THE WEISS FIBER SEQUENCE AND SOME APPLICATIONS

Abstract. These are Mauricio Bustamante's notes for his talk at the "Building Bridges Seminar: Invariants of embedding spaces", organized by Danica Kosanović in the middle of a pandemic in 2020-2021.

1. The Weiss fiber sequence

Throughout we fix a d-dimensional compact smooth manifold M and a codimension 0 submanifold $N \subset \partial M$ of its boundary, and we thicken it to an $N \times I \subset M$, where $I = [0,1]$.

The Weiss fiber sequence expresses $B\operatorname{Diff}_\partial(N \times I)$ as the difference between $B\operatorname{Diff}_\partial(M)$ and $B\operatorname{Emb}^{\cong}_{\partial/2}(M)$, where $\operatorname{Emb}^{\cong}_{\partial/2}(M)$ is the topological monoid of self-embeddings of M which are the identity on a neighborhood of $\partial M - \operatorname{int}(N)$, and are isotopic (through such embeddings) to a diffeomorphism that is the identity on a neighborhood of ∂M. More precisely, the Weiss fiber sequence is

$$B\operatorname{Diff}_\partial(N \times I) \to B\operatorname{Diff}_\partial(M) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M). \tag{1}$$

In order to obtain (1) we set $V := M - \operatorname{int}(N \times [0,1])$. The boundary of V decomposes as

$$\partial V = (\partial M - \operatorname{int}(N)) \cup \partial_1 V$$

for some other manifold $\partial_1 V$. We will use the notation $\partial/2 = \partial M - \operatorname{int}(N)$. By the isotopy extension theorem[1], restriction gives rise to a fibration

$$\operatorname{Diff}_\partial(M - \operatorname{int}(V)) \to \operatorname{Diff}_\partial(M) \to \operatorname{Emb}^{\mathrm{ext}}_{\partial/2}(V, M)$$

where the last term is the space of embeddings of V into M which restrict to the inclusion on $\partial/2$ and are isotopic to an embedding that extends to a self-diffeomorphism of M. Observe that the fiber of this fibration is exactly $\operatorname{Diff}_\partial(N \times I)$. The base of the fibration can be identified with $\operatorname{Emb}^{\cong}_{\partial/2}(M)$ as follows[2]: again we have a fibration sequence

$$\operatorname{Emb}^{\cong}_{\partial/2}(M) \to \operatorname{Emb}^{\mathrm{ext}}_{\partial/2}(V, M)$$

whose fiber over the inclusion is the space of self-embeddings of $N \times I$ which restrict to the identity on $N \times \{0\} \cup \partial N \times I$. This space is contractible by the existence and uniqueness of collars. So the previous restriction map is a homotopy equivalence. In total, we get a fibration sequence

$$\operatorname{Diff}_\partial(N \times I) \to \operatorname{Diff}_\partial(M) \to \operatorname{Emb}^{\cong}_{\partial/2}(M)$$

which deloops to (1) as all the maps are compatible with the operation of composition.

The delooped Weiss fiber sequence. It turns out that one can use the operation "stacking in the interval direction" to give a unital topological monoid model for $B\operatorname{Diff}_\partial(N \times I)$, and a right $B\operatorname{Diff}_\partial(N \times I)$-module model for $B\operatorname{Diff}_\partial(M)$. This gives rise to a delooping of $B\operatorname{Diff}_\partial(N \times I)$ and yields a *delooped Weiss fiber sequence* (established by A. Kupers)

$$B\operatorname{Diff}_\partial(M) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M) \to B^2\operatorname{Diff}_\partial(N \times I) \tag{2}$$

This sequence is the result of a more general fact: if X is a simplicial or topological right A-module, for A some path-connected unital topological monoid, then there is a fibration sequence of the form

$$X \to X /\!\!/ A \to BA.$$

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[1] It is perhaps safer to apply the isotopy extension theorem to proper embeddings. So one should change V by another isotopy equivalent manifold V' such that $\partial V' = \partial/2$. For example $V' = M - \operatorname{int}(N) \times I$.

[2] During the talk I realized that maybe an easier argument is just to show that V and M are isotopy equivalent.

In our case, $X = B\operatorname{Diff}_\partial(M)$ and $A = B\operatorname{Diff}_\partial(N \times I)$. The most involved step is to identify $B\operatorname{Diff}_\partial(M) /\!\!/ B\operatorname{Diff}_\partial(N \times I)$ with $B\operatorname{Emb}^{\cong}_{\partial/2}(M)$. This can be done by showing that there is a weak homotopy equivalence

$$B\operatorname{Diff}_\partial(M) /\!\!/ B\operatorname{Diff}_\partial(N \times I) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M) \times * /\!\!/ B\operatorname{Diff}_\partial(N \times I) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M).$$

In the end this also follows from the isotopy extension theorem.

2. Applications

The Weiss fiber sequence can be used to obtain "qualitative" information about $B\operatorname{Diff}_\partial(M)$.

Theorem 2.1 (A. Kupers)**.** *The homotopy groups $\pi_k(B\operatorname{Diff}_\partial(D^d))$ are finitely generated for all $k \geq 2$, $d \geq 6$ and $d \neq 7$. Furthermore, if M^d is 2-connected, then $\pi_k(B\operatorname{Diff}_\partial(M))$ is finitely generated for all $k \geq 2$.*

Theorem 2.2 (Bustamante–Krannich–Kupers)**.** *If $d = 2n \geq 6$ and $\pi_1(M)$ is finite then $\pi_k(B\operatorname{Diff}_\partial(M))$ is finitely generated for all $k \geq 2$.*

Remark. The case $k = 1$ (i.e. mapping class groups) in the previous theorems had been handled before: The group $\pi_0(\operatorname{Diff}_\partial(D^d))$ is isomorphic to the group Θ_{d+1} of homotopy $(d+1)$-spheres, which is finite by Kervaire–Milnor. When $\pi_1(M) = 0$ and $\partial M = \emptyset$, Sullivan shows that $\pi_0(\operatorname{Diff}(M))$ is a group commensurable up to finite kernel with an arithmetic group. In particular of type F_∞. Later Triantafillou extended Sullivan's result to closed manifolds with finite fundamental group.

Theorem 2.3 (Bustamante–Randal-Williams)**.** *Let p be a prime number. Then the group $\pi_{2p-3}(B\operatorname{Diff}_\partial(S^1 \times D^{2n-1}))$ contains a subgroup isomorphic to $\bigoplus^\infty \mathbb{Z}/p$, provided $2p-3 < n-1$.*

Remark. In the concordance stable range (roughly $2p - 3 < 2n/3$), this follows from Waldhausen's parametrized h-cobordism theorem and work of Hatcher, Igusa, Hesselholt, and Grunewald-Klein-Macko.

An example: finiteness for the moduli space of the even dimensional disk. Let us see how the delooped Weiss fiber sequence is actually used in showing that the higher homotopy groups of $B\operatorname{Diff}_\partial(D^{2n})$ are degreewise finitely generated if $2n \geq 6$, which is a special case of Kupers' result mentioned above. First we stabilize the disk by attaching g n-handles. The stabilized manifold is

$$W_{g,1} := D^{2n} \#(S^n \times S^n)^{\#g}.$$

This will be our M. We pick a $(2n-1)$-disk in $\partial W_{g,1} = S^{2n-1}$, which will be our N. In this case the Weiss fiber sequence takes the form

$$B\operatorname{Diff}_\partial(W_{g,1}) \to B\operatorname{Emb}^{\cong}_{\partial/2}(W_{g,1}) \to B^2\operatorname{Diff}_\partial(D^{2n}).$$

As the space $B^2\operatorname{Diff}_\partial(D^{2n})$ is 1-connected, it suffices to show that the homology groups of $B^2\operatorname{Diff}_\partial(D^{2n})$ are finitely generated. This will follow from the Serre spectral sequence of the previous fibration if

- The homology groups of $B\operatorname{Diff}_\partial(W_{g,1})$ are finitely generated.
- The homology groups of $B\operatorname{Emb}^{\cong}_{\partial/2}(W_{g,1})$ are finitely generated.

To show the first item we use the work of Galatius and Randal-Williams: the stable homology of $B\operatorname{Diff}_\partial(W_{g,1})$ is isomorphic to the homology of a component of the infinite loop space of the Thom spectrum of the inverse of $\theta^*_{2n}\gamma_{2n}$, where γ_{2n} is the canonical $2n$-plane bundle over $BO(2n)$, and $\theta : BO(2n)[n, \infty) \to BO(2n)$ is an n-connected cover of $BO(2n)$. Basic algebraic topology let us show that the homology groups of this infinite loop space are finitely generated. The statement for finite g follows from homological stability.

As for the second item we consider the "universal cover fibration"

$$B\operatorname{Emb}^{\mathrm{id}}_{\partial/2}(W_{g,1}) \to B\operatorname{Emb}^{\cong}_{\partial/2}(W_{g,1}) \to B\pi_0(\operatorname{Emb}^{\cong}_{\partial/2}(W_{g,1})).$$

A spectral sequence argument shows that the homology groups of $B\operatorname{Emb}^{\cong}_{\partial/2}(W_{g,1})$ are finitely generated if

(1) The homology groups $H_k(\pi_0(\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})); A)$ are finitely generated for all $\mathbb{Z}[\pi_0(\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})]$-modules A which are finitely generated as abelian groups.
(2) The homology groups of $B\,\mathrm{Emb}^{\mathrm{id}}_{\partial/2}(W_{g,1})$ are finitely generated.

To show (1) one can use the bottom part of the long exact sequence in homotopy groups of the Weiss fiber sequence to obtain an extension

$$1 \to F \to \pi_0(\mathrm{Diff}_\partial(W_{g,1})) \to \pi_0(\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})) \to 1$$

where F is a quotient of the group of homotopy $(2n+1)$-spheres, and so finite. The mapping class group of $W_{g,1}$ was studied by Kreck. He showed that this group is an extension of an arithmetic group by a finitely generated abelian group. In particular it is of type F_∞. It then follows that $\pi_0(\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1}))$ is a group of type F_∞. This implies (1).

To prove (2) we observe that $B\,\mathrm{Emb}^{\mathrm{id}}_{\partial/2}(W_{g,1})$ is a 1-connected space and hence its homology groups are finitely generated if its homotopy groups are finitely generated. Now the relative handle dimension of $W_{g,1}$ is n, whereas the geometric dimension of $W_{g,1}$ is $2n$. Thus from the point of view of embedding calculus, self-embeddings $W_{g,1} \to W_{g,1}$ relative to $\partial/2$ have codimension n. Therefore, if $n \geq 3$, the embedding calculus tower converges and in particular

$$\pi_k(B\,\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})) = \pi_{k-1}(\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})) \cong \pi_k(T_\infty\,\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})).$$

It's not hard to show that the homotopy groups of $T_1\,\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})$ and of the layers $L_r\,\mathrm{Emb}^{\cong}_{\partial/2}(W_{g,1})$ are finitely generated for all r. An induction over the embedding tower will complete the proof of (2). □

We won't comment on the other theorems here. But it is important to know that at the core of the proof of all of them lies the same "principle": the effect of stabilization by genus on the moduli space of $N \times I$ is measured by the classifying space of the monoid of self-embeddings (with a relaxed boundary condition) of the stabilized manifold.