

SECOND ORDER STABILITY FOR THE MONGE-AMPÈRE EQUATION AND STRONG SOBOLEV CONVERGENCE OF OPTIMAL TRANSPORT MAPS

GUIDO DE PHILIPPIS AND ALESSIO FIGALLI

ABSTRACT. The aim of this note is to show that Alexandrov solutions of the Monge-Ampère equation, with right hand side bounded away from zero and infinity, converge strongly in $W_{\text{loc}}^{2,1}$ if their right hand side converge strongly in L_{loc}^1 . As a corollary we deduce strong $W_{\text{loc}}^{1,1}$ stability of optimal transport maps.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Recently [3] the authors showed that convex Alexandrov solutions of

$$(1.1) \quad \begin{cases} \det D^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < \lambda \leq f \leq \Lambda$, are $W_{\text{loc}}^{2,1}(\Omega)$. More precisely, they were able to prove uniform interior $L \log L$ -estimates for $D^2 u$. This result has also been improved in [4, 8], where it is actually shown that $u \in W_{\text{loc}}^{2,\gamma}(\Omega)$ for some $\gamma = \gamma(n, \lambda, \Lambda) > 1$: more precisely, for any $\Omega' \subset\subset \Omega$,

$$(1.2) \quad \int_{\Omega'} |D^2 u|^\gamma \leq C(n, \lambda, \Lambda, \Omega, \Omega').$$

A question which naturally arises in view of the previous results is the following: choose a sequence of functions f_k with $\lambda \leq f_k \leq \Lambda$ which converges to f strongly in $L_{\text{loc}}^1(\Omega)$, and denote by u_k and u the solutions of (1.1) corresponding to f_k and f respectively. By the convexity of u_k and u , and the uniqueness of solutions to (1.1), it is immediate to deduce that $u_k \rightarrow u$ uniformly, and $\nabla u_k \rightarrow \nabla u$ in $L_{\text{loc}}^p(\Omega)$ for any $p < \infty$. What can be said about the strong convergence of $D^2 u_k$? Due to the highly nonlinear character of the Monge-Ampère equation, this question is nontrivial. (Note that weak $W_{\text{loc}}^{2,1}$ convergence is immediate by compactness, even under the weaker assumption that f_k converge to f weakly in $L_{\text{loc}}^1(\Omega)$.)

The aim of this short note is to prove that actually strong convergence holds. In fact our main result is the following:

Theorem 1.1. *Let $\Omega_k \subset \mathbb{R}^n$ be a family of convex domains, and let $u_k : \Omega_k \rightarrow \mathbb{R}$ be convex Alexandrov solutions of*

$$(1.3) \quad \begin{cases} \det D^2 u_k = f_k & \text{in } \Omega_k \\ u_k = 0 & \text{on } \partial\Omega_k \end{cases}$$

with $0 < \lambda \leq f_k \leq \Lambda$. Assume that Ω_k converge to some convex domain Ω in the Hausdorff distance, and $f_k \chi_{\Omega_k}$ converge to f in $L^1_{\text{loc}}(\Omega)$. Then, if u denotes the unique Alexandrov solution of

$$\begin{cases} \det D^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for any $\Omega' \subset\subset \Omega$ we have

$$(1.4) \quad \|u_k - u\|_{W^{2,1}(\Omega')} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

(Obviously, since the functions u_k are uniformly bounded in $W^{2,\gamma}(\Omega')$, this gives strong convergence in $W^{2,\gamma'}(\Omega')$ for any $\gamma' < \gamma$.)

As a consequence of the previous theorem we can prove the following stability result for optimal transport maps:

Theorem 1.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be two bounded domains with Ω_2 convex, and let f_k, g_k be a family of probability densities such that $0 < \lambda \leq f_k, g_k \leq \Lambda$ inside Ω_1 and Ω_2 respectively. Assume that $f_k \rightarrow f$ in $L^1(\Omega_1)$ and $g_k \rightarrow g$ in $L^1(\Omega_2)$, and let $T_k : \Omega_1 \rightarrow \Omega_2$ (resp. $T : \Omega_1 \rightarrow \Omega_2$) be the (unique) optimal transport map for the quadratic cost sending f_k onto g_k (resp. f onto g). Then $T_k \rightarrow T$ in $W^{1,\gamma'}_{\text{loc}}(\Omega_1)$ for some $\gamma' > 1$.*

We point out that, in order to prove (1.4) and the local $W^{1,1}$ stability of optimal transport maps, the interior $L \log L$ -estimates from [3] are sufficient. Indeed, the $W^{2,\gamma}$ -estimates are used just to improve the convergence from $W^{2,1}_{\text{loc}}$ to $W^{2,\gamma'}_{\text{loc}}$ with $\gamma' < \gamma$.

The paper is organized as follows: in the next section we collect some notation and preliminary results. Then in Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2.

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2. NOTATION AND PRELIMINARIES

Given a convex function $u : \Omega \rightarrow \mathbb{R}$, we define its *Monge-Ampère measure* as

$$\mu_u(E) := |\partial u(E)| \quad \forall E \subset \Omega \text{ Borel}$$

(see [7, Theorem 1.1.13]), where

$$\partial u(E) := \bigcup_{x \in E} \partial u(x).$$

Here $\partial u(x)$ is the subdifferential of u at x , and $|F|$ denotes the Lebesgue measure of a set F . In case $u \in C^{1,1}_{\text{loc}}$, by the Area Formula [5, Paragraph 3.3] the following representation holds:

$$\mu_u = \det D^2 u \, dx.$$

The main property of Monge-Ampère measure we are going to use is the following (see [7, Lemma 1.2.2 and Lemma 1.2.3]):

Proposition 2.1. *Let $u_k : \Omega \rightarrow \mathbb{R}$ be a sequence of convex functions converging locally uniformly to u . Then the associated Monge-Ampère measures μ_{u_k} converges to μ_u in duality with the space of continuous functions compactly supported in Ω . In particular*

$$\mu_u(A) \leq \liminf_{k \rightarrow \infty} \mu_{u_k}(A)$$

for any open set $A \subset \Omega$.

Given a Radon measure ν on \mathbb{R}^n and a bounded convex domain $\Omega \subset \mathbb{R}^n$, we say that a convex function $u : \Omega \rightarrow \mathbb{R}$ is an *Alexandrov solution* of the Monge-Ampère equation

$$\det D^2u = \nu \quad \text{in } \Omega$$

if $\mu_u(E) = \nu(E)$ for every Borel set $E \subset \Omega$.

If $v : \bar{\Omega} \rightarrow \mathbb{R}$ is a continuous function, we define its *convex envelope inside* Ω as

$$(2.1) \quad \Gamma_v(x) := \sup\{\ell(x) : \ell \leq v \text{ in } \Omega, \ell \text{ affine}\}.$$

In case Ω is a convex domain and $v \in C^2(\Omega)$, it is easily seen that

$$(2.2) \quad D^2v(x) \geq 0 \quad \text{for every } x \in \{v = \Gamma_v\} \cap \Omega$$

in the sense of symmetric matrices. Moreover the following inequality between measures holds in Ω :

$$(2.3) \quad \mu_{\Gamma_v} \leq \det D^2v \mathbf{1}_{\{v = \Gamma_v\}} dx,$$

(here $\mathbf{1}_E$ is the characteristic function of a set E).¹

We recall that a continuous function v is said to be *twice differentiable* at x if there exists a (unique) vector $\nabla v(x)$ and a (unique) symmetric matrix $\nabla^2 v(x)$ such that

$$v(y) = v(x) + \nabla v(x) \cdot (y - x) + \frac{1}{2} \nabla^2 v(x)[y - x, y - x] + o(|y - x|^2).$$

In case v is twice differentiable at some point $x_0 \in \{v = \Gamma_v\}$, then it is immediate to check that

$$(2.5) \quad \nabla^2 v(x_0) \geq 0.$$

By Alexandrov Theorem, any convex function is twice differentiable almost everywhere (see for instance [5, Paragraph 6.4]). In particular, (2.5) holds almost everywhere on $\{v = \Gamma_v\}$ whenever v is

¹To see this, let us first recall that by [7, Lemma 6.6.2], if $x_0 \in \Omega \setminus \{\Gamma_v = v\}$ and $a \in \partial\Gamma_v(x_0)$, then the convex set

$$\{x \in \Omega : \Gamma_v(x) = a \cdot (x - x_0) + \Gamma_v(x_0)\}$$

is nonempty and contains more than one point. In particular

$$\partial\Gamma_v(\Omega \setminus \{\Gamma_v = v\}) \subset \{p \in \mathbb{R}^n : \text{there exist } x, y \in \Omega, x \neq y \text{ and } p \in \partial\Gamma_v(x) \cap \partial\Gamma_v(y)\}.$$

This last set is contained in the set of nondifferentiability of the convex conjugate of Γ_v , so it has zero Lebesgue measure (see [7, Lemma 1.1.12]), hence

$$(2.4) \quad |\partial\Gamma_v(\Omega \setminus \{\Gamma_v = v\})| = 0.$$

Moreover, since $v \in C^1(\Omega)$, for any $x \in \{\Gamma_v = v\} \cap \Omega$ it holds $\partial\Gamma_v(x) = \{\nabla v(x)\}$. Thus, using (2.4) and (2.2), for any open set $A \subset \subset \Omega$ we have

$$\begin{aligned} \mu_{\Gamma_v}(A) &= |\partial\Gamma_v(A \cap \{\Gamma_v = v\})| = |\nabla v(A \cap \{\Gamma_v = v\})| \\ &\leq \int_{A \cap \{\Gamma_v = v\}} |\det D^2v| = \int_{A \cap \{\Gamma_v = v\}} \det D^2v, \end{aligned}$$

as desired. (The inequality above follows from the Area Formula in [5, Paragraph 3.3.2] applied to the C^1 map ∇v .)

the difference of two convex functions.

Finally we recall that, in case $v \in W_{\text{loc}}^{2,1}$, then the pointwise Hessian of v coincides almost everywhere with its distributional Hessian [5, Sections 6.3 and 6.4]. Since in the sequel we are going to deal with $W_{\text{loc}}^{2,1}$ convex functions, we will use D^2u to denote both the pointwise and the distributional Hessian.

3. PROOF OF THEOREM 1.1

We are going to use the following result:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $u, v : \bar{\Omega} \rightarrow \mathbb{R}$ be two continuous strictly convex functions such that $\mu_u = f dx$ and $\mu_v = g dx$, with $f, g \in L_{\text{loc}}^1(\Omega)$. Then*

$$(3.1) \quad \mu_{\Gamma_{u-v}} \leq \left(f^{1/n} - g^{1/n} \right)^n \mathbf{1}_{\{u-v=\Gamma_{u-v}\}} dx.$$

Proof. In case u, v are of class C^2 inside Ω , by (2.2) we have

$$0 \leq D^2u(x) - D^2v(x) \quad \text{for every } x \in \{u - v = \Gamma_{u-v}\},$$

so using the monotonicity and the concavity of the function $\det^{1/n}$ on the cone of non-negative symmetric matrices we get

$$0 \leq \det(D^2u - D^2v) \leq \left((\det D^2u)^{1/n} - (\det D^2v)^{1/n} \right)^n \quad \text{on } \{u - v = \Gamma_{u-v}\},$$

which combined with (2.3) gives the desired result.

Now, for the general case, we consider a sequence of smooth uniformly convex domains Ω_k increasing to Ω , two sequences of smooth functions f_k and g_k converging respectively to f and g in $L_{\text{loc}}^1(\Omega)$, and we solve

$$\begin{cases} \det D^2u_k = f_k & \text{in } \Omega_k \\ u_k = u * \rho_k & \text{on } \partial\Omega_k, \end{cases} \quad \begin{cases} \det D^2v_k = g_k & \text{in } \Omega_k \\ v_k = v * \rho_k & \text{on } \partial\Omega_k, \end{cases}$$

where ρ_k is a smooth sequence of convolution kernels. In this way both u_k and v_k are smooth on $\bar{\Omega}_k$ [6, Theorem 17.23], and $\|u_k - u\|_{L^\infty(\Omega_k)} + \|v_k - v\|_{L^\infty(\Omega_k)} \rightarrow 0$ as $k \rightarrow \infty$.² Hence, also $\Gamma_{u_k - v_k}$ converges locally uniformly to Γ_{u-v} . Moreover, it follows easily from the definition of contact set that

$$(3.2) \quad \limsup_{k \rightarrow \infty} \mathbf{1}_{\{u_k - v_k = \Gamma_{u_k - v_k}\}} \leq \mathbf{1}_{\{u - v = \Gamma_{u - v}\}}.$$

We now observe that the previous step applied to u_k and v_k gives

$$\mu_{\Gamma_{u_k - v_k}} \leq \left((\det D^2u_k)^{1/n} - (\det D^2v_k)^{1/n} \right)^n \mathbf{1}_{\{u_k - v_k = \Gamma_{u_k - v_k}\}} dx,$$

Thus, letting $k \rightarrow \infty$ and taking in account Proposition 2.1 and (3.2), we obtain (3.1). \square

Proof of Theorem 1.1. The L_{loc}^1 convergence of u_k (resp. ∇u_k) to u (resp. ∇u) is easy and standard, so we focus on the convergence of the second derivatives.

²Indeed, first of all it is easy to see that u_k (resp. v_k) converges uniformly to u (resp. v) both on $\partial\Omega_k$ and in any compact subdomain of Ω . Then, using for instance a contradiction argument, one exploits the convexity of u_k (resp. v_k) and Ω_k , and the uniform continuity of u (resp. v), to show that the convergence is actually uniform on the whole Ω_k .

Without loss of generality we can assume that Ω' is convex, and that $\Omega' \subset\subset \Omega_k$ (since $\Omega_k \rightarrow \Omega$ in the Hausdorff distance, this is always true for k sufficiently large). Fix $\varepsilon \in (0, 1)$, let $\Gamma_{u-(1-\varepsilon)u_k}$ be the convex envelope of $u - (1 - \varepsilon)u_k$ inside Ω' (see (2.1)), and define

$$A_k^\varepsilon := \{x \in \Omega' : u(x) - (1 - \varepsilon)u_k(x) = \Gamma_{u-(1-\varepsilon)u_k}(x)\}.$$

Since $u_k \rightarrow u$ locally uniformly, $\Gamma_{u-(1-\varepsilon)u_k}$ converges uniformly to $\Gamma_{\varepsilon u} = \varepsilon u$ (as u is convex) inside Ω' . Hence, by applying Proposition 2.1 and (3.1) to u and $(1 - \varepsilon)u_k$ inside Ω' , we get that

$$\begin{aligned} \varepsilon^n \int_{\Omega'} f &= \mu_{\Gamma_{\varepsilon u}}(\Omega') \\ &\leq \liminf_{k \rightarrow \infty} \mu_{\Gamma_{u-(1-\varepsilon)u_k}}(\Omega') \\ &\leq \liminf_{k \rightarrow \infty} \int_{\Omega' \cap A_k^\varepsilon} \left(f^{1/n} - (1 - \varepsilon)f_k^{1/n} \right)^n. \end{aligned}$$

We now observe that, since f_k converges to f in $L^1_{\text{loc}}(\Omega)$, we have

$$\left| \int_{\Omega' \cap A_k^\varepsilon} \left(f^{1/n} - (1 - \varepsilon)f_k^{1/n} \right)^n - \int_{\Omega' \cap A_k^\varepsilon} \varepsilon^n f \right| \leq \int_{\Omega'} \left| \left(f^{1/n} - (1 - \varepsilon)f_k^{1/n} \right)^n - \varepsilon^n f \right| \rightarrow 0$$

as $k \rightarrow \infty$. Hence, combining the two estimates above, we immediately get

$$\int_{\Omega'} f \leq \liminf_{k \rightarrow \infty} \int_{\Omega' \cap A_k^\varepsilon} f,$$

or equivalently

$$\limsup_{k \rightarrow \infty} \int_{\Omega' \setminus A_k^\varepsilon} f = 0.$$

Since $f \geq \lambda$ inside Ω (as a consequence of the fact that $f_k \geq \lambda$ inside Ω_k), this gives

$$(3.3) \quad \lim_{k \rightarrow \infty} |\Omega' \setminus A_k^\varepsilon| = 0 \quad \forall \varepsilon \in (0, 1).$$

We now recall that, by the results in [1, 3, 4, 8], both u and $(1 - \varepsilon)u_k$ are strictly convex and belong to $W^{2,1}(\Omega')$. Hence we can apply (2.5) to deduce that

$$D^2u - (1 - \varepsilon)D^2u_k \geq 0 \quad \text{a.e. on } A_k^\varepsilon.$$

In particular, by (3.3),

$$|\Omega' \setminus \{D^2u \geq (1 - \varepsilon)D^2u_k\}| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By a similar argument (exchanging the roles of u and u_k)

$$|\Omega' \setminus \{D^2u_k \geq (1 - \varepsilon)D^2u\}| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence, if we call $B_k^\varepsilon := \left\{ x \in \Omega' : (1 - \varepsilon)D^2u_k \leq D^2u \leq \frac{1}{1-\varepsilon}D^2u_k \right\}$, it holds

$$\lim_{k \rightarrow \infty} |\Omega' \setminus B_k^\varepsilon| = 0 \quad \forall \varepsilon \in (0, 1).$$

Moreover, by (1.2) applied to both u_k and u , we have³

$$\begin{aligned} \int_{\Omega'} |D^2u - D^2u_k| &= \int_{\Omega' \cap B_k^\varepsilon} |D^2u - D^2u_k| + \int_{\Omega' \setminus B_k^\varepsilon} |D^2u - D^2u_k| \\ &\leq \frac{\varepsilon}{1-\varepsilon} \int_{\Omega'} |D^2u| + \|D^2u - D^2u_k\|_{L^\gamma(\Omega')} |\Omega' \setminus B_k^\varepsilon|^{1-1/\gamma} \\ &\leq C \left(\frac{\varepsilon}{1-\varepsilon} + |\Omega' \setminus B_k^\varepsilon|^{1-1/\gamma} \right). \end{aligned}$$

Hence, letting first $k \rightarrow \infty$ and then sending $\varepsilon \rightarrow 0$, we obtain the desired result. \square

4. PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we will need the following lemma (note that for the next result we do not need to assume the convexity of the target domain):

Lemma 4.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be two bounded domains, and let f_k, g_k be a family of probability densities such that $0 < \lambda \leq f_k, g_k \leq \Lambda$ inside Ω_1 and Ω_2 respectively. Assume that $f_k \rightarrow f$ in $L^1(\Omega_1)$ and $g_k \rightarrow g$ in $L^1(\Omega_2)$, and let $T_k : \Omega_1 \rightarrow \Omega_2$ (resp. $T : \Omega_1 \rightarrow \Omega_2$) be the (unique) optimal transport map for the quadratic cost sending f_k onto g_k (resp. f onto g). Then*

$$\frac{f_k}{g_k \circ T_k} \rightarrow \frac{f}{g \circ T} \quad \text{in } L^1(\Omega_1).$$

Proof. By stability of optimal transport maps (see for instance [9, Corollary 5.23]) and the fact that $f_k \geq \lambda$ (and so $f \geq \lambda$), we know that $T_k \rightarrow T$ in measure (with respect to Lebesgue) inside Ω .

We claim that $g \circ T_k \rightarrow g \circ T$ in $L^1(\Omega_1)$. Indeed this is obvious if g is uniformly continuous (by the convergence in measure of T_k to T). In the general case we choose $g_\eta \in C(\overline{\Omega_2})$ such that $\|g - g_\eta\|_{L^1(\Omega_2)} \leq \eta$ and we observe that (recall that $f_k, f \geq \lambda$, $g_k, g \leq \Lambda$, and that by definition of transport maps we have $T_{\#}f_k = g_k$, $T_{\#}f = g$)

$$\begin{aligned} \int_{\Omega_1} |g \circ T_k - g \circ T| &\leq \int_{\Omega_1} |g_\eta \circ T_k - g_\eta \circ T| + \int_{\Omega_1} |g_\eta \circ T_k - g \circ T_k| \frac{f_k}{\lambda} + \int_{\Omega_1} |g_\eta \circ T - g \circ T| \frac{f}{\lambda} \\ &= \int_{\Omega_1} |g_\eta \circ T_k - g_\eta \circ T| + \int_{\Omega_2} |g_\eta - g| \frac{g_k}{\lambda} + \int_{\Omega_2} |g_\eta - g| \frac{g}{\lambda} \\ &\leq \int_{\Omega_1} |g_\eta \circ T_k - g_\eta \circ T| + 2 \frac{\Lambda}{\lambda} \eta. \end{aligned}$$

Thus

$$\limsup_{k \rightarrow \infty} \int_{\Omega_1} |g \circ T_k - g \circ T| \leq 2 \frac{\Lambda}{\lambda} \eta,$$

and the claim follows by the arbitrariness of η .

³If instead of (1.2) we only had uniform $L \log L$ a-priori estimates, in place of Hölder inequality we could apply the elementary inequality $t \leq \delta t \log(2+t) + e^{1/\delta}$ with $t = |D^2u - D^2u_k|$ inside $\Omega' \setminus B_k^\varepsilon$, and we would let first $k \rightarrow \infty$ and then send $\delta, \varepsilon \rightarrow 0$.

Since

$$\begin{aligned}
 \int_{\Omega_1} |g_k \circ T_k - g \circ T| &\leq \int_{\Omega_1} |g_k \circ T_k - g \circ T_k| \frac{f_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\
 &= \int_{\Omega_2} |g_k - g| \frac{g_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\
 &\leq \frac{\Lambda}{\lambda} \|g_k - g\|_{L^1(\Omega_2)} + \int_{\Omega_1} |g \circ T_k - g \circ T|,
 \end{aligned}$$

from the claim above we immediately deduce that also $g_k \circ T_k \rightarrow g \circ T$ in $L^1(\Omega_1)$.

Finally, since $g_k, g \geq \lambda$ and $f \leq \Lambda$,

$$\begin{aligned}
 \int_{\Omega_1} \left| \frac{f_k}{g_k \circ T_k} - \frac{f}{g \circ T} \right| &\leq \int_{\Omega_1} \left| \frac{f_k - f}{g_k \circ T_k} \right| + \int_{\Omega_1} f \left| \frac{1}{g_k \circ T_k} - \frac{1}{g \circ T} \right| \\
 &\leq \frac{1}{\lambda} \|f_k - f\|_{L^1(\Omega_1)} + \Lambda \int_{\Omega_1} \frac{|g_k \circ T_k - g \circ T|}{g_k \circ T_k g \circ T} \\
 &\leq \frac{1}{\lambda} \|f_k - f\|_{L^1(\Omega_1)} + \frac{\Lambda}{\lambda^2} \|g_k \circ T_k - g \circ T\|_{L^1(\Omega_1)},
 \end{aligned}$$

from which the desired result follows. \square

Proof of Theorem 1.2. Since T_k are uniformly bounded in $W^{1,\gamma}(\Omega'_1)$ for any $\Omega'_1 \subset\subset \Omega$, it suffices to prove that $T_k \rightarrow T$ in $W^{1,1}_{\text{loc}}(\Omega_1)$.

Fix $x_0 \in \Omega_1$ and $r > 0$ such that $B_r(x_0) \subset \Omega_1$. By compactness, it suffices to show that there is an open neighborhood \mathcal{U}_{x_0} of x_0 such that $\mathcal{U}_{x_0} \subset B_r(x_0)$ and

$$\int_{\mathcal{U}_{x_0}} |T_k - T| + |\nabla T_k - \nabla T| \rightarrow 0.$$

It is well-known [2] that T_k (resp. T) can be written as ∇u_k (resp. ∇u) for some strictly convex function $u_k : B_r(x_0) \rightarrow \mathbb{R}$ (resp. $u : B_r(x_0) \rightarrow \mathbb{R}$). Moreover, up to subtract a constant to u_k (which will not change the transport map T_k), one may assume that $u_k(x_0) = u(x_0)$ for all $k \in \mathbb{N}$.

Since the functions $T_k = \nabla u_k$ are bounded (as they take values in the bounded set Ω_2), by classical stability of optimal maps (see for instance [9, Corollary 5.23]) we get that $\nabla u_k \rightarrow \nabla u$ in $L^1_{\text{loc}}(B_r(x_0))$. (Actually, if one uses [2], ∇u_k are locally uniformly Hölder maps, so they converge locally uniformly to ∇u .) Hence, to conclude the proof we only need to prove the convergence of $D^2 u_k$ to $D^2 u$ in a neighborhood of x_0 .

To this aim, we observe that, by strict convexity of u , we can find a linear function $\ell(z) = a \cdot z + b$ such that the open convex set $Z := \{z : u(z) < u(x_0) + \ell(z)\}$ is non-empty and compactly supported inside $B_{r/2}(x_0)$. Hence, by the uniform convergence of u_k to u (which follows from the L^1_{loc} convergence of the gradients, the convexity of u_k and u , and the fact that $u_k(x_0) = u(x_0)$), and the fact that ∇u is transversal to ℓ on ∂Z , we get that $Z_k := \{z : u_k(z) < u_k(x_0) + \ell(z)\}$ are non-empty convex sets which converge in the Hausdorff distance to Z .

Moreover, by [2] the maps $v_k := u_k - \ell$ solve in the Alexandrov sense

$$\begin{cases} \det D^2 v_k = \frac{f_k}{g_k \circ T_k} & \text{in } Z_k \\ v_k = 0 & \text{on } \partial Z_k \end{cases}$$

(here we used that the Monge-Ampère measures associated to v_k and u_k are the same). Therefore, thanks to Lemma 4.1, we can apply Theorem 1.1 to deduce that $D^2u_k \rightarrow D^2u$ in any relatively compact subset of Z , which concludes the proof. \square

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SCUOLA NORMALE SUPERIORE, P.ZA DEI CAVALIERI 7, I-56126 PISA, ITALY
E-mail address: guido.dephilippis@sns.it

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION C1200, AUSTIN TX 78712, USA
E-mail address: figalli@math.utexas.edu