LECTURE 4

– Remove 0-handles Lemma – (Cor of Cancellation)

If W is connected, then any handle decomp. of $(W, \partial_0 W, \partial_1 W)$ can be modified to one in which either there are no 0-handles (if $\partial_0 W \neq \emptyset$) or there is precisely one 0-handle (if $\partial_0 W = \emptyset$).

proof. If $\partial_0 W \neq \emptyset$, then for any 0-handle h^0 of W there must be a 1-handle h^1 that attaches both to h^0 and $\partial_0 W$; otherwise, W would be disconnected (as handles of index ≥ 2 have connected att. regions). But then h^0 and h^1 are in cancelling position: $A_{h^1} \pitchfork B_{h^0} = \{pt\}$ so we can remove both.
If $\partial_0 W = \emptyset$, first attach one 0-handle and then apply the case $\partial_0 W \neq \emptyset$. □

– Remove n-handles Lemma –

If W is connected, then any handle decomp. of $(W, \partial_0 W, \partial_1 W)$ can be modified to one in which either there are no n-handles (if $\partial_1 W \neq \emptyset$) or there is precisely one n-handle (if $\partial_1 W = \emptyset$).

proof. Turn the handle decomposition upside down and apply – Remove 0-handles Lemma –. □

Now recall:

key **Thm [Smale 1961] – h-cobordism Theorem –**

Any simply connected h-cobordism $(W, \partial_0 W, \partial_1 W)$ with $\dim W \geq 6$ is trivial.
↖ means $\partial_i W \hookrightarrow W$ are homotopy equivalences.

In the case $\pi_1 W \cong \pi_1(\partial_i W)$ is not trivial, we need to additionally assume $\partial_i W \hookrightarrow W$ are simple homotopy eq. (↗ this defines s-cobordism)
This is measured by an invariant called the Whitehead torsion that will be explained later. $Wh(W, \partial_0 W) \in Wh(\pi_1 W)$

§ S-COBORDISM THEOREM

key **Thm [Smale 1961] – s-cobordism Theorem –**

If $(W, \partial_0 W, \partial_1 W)$ is an s-cobordism with $\dim W \geq 6$, then it is smoothly trivial, i.e. there is a diffeomorphism $(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

sketch of proof. Pick a handle decomposition of $(W, \partial_0 W, \partial_1 W)$.
Thanks to Remove 0- and n-handles Lemma, we can assume no 0- and n-handles.

Step 1. – Normal Form Lemma –
For every h-cobordism of dimension $n \geq 6$ and any $2 \leq \ell \leq n-3$ there is a handle decomposition of the form $\partial_0 W \times [0,1] \cup \bigcup_{i=1}^{r_\ell} h_i^\ell \cup \bigcup_{j=1}^{r_{\ell+1}} h_j^{\ell+1}$

using: Handle Trading Lemma

Step 2. Put handles into algebraically cancelling position:
using:
- $H_*(\widetilde{W}, \partial_0 \widetilde{W}; \mathbb{Z})$ is computed by the Morse chain complex
- & $H_*(\widetilde{W}, \partial_0 \widetilde{W}; \mathbb{Z}) = 0$ since $\partial_0 W \hookrightarrow W$ is a homotopy equivalence
- & $Wh(W, \partial_0 W) = 0$ since $\partial_0 W \hookrightarrow W$ is a simple h.e.
- & Handle Slides

Step 3. Cancel $\bigcup_{i=1}^{r_\ell} h_i^\ell \cup \bigcup_{j=1}^{r_{\ell+1}} h_j^{\ell+1}$ using: Whitney Trick Lemma [$\dim W \geq 6$ crucial]
⇝ improves algebraically cancelling into geometrically cancelling.

□

NOTATION: Given a handle decomposition of $(W, \partial_0 W, \partial_1 W)$ let $W^{\leq k} = \partial_0 W \cup$ handles of index $\leq k$. Then $W^{\leq k}$ is a cobordism from $\partial_0 W^{\leq k} = \partial_0 W$ to $\partial_1 W^{\leq k}$.

def. – Morse chain complex –

Given a handle decomposition $\{h_i^k\}_{\substack{0 \leq k \leq n \\ 1 \leq i \leq r_k}}$ of a cobordism $(W, \partial_0 W, \partial_1 W)$ we define a chain complex $(C_*^{\mathcal{M}}, \delta_*^{\mathcal{M}})$ over $\mathbb{Z}$ as follows:

for $0 \leq k \leq n$ let $C_k^{\mathcal{M}} := \mathbb{Z}\{H_1^k, \dots, H_{r_k}^k\}$ (the free ab. gp on r_k generators)

and $\delta_k^{\mathcal{M}} : C_k^{\mathcal{M}} \longrightarrow C_{k-1}^{\mathcal{M}}$ by $\delta_k^{\mathcal{M}}(H_i^k) := \sum_{1 \leq j \leq r_{k-1}} I(A_i^k \pitchfork B_j^{k-1}) \cdot H_j^{k-1}$

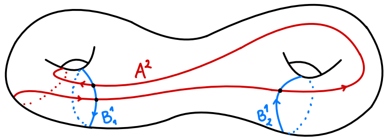
where $A_i^k : S^{k-1} \hookrightarrow \partial_1 W^{\leq k-1}$ is att. sphere of h_i^k

$B_j^{k-1} : S^{n-k} \hookrightarrow \partial_1 W^{\leq k-1}$ is belt sphere of h_j^{k-1} (Note: $k-1+n-k = n-1 = \dim \partial_1 W^{(k-1)}$).

$I(A_i^k \pitchfork B_j^{k-1}) := \sum_{p \in A \pitchfork B} \varepsilon_p \in \mathbb{Z}$ is the intersection number, where:

$$\varepsilon_p := \begin{cases} +1, & \text{if } dA_i^k(TS^{k-1})_a \oplus dB_j^{k-1}(TS^{n-k})_b \underset{\text{or. pres.}}{\cong} T(\partial_1 W^{(k-1)})_p \text{ at } p = A(a) = B(b) \\ -1, & \text{otherwise.} \end{cases}$$

Example.



$I(A^2 \pitchfork B_1^1) = 1 - 1 = 0$
$I(A^2 \pitchfork B_2^1) = 1$

Note: We fixed an orientation on $\mathbb{R}^k$ for all $k \geq 0$, so also on $\mathbb{D}^k$ and $S^{k-1} = \partial \mathbb{D}^k$ and thus on the core, att. and belt spheres of the k-handle $h^k \cong \mathbb{D}^k \times \mathbb{D}^{n-k}$.

Note: For $v \in C_k^{\mathcal{M}}$ we can write $\delta_k^{\mathcal{M}}(v) = I_k^{\mathcal{M}} \cdot v$ for $r_k \times r_{k-1}$-matrix $I_k^{\mathcal{M}} := (\partial_k^{\mathcal{M}}(H_i^k))_{1 \leq i \leq r_k}$

Thm. This defines a chain complex whose homology is $H_*(C_*^{\mathcal{M}}, \delta_*^{\mathcal{M}}) \cong H_*(W, \partial_0 W; \mathbb{Z})$.

proof. Recall that for a CW-complex X with k-skeleton $X^{\leq k} \subseteq X$, $H_*(X, X^{\leq 0}; \mathbb{Z})$ can be computed as homology of $(C_*^{CW}, \delta_*^{CW})$ defined by $C_k^{CW} := H_k(X^{\leq k}, X^{\leq k-1}; \mathbb{Z}) \cong$ free abelian on k-cells and $\delta_k^{CW} : H_k(X^{\leq k}, X^{\leq k-1}; \mathbb{Z}) \xrightarrow{\partial} H_{k-1}(X^{\leq k-1}; \mathbb{Z}) \xrightarrow{i_*} H_{k-1}(X^{\leq k-1}, X^{\leq k-2}; \mathbb{Z})$.

Lemma. Collapsing handles to cores is a deformation retraction of W on a CW complex X. ← Exercise.

Let $f_k : C_k^{\mathcal{M}}(W, \partial_0 W) \longrightarrow C_k^{CW}(X, X^{(0)})$ send H_i^k to the cell $c_i^k :=$ the core of h_i^k.

This is clearly an isomorphism for all k, so we just need to check that differentials agree, i.e. $f_{k-1}(\delta_k^{\mathcal{M}}(H_i^k)) = \delta_k^{CW}(f_k(H_i^k))$

$$f_{k-1}(\delta_k^{\mathcal{M}}(H_i^k)) = \sum_{1 \leq j \leq r_{k-1}} I(A_i^k \pitchfork B_j^{k-1}) \cdot \underbrace{f_{k-1}(H_i^{k-1})}_{= c_i^{k-1}}$$

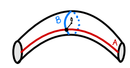
$$\delta_k^{CW}(f_k(H_i^k)) = \delta_k^{CW}(c_i^k) = \sum_{1 \leq j \leq r_{k-1}} \deg\left(S^{k-1} \xrightarrow{\partial c_i^k = A_i^k} X^{\leq k-1} \xrightarrow{\text{quot}_{c_j}} X^{\leq k-1} / X^{\leq k-1} \setminus c_j^{k-1} \simeq S^{k-1}\right) \cdot c_j^{k-1}$$

Indeed, the degree of a map can be computed as the sum of local degrees at points of a preimage set:

$$\deg(\text{quot}_{c_j} \circ A_i^k) = \sum_{a \in (\text{quot} \circ A_i^k)^{-1}(0)} \text{loc. deg. of } \text{quot}_{c_j} \circ A_i^k \text{ at } a = \sum_{\substack{a \in S^{k-1},\, b \in S^{n-k} \\ p = A_i^k(a) = B_j^{k-1}(b)}} \text{loc. deg of } \text{pr} \circ A_i^k|_{V_a \subseteq S^{k-1}}$$

$\iff a \in S^{k-1},\ A_i^k(a) \in \text{quot}^{-1}(0)$

$\iff a \in S^{k-1},\ b \in S^{n-k} = \partial(0 \times \mathbb{D}^{n-(k-1)})\ \ A_i^k(a) = B_j^{k-1}(b)$



where $\text{pr} : \nu_{B_j \subseteq W^{\leq k-1}} \longrightarrow B_j$ bundle projection □.

Let $\widetilde{W} \xrightarrow{p} W$ the universal cover of W, $\widetilde{\partial_0 W} := p^{-1}(\partial_0 W)$ the induced cover of $\partial_0 W$.

Then $W \simeq X$ CW complex and we have analogously $\widetilde{X}$, $\widetilde{X^{\leq k}}$ and

$C_*^{CW}(\widetilde{X}, \widetilde{X^{\leq 0}}; \mathbb{Z}) \cong C_*^{CW}(X, X^{\leq 0}; \mathbb{Z}[\pi])$

is the abelian group generated by all lifts $g \cdot \tilde{c}_i^k$ of cells c_i^k of $(X, X^{\leq 0})$ to $\widetilde{X}$.

It admits an action of $\pi = \pi_1 X$ (by deck transformations), which makes it into the free $\mathbb{Z}[\pi]$-module generated by some fixed lifts $\tilde{c}_i^k$, $1 \leq i \leq r_k$.

↳ the free abelian group generated by the set π.

Thm – Equivariant Morse chain complex –

The $\mathbb{Z}[\pi]$-chain complex $(C_*^{\widetilde{\mathcal{M}}}, \delta_*^{\widetilde{\mathcal{M}}})$ defined by: $C_k^{\widetilde{\mathcal{M}}} := \mathbb{Z}\{gH_i^k : g \in \pi, 1 \leq i \leq r_k\}$

$$\delta_k^{\widetilde{\mathcal{M}}}(gH_i^k) := \sum_{g' \in \pi,\, 1 \leq j \leq r_{k-1}} I(g\tilde{A}_i^k \pitchfork g'\tilde{B}_j^{k-1})\, g' H_j^{k-1}$$

computes $H_*(\widetilde{W}, \widetilde{\partial_0 W}; \mathbb{Z}) \cong H_*(W, \partial_0 W; \mathbb{Z}[\pi])$.

proof. Analogously to the preceding proof, define $f_k : C_k^{\widetilde{\mathcal{M}}}(W, \partial_0 W) \longrightarrow C_k^{CW}(\widetilde{X}, \widetilde{\partial_0 X})$, $gH_i^k \longmapsto g \cdot \tilde{c}_i^k$ □.

— Handle Slides Lemma —

The change of basis $\tilde{H}_j^{k+1} \longmapsto \tilde{H}_j^{k+1} + g\tilde{H}_{j'}^{k+1}$ (or $\tilde{H}_j^{k+1} - g\tilde{H}_{j'}^{k+1}$) in $C_*^{\tilde{\mathcal{U}}}$ for some $1 \le j, j' \le r_{k+1}$, $g \in \pi$, can be realised geometrically on the handle decomposition. More precisely, the att. sphere of h_j^{k+1} can be isotoped so that the resulting (k+1)-handle corresponds to $\tilde{H}_j^{k+1} \pm g\tilde{H}_{j'}^{k+1}$ in $C_*^{\tilde{\mathcal{U}}}$.

proof

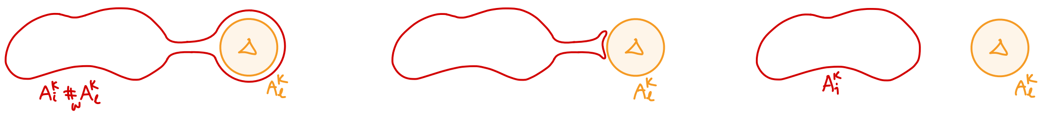
We can attach handles of the same index in any order, so consider $(W^{\le k} \cup h_j^{k+1}) \cup h_{j'}^{k+1}$.
In $\partial_1 (W^{\le k} \cup h_{j'}^{k+1})$ we have a push-off of $A_{j'}$, which bounds a disk Δ = push off of the core of $h_{j'}^{k+1}$.
Then we can form an ambient connected sum

$$A_j \#_{\gamma} A_{j'} = (A_j \setminus \nu(pt)) \cup \nu(w_j^{-1} \cdot \gamma \cdot w_{j'}) \cup (A_{j'} \setminus \nu(pt))$$

where $[\gamma] = g \in \pi$,
w_j = path from A_j to the basepoint

Since $A_\ell^k \setminus \nu(pt)$ is isotopic rel boundary to $\nu(pt)$, via Δ, we have isotopies

$$A_j \# A_{j'} \simeq A_i^k \cup \nu(w_j^{-1} \cdot \gamma \cdot w_{j'}) \cup \nu(pt) \simeq A_i^k .$$



On the other hand, we clearly have that a handle attached to $A_j \#_\gamma A_{j'}$ corresponds to $H_j^{k+1} + gH_{j'}^{k+1}$ (to get $-gH_j^{k+1}$ use oppositely oriented $A_{j'}$). □