Diffeomorphisms of odd-dimensional discs

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1. Concordances relate diffeos of odd- and even-dim'l discs.

$$BDiff_\partial(D^{d+1}) \longrightarrow B\underbrace{C(D^d)}_{Diff_{D^d}(D^{d+1})} \longrightarrow BDiff_\partial(D^d)$$

Hatcher: $\boxed{D^1 \times - : BC(D^d) \to BC(D^{d+1})}$ $\quad D^1 \times D^{d+1} \cong D^{d+2}$

Waldhausen: $\underset{d\to\infty}{\mathrm{hocolim}}\, BC(D^d) \simeq \Omega\, Wh^{Diff}(*) \underset{\mathbb{Q}}{\simeq} \Omega K(\mathbb{Z})$

Borel: $\pi_*(\Omega K(\mathbb{Z})) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = 4, 8, 12, \dots \\ 0 & \text{else} \end{cases}$

Igusa: Hatcher's map is $\sim \frac{d}{3}$-connected.

$$\Rightarrow \pi_* BC(D^d) \otimes \mathbb{Q} \simeq \begin{cases} \mathbb{Q} & * = 4, 8, 12, \dots \\ 0 & \text{else} \end{cases}$$

in degrees $* \lesssim \frac{d}{3}$ (Krannich '20 $\leq d$)

In the LES for the concordance fibration, these classes contribute to BDiff of the odd-dim'l disc (Farrell-Hsiang)

2. Watanabe

i) $\pi_{2n-2} BDiff_\partial(D^{2n+1}) \otimes \mathbb{Q}$ contains a class (different from the K-theory classes) for "many" $n \geq 2$.

ii) $\pi_{r(2n-2)} BDiff_\partial(D^{2n+1}) \otimes \mathbb{Q} \twoheadrightarrow A_r$, $r \geq 2$

$A_r = (\text{trivalent graphs with } 2r \text{ vertices}) / (IHX, \text{signs})$

has dimension

$r = 1 \quad 2 \quad 3 \quad 4 \quad \ldots$

$1, 1, 1, 2, 2, 3, 4, \ldots$

3. Weiss: "often" $p_n \neq e^2 \in H^{4n}(BTop(2n); \mathbb{Q})$ and $p_{n+i} \neq 0$.

As Pontryagin classes are stable, implies $p_{n+i} \neq 0 \in H^{4n}(BTop(2n+1); \mathbb{Q})$

$\Rightarrow$ produces classes $\pi_{2n-2+4i} BDiff_\partial(D^{2n+1}) \otimes \mathbb{Q}$, $i \geq 0$.

4. Theorem [Krannich-R-W]: For $2n \geq 6$ and in degrees $* < 3n-7$,

$$\pi_* BDiff_\partial(D^{2n+1}) \otimes \mathbb{Q} = \underbrace{\begin{cases} \mathbb{Q}^+ & * = 4, 8, \ldots \\ 0 & \text{else} \end{cases}}_{\text{K-theory}} \oplus \underbrace{\begin{cases} \mathbb{Q}^- & * = 2n-2, 2n+2, 2n+6, \ldots \\ 0 & \text{else} \end{cases}}_{\text{Weiss-Pontryagin}}$$

$$\underbrace{\oplus\ \mathbb{Q}^- \text{ in degree } 2n-2 \ \oplus\ \mathbb{Q}^- \text{ in degree } 2n-1}_{\text{new}}$$

5. Watanabe's classes of type (i) always exist, and agree with the lowest Weiss-Pontryagin class.

$$B\mathrm{hAut}^+(S^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n)$$

and the composition

$$B\mathrm{hAut}^+(S^{2n-1}) \xrightarrow{\ \Sigma\ } B\mathrm{hAut}^+(S^{2n}) \simeq K(\mathbb{Q}, 4n)$$

is e^2: call the generator $\widetilde{e^2}$.

$$\begin{tikzcd}
\underset{\widetilde{e^2} = p_n}{BSO(2n+1)} \arrow[r] & \overset{\widetilde{e^2}}{BSTop(2n+1)} \arrow[r] \arrow[d] & \overset{\widetilde{e^2}}{B\mathrm{hAut}^+(S^{2n})} \\
& BSTop &
\end{tikzcd}$$

$p_n \mapsfrom p_n$

$p_n \neq \widetilde{e^2}$ on $BSTop(2n+1)$ $\Rightarrow$ $\tau(p_n - \widetilde{e^2}) \in H^{4n-1}\left(\frac{STop(2n+1)}{SO(2n+1)}\right)$

Weiss.

$\Rightarrow$ $\Omega^{2n+1}(\tau(p_n - \widetilde{e^2})) \in H^{2n-2}\left(\Omega^{2n+1}\left(\frac{STop(2n+1)}{SO(2n+1)}\right)\right) \simeq B\mathrm{Diff}_\partial(D^{2n+1})$

$\neq 0$.

<u>Fact</u>: this coh class is $-4\xi_2$, ξ_2 = Kontsevich class for $\ominus$.

6. Rationalised embedding calculus gives a prediction

$$\pi_* BT_\infty^{\mathbb{Q}}\mathrm{Diff}_\partial(D^{2n+1}) = \begin{cases} \mathbb{Q} & * = 2n-5,\ 2n-9, \dots \\ 0 & \text{else} \end{cases} \quad (* \lesssim 3n)$$

(Fresse–Turchin–Willwacher) so the map

$$B\mathrm{Diff}_\partial(D^{2n+1}) \to BT_\infty^{\mathbb{Q}}\mathrm{Diff}_\partial(D^{2n+1})$$

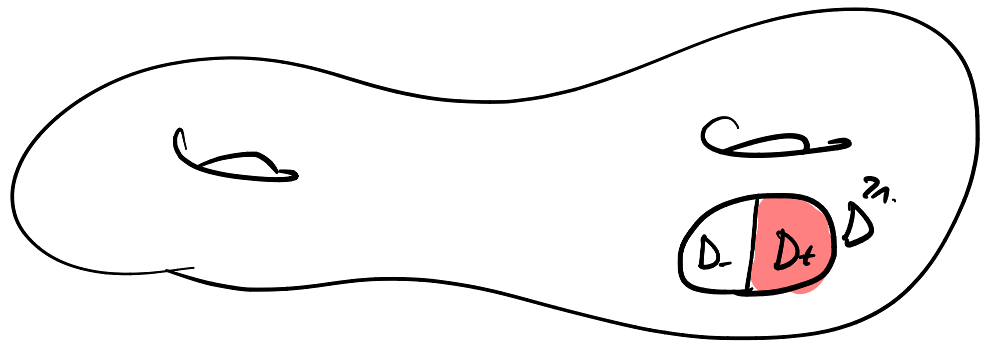
is zero on $\pi_*(-)\otimes\mathbb{Q}$ in this range.

[Also, the homotopy of this cannot be described just in terms of K + L-theory]

7. Don't look at $BDiff_\partial(D^{2n+1})$ directly, but at $BC(D^{2n})$ instead.

Use handlebodies

$$V_g = \natural_g S^n \times D^{n+1}$$



Weiss fibre sequence

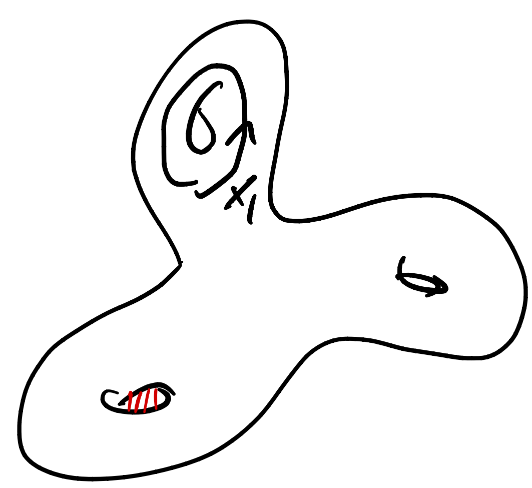
$$BC(D^{2n}_-) \longrightarrow \underbrace{BDiff^{fr}_{D^{2n}_+}(V_g)}_{\text{Accessible on } H^* \text{ by Botvinnik–Perlmutter. } \simeq_{\mathbb{Q}} *} \longrightarrow \underbrace{BEmb^{fr}_{D_+}(V_g, \underbrace{\partial V_g \setminus D_-}_{\text{fixed setwise}})}_{\text{Accessible using mult. disjunction } + \cdots}$$

D_+: fixed pointwise

8. Watanabe has a "Borromean" family of diffeos:

travel with gills along handle.

$$S^{n-2} \longrightarrow Diff_\partial(V_3)$$
$$\Updownarrow$$
$$S^{n-1} \longrightarrow BDiff^{fr}_\partial(V_3)$$



V_3

This is a construction.

1. ~~It is easy to~~ calculate $\pi_{n-1} BhAut_\partial(V_g) \otimes \mathbb{Q} \cong \begin{cases} \Lambda^3 H_n(V_g) & n \text{ odd} \\ Sym^3 H_n(V_g) & n \text{ even} \end{cases}$

One can.

2. Proposition: there is a unique

$$\alpha \in \pi_{n-1} BDiff^{fr}_\partial(V_3) \otimes \mathbb{Q} \quad \text{s.t.}$$

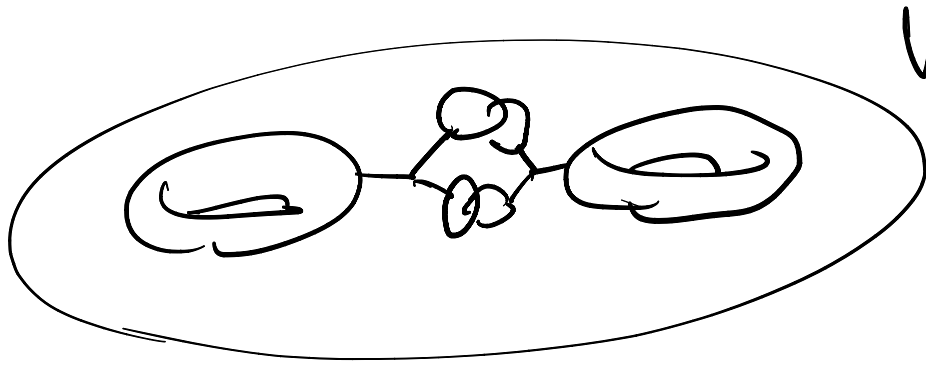
i) in $\pi_{n-1} BhAut_\partial(V_3)$ it maps to $[x_1 \otimes x_2 \otimes x_3]$ — $x_i \in H_n(V_3)$ std basis.

ii) α is null when extended along $V_3 \hookrightarrow V_2$ by filling any of the 3 handles.

[Up to a scalar, $\alpha = \alpha_{\text{Watanabe}}$].

__Note__: α is symmetric in the 3 handles; α_{Watanabe} is not obviously.

9. $V_2 \longleftarrow V_3 \sqcup V_3$.



$$\begin{tikzcd} S^{n-1} \vee S^{n-1} \arrow[r] & S^{n-1} \times S^{n-1} \arrow[r] \arrow[d] & BDiff_\partial^{fr}(V_2) \\ & S^{2n-2} \arrow[ur, "\alpha^{(2)}"'] & \end{tikzcd}$$

Using $V_2 \xhookrightarrow{e} V_g$ this gives a map

$$\left\{ \begin{array}{ll} Sym^2 H_n(V_g) & n \text{ odd} \\ \Lambda^2 H_n(V_g) & n \text{ ev.} \end{array} \right. \longrightarrow \pi_{2n-2} BDiff_\partial^{fr}(V_g).$$

as representations of $MCG(V_g)$