BBS TALK: EMBEDDING CALCULUS AND SMOOTH STRUCTURES

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Abstract. I will explain joint work with Ben Knudsen about the extent to which embedding calculus is sensitive to the smooth structures of the domain and target. In particular, we will prove that in dimension 4 the approximations provided by embedding calculus depends only on the underlying topological manifolds and the vector bundle reductions of their tangent microbundles provided by the smooth structures. If time permits, I will also give some examples of high-dimensional exotic spheres distinguished by embedding calculus and ask some questions.

This is joint work with Ben Knudsen, see [KK20].

Question 0.1. To what extent can invariants from embedding calculus be used to distinguish smooth structures?

1. Invariants of smooth manifolds

Homeomorphic but not diffeomorphic smooth manifolds are homotopy equivalent, so cannot be distinguished from each other by any of the usual invariants from algebraic topology (such as homology groups or homotopy groups). However, not all hope is lost: we could try to construct some other space from a manifold (and possibly some additional data), and show that these spaces are not homotopy equivalent using an invariant from algebraic topology.

The construction of these spaces had better depend on the smooth structure of d-dimensional smooth manifold M. An analytic approach to doing so is to take spaces of solutions of certain systems of differential equations on bundles over M; this leads to gauge theory. However, here we take geometric approach of considering spaces of smooth embeddings.

Embedding of points. The following is the simplest example:

Definition 1.1. Write $\underline{k} = \{1, \ldots, k\}$. The *configuration space* $\mathrm{Conf}_k(M)$ is the space of smooth embeddings $\underline{k} \hookrightarrow M$, in the C^∞-topology.

If there were a diffeomorphism $f\colon M \to M'$, then this would induce a map $f\colon \mathrm{Conf}_k(M) \to \mathrm{Conf}_k(M')$ with inverse the map induced by f^{-1}, so $\mathrm{Conf}_k(M)$ would be homotopy equivalent to $\mathrm{Conf}_k(M')$. Conversely, if $\mathrm{Conf}_k(M)$ and $\mathrm{Conf}_k(M')$ are not homotopy equivalent then M and M' can not be diffeomorphic.

The configuration spaces are not the best choice of space to extract from M, as $\mathrm{Conf}_k(M)$ is homeomorphic to the space of k-tuples of distinct points in M:

$$\mathrm{Conf}_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j \text{ if } i \neq j\},$$

so its homotopy type does not depend on the smooth structure of M. However, they don't *just* depend on the homotopy type of M; by a result of Longoni and Salvatore [LS05],

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$\mathrm{Conf}_2(L(7,1))$ is not homotopy equivalent to $\mathrm{Conf}_2(L(7,2))$ even though the two lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent.

Remark 1.2. Classically, lens spaces are distinguished by Reidemeister torion. On the other hand, Longoni and Salvatore distinguish them by Massey products on the universal cover of $\mathrm{Conf}_2(M)$. Can one in general detect Reidemeister torsion of manifolds in the homotopy type of their configuration spaces? What about Whitehead torsion of h-cobordisms?

Embeddings of open discs. We could try to amend this by replacing $\underline{k}$ by $\underline{k} \times \mathbb{R}^d$, which I will think of as a collection of open discs. Then evaluation at the origin gives a map

$$\mathrm{Emb}(\underline{k} \times \mathbb{R}^d, M) \to \mathrm{Conf}_k(M)$$

which by isotopy extension is a fiber bundle with fiber over a configuration $(m_1, \ldots, m_k)$ given by the space of embeddings k open discs sending the center of the ith open disc to m_i. This is homotopy equivalent to the space k-tuples of frames $\mathrm{Iso}(\mathbb{R}^d, T_{m_1}M) \times \cdots \mathrm{Iso}(\mathbb{R}^d, T_{m_k}M)$. In particular, for $k = 1$ we recover the homotopy type of the total space of the frame bundle of TM.

Embeddings of open discs and maps between these. Of course, these spaces of embeddings are natural in the domain: for any embedding $\underline{k'} \times \mathbb{R}^d \hookrightarrow \underline{k} \times \mathbb{R}^d$, precomposition induces a map

$$\mathrm{Emb}(\underline{k} \times \mathbb{R}^d, M) \longrightarrow \mathrm{Emb}(\underline{k'} \times \mathbb{R}^d, M).$$

This is continuous in the embeddings, and is associative. For example, this means $\mathrm{Emb}(\underline{k} \times \mathbb{R}^d, M)$ comes with actions of Σ_k (permuting the discs) and $O(d)^k$ (rotating each of the discs).

All these maps are convenient encoded in terms of enriched presheaves, an idea due to Boavida de Brito and Weiss [BdBW13]. Let Disc_d be the category whose objects are given by $\underline{k} \times \mathbb{R}^d$ for $k \geq 0$ and whose morphism spaces are given by

$$\mathsf{Disc}_d(\underline{k'} \times \mathbb{R}^d, \underline{k} \times \mathbb{R}^d) = \mathrm{Emb}(\underline{k'} \times \mathbb{R}^d, \underline{k} \times \mathbb{R}^d).$$

That $\mathrm{Emb}(-, M)$ is natural in the open discs we plug into the domain is then succinctly encoded as saying that it is a *enriched presheaf*

$$\mathrm{Emb}(-, M)\colon \mathsf{Disc}_d^{\mathrm{op}} \longrightarrow \mathsf{Top}.$$

Remark 1.3. A presheaf on Disc_d is the same as a right module over the operad with k-ary operations $\mathrm{Emb}(\underline{k} \times \mathbb{R}^d, \mathbb{R}^d)$, which is homotopy equivalent to the framed little k-disks operad or equivalently the k-dimensional Fulton–MacPherson operad (e.g. [Tur13]).

If there were a diffeomorphism $f\colon M \to M'$, then postcomposing by f gives a map

$$f \circ -\colon \mathrm{Emb}(-, M) \longrightarrow \mathrm{Emb}(-, M')$$

of presheaves on Disc_d, which yields a homotopy equivalence whenever we evaluate at any object $\underline{k} \times \mathbb{R}^d$. Thus whether $\mathrm{Emb}(-, M)$ and $\mathrm{Emb}(-, M')$ are weakly equivalent as presheaves—in the sense, that there is a zigzag of such "objectwise weak equivalences" between them—can serve to distinguish M and M'.

Question 1.4. How good of an invariant of the smooth manifold M is the presheaf $\mathrm{Emb}(-, M)$?

I will show that there is no clear-cut answer:

Theorem 1.5 (Knudsen–K.)**.** *If M and M' are homeomorphic 1-connected closed 4-manifolds, then the presheaves $\mathrm{Emb}(-, M)$ and $\mathrm{Emb}(-, M')$ are weakly equivalent.*

Theorem 1.6 (Knudsen–K.). *In infinitely-many dimensions d there exists an exotic sphere Σ^d so that the presheaves $\mathrm{Emb}(-, S^d)$ and $\mathrm{Emb}(-, \Sigma^d)$ are not weakly equivalent.*

2. Embedding calculus

Before explaining the proofs of Theorem 1.5 and Theorem 1.6, let me explain their relationship to embedding calculus and some consequences.

Let $\mathsf{PSh}(\mathsf{Disc}_d)$ denote the category of enriched presheaves on Disc_d, of which $\mathrm{Emb}(-.M)$ is an object. Then $\mathrm{Emb}(-, M)$ and $\mathrm{Emb}(-, M')$ are equivalent if and only if there is a homotopy-invertible element in the *derived* mapping space

$$\mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_d)}(\mathrm{Emb}(-, M), \mathrm{Emb}(-, M')).$$

Remark 2.1. Strictly speaking, a derived mapping space depends on a choice of class of weak equivalences; here they are the objectwise weak equivalences. One construction of the derived mapping space is the Dwyer–Kan hammock localisation, which is literally built from zigzags of objectwise weak equivalences [DK80].

We encountered this space in an earlier talk. Recall from the talks in the first block that embedding calculus provides a tower [Wei99]

$$\begin{array}{ccc} & & \cdots \\ & \nearrow & \downarrow \\ & & T_2\mathrm{Emb}(M, M') \\ & \nearrow & \downarrow \\ \mathrm{Emb}(M, M') & \longrightarrow & T_1\mathrm{Emb}(M, M') \end{array}$$

of approximations, which gives a map

$$\mathrm{Emb}(M, N) \longrightarrow T_\infty\mathrm{Emb}(M, N) \coloneqq \operatorname*{holim}_{k\to\infty} T_k\mathrm{Emb}(M, N).$$

This map is a homotopy equivalence when the handle dimension of M is $\leq d-3$ [GW99, GK15] (and when $d \leq 2$, in joint work with Manuel Krannich [KK21]), but the map always exists and is worth studying as a source of invariants.

Theorem 2.2 (Boavida de Brito–Weiss).

$$T_\infty\mathrm{Emb}(M, N) \simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_d)}(\mathrm{Emb}(-, M), \mathrm{Emb}(-, N)).$$

Thus these derived mapping spaces are models for embedding calculus; they may not be the most explicit models for embedding calculus, unlike say [GKW03], but it has excellent formal properties.

2.1. **Applications to (2-)knots in 4-manifolds.** In Theorem 2.2 we assumed that M and N both have dimension d. If M is of dimension $\ell < d$, we can restrict $\mathrm{Emb}(-, M)$ a presheaf on Disc_ℓ by taking the composition

$$i^*\mathrm{Emb}(-, M) \colon \mathsf{Disc}_\ell^{\mathrm{op}} \xrightarrow{i} \mathsf{Disc}_d^{\mathrm{op}} \xrightarrow{\mathrm{Emb}(-,M)} \mathsf{Top}$$

with i induced by taking the product with $\mathrm{id}_{\mathbb{R}^{d-\ell}}$. We then similarly have

$$T_\infty\mathrm{Emb}(M, N) \simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_\ell)}(\mathrm{Emb}(-, M), i^*\mathrm{Emb}(-, N)).$$

Let us deduce from this the following result for spaces of knots in 4-manifolds.

Corollary 2.3. *If M and M' are homeomorphic 1-connected closed 4-manifolds, then* $\mathrm{Emb}(S^1, M)$ *and* $\mathrm{Emb}(S^1, M')$ *are weakly equivalent. The same is true when we replace* S^1 *by* $\sqcup_r S^1$.

Proof. We use the previously stated result that the presheaves $\mathrm{Emb}(-, M)$ and $\mathrm{Emb}(-, M')$ on Disc_4 are weakly equivalent. If so, then the presheaves $i^*\mathrm{Emb}(-, M)$ and $i^*\mathrm{Emb}(-, N)$ on Disc_1 are also weakly equivalent.

We write

$$\begin{aligned}
\mathrm{Emb}(S^1, M) &\simeq T_\infty \mathrm{Emb}(S^1, M) && \text{convergence} \\
&\simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_1)}(\mathrm{Emb}(-, S^1), i^*\mathrm{Emb}(-, M)) && \text{Theorem 2.2} \\
&\simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_1)}(\mathrm{Emb}(-, S^1), i^*\mathrm{Emb}(-, M')) && \text{Theorem 1.5} \\
&\simeq T_\infty \mathrm{Emb}(S^1, M') && \text{Theorem 2.2} \\
&\simeq \mathrm{Emb}(S^1, M'). && \text{convergence}
\end{aligned}$$

For the generalization just replace S^1 by $\sqcup_r S^1$. □

Remark 2.4. This answers a question of Viro [Vir15], and improves on a result of Arone and Szymik (though they did a lot more) [AS19].

If we replace S^1 with a surface Σ and M, M' as above, we still get that a diagram

$$\begin{array}{ccc}
\mathrm{Emb}(\Sigma, M) & \longrightarrow & T_\infty \mathrm{Emb}(\Sigma, M) \\
 & & \big\downarrow \simeq \\
\mathrm{Emb}(\Sigma, M') & \longrightarrow & T_\infty \mathrm{Emb}(\Sigma, M')
\end{array}$$

with vertical map a weak equivalence. However, the horizontal maps are rarely a weak equivalence, if ever (this is why we've had so many talks about embeddings of surfaces into 4-manifolds). However, we do learn the following lesson: any invariant of embeddings of surfaces into 4-manifolds which factors over the limit of the embedding calculus Taylor tower does *not* depend on the smooth structure of the target.

Question 2.5. Do configuration space integrals factor over the limit of the Taylor tower? What about invariants based on Whitney towers?

2.2. **Proof of Theorem 1.6.** The relationship to embedding calculus also leads to a quick proof of Theorem 1.6. Let us focus on dimension 16 for concreteness, in which case we use [HLS65]:

Theorem 2.6 (Hsiang–Levine–Szczarba)**.** *There is an exotic* 16*-sphere* Σ^{16} *which does not embed in* $\mathbb{R}^{19}$.

Corollary 2.7. *The presheaves* $\mathrm{Emb}(-, S^{16})$ *and* $\mathrm{Emb}(-, \Sigma^{16})$ *on* Disc_{16} *are not weakly equivalent.*

Proof. Suppose that there were an equivalence $\mathrm{Emb}(-,\Sigma^{16}) \to \mathrm{Emb}(-,S^{16})$, then we have

$$\begin{aligned}
\mathrm{Emb}(S^{16},\mathbb{R}^{19}) &\simeq T_\infty \mathrm{Emb}(S^{16},\mathbb{R}^{19}) && \text{convergence} \\
&\simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_{16})}(\mathrm{Emb}(-,S^{16}), i^*\mathrm{Emb}(-,\mathbb{R}^{19})) && \text{Theorem 1.5} \\
&\simeq \mathrm{Map}^h_{\mathsf{PSh}(\mathsf{Disc}_{16})}(\mathrm{Emb}(-,\Sigma^{16}), i^*\mathrm{Emb}(-,\mathbb{R}^{19})) && \text{existence of equivalence} \\
&\simeq T_\infty \mathrm{Emb}(\Sigma^{16},\mathbb{R}^{19}) && \text{Theorem 1.5} \\
&\simeq \mathrm{Emb}(\Sigma^{16},\mathbb{R}^{19}) && \text{convergence.}
\end{aligned}$$

The domain is non-empty, but the target is empty. We get a contradiction. □

In fact, this happens for $d = 2^j$ for $j \geq 3$ (using work of Mahowald). The lowest dimension d in which I know an example is $d = 8$ (using work of Levine). All the examples have non-trivial coker(J)-component; I do not know whether there is an bP-sphere for which this is true (being a d-dimensional bP-sphere is equivalent to admitting an embedding into $\mathbb{R}^{d+2}$, due to Kervaire). More ambitiously, one can ask:

Question 2.8. For $\Sigma, \Sigma' \in \Theta_d$, is $\mathrm{Emb}(-,\Sigma) \simeq \mathrm{Emb}(-,\Sigma')$ if and only if Σ is diffeomorphic to $\pm\Sigma'$? What about other smooth manifolds which are homeomorphic but not diffeomorphic?

Also not that we don't actually need that there is a *homotopy-invertible* map $\mathrm{Emb}(-,\Sigma^{16}) \to \mathrm{Emb}(-,S^{16})$, just a map.

Question 2.9. If M, M' are closed manifolds of the same dimension, is any map of presheaves $\mathrm{Emb}(-,M) \to \mathrm{Emb}(-,M')$ homotopy-invertible?

3. Proof of Theorem 1.5

Let me now explain why, for homeomorphic 1-connected closed 4-manifolds M, M', we have a weak equivalence of presheaves $\mathrm{Emb}(-,M) \simeq \mathrm{Emb}(-,M')$ on Disc_4. More generally, we prove this is the case if we have a homeomorphism $f\colon M \to M'$ so that the topological derivative lifts to an isomorphism of vector bundles $f^*TM' \cong TM$. More precisely, the diagram of tangent microbundle classifiers

$$\begin{array}{ccc}
M & \longrightarrow & M' \\
{\scriptstyle T^{\mathrm{mb}}M} \searrow & & \swarrow {\scriptstyle T^{\mathrm{mb}}M'} \\
& B\mathrm{Top}(4) &
\end{array}$$

which we may assume commutes, should lift to a diagram

$$\begin{array}{ccc}
M & \longrightarrow & M' \\
{\scriptstyle TM} \searrow & & \swarrow {\scriptstyle TM'} \\
& BO(4) &
\end{array}$$

commuting up to homotopy over $B\mathrm{Top}(4)$. Since $\mathrm{Top}(4)/O(4) \to \mathrm{Top}/O$ is 5-connected by Freedman–Quinn [FQ90], the only obstruction to f having this property is an obstruction class in $H^3(M;\mathbb{Z}/2)$; if M is 1-connected this vanishes by Poincaré duality.

One of the reasons that I described embedding calculus in terms of derived mapping spaces of presheaves is that we may replace the domain Disc_d of the presheaves with a Dwyer–Kan equivalent category [Kör17].

A *Dwyer–Kan equivalence* $F\colon \mathsf{C} \to \mathsf{D}$ between enriched categories is an enriched functor that is essentially surjective on homotopy categories (i.e. replace the mapping spaces by their sets of path components) and has the property that the maps $\mathsf{C}(x,y) \to \mathsf{D}(F(x),F(y))$ are weak equivalences. Then $X, X' \in \mathsf{PSh}(\mathsf{D})$ are weakly equivalent if and only if $X \circ F, X' \circ F \in \mathsf{PSh}(\mathsf{D})$ are.

The idea is now to replace smooth manifolds M by pairs $(M, \mathcal{T}M)$ of a topological manifold with a vector bundle refinement of their tangent microbundle, and replace smooth embeddings by the homotopy pullback

$$\begin{array}{ccc} \mathrm{Emb}^{\mathrm{fs}}((M,\mathcal{T}M),(N,\mathcal{T}N)) & \longrightarrow & \mathrm{Bun}^{\mathrm{vb}}(\mathcal{T}M,\mathcal{T}N) \\ \downarrow & & \downarrow \\ \mathrm{Emb}^{\mathrm{Top}}(M,N) & \longrightarrow & \mathrm{Bun}^{\mathrm{mb}}(T^{\mathrm{mb}}M,T^{\mathrm{mb}}N), \end{array}$$

where $\mathrm{Bun}^{\mathrm{vb}}$ denotes vector bundle maps and $\mathrm{Bun}^{\mathrm{mb}}$ denotes microbundle maps. We call these "formally smooth manifolds" and "formally smooth embeddings". We write TM for $\mathcal{T}M$ if M is smooth and we pick the tangent bundle as the vector bundle refinement of its tangent microbundle.

Just like smooth embeddings of d-discs are determined by their germ near the origin, so are topological embeddings. This gives

$$\begin{array}{ccc} \mathrm{Emb}^{\mathrm{fs}}(\mathbb{R}^d,\mathbb{R}^d) \simeq O(d) & \longrightarrow & \mathrm{Bun}^{\mathrm{vb}}(T\mathbb{R}^d,T\mathbb{R}^d) \simeq O(d) \\ \downarrow & & \downarrow \\ \mathrm{Emb}^{\mathrm{Top}}(\mathbb{R}^d,\mathbb{R}^d) \simeq \mathrm{Top}(d) & \longrightarrow & \mathrm{Bun}^{\mathrm{mb}}(T^{\mathrm{mb}}\mathbb{R}^d,T^{\mathrm{mb}}\mathbb{R}^d) \times \mathrm{Top}(d) \end{array}$$

so the inclusion $\mathrm{Emb}(\mathbb{R}^d,\mathbb{R}^d) \to \mathrm{Emb}^{\mathrm{fs}}(\mathbb{R}^d,\mathbb{R}^d)$ is a weak equivalence. Then same is true replacing the domain by $\sqcup_k \mathbb{R}^d$ and the target by any formally smooth manifold M.

This means that

$$j \colon \mathsf{Disc}_d \longrightarrow \mathsf{Disc}_d^{\mathrm{fs}}$$

is an equivalence on mapping spaces, and as it is an isomorphism on homotopy categories it is a Dwyer–Kan equivalence. Similarly, one proves that $\mathrm{Emb}(-,M) \to j^*\mathrm{Emb}^{\mathrm{fs}}(-,M)$ and $\mathrm{Emb}(-,M') \to j^*\mathrm{Emb}^{\mathrm{fs}}(-,M')$ equivalences. It thus suffices to show that

$$\mathrm{Emb}^{\mathrm{fs}}(-,M) \simeq \mathrm{Emb}^{\mathrm{fs}}(-,M')$$

in $\mathsf{PSh}(\mathsf{Disc}_d^{\mathrm{fs}})$. Now we observe that a postcomposition with a homeomorphism $f \colon M \to M'$ preserving the tangent bundle induces a weak equivalence $\mathrm{Emb}^{\mathrm{fs}}(-,M) \to \mathrm{Emb}^{\mathrm{fs}}(-,M')$.

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