OPTIMAL REGULARITY FOR THE CONVEX ENVELOPE AND SEMICONVEX FUNCTIONS RELATED TO SUPERSOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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Dedicated to Luis Caffarelli on his 70^{th} Birthday.

ABSTRACT. In this paper we prove optimal regularity for the convex envelope of supersolutions to general fully nonlinear elliptic equations with unbounded coefficients. More precisely, we deal with coefficients and right hand sides (RHS) in L^q with $q \geq n$. This extends the result of L. Caffarelli on the $C^{1,1}_{loc}$ regularity of the convex envelope of supersolutions of fully nonlinear elliptic equations with bounded RHS. Moreover, we also provide a regularity result with estimates for $\omega-$semiconvex functions that are supersolutions to the same type of equations with unbounded RHS (i.e, RHS in $L^q, q \geq n$). By a completely different method, our results here extend the recent regularity results obtained by the first and third authors in [3] for $q > n$, as far as fully nonlinear PDEs are concerned. These results include, in particular, the apriori estimate obtained by L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck in [10] on the modulus of continuity of the gradient of $\omega-$semiconvex supersolutions (for linear equations and bounded RHS) that have a Hölder modulus of semiconvexity.

1. INTRODUCTION

The issue about the regularity of the convex envelope appears in many branches of Analysis and Geometry, among others. A list of few of them includes: Alexandrov-Bakelman-Pucci estimate in fully nonlinear equations [6, 24], Monge-Ampère equation [7, 29, 15, 20], geometric flows [12, 30], Hamilton-Jacobi equations [4, 19, 11], optimal transportation [22], calculus of variations and optimal control [11], etc. As shown in [36], the convex envelope can also be seen as a solution to an obstacle problem. Note that the convex envelope always enjoys local interior Lipschitz regularity (being a convex function), so the delicate question is to investigate under what conditions one can say that it is C^1 or more regular. In the papers [9, 28, 32, 36], results in this direction were established. Matters about the optimal regularity of the convex envelope are indeed harder and seem to be much less known.

Recently, the second author together with G. De Philippis studied questions related to this subject in [16]. More precisely, they investigated how the regularity of the boundary data and of the boundary itself make an influence on the regularity of the convex envelope. There, they presented a list of regularity results, and by a set of examples they show the sharpness of their conditions. As pointed out in [28], inquiring about smoothness of convex envelopes is not meant only to satisfy mathematical curiosity: the operation of convexification is fundamental, for instance, in the mathematical study of thermodynamic phase equilibria (see [28] and the references therein).

In [6], for the first time [1] in the context of viscosity solutions for fully nonlinear elliptic PDEs, L. Caffarelli proved the famous Alexandrov-Bakelman-Pucci estimate (ABP estimate). A key step in Caffarelli's proof is to show that the convex envelope of supersolutions $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f$ [2] is $C^{1,1}_{loc}$ when $f \in L^\infty$. A closely related subject to the regularity of the convex envelope is the regularity of $\omega-$semiconvex functions, where ω is a general modulus of semiconvexity. These functions also enjoy local Lipschitz regularity (see for instance [18, Lemma A.5] or Proposition 8.2), and the same questions about how much further regularity they eventually possess naturally arise.

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[1]We point out that around the same time, N. Trudinger in [39] (without explicitly stating the ABP maximum principle for viscosity supersolutions) pointed out that the ABP maximum principle holds for semi-concave functions using the standard proof. From there one can also obtain it for viscosity supersolutions by regularizing the supersolutions via inf-convolution.

[2]This is a Pucci extremal opertator with ellipticity constants $0 < \lambda < \Lambda$. We recall definition in section 2.

In [10], L. Caffarelli, J.J. Kohn, L. Nirenberg, and J. Spruck investigated an apriori estimates for the modulus of continuity of the gradient for classical supersolutions of linear second order uniformly elliptic equations with bounded RHS that are also ω_0-semiconvex for $\omega_0(t) = Ct^\alpha$ with $\alpha \in (0,1)$, and they obtained a logarithmic modulus of continuity for the gradient (see the Corollary of Lemma 2.2 in [10]).

The study of convexity properties related to supersolutions plays a relevant role in the theory of fully nonlinear PDEs (see for instance the classical paper by O. Alvarez, J.M. Lasry, and P. L. Lions [2] and the references therein). Moreover, in [31], C. Imbert obtained the convexity of solutions as well as $C^{1,1}$ estimates for convex supersolutions for some class of fully nonlinear equations.

Recently, in [3], the first and third authors obtained an optimal regularity result with estimates that improve the Corollary of Lemma 2.2 in [10]. The context in [3] involved general $\omega-$semiconvex functions that are supersolutions to fully nonlinear and quasilinear equations with unbounded RHS case (meaning RHS is in L^q with $q > n$).

The purpose of this current paper is two-fold: The first goal is to present the optimal interior regularity of the convex envelope of L^n viscosity supersolutions of the Pucci extremal operators with unbounded coefficients. Very roughly, one of our results says: if $u < 0$ in B_1 and $u = 0$ along ∂B_1 satisfies in the viscosity sense

$$\mathcal{P}^-_\gamma[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma(x)|\nabla u| \leq f(x) \text{ in } B_1$$

where $0 \leq \gamma \in L^p(B_1), f \in L^q(B_1)$ with $p \geq n$ and $q \geq n$, then the convex envelope of u, denoted by Γ_u, belongs to $C^{1,1-n/q}_{loc}(B_1)$ (here and in the sequel, $C^{1,1-n/q}_{loc}(B_1) = C^1(B_1)$ when $q = n$). It is worth emphasizing that also the case where $p = n$ is treated here, exploiting a simple result from measure theory in order to extend to this case some results that previously were only available for $p > n$. As we shall discuss later, our $C^{1,1-n/q}_{loc}$ regularity result is sharp. Under a more specific geometric setting for u, we obtain in fact a precise estimate for the corresponding Hölder semi-norm of the gradient of the convex envelope (see Theorem 2.8).

Our second goal is to provide a sharp regularity result for $\omega-$semiconvex supersolutions to the Pucci extremal operator $\mathcal{P}^-_\gamma$ given above. As far as fully nonlinear equations are concerned, this improves the corresponding result in [3]. In fact, the technique developed in [3] seems to be unable to treat the case where the coefficients of the equation are in L^n. The reason behind this is that the method in [3] is based on a new quantitative version for the Inhomogeneous Hopf-Oleĭnik Lemma (IHOL) that requires RHS in L^q with $q > n$. As a matter of fact, a stronger version of IHOL for fully nonlinear equations was obtained by B. Sirakov in a form of a boundary weak Harnack inequality in [38]. IHOL for quasilinear equations was also obtained in [3].

The key point in our argument is a new estimate on the growth rate of $\omega-$semiconvex functions that are below supersolutions. The philosophical idea here is that Harnack inequality allows solutions to reproduce their modulus of continuity by below also from above. Thus, semiconvexity together with the equation (Harnack type estimates) should imply regularity at the contact points (see Remark 4.1). As a matter of fact, our estimate is obtained by the use of the weak Harnack inequality for equations with unbounded coefficients proven by S. Koike and A. Swiech in [33][3] Note that in [6] or [8], weak Harnack inequality is obtained from (a clever use of) the ABP estimate which by its turn is proven by using the regularity of the convex envelope. There, the fact that RHS is in L^∞ seems to play an important role in L. Caffarelli's argument. Here instead, based on the new version of the weak Harnack inequality in [33], we are able to reverse this approach and show that the optimal regularity of the convex envelope can be obtained by using the weak Harnack inequality.

Our paper is organized as follows: In section 2, we describe the structural conditions about the PDEs studied in the paper and also present our main results. In section 3, we discuss some examples that show that our regularity results are sharp. Section 4 is devoted to the proof of our key estimate that yields the (pointwise) regularity inside the contact set. In sections 5 and 6, we give the proofs of our results related to the regularity of the convex envelope. In section 7, we present the proof of our regularity Theorem about $\omega-$semiconvex functions. In section 8, we provide a self contained proof of the Lipschitz regularity and $L^p - L^\infty$ estimates for $\omega-$semiconvex functions and their gradients that are needed in the paper. We believe these estimates may show to be useful in other circumstances and may be of independent interest.

[3]Although not explicitly stated in [33], as explained in Remark 4.6 below, the weak Harnack inequality for $p = q = n$ is a consequence of the results in that paper.

Finally, for completeness, in the appendix we give short proofs of some useful Lemma about the relation between $C^{1,\omega}$ pointwise regularity and the classical $C^{1,\omega}$ regularity. This is needed in the analysis of our estimates.

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2. Setting and Main Results

To state our main results, we first recall some basic definition from convex analysis and fully nonlinear PDEs.

2.1. **Convex envelope and $\omega-$semiconvex functions.** Let $u \in C^0(\overline{\Omega})$, where Ω is an open bounded set in $\mathbb{R}^n$. We define the *convex envelope* of u (and denote it by Γ_u) the function defined for every $x \in \overline{\Omega}$ as follows

$$\begin{aligned}\Gamma_u(x) \quad &:= \quad \sup\Big\{\varphi(x) \,:\, \varphi \text{ is convex in } \overline{\Omega} \text{ and } \varphi \le u \text{ in } \overline{\Omega}\Big\}\\ &= \quad \sup\Big\{L(x) \,:\, L \text{ is affine in } \overline{\Omega} \text{ and } L \le u \text{ in } \overline{\Omega}\Big\}.\end{aligned}$$

The set

$$\mathcal{C}(u) := \Big\{x \in \overline{\Omega} \,:\, u(x) = \Gamma_u(x)\Big\}$$

is called the *contact set* between u and the convex envelope Γ_u. In principle, this set may be empty or may be contained inside the boundary of Ω. These consideration will be relevant for our results here. We will return to this issue below.

We also recall some definition from nonsmooth analysis. Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set. We say that $u : \Omega \to \mathbb{R}$ is $\omega-$*semiconvex* if for any $x, y \in \Omega$ and any $t \in [0, 1]$

$$u(tx + (1-t)y) \le tu(x) + (1-t)u(y) + t(1-t)|x-y|\omega(|x-y|). \tag{2.1}$$

where $\omega : [0, \infty) \to [0, \infty)$ is a nondecreasing upper semicontinuous functions such that

$$\omega(0) = \lim_{t\to 0^+} \omega(t) = 0.$$

We recall the normal mapping at x_0 defined as

$$\partial_\omega u(x_0) := \Big\{p \in \mathbb{R}^n \,:\, u(x) \ge u(x_0) + p\cdot(x-x_0) - |x-x_0|\omega(|x-x_0|) \quad \forall x \in \Omega\Big\}.$$

In the case u is $\omega-$semiconvex, the sets $\partial_\omega u(x)$ are non-empty and compact (see Proposition 2.1 in [1]).

Remark 2.1 (Perturbation by affine functions). Let $u : \Omega \to \mathbb{R}$ be a $\omega-$semiconvex function defined in Ω a bounded open convex set. Assume that $L(x) = A\cdot(x-x_0) + B$ with $A \in \mathbb{R}^n$ and $B \in \mathbb{R}$ be an affine function in $\mathbb{R}^n$. Then, $\overline{\varphi}(x) := \varphi(x) + L(x)$ is also $\omega-$semiconvex in Ω. This elementary fact will be relevant in the proof of Theorem 2.9.

2.2. **Supersolutions and Pucci Operators.** We now introduce the structural conditions for the PDEs that appear in this paper. We start by recalling the Pucci extremal operators. Let us denote $\mathcal{S}^{n\times n}$ the space of symmetric matrices of order n. For $0 < \lambda \le \Lambda$, the operator $\mathcal{M}^-_{\lambda,\Lambda} : \mathcal{S}^{n\times n} \to \mathbb{R}$ is given by

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda\cdot\sum_{e_i>0} e_i + \Lambda\cdot\sum_{e_i<0} e_i = \lambda\cdot Tr(M^+) - \Lambda\cdot Tr(M^-), \tag{2.2}$$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda\cdot\sum_{e_i>0} e_i + \lambda\cdot\sum_{e_i<0} e_i = \Lambda\cdot Tr(M^+) - \lambda\cdot Tr(M^-), \tag{2.3}$$

where e_i are the eigenvalues of M and $Tr(M)$ denotes the trace of the matrix M. We recall that

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{M\in\mathcal{A}_{\lambda,\Lambda}} Tr(AM), \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{M\in\mathcal{A}_{\lambda,\Lambda}} Tr(AM), \tag{2.4}$$

where

$$\mathcal{A}_{\lambda,\Lambda} := \Big\{ A\in\mathcal{S}^{n\times n} \ : \ \lambda I_n \le A \le \Lambda I_n \Big\}. \tag{2.5}$$

For $\gamma \ge 0$ a measurable function, we define the *Pucci operator* $\mathcal{P}^-_{\lambda,\Lambda,\gamma} : \mathcal{S}^{n\times n}\times\mathbb{R}^n\times\Omega\to\mathbb{R}$ to be

$$\mathcal{P}^-_{\lambda,\Lambda,\gamma}(M,v,x) = \mathcal{M}^-_{\lambda,\Lambda}(M) - \gamma(x)|v|. \tag{2.6}$$

Throughout the paper, $\gamma, f \in L^r(\Omega)$ with $\gamma\ge 0$. Also, $r\ge n$ will be specified in each particular result. Recall that $u\in C^0(\Omega)$ is a solution to $\mathcal{P}^-_\gamma[u]\le f$ in Ω in the L^n-*viscosity sense* if for any $\phi\in W^{2,n}_{loc}(\Omega)$ such that $u-\phi$ has a local mimimum at $x_0\in\Omega$ we have

$$ess \liminf_{x\to x_0}\Big(\mathcal{P}^-_\gamma[\phi]-f(x)\Big) = ess \liminf_{x\to x_0}\Big(\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x)) - \gamma(x)|\nabla\phi(x)| - f(x)\Big) \le 0.$$

Subsolutions are defined similarly. For simplicity, we also make use of the following notation:

$$\mathcal{P}^-_\gamma[u](x) = \mathcal{P}^-_{\lambda,\Lambda,\gamma}[u](x) := \mathcal{P}^-_{\lambda,\Lambda,\gamma}(D^2u(x),\nabla u(x),x),$$

$$\overline{S}(\gamma,f) := \overline{S}(\lambda,\Lambda,\gamma,f) = \Big\{ u\in C^0(\Omega) \ : \ \mathcal{P}^-_\gamma[u](x)\le f(x) \ \text{ in } \ \Omega \ \text{ in the } L^n-\text{viscosity sense} \Big\},$$

$$\overline{S}_s(\gamma,f) := \overline{S}_s(\lambda,\Lambda,\gamma,f) = \Big\{ u\in W^{2,n}_{loc}(\Omega) \ : \ \mathcal{P}^-_\gamma[u](x)\le f(x)\, a.e. \ \text{ in } \ \Omega\Big\}.$$

Remark 2.2. We refer the reader to [13] for details about the L^p viscosity theory of fully nonlinear PDEs.

2.3. Moduli of continuity and a preliminary result from measure theory.

Definition 2.3. *A modulus of continuity is a nondecreasing function* $\omega : [0,\delta_\omega)\to[0,\infty)$ *such that* $\lim_{t\to 0^+}\omega(t)=\omega(0)=0$. *Here* $\delta_\omega\in(0,\infty]$.

Let a differentiable function u *be defined in an open set* Ω. *We define the following quantity for any* $K\subset\Omega$:

$$[\nabla u]_{C^{0,\omega}(K)} := \inf\Big\{ C>0 \ : \ |\nabla u(x)-\nabla u(y)|\le C\cdot\omega(|x-y|), \ \forall x,y\in K, |x-y|<\delta_\omega\Big\} \tag{2.7}$$

Note that, if ω *is strictly positive and* $[\nabla u]_{C^{0,\omega}(K)}<\infty$, *then it is easy to see that*

$$[\nabla u]_{C^{0,\omega}(K)} = \sup_{\substack{x,y\in K\\ x\ne y,\ |x-y|<\delta_\omega}} \frac{|\nabla u(x)-\nabla u(y)|}{\omega(|x-y|)}.$$

We say that a function $u\in C^{1,\omega}_{loc}(\Omega)$ *if it is differentiable in* Ω *and for any* $K\subset\subset\Omega$ *we have*

$$\|u\|_{C^{1,\omega}(K)} := \|u\|_{L^\infty(K)} + \|\nabla u\|_{L^\infty(K)} + [\nabla u]_{C^{0,\omega}(K)} < \infty.$$

For $\alpha\in(0,1]$, we recall the weighted $C^{1,\alpha}$ norm that will also appear in the sequel: If $u\in C^{1,\alpha}(B_r)$

$$\|u\|^*_{C^{1,\alpha}(B_r)} := \|u\|_{L^\infty(B_r)} + r\cdot\|\nabla u\|_{L^\infty(B_r)} + r^{1+\alpha}\cdot[\nabla u]_{C^\alpha(B_r)},$$

where

$$[\nabla u]_{C^\alpha(B_r)} = \sup_{\substack{x,y\in B_r\\ x\ne y}} \frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^\alpha}.$$

Generally speaking, in this paper, we will always have $\gamma\in L^p(\Omega)$ and $f\in L^q(\Omega)$. This will be specified precisely in each statement in the sequel. As matter of fact, throughout the paper, one of the following conditions with respect to these exponents will be always in place

$$\text{(E)} \qquad \begin{cases} p \geq q > n, \\ p > q \geq n, \\ p = q = n \qquad \text{and} \quad u \in \overline{S}_s(\gamma; f) \ \text{ in } \Omega. \end{cases}$$

In order to deal with the case $p = q = n$, the following result will be important (see Remarks 2.5 and 4.4).

Lemma 2.4. *Let $g \in L^1(\mathbb{R}^n)$. Then there exists a modulus of continuity $\Theta : [0, \infty) \to [0, \infty)$ such that*

$$\int_{B_\rho(x_0)} |g(x)|dx \leq \Theta(\rho) \qquad \forall x_0 \in \Omega,\ \forall \rho > 0.$$

In particular, if $\Omega \subset \mathbb{R}^n$ and $g \in L^1(\Omega)$, we have

$$\int_{B_\rho(x_0)\cap\Omega} |g(x)|dx \leq \Theta(\rho) \qquad \forall x_0 \in \Omega,\ \forall \rho > 0.$$

Proof. By the absolute continuity property of the integral, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_{B_\rho(x_0)} |g(x)|dx < \varepsilon \qquad \forall x_0 \in \mathbb{R}^n,\ \forall \rho \leq \delta(\varepsilon). \tag{2.8}$$

Thanks to this fact, it suffices to set

$$\Theta(\rho) := \sup_{x_0 \in \mathbb{R}^n} \int_{B_\rho(x_0)} |g(x)|dx \qquad \forall \rho > 0.$$

In the case $g \in L^1(\Omega)$, we apply the previous result to the extension of g by zero outside Ω. □

2.4. **Statement of the main results.** Before we state our results, we make a Remark.

Remark 2.5. In Theorems 2.6, 2.8 and 2.9, condition (E) will be always in place. Moreover, whenever $q = n$, $C^{1,1-n/q}_{loc}(\Omega)$ simply means $C^1(\Omega)$. Note that, when $p = q = n$, it follows by Lemma 2.4 applied to $g(x) = \gamma(x)^n$ that there exists a modulus of continuity Θ_γ such that

$$||\gamma||_{L^n(B_\rho(x_0)\cap\Omega)} \leq \Theta_\gamma(\rho) \quad \forall x_0 \in \Omega.$$

In the next statements, Θ_γ will refer to this modulus of continuity. Analogously, if $q = n$, Lemma 2.4 applied to $g(x) = |f(x)|^n$ guarantees the existence of a modulus of continuity ϑ such that

$$\max\left\{\|f\|_{L^n(B_r(x_0)\cap\Omega)}, r^{1-\frac{n}{p}} \cdot \|\gamma\|_{L^p(B_r(x_0)\cap\Omega)}\right\} \leq \vartheta(r) \quad \forall x_0 \in \Omega, \quad \forall r > 0.$$

Theorem 2.6. *Let Ω be an open, bounded and convex set in $\mathbb{R}^n$ and $u \in C^0(\overline{\Omega}) \cap \overline{S}(\gamma; f)$ in Ω, with $\gamma \in L^p(\Omega)$ and $f \in L^q(\Omega)$. Suppose that $u = 0$ on $\partial\Omega$ and that $u < 0$ in Ω. Then, $\Gamma_u \in C^{1,1-n/q}_{loc}(\Omega)$ and this regularity is optimal. Moreover, in the case $q = n$, let ϑ be a modulus of continuity such that*

$$\max\left\{\|f\|_{L^n(B_r(x_0))}, r^{1-\frac{n}{p}} \cdot \|\gamma\|_{L^p(B_r(x_0))}\right\} \leq \vartheta(r) \quad \forall x_0 \in \Omega, \quad \forall r \leq dist(x_0, \partial\Omega). \tag{2.9}$$

Then $\Gamma_u \in C^{1,\overline{\vartheta}}_{loc}(\Omega)$, where for any $\Omega' \subset\subset \Omega$

$$\overline{\vartheta}(r) = \overline{\vartheta}_{\Omega'}(r) = C\vartheta(\mu r) \quad \forall r \in [0, r_*].$$

Here

$$r_* = r_*(n, \inf_\Omega u, \Omega', dist(\Omega', \partial\Omega)) > 0, \quad \mu = \mu(n, \inf_\Omega u, \Omega') > 0,$$

$$C = C(n, p, q, \lambda, \Lambda, \|\gamma\|_{L^p(\Omega)}, \inf_\Omega u, \Omega', dist(\Omega', \partial\Omega), \Theta_\gamma) > 0. \tag{2.10}$$

There is a related result to Theorem 2.6 that can be found in the classical book of C. Gutierrez on the Monge-Ampère Equation (Proposition 6.6.1 in [29]). There, like in the statement above, u vanishes on the boundary and it is negative inside the domain. However, it has regularity up to the boundary, i.e, $u \in C^2(\overline{\Omega})$. As a matter of fact, Proposition 6.6.1 in [29] follows from Proposition 4.1 in [16] since the C^2 regularity may be replaced only by the $(1+\alpha)$ semi-convavity of u. Observe also that Proposition 6.6.1 in [29] follows from our Theorem 2.6 above since $\Delta u \leq \|\Delta u\|_{L^\infty(\Omega)}$ and thus the supersolution condition is satisfied.

Remark 2.7. Roughly speaking, the idea of the proof of Theorem 2.6 is the following: We use the PDE (Pucci extremal operator) to prove some kind of semi-concavity property via our key estimate (Proposition 4.2 below). This gives the regularity inside the contact set. Then, we use the prescribed geometry of u to propagate the regularity from the contact set to the rest of the domain. We point out that the geometry of u plays an important role here, as it forces the contact between u and its convex envelope Γ_u to happen inside the domain and not only on the boundary (see Remark 5.3). In the absence of this geometry, the contact set may be totally contained on the boundary and even the differentiability of the convex envelope may be lost, as shown in Example 3.1 below.

Under some more specific geometry, we obtain the following result which can be seen as an extension of Lemma 3.5 in book [8] by L. Caffarelli and X. Cabré. We note that the regularity provided by (2.11) is optimal.

Theorem 2.8. *Let $u \in C^0(\overline{B}_r) \cap \overline{S}(\gamma; f)$ in B_r, with $\gamma \in L^p(B_r)$ and $f \in L^q(B_r)$. Assume that $u \geq 0$ along ∂B_r, and let Γ_u be the convex envelope of $-u^-$ with respect to $\overline{B}_{2r}$, where u^- is extended by zero outside B_r*[4]*. Then, $\Gamma_u \in C^{1,1-n/q}_{loc}(B_r)$. Also, in the case $q > n$, given $R_0 > 0$ the following estimate holds for $0 < r \leq R_0$*

$$\|\Gamma_u\|^*_{C^{1,1-\frac{n}{q}}(B_r)} \leq C\Big(\Big(1 + \|\gamma\|_{L^p(B_r)}\Big)\|u\|_{L^\infty(B_r)} + \|f^+\|_{L^q(B_r)}\Big) \tag{2.11}$$

where $C = C(n, \lambda, \Lambda, p, q, R_0, \|\gamma\|_{L^p(B_{R_0})}) > 0$.

Moreover, in the case $q = n$, let ϑ be a modulus of continuity such that

$$\max\left\{\|f\|_{L^n(B_\rho(x_0))}, \rho^{1-\frac{n}{p}}\|\gamma\|_{L^p(B_\rho(x_0))}\right\} \leq \vartheta(\rho) \quad \forall x_0 \in B_r, \quad 0 < \rho < r \leq R_0. \tag{2.12}$$

Then $u \in C^{1,\overline{\vartheta}}(\overline{B}_{r/384n})$ with the following estimate

$$[\nabla \Gamma_u]_{C^{0,\overline{\vartheta}}(B_{r/384n})} \leq C\left(1 + \frac{\|u\|_{\infty(B_r(x_0))}}{r}\right). \tag{2.13}$$

Here,

$$\overline{\vartheta}(\rho) := \vartheta(6n\rho) \text{ for } \rho \in [0, r/48n],$$

and

$$C = C(n, p, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0. \tag{2.14}$$

Finally, our last result concerns the optimal regularity of supersolutions that are $\omega-$semiconvex.

Theorem 2.9. *Let $\varphi \in \overline{S}(\gamma, f)$ in B_r be a bounded $\omega-$semiconvex function in B_r with $0 < r \leq R_0$. Assume that $\gamma \in L^p(B_r)$ and $f \in L^q(B_r)$, and let $\alpha = 1 - n/q \geq 0$, $\beta := n(q^{-1} - p^{-1}) \geq 0$. Then $\varphi \in C^{1,\zeta}(B_{r/64})$, where*

$$\zeta(s) := \omega(4s) + \left(\|f\|_{L^q(B_{4s})} + s^\beta\left(\frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\right)\|\gamma\|_{L^p(B_{4s})}\right)s^\alpha \quad \text{for } s \in (0, r/8). \tag{2.15}$$

In the case $q = n$, let ϑ be a modulus of continuity such that

$$\max\left\{\|f\|_{L^n(B_\rho(x_0))}, \rho^{1-\frac{n}{p}}\|\gamma\|_{L^p(B_\rho(x_0))}\right\} \leq \vartheta(\rho) \quad \forall x_0 \in \overline{B}_r, \quad 0 < \rho < r - |x_0| \leq r \leq R_0. \tag{2.16}$$

Let Υ be given by the following formula for any $s \in (0, r/8)$:

$$\Upsilon(s) := \begin{cases} \omega(4s) + s^\alpha & \text{if } n > q, \\ \omega(4s) + \vartheta(4s) & \text{if } n = q, \text{ with } \vartheta \text{ as in (2.16) above.} \end{cases}$$

[4] Here, γ and f are also extended by zero outisde B_r. See the proof for details

Then, $\varphi \in C^{1,\Upsilon}(B_{r/64})$ with the following estimates in the respective cases as the ones above:

$$[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})} \leq C\bigg(1 + \|f\|_{L^q(B_r)} + \bigg(\frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\bigg)\|\gamma\|_{L^p(B_r)}\bigg),$$

$$[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})} \leq C\bigg(1 + \frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\bigg).$$

Here,

$$C = C(n, p, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0. \tag{2.17}$$

Remark 2.10. The dependence on Θ_γ in the constants (2.10), (2.14) and (2.17) only takes place in the case $p = q = n$. Furthermore, in that case, $R_0 > 0$ can be dropped in (2.14) and (2.17).

3. EXAMPLES

We briefly point out examples showing the sharpness ine Theorems 2.6 and 2.8 (see also [3]).

Example 3.1 (Breaking down the geometry of u in Theorem 2.6). This example is inspired by the very nice one constructed by A. Fathi and M. Zavidovique in Remark 2.4 in [19]. The authors were interested in showing that the condition of coercivity on the boundary is not artificial if one is interested in obtaining higher regularity (beyond local Lipschitz regularity) of the convex envelope. See Theorem 2.3 in [19]. Here, we perform a simple modification of their example to fit our purposes, i.e, to show that the geometry imposed in Theorem 2.6 is relevant for some degree of smoothness beyond local Lipschitz regularity of the convex envelope to hold. We give a detailed construction below.

Let $R = [0,1] \times [0,1]$ and $u : R \to \mathbb{R}$ such that

$$u(x,y) := \begin{cases} x - y + 1, & \text{for } x \leq y, \\ y - x + 1, & \text{for } x > y. \end{cases}$$

It is easy to see that u defined above is concave since it has a supporting plane from above at all points in R (see figure 1 below). Thus,

$$\Delta u \leq 0 \quad \text{in} \quad R^\circ := int(R) \quad \text{in the viscosity sense.}$$

Now let us consider $\varphi : R \to \mathbb{R}$ given by

$$\varphi(x,y) := \begin{cases} -x - y + 1, & \text{for } x + y \leq 1, \\ x + y - 1, & \text{for } x + y > 1. \end{cases}$$

By a similar argument (see figure 2 below), it is also easy to check that φ is convex and $\varphi \leq u$ in R. Thus,

$$\varphi \leq \Gamma_u \quad \text{in} \quad R. \tag{3.1}$$

We claim that $\varphi = \Gamma_u$. Indeed, let us consider the sets

$$T_1 = \Big\{(x,y) \in R : x + y \leq 1\Big\} \quad \text{and} \quad T_2 = \Big\{(x,y) \in R : x + y \geq 1\Big\}.$$

If we denote Γ_u^i the convex envelope of $u_{|_{T_i}} : T_i \to \mathbb{R}$ for $i = 1, 2$, then we have

$$\Gamma_u \leq \Gamma_u^i \quad \text{in} \quad T_i \quad \text{for} \quad i = 1, 2. \tag{3.2}$$

Let now, $A_i(x,y)$ to be a generic affine function such that $A_i \leq u_{|_{T_i}}$ in T_i for $i = 1, 2$. In particular, by construction of u

$$u(1,0) = u(0,1) = 0.$$

Observe that $\varphi \equiv 0$ along the edge $x + y = 1$, Moreover, since A_i is affine in T_i,

$$(x,y) \in R, \quad x + y = 1 \quad \Longrightarrow \quad A_i(x,y) \leq \max\{u(1,0), u(0,1)\} = 0 \leq \varphi(x,y).$$

Also, both φ and u are affine functions along the edges of R and they coincide on the vertices of R. Thus, we conclude that $A_i \leq \varphi$ on ∂T_i, and therefore $A_i \leq \varphi$ in T_i (since they are both affine). This proves that $\Gamma_u^i \leq \varphi$ in T_i. By (3.1) and (3.2) we conclude that $\varphi = \Gamma_u$ in R.

We now set $v := u - 2$. Then $\Gamma_v = \Gamma_u - 2$, $\Delta v \leq 0$ in R°, and $v \leq -1$ in R, but $v \not\equiv 0$ along ∂R. Here, Γ_v is not differentiable at any point on the diagonal $x + y = 1$. Also, $\mathcal{C}(v) \subset \partial R$ and the optimal

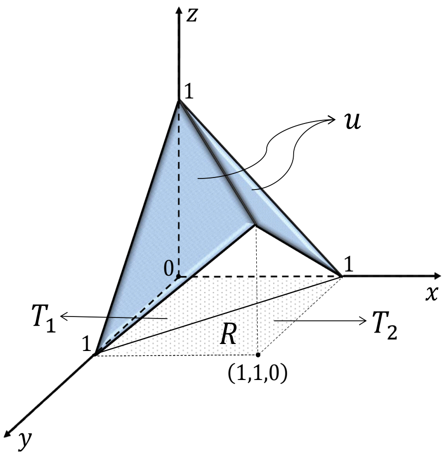

FIGURE 1. The graph of u

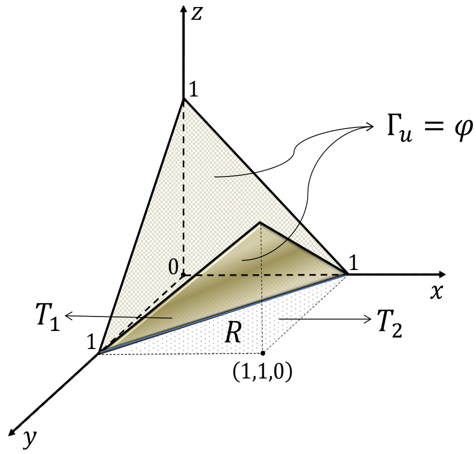

FIGURE 2. The graph of Γ_u

regularity of Γ_v is only locally Lipschitz continuous. This show the necessity of the geometry of u required in Theorem 2.6.

Example 3.2 (Sharpness of the exponent $1 - n/q$, when RHS is in L^q with $q > n$, for Theorem 2.6). Let $q > 2$ and consider the functions $I : (0, 3/4] \to [0, \infty)$ and $w : [0, 3/4] \to [0, \infty)$ given by

$$I(t) := \frac{t^{1-2/q}}{|\ln(t)|^{2/q}}, \qquad w(t) := \int_0^t I(s)\, ds.$$

We observe that $w \in C^{1,1-2/q}([0, 1/2]) \cap C^\infty(0, 3/4)$. Also, w is nonnegative, convex, and (strictly) increasing in $[0, 1/2]$. Now, we define in two dimensions

$$u(x) := w(|x|) - w(1/2) \quad \text{for} \quad x \in \overline{B}_{1/2} \subset \mathbb{R}^2.$$

Then u is nonpositive (since w attains its maximum on $[0, 1/2]$ at $t = 1/2$) and convex, with $u \equiv 0$ on $\partial B_{1/2}$ and $u < 0$ in $B_{1/2}$. One can check that $u \in C^{1,1-2/q}_{loc}(B_{1/2}) \cap C^\infty(B_{1/2} \setminus \{0\})$. Moreover, a direct computation shows that $\Delta u = f$ where

$$f(x) := \frac{(2 - 2/q)|x|^{-2/q}}{|\ln|x||^{2/q}} + \frac{2}{q}\frac{|x|^{-2/q}}{|\ln|x||^{1+2/q}} \in L^q(B_{1/2}).$$

In particular, by the Calderon-Zygmund theory, $u \in W^{2,q}_{loc}(B_{1/2})$ with $q > 2$ (this can also be checked by a direct computation). In particular, u is a L^q-strong solution, hence a L^q-viscosity solution to $\Delta u = f$ in $B_{1/2}$ by Theorem 2.1 in [14]. Since u is convex in $B_{1/2}$ it coincides with its convex envelope, i.e, $\Gamma_u \equiv u$ in $B_{1/2}$. Hence,

$$|\nabla \Gamma_u(x)| = |\nabla u(x)| = I(|x|) \in C^{0,1-2/q}(B_{1/2}) \setminus C^{0,1+\epsilon-2/q}(B_{1/2}) \qquad \forall \epsilon > 0.$$

Example 3.3 (Sharpness of the exponent $1 - n/q$, when RHS is in L^q with $q > n$, for Theorem 2.8). We define $\widetilde{u}(x) := -(u(x))^-$, where $u(x) := 4w(|x|) - w(1/2)$ for $x \in \overline{B}_{1/2}$ with w as in Example 3.2. Clearly $\Delta u = 4f \in L^q(B_{1/2})$ (f defined in Example 3.2) with $q > 2$. Since w is strictly increasing in $[0, 1/2]$, there exists $r_0 < 1/2$ such that $u < 0$ in B_{r_0} and $u \geq 0$ in $\overline{B}_{1/2} \setminus B_{r_0}$. In particular, $\widetilde{u} < 0$ in B_{r_0} and $\widetilde{u} \equiv 0$ in $\overline{B}_{1/2} \setminus B_{r_0}$. Now, set $v(x) := w(|x|) - w(1/2)$. Then, $\widetilde{u}(0) = v(0) = -w(1/2) < 0$. Moreover, $v \leq \widetilde{u}$ in $\overline{B}_{1/2}$. Since v is convex in $\overline{B}_{1/2}$, we have $v \leq \Gamma_{\widetilde{u}} \leq \widetilde{u}$ in $\overline{B}_{1/2}$. Since $v, \widetilde{u} \in C^{1,1-2/q}(B_{r_0/2})$ in a sharp way, we conclude that $\Gamma_{\widetilde{u}}$ cannot have better regularity than $\Gamma_{\widetilde{u}} \in C^{1,1-2/q}(B_{1/2})$. This shows that regularity in Theorem 2.8 is sharp.

Example 3.4 (Counterexample with RHS in L^q with $q < n$). Let us consider the convex function $u(x) = |x| - 1$ defined in $B_1 \subset \mathbb{R}^n$ for $n \geq 2$. Direct computation shows that $\Delta u(x) = (n-1)|x|^{-1} =: f(x)$

for $x \neq 0$. Now, it follows that $f \in L^q(B_1)$ if and only if $0 < q < n$. Taking $q \in (p_0, n)$ where $p_0 = p_0(n) \in [n/2, n)$ is the exponent defined in [33]. We conclude from Theorem 3.1 in [33] that u is a L^q viscosity solution to $\Delta u = f(x)$ in B_1. However, u is not differentiable at the origin. This shows that neither Theorem 2.6 nor Theorem 2.9 hold for $q < n$.

4. Key estimate and the Regularity on the contact set

Now we present an estimate which is the starting point for all the results in this paper. Before, a Remark to explain the ideas.

Remark 4.1. An important feature of the Harnack inequality is that control by below implies control by above and vice-versa. As a matter of fact, Harnack inequality allows solutions to replicate their modulus of continuity from below to above (and vice-versa also). Moreover, if the equation "does not see" linear functions, this can go up to $C^{1,1}$ regularity once one can choose the tangent plane of the solution to be the canonical reference axis for the estimates. Let us now formalize the argument.

Assume $u \in C^0(B_1)$ satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le 0 \le \mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ in B_1 in the viscosity sense. Let l be an affine function so that $u(0) = l(0)$. Assume that u separates from l by below with modulus of continuity given by $\omega \ge 0$. By this, we mean

$$\inf_{B_r}(u - l) \ge -\omega(r) \quad \forall r \in (0, 1).$$

Thus, by setting $v_r(x) := u(x) - l(x) + \omega(r)$ for $x \in B_1$, we see that $0 \le v_r \in C^0(B_r)$ and moreover $\mathcal{M}^-_{\lambda,\Lambda}(D^2v_r) \le 0 \le \mathcal{M}^+_{\lambda,\Lambda}(D^2v_r)$ in B_r in the viscosity sense. Thus, by Harnack inequality, there exists a universal $C = C(n, \lambda, \Lambda) \ge 1$ so that

$$\sup_{B_{r/2}}(u - l) \le \sup_{B_{r/2}}(u - l + \omega(r)) = \sup_{B_{r/2}} v_r \le C v_r(0) = C\omega(r) \quad \forall r \in (0, 1).$$

In the case we are dealing with supersolutions, only half-Harnack inequality (weak Harnack inequality) is at our disposal. In order to reproduce the type of argument above we need some kind of $L^\infty - L^\varepsilon$ estimate for $\varepsilon > 0$ small in order to pass from the average control to a uniform control. Moreover, if equation "sees" linear functions, the slope of the tangent plane can be incorporated to the RHS of the equation and thus we also need gradient estimates in order to control the error produced by the tilt of the tangent plane in our estimates.

We apply these lines of ideas for $\omega-$semiconvex functions that are below supersolutions. They have a $C^{1,\omega}$ modulus of continuity by below, they do satisfy the $L^\infty - L^\varepsilon$ estimate as well as gradient estimates as proven in Proposition 8.2. Thus, by using the equation to transfer information from the supersolution to the $\omega-$semiconvex function, we may expect $\omega-$semiconvex functions that are below supersolutions to be regular, once they can reproduce a regular modulus of continuity by above at the contact points. This modulus of continuity should be dictated by ω and by the RHS. This is indeed the case and the details are presented in our key estimate below (zero tangent plane case) as well as in Proposition 7.1 where these kind of ideas are used in full generality.

Proposition 4.2 (Key estimate). *Let $u \in \overline{S}(\gamma; f)$ in B_r where $\gamma \in L^p(B_r)$, $f \in L^q(B_r)$, and $r \le R_0$. Suppose that φ is a $\omega-$semiconvex function in B_r such that $0 \le \varphi \le u$ in B_r. Then, for any $\rho \in (0, r)$,*

$$\|\varphi\|_{L^\infty(B_{\rho/2})} \le C\Big(\inf_{B_{\frac{3\rho}{4}}} u + \vartheta(\rho)\rho\Big) \tag{4.1}$$

where $\alpha = 1 - \frac{n}{q} \ge 0$ and for $\rho \in (0, r)$

$$\vartheta(\rho) := \rho^\alpha \|f\|_{L^q(B_\rho)} + \omega(\rho), \tag{4.2}$$

and

$$C = C(n, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0 \tag{4.3}$$

is a universal constant. Also we have

$$\vartheta(\rho) = o(1) \quad \text{as} \quad \rho \to 0. \tag{4.4}$$

In the case that $u(0) = 0$, i.e, if 0 is a contact point, then

$$\|\varphi\|_{L^\infty(B_{\rho/2})} \leq C\vartheta(\rho)\rho. \tag{4.5}$$

In particular, in this case, φ is differentiable at the origin and $\nabla\varphi(0) = 0$.

Remark 4.3. Our key estimate above is an extension of L. Caffarelli's crucial estimate on the $C^{1,1}$ regularity of the convex envelope inside the contact set given in Lemma 3.3 in the book [8]. Indeed, in Lemma 3.3 in [8], $\gamma = 0, f \in L^\infty, \omega = 0$ (convex functions setting) and $0 = \varphi(0) = u(0)$. This way, we recover the quadratic growth at the contact points, since in this case (4.5) reads

$$\sup_{B_{\rho/2}} \varphi \leq C||f||_{L^\infty(B_r)}\rho^2, \quad \forall \rho \in (0, r).$$

Remark 4.4. The case $p = q = n$ in (E) is allowed in Proposition 4.2, thanks to Lemma 2.4 and Remark 2.5. Indeed, as explained in Remark 4.6 below, the weak Harnack inequality for L^n-strong supersolution holds in this case and so does our Proposition 4.2. As in Remark 2.10, the dependence of $C > 0$ given in (4.3) on the modulus of continuity Θ_γ takes place only in the case $p = q = n$. Moreover, in this case, the dependence of $C > 0$ given in (4.3) on both R_0 and $||\gamma||_{L^n(B_{R_0})}$ can be dropped.

Remark 4.5. In [37], the Harnack inequality is obtained for linear equations in nondivergence form with zero RHS and drift term in L^n. Also, in [35], a similar result is obtained in the situation where equation holds in the regions where gradient is large.

Remark 4.6. In the case $p = q = n$, the weak Harnack inequality holds for L^n-strong supersolutions, namely, $0 \leq u \in \overline{S}_s(\gamma; f)$ with $\gamma, f \in L^n$. Moreover, in this case, the (universal) constant in the weak Harnack inequality also depends on Θ_γ. Besides, there is no dependence neither on $||\gamma||_{L^n(B_{R_0})}$ nor on R_0. Indeed, by Remark 4.4 in [33], weak Harnack inequality holds for nonnegative L^n-strong supersolutions in $\overline{S}_s(\gamma; f)$ under the assumption that $||\gamma||_{L^n}$ is (universally) small (see also Lemma 4.3 in [33]). The argument in the proof of Theorem 4.5 in [33] shows that weak Harnack inequality follows from Lemma 4.3 and Remark 4.4 in [33], using and a covering argument (like for instance, the one pointed out by X. Cabré in Remark 3.2 in [5]) provided it holds in small cubes (or balls). This is indeed the case, as one can see by following an argument similar to the one used in the proof of Theorem 4.5 in [33], where condition $p > n$ should be replaced by the use of Lemma 2.4 applied to $g = \gamma^n$. We also refer the reader to Theorem 2.3, Remark 1 and Corollary 1 in [34] for general statement and proofs about the weak Harnack inequality.

Remark 4.7. In case $p = q = n$, it is not clear if a function in $\overline{S}(\gamma, f)$ which is also in $W^{2,n}$ is a L^n-strong solution. Therefore one has to require in (E) that our function is a strong supersolution. Essentially, the difficulty here comes from the lack of existence results for L^n-strong solutions for (Pucci) extremal equations when $p = q = n$. Thus, one cannot implement an argument like in Theorem 3.4 and Corollary 3.7 in [13] (bounded drift term) or, more generally, Proposition 9.1 in [33].

Proof of Proposition 4.2. Since $\varphi \leq u$, it follows from the weak Harnack inequality (see Theorem 4.5 and Remarks 4.4 and 5.2 in [33]) applied to u that, for any $\rho \in (0, r)$,

$$\left(\fint_{B_{3\rho/4}} |\varphi(x)|^\varepsilon dx\right)^{\frac{1}{\varepsilon}} \leq \left(\fint_{B_{3\rho/4}} |u(x)|^\varepsilon dx\right)^{\frac{1}{\varepsilon}} \leq \overline{C}\Big(\inf_{B_{\frac{3\rho}{4}}} u + \rho^{2-\frac{n}{q}}\|f\|_{L^q(B_\rho)}\Big) \tag{4.6}$$

where $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$ and $\overline{C} = \overline{C}(n, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$ (see also Remark 4.6) are universal constants. Also, we can apply the $L^p - L^\infty$ estimate for $\omega-$semiconvex functions (see Proposition 8.2) with $p = \varepsilon$ to get

$$\sup_{B_{\rho/2}} |\varphi| \leq C_2\left[\left(\fint_{B_{3\rho/4}} |\varphi|^\varepsilon dx\right)^{1/\varepsilon} + \rho\omega(\rho)\right] \quad \text{for some} \quad C_2 = C_2(n, \lambda, \Lambda) > 0. \tag{4.7}$$

Combining (4.6) and (4.7) above we obtain

$$\begin{aligned}\sup_{B_{\rho/2}}|\varphi| &\le \overline{C}C_2\Big(\inf_{B_{\frac{3\rho}{4}}} u+\rho^{2-\frac{n}{q}}\|f\|_{L^q(B_\rho)}\Big)+C_2\rho\omega(\rho)\\ &\le C_2(\overline{C}+1)\inf_{B_{\frac{3\rho}{4}}} u+C_2(\overline{C}+1)\Big(\rho^\alpha\|f\|_{L^q(B_\rho)}+\omega(\rho)\Big)\rho\\ &= C\Big(\inf_{B_{\frac{3\rho}{4}}} u+\Big(\rho^\alpha\|f\|_{L^q(B_\rho)}+\omega(\rho)\Big)\rho\Big)\\ &= C\Big(\inf_{B_{\frac{3\rho}{4}}} u+\vartheta(\rho)\rho\Big),\end{aligned}$$

for $C:=C_2(\overline{C}+1)$. Finally, (4.4) is obvious if $q>n$, while in the case $q=n$ it follows from Lemma 2.4 applied to $g(x)=|f(x)|^n$. □

5. Proof of Theorem 2.6

Before we proceed with the proof of Theorem 2.6, we recall the following result from convex analysis.

Lemma 5.1 (Affine components with respect to the contact set - Lemma 6.6.2 in [29]). *Let Ω be an open bounded convex domain. Suppose $u\in C^0(\overline{\Omega})$. Let $x_0\in\Omega\setminus\mathcal{C}(u)$ and L be a supporting hyperplane to Γ_u at x_0. Then, there exist at most $n+1$ points $x_i\in\mathcal{C}(u)$ such that*

$$x_0=\sum_{i=1}^{n+1}\lambda_i x_i \quad \text{with} \quad \lambda_i\ge 0 \quad \text{and} \quad \sum_{i=1}^{n+1}\lambda_i=1.$$

Furthermore,

$$L(x_i)=\Gamma_u(x_i)=u(x_i)\quad \forall i\in\Big\{1,\cdots,n+1\Big\}. \tag{5.1}$$

In particular, L is a supporting hyperplane to Γ_u at any $x_i\in\{x_1,\cdots,x_{n+1}\}$. Moreover,

$$L\equiv\Gamma_u \quad \text{in} \quad \text{conv}\Big\{x_1,\cdots,x_{n+1}\Big\} \tag{5.2}$$

Remark 5.2. Although (5.2) is not explicitly stated in Lemma 6.6.2 in [29], it is easy to verify it holds. Indeed, it is a direct consequence of (5.1) and convexity. In order to see this, let $x\in\text{conv}\big\{x_1,\cdots,x_{n+1}\big\}$. Since $L\le\Gamma_u$ in Ω and $x=\sum_{i=1}^{n+1}\alpha_i x_i$ for some $\alpha_i\ge 0$ with $\sum_{i=1}^{n+1}\alpha_i=1$, we have

$$L(x)\le\Gamma_u(x)=\Gamma_u\Big(\sum_{i=1}^{n+1}\alpha_i x_i\Big)\le\sum_{i=1}^{n+1}\alpha_i\Gamma_u(x_i)=\sum_{i=1}^{n+1}\alpha_i L(x_i)=L(x).$$

Proof of Theorem 2.6

Proof. We divide the proof in two cases. The regularity inside and outside the contact set. So, let

$$\Omega'\subset\subset\Omega \quad \text{and} \quad \delta:=dist(\Omega',\partial\Omega)/4>0. \tag{5.3}$$

Claim 1: Let $x_0\in\Omega'\cap\mathcal{C}(u)$ and L_{x_0} be a supporting hyperplane for Γ_u at x_0. Then,

$$\|\Gamma_u-L_{x_0}\|_{L^\infty(B_{r/2}(x_0))}\le\vartheta_{x_0}(r)r \quad \forall r\in(0,\delta], \tag{5.4}$$

where

$$\vartheta_{x_0}(r):=C\Bigg(\|f\|_{L^q(B_r(x_0))}+\Bigg(\frac{\|u\|_{L^\infty(\Omega)}}{dist(\Omega',\partial\Omega)}\|\gamma\|_{L^p(B_r(x_0))}\Bigg)r^\beta\Bigg)r^\alpha, \tag{5.5}$$

$$\alpha=1-\frac{n}{q}\ge 0, \quad \beta=n\Big(\frac{1}{q}-\frac{1}{p}\Big)\ge 0,$$

and $C=C(n,q,p,\lambda,\Lambda,\delta,\|\gamma\|_{L^p(\Omega)},\Theta_\gamma)>0$ is a universal constant.

Proof of claim 1: Since L_{x_0} is a supporting plane of Γ_u at x_0, we have

$$0\le\Gamma_u-L_{x_0}\le u-L_{x_0} \text{ in } B_\delta(x_0) \qquad \text{and} \qquad 0=\Gamma_u(x_0)-L_{x_0}(x_0)=u(x_0)-\Gamma_u(x_0).$$

Observe that $u - L_{x_0} \in \overline{S}(\gamma; f + \gamma|\nabla L_{x_0}|)$ in $B_{2\delta}(x_0)$. By Proposition 4.2 applied to the convex function $\varphi = \Gamma_u - L_{x_0}$ (and thus $\omega \equiv 0$) we obtain (since x_0 is a contact point)

$$\|\Gamma_u - L_{x_0}\|_{L^\infty(B_{r/2}(x_0))} \leq \widehat{\vartheta}_{x_0}(r)r \quad \forall\, r \in (0, \delta],$$

where

$$\widehat{\vartheta}_{x_0}(r) = Cr^\alpha \|f - \gamma|\nabla L_{x_0}|\|_{L^q(B_r(x_0))}, \quad \alpha = 1 - \frac{n}{q} \geq 0,$$

and $C = C(n, q, \lambda, \Lambda, \delta, \|\gamma\|_{L^p(\Omega)}, \Theta_\gamma) > 0$ is a universal constant as before. From the conclusion of Proposition 4.2, we also conclude that Γ_u is differentiable at x_0 with $\nabla\Gamma_u(x_0) = \nabla L_{x_0}$. Now, we observe that the constant function $L \equiv -\|u\|_{L^\infty(\Omega)}$ is affine and below u everywhere in $\overline{\Omega}$. Thus, from the definition of Γ_u,

$$-\|u\|_{L^\infty(\Omega)} \leq \Gamma_u(x) \quad \forall x \in \overline{\Omega}. \tag{5.6}$$

Now, since Γ_u is convex and $\Gamma_u \leq u = 0$ on $\partial\Omega$, by the gradient estimates on convex functions (see Lemma 3.2.1 in [29], for instance) we obtain

$$|\nabla L_{x_0}| = |\nabla\Gamma_u(x_0)| \leq \frac{-\Gamma_u(x_0)}{dist(x_0, \partial\Omega)} \leq \frac{\|u\|_{L^\infty(\Omega)}}{dist(\Omega', \partial\Omega)}. \tag{5.7}$$

This implies that

$$\begin{aligned} \widehat{\vartheta}_{x_0}(r) &\leq C\left(\|f\|_{L^q(B_r(x_0))} + \frac{\|u\|_{L^\infty(\Omega)}}{dist(\Omega', \partial\Omega)}\|\gamma\|_{L^q(B_r(x_0))}\right)r^\alpha && (5.8)\\ &\leq C'\left(\|f\|_{L^q(B_r(x_0))} + \left(\frac{\|u\|_{L^\infty(\Omega)}}{dist(\Omega', \partial\Omega)}\|\gamma\|_{L^p(B_r(x_0))}\right)r^\beta\right)r^\alpha && (5.9)\end{aligned}$$

where $C' = C(1 + |B_1|^{(\frac{1}{q} - \frac{1}{p})})$. This finishes the proof of Claim 1.

Let us set, for each $m \in \mathbb{N}^*$,

$$K_m := \text{conv}\Big(L_m\Big) \quad \text{where} \quad L_m := \left\{x \in \Omega \,:\, u(x) \leq -\frac{1}{m}\right\}.$$

Clearly, $L_m \subset K_m$ for all $m \in \mathbb{N}$. Note that, due to the boundary condition $u = 0$ on $\partial\Omega$, we also have

$$L_m = \left\{x \in \overline{\Omega} \,:\, u(x) \leq -\frac{1}{m}\right\}.$$

Thus, $L_m \subset \Omega$ is compact for every $m \in \mathbb{N}^*$. Also, since $u < 0$ in Ω,

$$L_m \subset L_{m+1} \quad \forall m \in \mathbb{N}^* \quad \text{and} \quad \bigcup_{m\in\mathbb{N}^*} L_m = \Omega. \tag{5.10}$$

Hence, since Ω is convex, K_m is convex and compact (see Corollary A.1.7 in the appendix of [11]) and $K_m \subset \Omega$ for all $m \in \mathbb{N}^*$. Moreover,

$$K_m \subset K_{m+1} \quad \forall m \in \mathbb{N}^* \quad \text{and} \quad \bigcup_{m\in\mathbb{N}^*} K_m = \Omega \tag{5.11}$$

and thus

$$d_m := dist(K_m, \partial\Omega) > 0 \quad \forall m \in \mathbb{N}^*. \tag{5.12}$$

Claim 2: Let $x_0 \in K_m \setminus \mathcal{C}(u)$ and $\big\{\lambda_i\big\}_{i=1}^{n+1}$ and $\big\{x_i\big\}_{i=1}^{n+1}$ as in Lemma 5.1. Let us set

$$\overline{\lambda} := \frac{1}{4m(n+1)(-\inf_\Omega u)} > 0. \tag{5.13}$$

Then there exists at least one index $j \in \big\{1, \cdots, n+1\big\}$ such that $x_j \in K_{2m}$ and $\lambda_j \geq \overline{\lambda}$.

Proof of claim 2 (see also Remark 5.3): Since $\Gamma_u(x) \leq u(x) \leq -1/m$ for all $x \in L_m$, we conclude by convexity of Γ_u that

$$\Gamma_u \leq -1/m \quad \text{in} \quad K_m. \tag{5.14}$$

This implies, in particular, that

$$u \leq -1/m \quad \text{in} \quad K_m \cap \Big\{u = \Gamma_u\Big\}. \tag{5.15}$$

We claim that for at least one index $j \in \Big\{1, \cdots, n+1\Big\}$ we have $x_j \in K_{2m}$.

Indeed, if this is not the case, since $\Omega \setminus K_{2m} \subset \Omega \setminus L_{2m}$, we have

$$u(x_j) > -\frac{1}{2m} \quad \forall j \in \{1, \cdots, n+1\}. \tag{5.16}$$

Thus, by (5.14) and (5.16), we have for L a supporting hyperplane to Γ_u at x_0 that

$$-\frac{1}{m} \geq \Gamma_u(x_0) = L(x_0) = L\Big(\sum_{i=1}^{n+1} \lambda_i x_i\Big) = \sum_{i=1}^{n+1} \lambda_i L(x_i) = \sum_{i=1}^{n+1} \lambda_i u(x_i) > -\frac{1}{2m}, \tag{5.17}$$

which is clearly a contradiction.

Now, we claim that for at least one index $j \in \Big\{1, \cdots, n+1\Big\}$ for which the corresponding $x_j \in K_{2m}$ we necessarily have $\lambda_j \geq \overline{\lambda}$. Indeed, suppose this is not the case. Then, by (5.17) and (5.16), we estimate

$$\begin{aligned}
-\frac{1}{m} \geq \sum_{i=1}^{n+1} \lambda_i u(x_i) &= \sum_{x_i \in K_{2m}} \lambda_i u(x_i) + \sum_{x_i \notin K_{2m}} \lambda_i u(x_i) \\
&\geq \sum_{x_i \in K_{2m}} \overline{\lambda} \inf_{\Omega} u + \sum_{x_i \notin K_{2m}} -\frac{\lambda_i}{2m} \\
&\geq \sum_{i=1}^{n+1} \overline{\lambda} \inf_{\Omega} u + \sum_{i=1}^{n+1} -\frac{\lambda_i}{2m} \\
&= \overline{\lambda}(n+1) \inf_{\Omega} u - \frac{1}{2m} = -\frac{3}{4m}
\end{aligned}$$

which is impossible. Thus, claim 2 is now proven. Before we proceed, we observe that once $u < 0$ in Ω, it follows from (5.10) and (5.11) that there exists $m_0 \in \mathbb{N}^*$ such that $\overline{\Omega'} \subset L_{m_0} \subset K_{m_0} \subset K_{2m_0}$. From this and recalling (5.3) and (5.12) we conclude that

$$0 < d_{2m_0} \leq dist(\Omega', \partial\Omega) = 4\delta. \tag{5.18}$$

We define the following constants:

$$\delta_0 := \frac{1}{4} d_{2m_0} \leq \delta, \tag{5.19}$$

$$\overline{\lambda}_0 := \frac{1}{4m_0(n+1)(-\inf_{\Omega} u)} > 0\,^{5}, \tag{5.20}$$

$$\varrho_0 := \min\left\{\delta_0, \frac{\delta_0 \cdot \overline{\lambda}_0}{4}\right\} \leq \delta_0 \leq \delta. \tag{5.21}$$

Claim 3: Let $x_0 \in \Omega' \setminus \mathcal{C}(u)$ and L_{x_0} be a supporting hyperplane to Γ_u at x_0. Then,

$$\|\Gamma_u - L_{x_0}\|_{L^\infty(B_r(x_1))} \leq \vartheta^*_{x_1}\left(\frac{2r}{\overline{\lambda}_0}\right) r \quad \forall r \in (0, \varrho_0] \tag{5.22}$$

[5] We assume that $\overline{\lambda}_0 \in (0, 1)$ by taking $m_0 = m_0(n, \Omega', \inf_{\Omega} u)$ large enough.

for some $x_1 \in \mathcal{C}(u) \cap K_{2m_0}$, where

$$\vartheta^*_{x_1}(r) := \overline{C}\left(\|f\|_{L^q(B_r(x_1))} + \left(\frac{\|u\|_{L^\infty(\Omega)}}{d_{2m_0}}\|\gamma\|_{L^p(B_r(x_1))}\right)r^\beta\right)r^\alpha, \quad \text{for } r \in [0, \delta_0]. \tag{5.23}$$

Again, here

$$\alpha = 1 - \frac{n}{q} \geq 0, \quad \beta = n\left(\frac{1}{q} - \frac{1}{p}\right) \geq 0,$$

and $\overline{C} = \overline{C}(n, q, p, \lambda, \Lambda, d_{2m_0}, \|\gamma\|_{L^p(\Omega)}, \Theta_\gamma) > 0$ is universal.

Proof of claim 3: From Lemma 5.1, $x_0 = \sum_{i=1}^{n+1}\lambda_i x_i$ where $x_i \in \mathcal{C}(u)$ and $\lambda_i \geq 0$ with $\sum_{i=1}^{n+1}\lambda_i = 1$. Also, by claim 2, we can assume, up to relabeling the indices if necessary, that $\lambda_1 \geq \overline{\lambda}_0$ and $x_1 \in K_{2m_0}$, where $\overline{\lambda}_0$ is given above in (5.20). In order to simplify the notation, we denote $L := L_{x_0}$ and recall that L is a supporting plane for Γ_u at all points x_i for $i \in \{1, \cdots, n+1\}$, as indicated in Lemma 5.1.

Now, by applying claim 1 (the regularity case for points inside the contact set) with Ω' replaced by K_{2m_0} and x_0 replaced by x_1, we obtain an analogue of estimate (5.4) which is

$$\|\Gamma_u - L\|_{L^\infty(B_{r/2}(x_1))} \leq \vartheta^*_{x_1}(r)r \quad \forall r \in (0, \delta_0], \tag{5.24}$$

where for $\overline{D} = \overline{D}(n, q, p, \lambda, \Lambda, d_{2m_0}, \|\gamma\|_{L^p(\Omega)}, \Theta_\gamma) > 0$ and

$$\widehat{\vartheta}_{x_1}(r) := \overline{D}\left(\|f\|_{L^q(B_r(x_1))} + \left(\frac{\|u\|_{L^\infty(\Omega)}}{d_{2m_0}}\|\gamma\|_{L^p(B_r(x_1))}\right)r^\beta\right)r^\alpha, \tag{5.25}$$

which is clearly well defined in $(0, \delta_0]$. Observe that

$$|h| \leq \varrho_0 \Longrightarrow x_1 + \frac{h}{\lambda_1} \in \overline{B}_{\frac{|h|}{\lambda_1}}(x_1) \subset\subset B_{\frac{|2h|}{\lambda_1}}(x_1) \subset\subset \Omega,$$

since by (5.21), we have

$$\frac{2|h|}{\lambda_1} \leq \frac{2|h|}{\overline{\lambda}_0} \leq \frac{2\varrho_0}{\overline{\lambda}_0} \leq \delta_0 = \frac{1}{4}d_{2m_0}.$$

Thus, by (5.24), for any $|h| \leq \varrho_0$ we obtain

$$\left|\Gamma_u\left(x_1 + \frac{h}{\lambda_1}\right) - L\left(x_1 + \frac{h}{\lambda_1}\right)\right| \leq \|\Gamma_u - L\|_{L^\infty(B_{|h|/\lambda_1}(x_1))} \leq \widehat{\vartheta}_{x_1}\left(\frac{2|h|}{\lambda_1}\right)\frac{2|h|}{\lambda_1}. \tag{5.26}$$

Also, observe that for $|h| \leq \varrho_0$ we have $x_0 + h \in \Omega$, thanks to (5.21). Thus

$$\begin{aligned}
L(x_0 + h) \leq \Gamma_u(x_0 + h) &= \Gamma_u\left(\sum_{i=2}^{n+1}\lambda_i x_i + \lambda_1\left(x_1 + \frac{h}{\lambda_1}\right)\right) \\
&\leq \sum_{i=2}^{n+1}\lambda_i\Gamma_u(x_i) + \lambda_1\Gamma_u\left(x_1 + \frac{h}{\lambda_1}\right) \qquad \text{(by convexity)} \\
&\leq \sum_{i=2}^{n+1}\lambda_i L(x_i) + \lambda_1\left(L\left(x_1 + \frac{h}{\lambda_1}\right) + \widehat{\vartheta}_{x_1}\left(\frac{2|h|}{\lambda_1}\right)\frac{2|h|}{\lambda_1}\right) \qquad \text{(by (5.26))} \\
&\leq \sum_{i=2}^{n+1}\lambda_i L(x_i) + \lambda_1 L\left(x_1 + \frac{h}{\lambda_1}\right) + 2\widehat{\vartheta}_{x_1}\left(\frac{2|h|}{\lambda_1}\right)|h| \\
&\leq L(x_0 + h) + 2\widehat{\vartheta}_{x_1}\left(\frac{2|h|}{\overline{\lambda}_0}\right)|h| \qquad (L \text{ is affine and } \widehat{\vartheta}_{x_1} \text{ is nondecreasing}).
\end{aligned}$$

This implies that, for $\vartheta^*_{x_1}(t) := 2\widehat{\vartheta}_{x_1}(t)$,

$$\|\Gamma_u - L\|_{L^\infty(B_r(x_0))} \leq \vartheta^*_{x_1}\left(\frac{2r}{\overline{\lambda}_0}\right)r \quad \forall r \in (0, \varrho_0], \tag{5.27}$$

and thus, claim 3 is proven.

Now, we summarize our findings. For any $x_0 \in \Omega' \subset\subset \Omega$ and for all $r \in (0, \varrho_0]$, we define

$$\vartheta^{\#}_{x_0}(r) := \begin{cases} \vartheta_{x_0}(r), & \text{if } x_0 \in \mathcal{C}(u) \\ \vartheta^*_{x_1}\left(\frac{2r}{\overline{\lambda}_0}\right) & \text{if } x_0 \notin \mathcal{C}(u) \quad (x_1 \text{ as given in claim 3}), \end{cases} \tag{5.28}$$

where $\vartheta_{x_0}(r)$ and $\vartheta^*_{x_1}(r)$ are given respectively by (5.5) and (5.23). Then we have the following estimate

$$\|\Gamma_u - L_{x_0}\|_{L^\infty(B_r(x_0))} \le \vartheta^{\#}_{x_0}(r)r \quad \forall r \in (0, \varrho_0], \tag{5.29}$$

where L_{x_0} is a supporting plane for Γ_u at x_0.

By (5.18) and recalling $\alpha = 1 - \frac{n}{q} \ge 0$ and $\beta = n(q^{-1} - p^{-1}) \ge 0$, we have

$$\vartheta_{x_0}(r) \le D_1\Big(\|f\|_{L^q(B_r(x_0))} + r^\beta \|\gamma\|_{L^p(B_r(x_0))}\Big)r^\alpha, \qquad \forall r \in (0, \varrho_0], \tag{5.30}$$

$$\vartheta^*_{x_1}\left(\frac{2r}{\overline{\lambda}_0}\right) \le D_2\Big(\|f\|_{L^q(B_{\frac{2r}{\overline{\lambda}_0}}(x_1))} + r^\beta \|\gamma\|_{L^p(B_{\frac{2r}{\overline{\lambda}_0}}(x_1))}\Big)r^\alpha \qquad \forall r \in (0, \varrho_0], \tag{5.31}$$

where

$$D_1 := C\Big(1 + ||u||_{L^\infty(\Omega)} d^{-1}_{2m_0}\Big), \quad D_2 := \overline{C}\Big(1 + ||u||_{L^\infty(\Omega)} d^{-1}_{2m_0}\Big)(1 + 2^\beta \overline{\lambda}_0^{-\beta})(1 + 2^\alpha \overline{\lambda}_0^{-\alpha}) > 0.$$

Here, C and $\overline{C}$ are given in (5.5) and (5.23). Also note that, since $u < 0$ in Ω, $||u||_{L^\infty(\Omega)} = -\inf_\Omega u$.

The Theorem follows from the estimates (5.30) and (5.31). Indeed, if $q > n$, and thus $\alpha \in (0, 1)$, we have

$$\vartheta^{\#}_{x_0}(r) \le \max\{D_1, D_2\}(1 + \varrho_0^\beta)(\|f\|_{L^q(\Omega)} + \|\gamma\|_{L^p(\Omega)})r^\alpha, \qquad \forall r \in (0, \varrho_0].$$

Thus, $u \in C^{1,1-\frac{n}{q}}_{loc}(\Omega)$ by (5.29) and Corollary 9.2. Now we study the case $q = n$. In this case, $\alpha = 0$. Then, it follows from (5.30) and (5.31) and the absolute continuity property of the integral that

$$\vartheta^{\#}_{x_0}(r) = o(1) \quad \text{as } r \to 0. \tag{5.32}$$

Thus, Γ_u is differentiable everywhere in Ω. Since it is also convex, then $\Gamma_u \in C^1(\Omega)$ (see Theorem A.1.13 in [11]). Finally, still in the case $q = n$, it follows from (2.9), (5.30), and (5.31), that for $r \in [0, \varrho_0]$ (recall that $\overline{\lambda}_0 = \overline{\lambda}_0(m_0, n, \inf_\Omega u) \in (0, 1)$)

$$\vartheta^{\#}_{x_0}(r) \le \begin{cases} 2D_1\vartheta(r) & \text{if } x_0 \in \mathcal{C}(u), \\ 2D_2\vartheta\left(\frac{2r}{\overline{\lambda}_0}\right) & \text{if } x_0 \notin \mathcal{C}(u). \end{cases} \tag{5.33}$$

In particular,

$$\vartheta^{\#}_{x_0}(r) \le 2(D_1 + D_2)\vartheta\left(\frac{2r}{\overline{\lambda}_0}\right), \quad \forall r \in [0, \varrho_0].$$

Once more, the result follows from (5.29) and Corollary 9.2. This finishes the proof of Theorem 2.6. □

Remark 5.3. Observe that claim 2 above actually shows that $\mathcal{C}(u) \cap \Omega \neq \varnothing$. This is a consequence of the geometry imposed on u. It is encoded in the proof in the properties of the sublevelsets L_m and their convex hull K_m.

6. Proof of Theorem 2.8

In this section we give the proof of Theorem 2.8. Essentially, it follows the ideas of the proof of Theorem 2.6 but in this case we have a better control on the affine projection x_1 and its coefficient $\overline{\lambda}$ from the previous proof. This control was already studied by L. Caffarelli and X. Cabré in their book [8] and this analysis will replace claim 2 of the previous proof. This better control indeed allows us to get a precise estimate for the weighted $C^{1,\alpha}$ norm of Γ_u whenever $\alpha = 1 - n/q > 0$. The details follow below.

Proof of Theorem 2.8. First, let us extend γ and f by zero outside B_r. It is not hard to check that $-u^- = \min\{u, 0\} \in \overline{S}(\gamma; f^+)$ in B_{2r}. We recall that Γ_u denotes the convex envelope of $-u^-$ in $\overline{B}_{2r}$. As before, we divide the proof in two cases: estimates inside and outside the contact set. The contact set here is given by

$$\mathcal{C}(u) := \Big\{x \in \overline{B}_{2r} \,:\, -u^-(x) = \Gamma_u(x)\Big\}.$$

In what follows, L_{x_0} denotes a supporting plane for Γ_u at $x_0 \in B_r$.

Case 1: $x_0 \in \overline{B}_r \cap \mathcal{C}(u)$.

By following the proof of claim 1 in the proof of Theorem 2.6, we conclude that for any $x_0 \in \overline{B}_r \cap \mathcal{C}(u)$ we have

$$\|\Gamma_u - L_{x_0}\|_{L^\infty(B_{\rho/2}(x_0))} \leq \vartheta_{x_0}(\rho)\rho \quad \forall \rho \in (0, \delta], \tag{6.1}$$

where $\delta := r/4$ and

$$\vartheta_{x_0}(\rho) := C\Bigg(\|f^+\|_{L^q(B_\rho(x_0))} + \bigg(\frac{\|u\|_{L^\infty(B_r)}}{r}\|\gamma\|_{L^p(B_\rho(x_0))}\bigg)\rho^\beta\Bigg)\rho^\alpha, \tag{6.2}$$

with

$$\alpha = 1 - \frac{n}{q} \geq 0, \quad \beta = n\bigg(\frac{1}{q} - \frac{1}{p}\bigg) \geq 0$$

and

$$C = C(n, p, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0 \quad \text{is a universal constant.} \tag{6.3}$$

Case 2: $x_0 \in \overline{B}_r \setminus \mathcal{C}(u)$.

Here, we are exactly in the same conditions of Step 1 in the proof of Lemma 3.5 in [8]. Thus from observations (a) and (b) of that proof, it follows that $x_0 \in \mathrm{conv}\big\{x_1, \cdots, x_{n+1}\big\}$, where these points need not to be all distinct and they lie in $B_r \cap \big\{u = \Gamma_u\big\}$ except possibly one that may belong to ∂B_{2r}. Also, $L_{x_0} \equiv \Gamma_u$ in $\mathrm{conv}\big\{x_1, \cdots, x_n\big\}$, and if we write $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, then there is at least one index $i \in \big\{x_1, \cdots, x_n\big\}$ for which $\lambda_i \geq 1/3n$ and $x_i \in B_r \cap \mathcal{C}(u)$. Without losing generality, relabeling the indices if necessary, we assume that such pair is (λ_1, x_1).

The discussion above plays the role of claim 2 in the proof of the Theorem 2.6. This allows us to repeat directly the proof of claim 3 taking $d_{2m_0} = r$, $\overline{\lambda}_0 = 1/3n$, and $\varrho_0 = r/48n$. Observe that here, in the notation of Theorem 2.6, we have $\delta = \delta_0 = r/4$. As before, we have for any $x_0 \in \overline{B}_r$ that

$$\|\Gamma_u - L_{x_0}\|_{L^\infty(B_\rho(x_0))} \leq \vartheta^{\#}_{x_0}(\rho)\rho \quad \forall r \in (0, \varrho_0], \tag{6.4}$$

where

$$\vartheta^{\#}_{x_0}(\rho) := \begin{cases} \vartheta_{x_0}(\rho), & \text{if } \ x_0 \in \mathcal{C}(u) \\ \vartheta^*_{x_1}(6n\rho) & \text{if } \ x_0 \notin \mathcal{C}(u) \quad (x_1 \text{ as in the previous discussion}). \end{cases} \tag{6.5}$$

Here, $\vartheta_{x_0}(\rho)$ is given by (6.2) and

$$\vartheta_{x_1}(6n\rho) := C_2\Bigg(\|f^+\|_{L^q(B_{6n\rho}(x_1))} + \bigg(\frac{\|u\|_{L^\infty(B_r)}}{r}\|\gamma\|_{L^p(B_{6n\rho}(x_1))}\bigg)\rho^\beta\Bigg)\rho^\alpha, \tag{6.6}$$

where $C_2 = C_2(n, q, p, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$. In the case $q > n$, proceeding as before, we obtain

$$\vartheta^{\#}_{x_0}(\rho) \quad\leq\quad C_3\left(\|f^+\|_{L^q(B_r)} + \frac{\|u\|_{L^\infty(B_r)}}{r}\|\gamma\|_{L^p(B_r)}\right)\rho^\alpha \quad \text{for} \quad \rho \leq r/4 \tag{6.7}$$

with

$$C_3 = C_3(n, p, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0.$$

Moreover, thanks to the gradient estimate

$$|\nabla\Gamma_u(x)| \leq \frac{-\Gamma_u(x)}{dist(x, \partial B_r)} \leq \frac{\|u\|_{L^\infty(B_r)}}{r} \quad \forall x \in B_r,$$

it follows directly from (6.4), (6.7) and Lemma 9.1 that

$$\|u\|^*_{C^{1,1-\frac{n}{q}}(B_{r/2})} \leq C_4\left(\left(1 + \|\gamma\|_{L^p(B_r)}\right)\|u\|_{L^\infty(B_r)} + \|f^+\|_{L^q(B_r)}\right)$$

where $C_4 = C_4(n, p, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$.

As before, in the case where $q = n$, it follows that $\alpha = 0$ and that

$$\vartheta^{\#}_{x_0}(\rho) = o(1) \quad \text{ as } \rho \to 0.$$

This implies that Γ_u is differentiable everywhere in B_r. By convexity, as before, $\Gamma_u \in C^1(B_r)$.

Finally, still in the case $q = n$, we recall that (2.12) holds. Then from (6.2) and (6.6) we get

$$\vartheta^{\#}_{x_0}(\rho) \leq C_5\left(1 + \frac{\|u\|_{\infty(B_r(x_0))}}{r}\right)\vartheta(6n\rho) =: C_5\left(1 + \frac{\|u\|_{\infty(B_r(x_0))}}{r}\right)\overline{\vartheta}(\rho) \quad \forall\rho \in [0, \varrho_0],$$

where $C_5 = C_5(n, q, p, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$.

Thus, from (6.4) and Lemma 9.1 with $r_0 = r/48n$ we see that $\Gamma_u \in C^{1,\overline{\vartheta}}(\overline{B}_{r/384n})$ with

$$[\nabla\Gamma_u]_{C^{0,\overline{\vartheta}}(B_{r/384n})} \leq C_6\left(1 + \frac{\|u\|_{\infty(B_r(x_0))}}{r}\right),$$

where $C_6 = C_6(n, q, p, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$. ☐

7. Interior Regularity of $\omega-$semiconvex supersolutions and Proof of Theorem 2.9

Now we discuss the regularity of $\omega-$semiconvex supersolutions. We refer the reader again to Remark 4.1 for the ideas in the following proof.

Proposition 7.1 (Pointwise regularity for nonnegative $\omega-$semiconvex supersolutions). *Let $\varphi \in \overline{S}(\gamma; f)$ in B_r with $\gamma \in L^p(B_r)$, $f \in L^q(B_r)$, and $r \leq R_0$. Additionally, assume that φ is a bounded $\omega-$semiconvex function in B_r. Then, $\partial_\omega\varphi(0) = \{p\}$ is a singleton and if we set $L(x) := p \cdot x + \varphi(0)$ for $x \in \mathbb{R}^n$, then*

$$\|\varphi - L\|_{L^\infty(B_{\rho/4})} \leq C\overline{\zeta}(\rho)\rho \quad \forall\rho \in (0, r), \tag{7.1}$$

where

$$\overline{\zeta}(\rho) := \omega(\rho) + \rho^\alpha\left(\|f\|_{L^q(B_\rho)} + \rho^\beta\left(\frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\right)\|\gamma\|_{L^p(B_\rho)}\right) \tag{7.2}$$

with

$$\alpha = 1 - \frac{n}{q} \geq 0, \quad \beta = n\left(\frac{1}{q} - \frac{1}{p}\right) \geq 0,$$

and $C = C(n, p, q, \lambda, \Lambda, R_0, \|\gamma\|_{L^p(B_{R_0})}, \Theta_\gamma) > 0$. In particular, φ is differentiable at 0 with $\nabla\varphi(0) = p$.

Proof. Let $\rho \in (0,r)$. From the semiconvexity of φ, there exists $p \in \partial_\omega\varphi(0)$. We now set $L(x) := p\cdot x + \varphi(0)$ for all $x \in \mathbb{R}^n$. Once more from the semiconvexity of φ, we conclude that $\psi(x) := \varphi(x) - L(x) + \rho\omega(\rho) \geq 0$ in B_ρ. From Remark 2.1, ψ is $\omega-$semiconvex in B_r. Moreover, $\psi \in \overline{S}(\gamma; f + |\nabla L|\gamma)$ in B_r. Now, observe that for $C_1 = C_1(n,p,q) > 0$ and $0 < \rho < r$

$$\begin{aligned}
\rho^\alpha \|f + |\nabla L|\gamma\|_{L^q(B_\rho)} + \omega(\rho) &\leq \rho^\alpha\Big(\|f\|_{L^q(B_\rho)} + |\nabla L|\cdot\|\gamma\|_{L^q(B_\rho)}\Big)\\
&\leq \rho^\alpha\Big(\|f\|_{L^q(B_\rho)} + |B_1|^{\frac{1}{q}-\frac{1}{p}}|\nabla L|\cdot\|\gamma\|_{L^p(B_\rho)}\cdot\rho^\beta\Big)\\
&\leq C_1\rho^\alpha\Big(\|f\|_{L^q(B_\rho)} + |\nabla L|\cdot\|\gamma\|_{L^p(B_\rho)}\cdot\rho^\beta\Big) =: \chi(\rho) \qquad (7.3)
\end{aligned}$$

Hence, we can apply Proposition 4.2 in B_s to obtain, for all $s \in (0,\rho)$

$$\|\psi\|_{L^\infty(B_{s/2})} \leq C'\Big(\inf_{B_{\frac{3s}{4}}} \psi + \chi(s)s\Big).$$

with $C' = C'(n,p,q,\lambda,\Lambda,R_0,\|\gamma\|_{L^p(B_{R_0})},\Theta_\gamma) > 0$. Now, by taking $s = \rho/2$, we arrive to

$$\begin{aligned}
\|\varphi - L + \rho\omega(\rho)\|_{L^\infty(B_{\rho/4})} = \|\psi\|_{L^\infty(B_{\rho/4})} &\leq C'\Big(\inf_{B_{\frac{3\rho}{8}}} \psi + \chi\Big(\frac{\rho}{2}\Big)\rho\Big)\\
&\leq C'\Big(\psi(0) + \chi(\rho)\rho\Big)\\
&\leq C'\Big(\omega(\rho)\rho + \chi(\rho)\rho\Big).
\end{aligned}$$

From this, we conclude

$$\|\varphi - L\|_{L^\infty(B_{\rho/4})} \leq C''\Big(\omega(\rho)\rho + \chi(\rho)\rho\Big) \quad \text{with } C'' = C' + 1. \qquad (7.4)$$

In particular, by the absolute continuity property of the integral, we see that $\chi(\rho) = o(1)$ as $\rho \to 0$. Thus, φ is differentiable at zero and $\nabla\varphi(0) = \nabla L = p$. This proves that $\partial_\omega\varphi(0) = \{p\}$.
In order to control $\chi(\rho)$, we need an estimate on $|\nabla L|$ appearing in (7.3). We can assume without losing generality that $\nabla\varphi(0) \neq 0$. Fix $s < r$. By the $\omega-$semiconvexity of φ to obtain

$$\|\varphi\|_{L^\infty(B_r)} \geq \varphi(x) \geq \varphi(0) + \nabla\varphi(0)\cdot x - |x|\omega(|x|) \geq -\|\varphi\|_{L^\infty(B_r)} + \nabla\varphi(0)\cdot x - r\omega(r) \quad \forall x \in \overline{B}_s. \qquad (7.5)$$

Hence, choosing $x = |\nabla\varphi(0)|^{-1}\nabla\varphi(0)s$, we arrive at

$$r\omega(r) + 2\|\varphi\|_{L^\infty(B_r)} \geq |\nabla\varphi(0)|s. \qquad (7.6)$$

Now, letting $s \nearrow r$, we obtain

$$|\nabla L| \leq 2\|\varphi\|_{L^\infty(B_r)}/r + \omega(r). \qquad (7.7)$$

Hence, recalling (7.3), we conclude that the right hand side in (7.4) can be bounded by

$$2C''(1+C_1)\Bigg(\omega(\rho) + \rho^\alpha\bigg(\|f\|_{L^q(B_\rho)} + \rho^\beta\bigg(\frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\bigg)\|\gamma\|_{L^p(B_\rho)}\bigg)\Bigg)\cdot\rho,$$

as desired. □

We are now ready to prove Theorem 2.9.

Proof of Theorem 2.9. We apply Proposition 7.1 in $B_{r/2}(x_0)$ for every $x_0 \in \overline{B}_{r/2}$ to deduce that φ is differentiable at x_0. Moreover, denoting $L_{x_0}(x) := \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0)$ for all $x \in \mathbb{R}^n$, we have

$$\|\varphi - L_{x_0}\|_{L^\infty(B_{\rho/4}(x_0))} \leq C\overline{\zeta}(\rho)\rho \quad \forall x_0 \in B_{r/2}, \quad \forall \rho \in (0, r/2), \qquad (7.8)$$

where

$$\overline{\zeta}(\rho) := \omega(\rho) + \bigg(\|f\|_{L^q(B_\rho(x_0))} + \rho^\beta\bigg(\frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\bigg)\|\gamma\|_{L^p(B_\rho(x_0))}\bigg)\rho^\alpha,$$

and $C = C(n,p,q,\lambda,\Lambda,R_0,\|\gamma\|_{L^p(B_{R_0})},\Theta_\gamma) > 0$. Hence, performing the change of variables $s = \rho/4$, we arrive to

$$\|\varphi - L_{x_0}\|_{L^\infty(B_s(x_0))} \leq 4C\overline{\zeta}(4s)s =: \overline{C}\zeta(s)s \quad \forall x_0 \in \overline{B}_{r/2}, \quad \forall s \in (0, r/8). \qquad (7.9)$$

Recalling Remark 2.5 (absolute continuity of the integral), we see that $\zeta(s) = o(1)$ as $s \to 0$. Thus, u is differentiable and also $C^1(B_r)$ (see Proposition 3.3.4 in [11]). Then, Lemma 9.1 applied with $r_0 = r/8$

gives the first part of the Theorem. For the second part we observe that, in any of the cases described in the definition of Υ, we have that $\zeta(s) \leq \Upsilon(s)$ for any $s \in (0, r/8)$. Once more, Lemma 9.1 implies the result. This finishes the proof. □

8. $L^p - L^\infty$ ESTIMATES FOR $\omega-$SEMICONVEX FUNCTION FOR $p > 0$

In this section we present the (local) Lipschitz regularity as well as the $L^p - L^\infty$ estimates for $\omega-$semiconvex functions and their gradients. Here, the L^∞ norm of the function and of the gradient are controlled by average values of u and the modulus of semi-convexity. Some versions of these estimates can be found in Theorem 6.7 in [17] for convex functions in the case $p = 1$. Here we give a real analysis proof of the general $L^p - L^\infty$ estimates based on classical ideas from [17] and [23] (see for instance, Proposition 8.19). We refer the reader also to Theorem 8.1.9 in the recent book [25].

Lemma 8.1. *Let $\phi : [\varrho, R] \to \mathbb{R}$ be a nonnegative bounded function. Assume that forall $\varrho \leq t < s \leq R$*

$$\phi(t) \leq A(s-t)^{-\alpha} + B(s-t)^{-\beta} + C + \theta\phi(s),$$

for some $A, B, C \geq 0$ and $\alpha > \beta > 0$ and $\theta \in [0,1)$. Then we have

$$\phi(\varrho) \leq D\Big(A(R-\varrho)^{-\alpha} + B(R-\varrho)^{-\beta} + C\Big) \quad \text{for some} \quad D = D(\alpha, \theta) > 0.$$

Proof. See Lemma 6.1 in [27]. □

Proposition 8.2. [6] *Let $u \in L^1(B_r)$ be a $\omega-$semiconvex function and $p \in (0, \infty)$. Then the following hold:*

(a) $u \in C^{0,1}_{loc}(B_r)$;

(b) There exists $C_1 = C_1(n,p) > 0$ such that

$$\sup_{B_{r/2}} |u| \leq C_1 \left[\left(\fint_{B_r} |u|^p dx \right)^{1/p} + r\omega(r) \right]. \tag{8.1}$$

Equivalently, for some $C_2 = C_2(n,p) > 0$, for every $0 < \rho < r$ we have

$$\sup_{B_\rho} |u| \leq C_2 \left[\frac{1}{(r-\rho)^{n/p}} \left(\int_{B_r} |u|^p dx \right)^{1/p} + (r-\rho)\omega(r-\rho) \right]. \tag{8.2}$$

(c) For some $C_3 = C_3(n,p) > 0$ we have

$$ess \sup_{B_{r/2}} |\nabla u| \leq \frac{C_3}{r} \left[\left(\fint_{B_r} |u|^p dx \right)^{1/p} + r\omega(r) \right]. \tag{8.3}$$

Equivalently, for every $0 < \rho < r$, there exists $C_4 = C_4(n,p) > 0$ such that

$$ess \sup_{B_\rho} |\nabla u| \leq \frac{C_4}{r-\rho} \left[\frac{1}{(r-\rho)^{n/p}} \left(\int_{B_r} |u|^p dx \right)^{1/p} + (r-\rho)\omega(r-\rho) \right]. \tag{8.4}$$

(d) If $u \in L^\infty(B_r)$ then, for $0 < \rho < r$,

$$ess \sup_{B_\rho} |\nabla u| \leq 2 \left(\frac{||u||_{L^\infty(B_r)}}{r-\rho} + \omega(r-\rho) \right). \tag{8.5}$$

In particular [7]

$$ess \sup_{B_{r/2}} |\nabla u| \leq 4 \left(\frac{||u||_{L^\infty(B_r)}}{r} + \omega(r) \right). \tag{8.6}$$

[6]The estimates given in (b) and (c) for $p > 1$ are relevant only when $u \in L^p(B_r)$, as otherwise they are trivially true.

[7]As a matter of fact, by proceeding as in the proof of items (b) and (c) of Proposition 8.2, we see that the estimates in (8.5) and (8.6) are indeed equivalent, up to modifying the (universal) constants appearing in their statements.

Proof. **Case I:** We assume initially that $u \in C^1(B_r)$ and prove all the estimates first.

Fix a point $z \in B_{r/2}$. Then, for any $y \in B_{r/2}(z)$,

$$u(y) \geq u(z) + \nabla u(z)\cdot(y-z) - |y-z|\omega(|y-z|) \geq u(z) + \nabla u(z)\cdot(y-z) - r\omega(r). \tag{8.7}$$

Now, integrating (8.7) with respect to y in $B_{r/2}(z)$,

$$u(z) \leq \fint_{B_{r/2}(z)} |u(y)|dy + r\omega(r) \leq 2^n\left[\fint_{B_r} |u|dy + r\omega(r)\right]. \tag{8.8}$$

This gives an estimate from above for $u(z)$. We now proceed towards an estimate from below.

We choose a cutoff function $\xi \in C_c^\infty(B_r)$ such that

$$0 \leq \xi \leq 1, \quad |\nabla \xi| \leq \frac{C}{r}, \quad \xi \equiv 1 \quad \text{in } B_{3r/16} \quad \text{and} \quad \xi \equiv 0 \quad \text{in } B_r \setminus B_{\frac{3r}{8}} \tag{8.9}$$

for a universal constant $C > 0$. Note that, for any $y \in B_{\frac{3r}{8}}$, we have

$$u(z) \geq u(y) + \nabla u(y)\cdot(z-y) - |z-y|\omega(|z-y|) \geq u(y) + \nabla u(y)\cdot(z-y) - r\omega(r). \tag{8.10}$$

Observe now that, by the Divergence Theorem, since $\xi \equiv 0$ on $\partial B_{\frac{3r}{8}}$ we have

$$0 = \int_{B_{\frac{3r}{8}}} \operatorname{div}_y\big(u(y)(\xi(y)(z-y))\big)dy = \int_{B_{\frac{3r}{8}}} \nabla u(y)\cdot\xi(y)(z-y)dy + \int_{B_{\frac{3r}{8}}} u(y)\operatorname{div}_y\big(\xi(y)(z-y)\big)dy. \tag{8.11}$$

Note that, thanks to (8.9), there exists a universal constant $D > 0$ (independent of r) such that

$$\big\|\operatorname{div}_y\big(\xi(y)(y-z)\big)\big\|_{L^\infty(B_{\frac{3r}{8}})} \leq \|\nabla\xi(y)\cdot(y-z)\|_{L^\infty(B_{\frac{3r}{8}})} + \|\xi(y)\operatorname{div}_y(y-z)\|_{L^\infty(B_{\frac{3r}{8}})} \leq D. \tag{8.12}$$

Hence, multiplying (8.10) by $\xi(y)$, integrating over $B_{3r/8}$ with respect to y, and using (8.11), we obtain

$$\begin{aligned} u(z)\int_{B_{\frac{3r}{8}}}\xi(y)dy &\geq \int_{B_{\frac{3r}{8}}} u(y)\xi(y)dy + \int_{B_{\frac{3r}{8}}}\xi(y)\nabla u(y)\cdot(z-y)dy - \int_{B_{\frac{3r}{8}}} r\omega(r)dy \\ &= \int_{B_{\frac{3r}{8}}} u(y)\Big[\xi(y) - div_y\big(\xi(y)(y-z)\big)\Big]dy - \int_{B_{\frac{3r}{8}}} r\omega(r)dy \\ &\geq -(D+1)\left[\int_{B_{\frac{3r}{8}}}|u|dy + \int_{B_{\frac{3r}{8}}} r\omega(r)dy\right]. \end{aligned} \tag{8.13}$$

Since,

$$I := \int_{B_{\frac{3r}{8}}}\xi(y)dy \geq \int_{B_{3r/16}}\xi(y)dy \geq |B_{3r/16}| = 2^{-n}|B_{3r/8}|,$$

we have

$$I^{-1}\left[\int_{B_{\frac{3r}{8}}}|u|dy + \int_{B_{\frac{3r}{8}}} r\omega(r)dy\right] \leq 2^n\left[\fint_{B_{\frac{3r}{8}}}|u|dy + r\omega(r)\right].$$

So, it follows by (8.13) that

$$u(z) \geq -2^n(D+1)\left[\fint_{B_{\frac{3r}{8}}}|u|dy + r\omega(r)\right]. \tag{8.14}$$

Thus, (8.8) and (8.14) together prove (8.1) for $p = 1$ with $C_1 = 2^n(D+1)$.
We observe since $u \in L^1(B_1)$ then $u \in L^p(B_1)$ for $p \in (0,1]$ since in this case, $L^1(B_r) \hookrightarrow L^p(B_r)$ by Proposition 6.12 in [21]. In the sequel, whenever proving any of the estimates appearing in (b) or (c) involving the $p-$average of u with $p > 1$, we will assume without loss of generality that $u \in L^p(B_r)$ since otherwise the estimate is trivial. Note also that, since the average integral is a monotone function of the

exponent, (8.1) is also proven for any $p \in [1, \infty)$.
Now, let $0 < \rho < r$ and $x_0 \in B_\rho$ and $p \geq 1$. Thus, by (8.1) applied to $B_{r-\rho}(x_0)$, we have

$$\begin{aligned} |u(x_0)| &\leq \sup_{B_{(r-\rho)/2}(x_0)} |u| \\ &\leq G\left[\left(\fint_{B_{r-\rho}(x_0)} |u|^p dy\right)^{1/p} + (r-\rho)\omega(r-\rho)\right] \\ &\leq \overline{G}\left[\frac{1}{(r-\rho)^{n/p}}\left(\int_{B_r} |u|^p dy\right)^{1/p} + (r-\rho)\omega(r-\rho)\right] \end{aligned} \tag{8.15}$$

where $G, \overline{G} > 0$ are dimensional constants. Taking the supremum over $x_0 \in B_\rho$, the estimate above yields (8.2) for $p \geq 1$. We now study the case $p \in (0,1)$. Let $0 < \rho \leq t < s \leq r$. Assume $x_0 \in B_t$. Clearly, $B_{s-t}(x_0) \subset B_s$. Proceeding as in the estimate (8.15) applied to the ball $B_{s-t}(x_0)$, we obtain

$$\begin{aligned} |u(x_0)| &\leq \sup_{B_{(s-t)/2}(x_0)} |u| \\ &\leq \overline{G}\left[\frac{1}{(s-t)^n}\left(\int_{B_s} |u| dy\right) + (s-t)\omega(s-t)\right] \\ &\leq \overline{G}\left[\frac{1}{(s-t)^n}\left(\int_{B_s} |u|^p dy\right) \cdot \sup_{B_s} |u|^{1-p} + (s-t)\omega(s-t)\right] \\ &\leq \varepsilon \sup_{B_s} |u| + \frac{c(\varepsilon, p, n)}{(s-t)^{n/p}}\left(\int_{B_r} |u|^p dy\right)^{1/p} + \overline{G}(r-\rho)\omega(r-\rho), \end{aligned} \tag{8.16}$$

where here we used Young's inequality with conjugate exponents $(1-p)^{-1} \geq 1$ and $p^{-1} \geq 1$, namely

$$ab \leq \varepsilon a^{\frac{1}{1-p}} + c(\varepsilon, p) b^{1/p} \quad \text{for} \quad a, b \geq 0 \text{ and any } \varepsilon \in (0,1).$$

Thus, defining $\phi : [\rho, r] \to \mathbb{R}$ as $\phi(t) = \sup_{B_t} |u|$, the chain of inequalities above implies, by taking the supremum over B_t, that

$$\phi(t) \leq \frac{A_0}{(s-t)^{n/p}} + C_0 + \varepsilon\phi(s) \tag{8.17}$$

where

$$A_0 = c(\varepsilon, p, n)\left(\int_{B_r} |u|^p dy\right)^{1/p} \quad \text{and} \quad C_0 := \overline{G}(r-\rho)\omega(r-\rho). \tag{8.18}$$

Hence, choosing for instance $\varepsilon = 1/2$, Lemma 8.1 implies that, for some $D = D(n,p) > 0$, we have

$$\phi(\rho) \leq D\big(A_0 (r-\rho)^{-n/p} + C_0\big). \tag{8.19}$$

This proves (8.2), and thus (8.1), for $p \in (0,1)$. Hence (b) is proven.

We now prove (c). To this aim, given $z \in B_{r/2}$, we define

$$S_z := \left\{ y \in \mathbb{R}^n : \frac{r}{8} \leq |y-z| \leq \frac{r}{4} \text{ and } \nabla u(z) \cdot (y-z) \geq \frac{1}{2}|\nabla u(z)||y-z| \right\} \subset \overline{B}_{3r/4}.$$

Since u is $\omega-$semiconvex and of class C^1, we have

$$u(y) \geq u(z) + \nabla u(z) \cdot (y-z) - |y-z|\omega(|y-z|) \quad \forall y \in S_z.$$

In particular, by the definition of S_z, this implies that

$$u(y) \geq u(z) + \frac{r}{16}|\nabla u(z)| - r\omega(r) \quad \forall y \in S_z.$$

Integrating over S_z with respect to y and using (b), we obtain

$$\begin{aligned}|\nabla u(z)| &\leq \frac{16}{r}\left[\fint_{S_z}|u(y)-u(z)|dy + r\omega(r)\right]\\ &\leq \frac{16}{r}\left[2\sup_{B_{\frac{3r}{4}}}|u| + r\omega(r)\right]\\ &\leq \frac{C_3}{r}\left[\left(\fint_{B_r}|u|^p dx\right)^{1/p} + r\omega(r)\right].\end{aligned}$$

This proves (8.3). In order to show (8.4), it suffices to apply (8.3) in $B_{r-\rho}(x_0)$ for $x_0 \in B_\rho$ and proceed as done in (8.15).

Case II: Assume that u is merely $\omega-$semiconvex (i.e, it may not be necessarily C^1).

In this case, we proceed by regularizing u. Let $s < r$. Then, for $\varepsilon \in (0, r-s)$, we set

$$u_\varepsilon(x) := (u * \eta_\varepsilon)(x) = \int_{B_r}\eta_\varepsilon(x-y)u(y)dy = \int_{B_\varepsilon}u(x-y)\eta_\varepsilon(y)dy \quad \text{for} \quad x \in B_{r-\varepsilon} \supset \overline{B}_s.$$

Here $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon)$ where $0 \leq \eta \leq 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\eta(x)dx = 1$ and $supp(\eta) \subset \overline{B}_1$ is a standard mollifier. We claim that u_ε is $\omega-$semiconvex in $B_{r-\varepsilon}$. Indeed, we first observe that for $x, y \in B_{r-\varepsilon}$, $z \in B_\varepsilon$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}u\big(\lambda x + (1-\lambda)y - z\big) &= u(\lambda(x-z) + (1-\lambda)(y-z))\\ &\leq \lambda u(x-z) + (1-\lambda)u(y-z) + \lambda(1-\lambda)|x-y|\omega(|x-y|).\end{aligned}$$

Multiplying the estimate above by $\eta_\varepsilon(z) \geq 0$ and integrating over B_ε with respect to z we obtain

$$\begin{aligned}u_\varepsilon\big(\lambda x + (1-\lambda)y\big) &= \int_{B_\varepsilon}u\big(\lambda x + (1-\lambda)y - z\big)\eta_\varepsilon(z)dz\\ &\leq \lambda\int_{B_\varepsilon}u(x-z)\eta_\varepsilon(z)dz + (1-\lambda)\int_{B_\varepsilon}u(y-z)\eta_\varepsilon(z)dz\\ &+ \lambda(1-\lambda)\int_{B_\varepsilon}|x-y|\omega(|x-y|)\eta_\varepsilon(z)dz\\ &= \lambda u_\varepsilon(x) + (1-\lambda)u_\varepsilon(y) + \lambda(1-\lambda)|x-y|\omega(|x-y|),\end{aligned}$$

since $\int_{B_{\varepsilon r}}\eta_\varepsilon(z)dz = 1$. Moreover, $u_\varepsilon \in C^\infty(B_{r-\varepsilon})$. Now, let $0 < \rho < s < r$. For $\varepsilon < r-s$, we have $0 < \rho < s < r-\varepsilon < r$ and by the estimates in **Case I** (since $u \in C^1(B_s)$ is $\omega-$semiconvex) we obtain

$$\sup_{B_\rho}|u_\varepsilon| \leq C_2 \cdot I(u_\varepsilon, \rho, s, s, p), \tag{8.20}$$

$$\sup_{B_\rho}|\nabla u_\varepsilon| \leq \frac{C_4}{s-\rho} \cdot I(u_\varepsilon, \rho, s, s, p), \tag{8.21}$$

where

$$I(\xi, \rho, s, t, p) := \left[\frac{1}{(s-\rho)^{n/p}}\left(\int_{B_t}|\xi|^p dx\right)^{1/p} + (s-\rho)\omega(s-\rho)\right]. \tag{8.22}$$

Now, let $p \in (0, \infty)$. Then,

$$\left(\int_{B_s}|u_\varepsilon|^p dx\right)^{1/p} \leq |B_s|^{\frac{1}{p}-1}\int_{B_s}|u_\varepsilon|dx \leq |B_s|^{\frac{1}{p}-1}\int_{B_r}|u|dx \quad \text{for} \quad p \in (0,1), \tag{8.23}$$

$$\left(\int_{B_s} |u_\varepsilon|^p dx\right)^{1/p} \leq \left(\int_{B_r} |u|^p dx\right)^{1/p} \quad \text{for } p \in [1,\infty). \tag{8.24}$$

Above, we used once more that $L^1(B_r) \hookrightarrow L^p(B_r)$ if $p \in (0,1)$ (Proposition 6.12 in [21]) and that the mollification never increases the L^p-norm for $p \geq 1$ (Theorem C.19 in [26]). Plugging the information contained in (8.23) and (8.24) in the definition of $I(u_\varepsilon, \rho, s, s, p)$ in (8.22) and adding them up, we have that, for all $p \in (0,\infty)$,

$$I(u_\varepsilon, \rho, s, s, p) \leq \Big(1 + |B_s|^{\frac{1}{p}-1}(s-\rho)^{(1-\frac{n}{p})}\Big) I(u, \rho, s, r, 1) + I(u, \rho, s, r, p) < \infty. \tag{8.25}$$

Thus, by Ascoli-Arzelà Theorem and the properties of mollifiers (Theorem C.19 in [26]), we obtain from (8.20) and (8.21) that $u_\varepsilon \to u \in C^0(B_\rho)$ uniformly in B_ρ. Moreover, from the uniform convergence, we have

$$I(u_\varepsilon, \rho, s, s, p) \to I(u, \rho, s, s, p) \quad \text{as} \quad \varepsilon \to 0.$$

Thus from (8.20), by letting $\varepsilon \to 0$, we conclude that $\sup_{B_\rho} |u| \leq C_2 \cdot I(u, \rho, s, s, p)$. Now, letting $s \nearrow r$, we arrive to the estimate in (8.2) which by its turn implies (8.1). For the gradient estimate, we observe that by (8.21)

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_4 \cdot \frac{I(u_\varepsilon, \rho, s, s, p)}{s-\rho} \cdot |x-y|, \quad \forall x, y \in \overline{B}_s.$$

As before, letting $\varepsilon \to 0$ and $s \nearrow r$ afterwards, we obtain that $u \in C^{0,1}(\overline{B}_s)$ with estimate (8.4) that also implies (8.3). A simple covering argument together with the use of the previous estimates yield $u \in C^{0,1}_{loc}(B_r)$.

Finally, we observe that estimate (8.5) in (d) follows from the estimates (7.5), (7.6) and (7.7) applied to the ball $B_{r-\rho}(x_0)$ (instead of B_r) for any point of differentiability $x_0 \in B_\rho$ of u. □

9. Appendix: Pointwise $C^{1,\omega}$ *vs.* global $C^{1,\omega}$

In this Appendix, we present some estimates relating pointwise $C^{1,\omega}$ behavior and classical $C^{1,\omega}$ regularity in the interior. These estimates are well known, specially in the $C^{1,\alpha}$ case. However, it is not so easy to find a reference for their proofs, in particular in the generality discussed here. For completeness, we include the result here. The proof and ideas presented below are originally due to Lihe Wang.

Lemma 9.1 (**$C^{1,\omega}-$ interior regularity by uniform control on Taylor's expansion).** *Let u be defined in B_r and $\omega : [0, \delta_\omega) \to [0,\infty)$ a modulus of continuity. Moreover, let $r_0 \leq \min\{r/2, \delta_\omega\}$. Assume that for every $x_0 \in \overline{B}_{r/2}$ there exists an affine function P_{x_0} such that*

$$|u(x) - P_{x_0}(x)| \leq T|x-x_0|\omega(|x-x_0|) \qquad \forall x \in B_{r_0}(x_0). \tag{9.1}$$

Then, $u \in C^{1,\omega}(\overline{B}_{r_0/8})$ with the estimate

$$[\nabla u]_{C^{0,\omega}(\overline{B}_{r_0/8})} \leq ET. \tag{9.2}$$

Moreover if

$$\sup_{x_0 \in \overline{B}_{r/2}} |\nabla P_{x_0}| \leq T \tag{9.3}$$

and ω is strictly positive, then $u \in C^{1,\omega}(\overline{B}_{r/2})$ and the following estimate holds:

$$[\nabla u]_{C^{0,\omega}(\overline{B}_{r/2})} \leq \left(1 + \frac{1}{\omega(r_0/4)}\right) ET. \tag{9.4}$$

In both estimates above, $E > 0$ is a dimensional constant.

Proof. Clearly, (9.1) implies that u is differentiable at any point in $\overline{B}_{r/2}$. Now, assume $x_0, y_0 \in \overline{B}_{r/2}$ with $x_0 \neq y_0$ and $2d_0 := 2|x_0 - y_0| \leq r_0/2$. Set $z_0 := (x_0+y_0)/2 \in \overline{B}_{r/2}$. Clearly, $|z_0 - x_0| = |z_0 - y_0| = d_0/2$. Therefore $\overline{B}_{d_0/2}(z_0) \subset \overline{B}_{d_0}(x_0) \cap \overline{B}_{d_0}(y_0) \subset B_r$. By assumption (9.1) we have

$$||u - P_{x_0}||_{L^\infty(\overline{B}_{\frac{d_0}{2}}(z_0))} \leq ||u - P_{x_0}||_{L^\infty(\overline{B}_{d_0}(x_0))} \leq T d_0 \omega(d_0) \tag{9.5}$$

and

$$||u - P_{y_0}||_{L^\infty(\overline{B}_{\frac{d_0}{2}}(z_0))} \le ||u - P_{y_0}||_{L^\infty(\overline{B}_{d_0}(y_0))} \le T d_0 \omega(d_0). \tag{9.6}$$

Now, since $P_{x_0} - P_{y_0}$ is affine, we have for a dimensional constant $\overline{C} > 0$ that

$$\begin{aligned}\Big|\nabla u(x_0) - \nabla u(y_0)\Big| &= \Big|\nabla P_{x_0} - \nabla P_{y_0}\Big| \le \frac{2\overline{C}}{d_0}\Big|\Big|P_{x_0} - P_{y_0}\Big|\Big|_{L^\infty(\overline{B}_{d_0/2}(z_0))} \\ &\le \frac{2\overline{C}}{d_0}\left(\Big|\Big|u - P_{x_0}\Big|\Big|_{L^\infty(\overline{B}_{d_0}(x_0))} + \Big|\Big|u - P_{y_0}\Big|\Big|_{L^\infty(\overline{B}_{d_0}(y_0))}\right) \\ &\le 4\overline{C}T\omega(d_0) = 4\overline{C}T\omega(|x_0 - y_0|), \end{aligned} \tag{9.7}$$

where we added (9.5) and (9.6) in the last inequality. Now, we observe that if $x_0, y_0 \in \overline{B}_{r_0/8}$ with $x_0 \neq y_0$ then $2d_0 := 2|x_0 - y_0| \le r_0/2$. Thus, estimate (9.2) follows readily from (9.7). Let $x_0, y_0 \in \overline{B}_{r/2}$ be such that $d_0 := |x_0 - y_0| > r_0/4 > 0$. Then by (9.3), we have

$$\frac{|\nabla u(x_0) - \nabla u(y_0)|}{\omega(|x_0 - y_0|)} = \frac{|\nabla P_{x_0} - \nabla P_{y_0}|}{\omega(|x_0 - y_0|)} \le \frac{2T}{\omega(r_0/4)}. \tag{9.8}$$

Now, estimate (9.4) follows immediately by adding up the estimates (9.7) and (9.8). We can take $E := 4\overline{C} + 2$ and this finishes the proof.

□

By a standard covering argument, we obtain the following Corollary.

Corollary 9.2 (Pointwise Taylor's expansion everywhere). *Let $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is an open set, and let $\omega : [0, \delta_\omega) \to [0, \infty)$ be a strictly positive modulus of continuity. Assume that for any $\Omega' \subset\subset \Omega$ and for every $x_0 \in \Omega'$ there exist an affine function P_{x_0} and positive constants $r_{\Omega'}, C_{\Omega'}, G_{\Omega'}$ such that*

$$||u - P_{x_0}||_{L^\infty(B_r(x_0))} \le C_{\Omega'} \cdot r\omega(r) \quad \forall r \le r_{\Omega'} \le \big\{\delta_\omega, dist(\Omega', \partial\Omega)/2\big\},$$
$$||\nabla P_{x_0}||_{L^\infty(\Omega')} \le C_{\Omega'}.$$

Then, $u \in C^{1,\omega}_{loc}(\Omega)$.

References

[1] Alberti, G.; Ambrosio, L.; Cannarsa, P. On the singularities of convex functions. Manuscripta Math. 76 (1992), no. 3-4, 421-435.

[2] Alvarez, O.; Lasry, J.-M.; Lions, P.-L. Convex viscosity solutions and state constraints. *J. Math. Pures Appl.* (9) 76 (1997), no. 3, 265-288.

[3] Braga, J. Ederson M.; Moreira, Diego. Inhomogeneous Hopf-Oleinik Lemma and regularity of semiconvex supersolutions via new barriers for the Pucci extremal operators. Adv. Math. 334 (2018), 184-242.

[4] Bernard, P. Lasry-Lions regularization and a Lemma of Ilmanen. *Rend. Semin. Mat. Univ. Padova* 124 (2010), 221-229. ISBN: 978-88-7784-325-8

[5] Cabré, Xavier On the Alexandroff-Bakelḿan-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 48 (1995), no. 5, 539-570.

[6] Caffarelli, L. A. Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math.* (2) 130 (1989), no. 1, 189-213.

[7] Caffarelli, L. A. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math.* (2) 131 (1990), no. 1, 135-150.

[8] Caffarelli, L. A.; Cabré, X. Fully Nonlinear Elliptic Equations. *Amer. Math. Soc.* Coll. Publ. 43. Providence (RI): Amer. Math. Soc. 1995.

[9] Caffarelli, L.; Nirenberg, L.; Spruck, J. The Dirichlet problem for the degenerate Monge-Ampère equation. *Rev. Mat. Iberoamericana* 2 (1986), no. 1-2, 19-27.

[10] Caffarelli, L.; Kohn, J. J.; Nirenberg, L.; Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.* 38 (1985), no. 2, 209-252.

[11] Cannarsa, P.; Sinestrari, C. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston, Inc., Boston, MA, 2004. xiv+304 pp. ISBN: 0-8176-4084-3

[12] Cardaliaguet, P. Front propagation problems with nonlocal terms. II. *J. Math. Anal. Appl.* 260 (2001), no. 2, 572-601.

[13] Caffarelli, L.; Crandall, M. G.; Kocan, M.; Święch, A. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.* 49 (1996), no. 4, 365-397.

[14] Crandall, M. G.; Kocan, M.; Soravia, P.; Święch, A. On the equivalence of various weak notions of solutions of elliptic PDEs with measurable ingredients, *Progress in elliptic and parabolic partial differential equations* (Capri, 1994), 136 - 162. Pitman Res. Notes Math. Ser., 350, *Longman, Harlow,* 1996.

[15] De Philippis, G.; Figalli, A. Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps. *Anal. PDE* 6 (2013), no. 4, 993-1000.

[16] De Philippis, G.; Figalli, A. Optimal regularity of the convex envelope. *Trans. Amer. Math. Soc.* 367 (2015), no. 6, 4407-4422.
[17] Evans, Lawrence C.; Gariepy, Ronald F. Measure theory and fine properties of functions. Revised edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015. xiv+299 pp. ISBN: 978-1-4822-4238-6
[18] Fathi, A.; Figalli, A. Optimal transportation on non-compact manifolds. *Israel J. Math.* 175 (2010), no. 1, 1-59.
[19] Fathi, A.; Zavidovique, M. Ilmanen's Lemma on insertion of $C^{1,1}$ functions. *Rend. Semin. Mat. Univ. Padova* 124 (2010), 203-219. ISBN: 978-88-7784-325-8
[20] Figalli, A. *The Monge-Ampère equation and its applications.* Zürich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017. x+200
[21] Folland, Gerald B. *Real analysis. Modern techniques and their applications.* Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. xvi+386 pp. ISBN: 0-471-31716-0
[22] Gangbo, W.; McCann, R. J. The geometry of optimal transportation. *Acta Math.* 177 (1996), no. 2, 113-161.
[23] Giaquinta, M.; Martinazzi, L. *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs.* Seconda edizione. Scuola Normale Superiore Pisa, 2012. ISBN: 978-88-7642-443-4 (eBook).
[24] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
[25] Han, Qing Nonlinear elliptic equations of the second order. Graduate Studies in Mathematics, 171. American Mathematical Society, Providence, RI, 2016. viii+368 pp. ISBN: 978-1-4704-2607-1.
[26] Leoni, Giovanni A first course in Sobolev spaces. Graduate Studies in Mathematics, 105. American Mathematical Society, Providence, RI, 2009. xvi+607 pp. ISBN: 978-0-8218-4768-8
[27] Giusti, E. *Direct methods in the calculus of variations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003. viii+403 pp. ISBN: 981-238-043-4
[28] Griewank, A; Rabier, P. J. On the smoothness of convex envelopes. *Trans. Amer. Math. Soc.* 322 (1990), no. 2, 691-709.
[29] Gutiérrez, C. E. *The Monge-Ampère equation.* Progress in nonlinear differential equations and their applications, v. 44. Birkäuser, Boston, 2001.
[30] Ilmanen, T. The level-set flow on a manifold. Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), 193-204, *Proc. Sympos. Pure Math.,* 54, Part 1, Amer. Math. Soc., Providence, RI, 1993.
[31] Imbert, C. Convexity of solutions and $C^{1,1}$ estimates for fully nonlinear elliptic equations. *J. Math. Pures Appl.* (9) 85 (2006), no. 6, 791-807.
[32] Kirchheim, B; Kristensen, J. Differentiability of convex envelopes. *C. R. Acad. Sci. Paris Sér. I Math.* 333 (2001), no. 8, 725-728.
[33] Koike, S.; Święch. A. Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients. *J. Math. Soc. Japan* 61 (2009), no. 3, 723-755.
[34] Koike, S.; Święch, A. Local maximum principle for Lp-viscosity solutions of fully nonlinear elliptic PDEs with unbounded coefficients.*Comm. Pure Appl. Anal.* 11 (2012), no. 5, 1897-1910.
[35] Mooney, C. Harnack inequality for degenerate and singular elliptic equations with unbounded drift. *J. Differential Equations* 258 (2015), no. 5, 1577-1591.
[36] Oberman, A. M.; Silvestre, L. The Dirichlet problem for the convex envelope. *Trans. Amer. Math. Soc.* 363 (2011), no. 11, 5871-5886.
[37] Safonov, M. V. Non-divergence elliptic equations of second order with unbounded drift. *Nonlinear partial differential equations and related topics*, 211-232, Amer. Math. Soc. Transl. Ser. 2, 229, Adv. Math. Sci., 64, *Amer. Math. Soc., Providence, RI,* 2010.
[38] Sirakov, B. Boundary Harnack Estimates and Quantitative Strong Maximum Principles for Uniformly Elliptic PDE. *Int. Math. Res. Not. IMRN* 2018, no. 24, 7457-7482. .
[39] Trudinger, N. S. Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations. Rev. Mat. Iberoamericana 4 (1988), no. 3-4, 453-468.

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