))\,.$$

Since u_h is not identically 0 on $\mathbb{S}^{n-1}$, this contradicts $E_h \in \mathcal{M}(m_h)$ concluding the proof. □

5. Diameter of isoperimetric sets in the small weighted volume regime

In this section we prove that, when m is sufficiently small, the diameter of minimizers to (1.1) goes to zero in a quantitative way with respect to m. More precisely, we aim to show the following result:

Theorem 5.1 (Uniform diameter decay)**.** *Given $n \ge 2$ and positive constants α and R, there exist positive constants m_0 and C_0, depending on n, α, and R only with the following property: If $v : [0, \infty) \to [0, \infty)$ is locally Lipschitz and increasing, with $v(0) = 0$ and*

$$\operatorname*{ess\,sup}_{[0, 2\, R]} v' \le \alpha\,,$$

then for every $E \in \mathcal{M}_V(m)$ with $E \subset B_R$ and $m \le m_0$ we have

$$\mathrm{diam}\, E \le C_0\, m^{1/2n}\,. \tag{5.1}$$

Before discussing the proof of Theorem 5.1, we prove the following lemma. Let us recall that $\mathcal{H}^{n-1}_V(F) = \int_F e^V\, d\mathcal{H}^{n-1}$ for every $F \subset \mathbb{R}^n$.

Lemma 5.1. *If $r > 0$ and $F \subset \mathbb{R}^n$ is a Borel set with $|F| < \infty$, then there exists a partition $\mathcal{Q}$ of $\mathbb{R}^n$ into parallel cubes of side length r such that*

$$\mathrm{Vol}(F) \ge \frac{r}{2\, n} \sum_{Q \in \mathcal{Q}} \mathcal{H}^{n-1}_V(F \cap \partial Q)\,. \tag{5.2}$$

Proof. By Fubini's theorem, for every $r > 0$ and $1 \le h \le n$ we have

$$\begin{aligned} \mathrm{Vol}(F) &= \sum_{k \in \mathbb{Z}} \int_{k\, r}^{(k+1)\, r} \mathcal{H}^{n-1}_V\Big(F \cap \{x_h = t\}\Big)\, dt \\ &= r \int_0^1 \sum_{k \in \mathbb{Z}} \mathcal{H}^{n-1}_V\Big(F \cap \Big\{x_h = r(k+s)\Big\}\Big)\, ds\,. \end{aligned}$$

Thus, there exists a vector $z = (z_1, \dots, z_n) \in (0,1)^n$ such that

$$\mathrm{Vol}(F) \ge r \sum_{k \in \mathbb{Z}} \mathcal{H}^{n-1}_V\Big(F \cap \Big\{x_h = r(k + z_h)\Big\}\Big)\,, \qquad 1 \le h \le n\,.$$

So, if we define $\mathcal{Q}$ to be the partition of $\mathbb{R}^n$ into cubes of side length r such that

$$\bigcup_{Q \in \mathcal{Q}} \partial Q = \bigcup_{h=1}^{n} \bigcup_{k \in \mathbb{Z}} \Big\{x_h = r(k + z_h)\Big\}\,,$$

we find

$$r\sum_{Q\in\mathcal{Q}}\mathcal{H}^{n-1}_V(F\cap\partial Q)=2\,r\sum_{h=1}^n\sum_{k\in\mathbb{Z}}\mathcal{H}^{n-1}_V\Big(F\cap\Big\{x_h=r(k+z_h)\Big\}\Big)\le 2\,n\,\mathrm{Vol}(F)\,,$$

where the factor 2 in the first equality comes from the fact each facet of the cubes is counted twice. □

Proof of Theorem 5.1. In the following proof, C denotes a positive constant depending only on n, α, and R.

• *Step one:* We show that if m_0 is small enough and $E\in\mathcal{M}_V(m)$ with $m\le m_0$ and $E\subset B_R$, then there exists a cube Q of side-length $m^{1/2n}$ such that

$$\mathrm{Vol}(E\setminus Q)\le C\,m^{1+1/2n}\,. \tag{5.3}$$

Indeed, let $\mathcal{Q}$ be a partition of $\mathbb{R}^n$ by cubes of side-length r (r to be chosen) such that (5.2) holds true. Then by the classical isoperimetric inequality applied to $E\cap Q$ we find

$$P(E\cap Q)\ge n\omega_n^{1/n}|E\cap Q|^{(n-1)/n}.$$

Since $E\subset B_R$ and $|v'|\le\alpha$ on $[0,R]$, there exists C_0 such that $\mathrm{osc}_{Q\cap E}V\le C_0 r$; hence

$$n\omega_n^{1/n}\mathrm{Vol}(E\cap Q)^{(n-1)/n}\le e^{2C_0r}\mathrm{Per}(E\cap Q)=e^{2C_0r}\left(\mathrm{Per}(E;Q)+\mathcal{H}^{n-1}_V(E\cap\partial Q)\right).$$

Adding up over $Q\in\mathcal{Q}$ and using Lemma 5.1, we infer

$$n\omega_n^{1/n}\sum_{Q\in\mathcal{Q}}\mathrm{Vol}(E\cap Q)^{(n-1)/n}\le e^{2C_0r}\mathrm{Per}(E)+e^{2C_0r}\frac{2\,n\,m}{r}\,. \tag{5.4}$$

Since $v\ge 0$ we have $|B_{r(m)}|\le\mathrm{Vol}(B_{r(m)})=m$, that is $r(m)\le(m/\omega_n)^{1/n}$. Hence, if m_0 is such that $r(m_0)\le R$ we find

$$\begin{aligned}\mathrm{Per}(E)&\le\mathrm{Per}(B_{r(m)})\le n\omega_n r(m)^{(n-1)/n}e^{v(r(m))}\le n\omega_n^{1/n}m^{(n-1)/n}e^{(m/\omega_n)^{1/n}\alpha}\\&\le n\omega_n^{1/n}m^{(n-1)/n}(1+C\,m^{1/n})\,.\end{aligned} \tag{5.5}$$

Combining this estimate with (5.4), for r sufficiently small we get

$$n\omega_n^{1/n}\sum_{Q\in\mathcal{Q}}\mathrm{Vol}(E\cap Q)^{(n-1)/n}\le n\omega_n^{1/n}m^{(n-1)/n}+C\left(rm^{(n-1)/n}+\frac{m}{r}\right),$$

or equivalently

$$\sum_{Q\in\mathcal{Q}}\left(\frac{\mathrm{Vol}(E\cap Q)}{m}\right)^{(n-1)/n}-1\le C\left(r+\frac{m^{1/n}}{r}\right).$$

Hence, if we set $r=m^{1/2n}$ we obtain

$$\sum_{Q\in\mathcal{Q}}\left(\frac{\mathrm{Vol}(E\cap Q)}{m}\right)^{(n-1)/n}-1\le Cm^{1/2n}. \tag{5.6}$$

To conclude the proof, it suffices now to exploit the uniform concavity of the function $t\mapsto\Psi(t)=t^{(n-1)/n}+(1-t)^{(n-1)/n}-1$: first of all, since

$$\sum_{Q\in\mathcal{Q}}\frac{\mathrm{Vol}(E\cap Q)}{m}=1\,,$$

by (5.6) and the concavity of $t \mapsto t^{(n-1)/n}$ we get that, for any subfamily $\mathcal{Q}' \subset \mathcal{Q}$,

$$\begin{aligned}
&\Psi\Big(\sum_{Q\in\mathcal{Q}'}\frac{\mathrm{Vol}(E\cap Q)}{m}\Big)\\
&= \Big(\sum_{Q\in\mathcal{Q}'}\frac{\mathrm{Vol}(E\cap Q)}{m}\Big)^{(n-1)/n} + \Big(\sum_{Q\in\mathcal{Q}\setminus\mathcal{Q}'}\frac{\mathrm{Vol}(E\cap Q)}{m}\Big)^{(n-1)/n} - 1 \qquad (5.7)\\
&= \sum_{Q\in\mathcal{Q}}\Big(\frac{\mathrm{Vol}(E\cap Q)}{m}\Big)^{(n-1)/n} - 1 \le Cm^{1/2n}.
\end{aligned}$$

Moreover, as noticed for instance in [FiMP, Equation (3.13)], there exists a constant $c(n) > 0$ such that

$$\Psi(t) \ge c(n)\, t^{(n-1)/n} \qquad \forall\, t \in [0, 1/2].$$

This estimate combined with (5.7) implies the validity of the following implication:

$$\sum_{Q\in\mathcal{Q}'}\frac{\mathrm{Vol}(E\cap Q)}{m} \le \frac{1}{2} \quad\Rightarrow\quad \sum_{Q\in\mathcal{Q}'}\frac{\mathrm{Vol}(E\cap Q)}{m} \le \Big(\frac{C}{c(n)}m^{1/2n}\Big)^{1+1/n} \le Cm^{1/2n}.$$

By the arbitrariness of the subfamily $\mathcal{Q}'$ is is easy to deduce that the existence of a cube Q such that $\mathrm{Vol}(E \setminus Q) \le C\, m^{1+1/2n}$, concluding the proof of the first step.

• *Step two:* The argument here is very similar to the one in step two of the proof of Theorem 3.3. However, since some estimates are different and the proof is not too long, for completeness we provide all the details.

Let Q be given depending on E as in step one, set $Q(t) = (1 + t)Q$, and $w(t) = \mathrm{Vol}(E \setminus Q(t))$. Notice that $w'(t) = -\mathcal{H}^{n-1}_V(E \cap \partial Q(t))$ for a.e. $t \in (0, 1)$. By the classical isoperimetric inequality and using that the oscillation of v over $Q(t)$ is bounded by $C\, m^{1/2n}$ for $t \in (0, 1)$, for m sufficiently small we get

$$\begin{aligned}
\frac{n\omega_n^{1/n} w(t)^{(n-1)/n}}{2} &\le e^{-Cm^{1/2n}} n\omega_n^{1/n} w(t)^{(n-1)/n}\\
&\le \mathrm{Per}(E \setminus Q(t)) = \mathrm{Per}(E; \mathbb{R}^n \setminus Q(t)) + |w'(t)|\,.
\end{aligned}$$

Consider now the set $E_\lambda(t) = (1 + \lambda)\,(E \cap Q(t))$, $\lambda \ge 0$. Since $\mathrm{Vol}(E_0(t)) = m - w(t)$ and V is radially increasing, arguing as in (3.36) we get

$$\mathrm{Vol}(E_\lambda(t)) \ge (1 + \lambda)^n \big[m - w(t)\big]\,,$$

which implies by continuity the existence of some value $\lambda(t)$, with

$$0 < \lambda(t) \le C\frac{w(t)}{m},$$

such that $\mathrm{Vol}(E_{\lambda(t)}(t)) = m$. So, by minimality of E and by the local Lipschitz continuity of v, we easily obtain

$$\begin{aligned}
\mathrm{Per}(E; Q(t)) + \mathrm{Per}(E; \mathbb{R}^n \setminus Q(t)) &= \mathrm{Per}(E) \le \mathrm{Per}(E_{\lambda(t)}(t))\\
&\le e^{C\lambda(t)}(1 + \lambda(t))^{n-1}\mathrm{Per}(E \cap Q(t))\\
&\le \Big(1 + C\,\frac{w(t)}{m}\Big)\Big(\mathrm{Per}(E; Q(t)) + |w'(t)|\Big)\\
&\le \mathrm{Per}(E; Q(t)) + C\,\frac{w(t)}{m}\mathrm{Per}(E) + 2|w'(t)|\,.
\end{aligned}$$

We now observe that (5.5) implies the bound $\mathrm{Per}(E)\le C m^{(n-1)/n}$, which combined with the above estimates gives

$$\mathrm{Per}(E;\mathbb{R}^n\setminus Q(t))\le C\,\frac{w(t)}{m^{1/n}}+2|w'(t)|\,,\qquad\text{for a.e. } t\in(0,1).$$

Adding up $|w'(t)|$ to both sides, and noticing that for a.e. $t>0$

$$\begin{aligned}\mathrm{Per}(E;\mathbb{R}^n\setminus Q(t))+|w'(t)| &= \mathrm{Per}(E\setminus Q(t))\ge P(E\setminus Q(t))\\ &\ge n\omega_n^{1/n}|E\setminus Q(t)|^{(n-1)/n}\\ &\ge n\omega_n^{1/n}e^{-(n-1)v(R)/n}w(t)^{(n-1)/n}\,,\end{aligned}$$

(here we used that $v\ge 0$, the Euclidean isoperimetric inequality, and the inclusion $E\subset B_R$), we deduce that

$$n\omega_n^{1/n}e^{-(n-1)v(R)/n}\,w(t)^{(n-1)/n}-C\,\frac{w(t)}{m^{1/n}}\le 3|w'(t)|\,,\qquad\text{for a.e. } t\in(0,1).$$

Since $w(t)\le C\,m^{1+1/2n}$, for m small enough we get

$$n\omega_n^{1/n}e^{-(n-1)v(R)/n}w(t)^{(n-1)/n}\le 6|w'(t)|\,,$$

or equivalently, since $t\mapsto w(t)$ is decreasing,

$$w'(t)\le -\frac{6\,e^{(n-1)v(R)/n}}{\omega_n^{1/n}}\,.$$

Since $w(0)\le C\,m^{1+1/2n}$ this implies that $w(t)=0$ for every $t\ge C\,m^{1+1/2n}$, that is

$$E\subset\left(1+C\,m^{1+1/2n}\right)Q\,,$$

where the diameter of Q is equal to $\sqrt{n}\,m^{1/2n}$ (recall that in step one we chose the side of Q to be $m^{1/2n}$). This concludes the proof. □

6. Proof of Theorem 1.1-(ii)

We now combine the results of sections 2, 3, and 5 to prove Theorem 1.1-(ii). We first show that isoperimetric sets with sufficiently small weighted volume are necessarily C^1-small radial perturbations of balls with centers converging to the origin. We introduce the notation

$$\mathrm{bar}(E)=\frac{1}{\mathrm{Vol}(E)}\int_E x\,e^{V(x)}\,dx\,,$$

for the weighted barycenter of a set $E\subset\mathbb{R}^n$ satisfying $\mathrm{Vol}(E)>0$ and $\int_E|x|\,e^{V(x)}\,dx<\infty$.

Theorem 6.1. *If $n\ge 2$, and $\alpha:[0,\infty)\to[0,\infty)$ and $\psi:[0,\infty)\to[0,\infty)$ are strictly increasing functions with*

$$\psi(0)=0\,,\qquad\lim_{r\to\infty}\psi(r)=+\infty\,,$$

then for every $\rho>0$ there exist a positive constant $\hat m$ depending on n, α, ψ, and ρ only, with the following property: if $v:[0,\infty)\to[0,\infty)$ is locally Lipschitz, increasing, with $v(0)=0$, and such that

$$\operatorname*{ess\,sup}_{[0,r]}|v'|\le\alpha(r)\,,\qquad v(r)\ge\psi(r)\,,\qquad\forall\,r>0\,,\tag{6.1}$$

and if $E\in\mathcal{M}_V(m)$ with $m\le\hat m$, then there exist $u\in C^1(\mathbb{S}^{n-1};[-1,\infty))$, $x_0\in\mathbb{R}^n$, and $r>0$, with

$$E=x_0+\left\{t\,(1+u(\omega))\,\omega:\omega\in\mathbb{S}^{n-1}\,,0\le t<r\right\},\quad r+|x_0|+\|u\|_{C^1}\le\rho\,,\tag{6.2}$$

$\mathrm{Vol}(B(x_0,r))=\mathrm{Vol}(E)$, and $\mathrm{bar}(B(x_0,r))=\mathrm{bar}(E)$.

Proof. To prove the result it suffices to show that, if $\{v_h\}_{h\in\mathbb{N}}$ is a sequence of locally Lipschitz and increasing functions satisfying $v_h(0)=0$ and (6.1), and $E_h \in \mathcal{M}_{V_h}(m_h)$ is a sequence of sets with $V_h(x)=v_h(|x|)$ and $m_h \to 0$ as $h\to\infty$, then for m_h sufficiently small (where the smallness has to depend on n, α, ψ, and ρ only) there exist $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{S}^{n-1};[-1,\infty))$, $\{x_h\}_{h\in\mathbb{N}}\subset\mathbb{R}^n$, and $\{r_h\}_{h\in\mathbb{N}}\subset(0,\infty)$, such that

$$\mathrm{Vol}_h(B(x_h,r_h)) = \mathrm{Vol}_h(E_h)\,,\qquad \mathrm{bar}_h(B(x_h,r_h)) = \mathrm{bar}_h(E_h)\,,$$

and

$$E_h = x_h + \Big\{t\,(1+u_h(\omega))\,\omega : \omega\in\mathbb{S}^{n-1}\,, 0\le t<r_h\Big\}\,,\qquad \lim_{h\to\infty} r_h+|x_h|+\|u_h\|_{C^1}=0\,. \tag{6.3}$$

(Here and in the following, we shall denote by Per_h, Vol_h, and bar_h, the weighted perimeter, volume, and barycenter of a set with respect to the density e^{V_h}.)

In order to prove (6.3) we first need some preliminary estimates on the sets E_h.

• *Step one:* Let us fix $\bar m>0$ and apply Theorem 3.3 to find $R_1=R_1(n,\alpha,\psi,\bar m)$ such that $E\subset B_{R_1}$ for every $E\in\mathcal{M}_V(m)$, $m\le\bar m$. Again by Theorem 3.3, there exists a positive constant $\bar C$ depending on n, α, R_1, and $\bar m$ only, such that

$$P(E;B(y,s)) \le P(G;B(y,s)) + \frac{\bar C}{m^{1/n}}\,s^n\,, \tag{6.4}$$

whenever $E\in\mathcal{M}_V(m)$, $m\le\bar m$, $s\le\bar r=\min\{1,m^{1/n}/\bar C\}$, and $E\Delta G\subset\subset B(y,s)$. By Theorem 5.1, there exist positive constants $m_0\le\bar m$ and C_0, depending on n, α, and R_1 only, such that, if $E\in\mathcal{M}_V(m)$ and $m\le m_0$, then

$$\mathrm{diam}\,E \le C_0\,m^{1/2n}\,.$$

• *Step two:* Given $E\subset\mathbb{R}^n$ with $0<|E|<\infty$, let us set

$$\lambda_E = \frac{1}{|E|^{1/n}}\,,\qquad E^* = \lambda_E\,E\,,$$

so that $|E^*|=1$. We claim that, if $E\in\mathcal{M}_V(m)$ and $m\le\min\{\bar m,1/\bar C\}$, then

$$P(E^*;B(x,r)) \le P(F;B(x,r)) + \hat C\,r^n\,, \tag{6.5}$$

whenever $E^*\Delta F\subset\subset B(x,r)$ and $r\le 1/\bar C$. Indeed, since $v\ge0$ we have $m^{1/n}\ge|E|^{1/n}$; thus, if we set $y=x/\lambda_E$, $G=F/\lambda_E$, and $s=r/\lambda_E$, then $E\Delta G\subset\subset B(y,s)$ with $s\le r\,m^{1/n}\le m^{1/n}/\bar C=\min\{1,m^{1/n}/\bar C\}=\bar r$, so that, by (6.4),

$$\begin{aligned} P(E;B(x/\lambda_E,s)) &\le P(G;B(x/\lambda_E,s)) + \frac{\bar C}{m^{1/n}}\,s^n \\ &= P(G;B(x/\lambda_E,s)) + \bar C\,\lambda_E\,s^n\,; \end{aligned}$$

scaling back to E^*, we find (6.5).

• *Step three:* We now prove that, for h sufficiently large, (6.3) holds but with a ball which a priori may not have the same weighted barycenter as E_h. Up to discarding finitely many terms we can assume $m_h\le m_0$, so that by step one there exists $\{y_h\}_{h\in\mathbb{N}}\subset B_{R_1}$ such that $E_h\subset B(y_h,C_0\,m_h^{1/2n})$. Hence, up to extracting a subsequence, we have

$$y_h\to y_0\,,\qquad |y_0|\le R_1\,,\qquad E_h\subset B(y_0,C\,m_h^{1/2n})\,,$$

where C denotes a positive constant depending on n, α, ψ, and $\bar m$. Defining s_h and t_h by $\mathrm{Vol}_h(B(y_0,s_h))=\mathrm{Vol}_h(E_h)$ and $\omega_n\,t_h^n=|E_h|$, we see that

$$\begin{aligned} \omega_n s_h^n\,e^{V_h(y_0)-C\,m_h^{1/2n}} &\le \mathrm{Vol}_h(B(y_0,s_h)) \le \omega_n\,s_h^n\,e^{V_h(y_0)+C\,m_h^{1/2n}}\,, \\ \omega_n t_h^n\,e^{V_h(y_0)-C\,m_h^{1/2n}} &\le \mathrm{Vol}_h(E_h) \le \omega_n\,t_h^n\,e^{V_h(y_0)+C\,m_h^{1/2n}}\,; \end{aligned}$$

which gives $|(s_h/t_h)^n - 1| \le C\, m_h^{1/2n}$; at the same time, by minimality of E_h,

$$e^{V_h(y_0) - C\, m_h^{1/2n}}\, P(E_h) \le \mathrm{Per}_h(E_h) \le \mathrm{Per}_h(B(y_0,s_h)) \le e^{V_h(y_0)+C\, m_h^{1/2n}}\, n\omega_n\, s_h^{n-1}\,;$$

we thus find

$$P(E_h) \le (1 + C\, m_h^{1/2n}) n\omega_n s_h^{n-1} \le (1 + C\, m_h^{1/2n}) n\omega_n^{1/n} |E_h|^{(n-1)/n}\,.$$

Hence, by the quantitative isoperimetric inequality [FiMP] there exists $\{w_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^n$ such that, if t_h is defined as above, then

$$C\, m_h^{1/2n} \ge \frac{P(E_h)}{n\omega_n^{1/n}|E_h|^{(n-1)/n}} - 1 \ge \left(\frac{|E_h \Delta B(w_h,t_h)|}{C\,|E_h|}\right)^2\,. \tag{6.6}$$

(Note that clearly $w_h \to y_0$.) Setting $\lambda_h = 1/|E_h|^{1/n} = 1/(\omega_n^{1/n} t_h)$, $E_h^* = \lambda_h\, E_h$, and $z_h = \lambda_h\, w_h$, by scaling and translation invariance we find

$$C\, m_h^{1/4n} \ge \left| \Big(E_h^* - z_h\Big) \Delta B_{\omega_n^{-1/n}} \right|\,. \tag{6.7}$$

By step two, $\{E_h^* - z_h\}_{h\in\mathbb{N}}$ is a sequence of uniform almost-minimizers of the perimeter which (by (6.7)) is converging in L^1 to a ball. Thus by [T1, Theorem 1.9] there exists $\{\hat u_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{S}^{n-1};[-1,\infty))$ such that

$$E_h^* = z_h + \Big\{ t\,(1+\hat u_h(\omega))\,\omega : \omega \in \mathbb{S}^{n-1}\,, 0 \le t < \omega_n^{-1/n} \Big\}\,, \qquad \lim_{h\to\infty} \|\hat u_h\|_{C^1} = 0\,.$$

Scaling back to the sets E_h we find

$$E_h = w_h + \Big\{ t\,(1+\hat u_h(\omega))\,\omega : \omega \in \mathbb{S}^{n-1}\,, 0 \le t < t_h \Big\}\,, \qquad \lim_{h\to\infty} \|\hat u_h\|_{C^1} = 0\,. \tag{6.8}$$

• *Step four:* Let us now show that $w_h \to 0$, that is, let us prove that $y_0 = 0$. To this end, let us notice that, by $v_h(0) = 0$ and since $\{v_h\}_{h\in\mathbb{N}}$ are locally uniformly Lipschitz, there exists a locally Lipschitz, increasing function $v : [0,\infty) \to [0,\infty)$ such that $v_h \to v$ uniformly on compact subsets of $[0,\infty)$; in particular, $v(0) = 0$ and v satisfies (6.1). Now, by (6.8), using that $s_h/t_h \to 1$ and $w_h \to y_0$ we get

$$\begin{aligned}
\lim_{h\to\infty} \frac{\phi_{V_h}(m_h)}{m_h^{(n-1)/n}} &= \lim_{h\to\infty} \frac{\mathrm{Per}_h(E_h)}{\mathrm{Vol}_h(B(y_0,s_h))^{(n-1)/n}} \\
&= \lim_{h\to\infty} \frac{\int_{\mathbb{S}^{n-1}} (t_h(1+\hat u_h))^{n-1} e^{V_h(w_h + t_h(1+\hat u_h))} \sqrt{1 + \frac{|\nabla \hat u_h|^2}{(1+\hat u_h)^2}}}{\Big(\int_{\mathbb{S}^{n-1}} \int_0^{s_h} r^{n-1} e^{V_h(y_0 + r\,\omega)}\, dr\Big)^{(n-1)/n}} \\
&= \lim_{h\to\infty} \frac{n\omega_n t_h^{n-1} e^{V_h(w_h)}}{\Big(\int_{\mathbb{S}^{n-1}} \int_0^{s_h} r^{n-1} e^{V_h(y_0)} (1 + O(r))\, dr\Big)^{(n-1)/n}} \\
&= n\omega_n^{1/n} e^{V(y_0)/n}\,;
\end{aligned} \tag{6.9}$$

at the same time, by comparing E_h with $B(m_h)$ we see

$$\lim_{h\to\infty} \frac{\phi_{V_h}(m_h)}{m_h^{(n-1)/n}} \le \lim_{h\to\infty} \frac{\mathrm{Per}_h(B(m_h))}{\mathrm{Vol}_h(B(m_h))^{(n-1)/n}} = n\omega_n^{1/n} e^{v(0)/n} = n\omega_n^{1/n}\,. \tag{6.10}$$

Combining (6.9) and (6.10) we thus find $e^{\psi(|y_0|)/n} \le 1$; however, being ψ strictly increasing and since $\psi(0) = 0$, we find $y_0 = 0$ (and, in particular, $s_h = r(m_h)$).

• *Step five:* We finally prove (6.3), that is, we want to show that E_h is a C^1-small radial perturbation of a ball $B(x_h, r_h)$ having its same weighted volume and weighted barycenter. To this aim we observe that (6.8) implies

$$\frac{|w_h - \mathrm{bar}(E_h)|}{t_h} \le C\,\|\hat{u}_h\|_{C^0}\,.$$

Hence, by a simple continuity argument within the family of balls with weighted volume m_h, it is not difficult to see that there exists $x_h \in \mathbb{R}^n$ and $r_h > 0$ such that $B(x_h, r_h)$ has the same weighted volume and weighted barycenter as E_h, and moreover

$$\frac{|w_h - x_h|}{r_h} \le C\,\|\hat{u}_h\|_{C^0}\,, \qquad \left|\frac{t_h}{r_h} - 1\right| \le C\,\|\hat{u}_h\|_{C^0}\,.$$

Thanks to this last estimate, the fact that E_h is still a C^1-small perturbation of $B(x_h, r_h)$ follows easily from (6.8). □

Proof of Theorem 1.1-(ii). Without loss of generality, up to adding a constant to v (which does not change the variational problem, since this amounts to multiply e^V by a positive constant) we may assume that $v(0) = 0$. Since v is strictly convex and increasing, it is strictly increasing and satisfies $v(r) \to \infty$ as $r \to \infty$. By Theorem 6.1, every isoperimetric set for e^V with sufficiently small weighted volume is an arbitrarily C^1-small perturbation of a ball having its same weighted volume and barycenter, and with center arbitrarily close to the origin; hence, by Theorem 2.5, every isoperimetric set for e^V with sufficiently small weighted volume is a ball centered at the origin. □

7. Proof of Theorem 1.2

We now show how to combine the validity of the isoperimetric log-convex density conjecture both for small and large volumes, Theorem 2.3, and the fact that the strong form of the conjecture holds for all masses when $v(r) = c\,r^2$ ($c > 0$), to prove Theorem 1.2.

Proof of Theorem 1.2. Let $n \ge 2$, $c > 0$, and let $v \in C^2([0,\infty),[0,\infty))$ be an increasing convex function with

$$\|v - c\,r^2\|_{C^2([0,\infty))} < \delta\,, \qquad \Omega[v''](R,\sigma) \le \Omega_0(R,\sigma)\,, \qquad \forall\, R\,,\sigma > 0\,,$$

where Ω_0 is as in the statement of the theorem. Up to replacing v with $v - v(0)$ and δ with 2δ, we may directly assume that $v(0) = 0$ (as already observed before this does not change the variational problem, since adding a constant to v amounts to multiply e^V by a positive constant). Moreover, since v is increasing, provided δ is small enough we may ensure that

$$v''(r) \ge \frac{c}{2}\,, \qquad \forall\, r > 0\,, \tag{7.1}$$

$$\operatorname*{ess\,sup}_{[0,r]} |v'| \le (c+1)\max\{r,\delta\}\,, \quad v(r) \ge \frac{cr^2}{4}\,, \qquad \forall\, r > 0\,. \tag{7.2}$$

By (7.1) and Kolesnikov-Zhdanov theorem [KZ, Proposition 6.7], there exists $\bar{m} > 0$, depending on n and c only, such that if $E \in \mathcal{M}_V(m)$ and $m > \bar{m}$ then $E = B(m)$. Moreover, by Theorem 6.1 there exists $\hat{m} \le \bar{m}$ (depending on n and c only) such that, if $E \in \mathcal{M}_V(m)$ with $m \le \hat{m}$, then E is a C^1-small perturbation of a ball with its same weighted volume and barycenter. Furthermore, since the modulus of continuity of v'' is controlled in terms of Ω_0, up to further decrease the value of $\hat{m}$ in terms of n, c, and Ω_0 only, we may ensure by Theorem 2.5 that every isoperimetric set $E \in \mathcal{M}_V(m)$ with $m \le \hat{m}$ is in fact a ball centered at the origin. Setting $m_1 = \hat{m}/2 > 0$ and $m_2 = 2\,\bar{m} > m_1$, we are thus left to prove that, provided δ is small enough depending on m_1 and m_2 (thus, depending on n, c, and Ω_0), then every $E \in \mathcal{M}_V(m)$ with $m \in [m_1, m_2]$ is a ball centered at the origin.

Assume on the contrary the existence of sequences $\{v_h\}_{h\in\mathbb{N}} \subset C^2([0,\infty),[0,\infty))$ of increasing convex functions with $v_h(0)=0$ and $\|v_h - c\,r^2\|_{C^2([0,\infty))} \to 0$ as $h\to\infty$, and of sets $\{E_h\}_{h\in\mathbb{N}}$ with $E_h \in \mathcal{M}_{V_h}(m_h)$, $m_h \to m \in [m_1,m_2]$ and $|E_h \Delta B(r_h(m_h))| > 0$ for every $h\in\mathbb{N}$ (here, $r_h(m_h)$ is the radius of the ball centered at the origin with weighted volume m_h with respect to e^{V_h}). Exploiting that for $v = cr^2$ balls centered at the origin at the unique isoperimetric sets [RCBM, Theorem 5.2], Theorem 3.3, and [T1, Theorem 1.9], we easily see that, up to extracting subsequences, each E_h is an (arbitrarily) C^1-small perturbation of the ball $B_{r(m)}$, where $r(m)>0$ is such that

$$\int_{B_{r(m)}} e^{c\,|x|^2}\,dx = m\,.$$

Hence, as in the proof of Theorem 1.1-(i) we obtain that E_h is an (arbitrarily) C^1-small perturbation of the ball $B_{r_h(m_h)}$. Exploiting again Theorem 2.3, from $|E_h \Delta B(r_h(m_h))| > 0$ we obtain

$$\int_{\partial E_h} e^{V_h} > \int_{\partial B_{r_h(m_h)}} e^{V_h}\,,$$

provided h is large enough, thus contradicting the minimality of E_h. □

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