Embedding calculus in codimension zero

(ongoing work with Alexander Kupers)

All manifolds are compact and smooth. (except $\mathbb{R}^d$)

Fix a closed manifold P^{d-1}.

The category Man_P has objects: d-mfds M with $\partial M \cong P$

morphisms: $\mathrm{Diff}_\partial(M,N) = \mathrm{Emb}_\partial(M,N)$ (with topology)

Many questions in geom. topology are questions about $BMan_P$

classifying space

Ex: $\pi_0(BMan_P) = \{(M, \partial M \cong P)\}/_{\text{diffeo}}$

- $\Omega_M BMan_P \simeq \mathrm{Diff}_\partial(M)$
- $\mathrm{Emb}(W^d, M^d) \simeq \mathrm{hofib}_M \left(BMan_{\partial W \amalg \partial M} \xrightarrow{(-)\cup_{\partial W} W} BMan_{\partial M} \right)$

1 The classical approach to $BMan_P$

Slogan: "compare M to its homotopy type $(+\varepsilon)$"

For simplicity: $P = \emptyset$. Write $Man_P = Man_d$.

$$BMan_d \longrightarrow BPoinc_d$$

obj: d-dim simple Poincaré complexes

morphisms: $hAut^s(X,Y)$ (with topology)

$BPoinc_d$ potentially accessible by unstable htp theory.

↳ study difference:

Def: For $X \in Poinc_d$, define <u>simple structure space:</u>

$$S^s(M) = \mathrm{hofib}_X\left(BMan_d \longrightarrow BPoinc_d\right)$$

$$\rotatebox{90}{$\subseteq$}$$

$$\coprod_{\{(M^d,\, M^d \xrightarrow{\simeq_s} X)\} \Big/ \begin{tikzcd} M \arrow[r, "\simeq_s"] \arrow[d, "\exists \cong"'] \arrow[dr, phantom, "\circlearrowleft"] & X \\ M' \arrow[ur, "\simeq_s"'] & \end{tikzcd}} \left(hAut^s(M) \big/ Diff(M) \right)_{id}$$

path components of $S^s(M)$: <u>simple structure set.</u>

To access $S^s(M)$, one considers the following factorisation

$$BMan_d \longrightarrow \underbrace{B\widetilde{Man}_d}_{\substack{\text{objects: } M^d \text{ mfd} \\ \text{morphisms: } \widetilde{Diff}(M,N)}} \longrightarrow BPoinc_d$$

simplified version of diffeomorphism group

$$\begin{tikzcd}
\mathrm{hofib}_M(BMan_d \to B\widetilde{Man}_d) \arrow[d, phantom, "\rotatebox{90}{$=:$}"] & \mathrm{hofib}_X(BMan_d \to BPoinc_d) \arrow[d, phantom, "\rotatebox{90}{$=:$}"] & \mathrm{hofib}_X(B\widetilde{Man}_d \to BPoinc_d) \arrow[d, phantom, "\rotatebox{90}{$=:$}"] \\
S^{\widetilde{Man}/Man}(M) \arrow[r] \arrow[d, "(*)"] & S^s(X) \arrow[r] & \widetilde{S}^s(X) \ni (M, M \xrightarrow{\simeq} X) \arrow[d] \\
\Omega^{\infty+1} Wh^{Diff}(M) /\!/ \mathbb{Z}/2 & & \mathcal{N}(X) \simeq Map(M, G/O) \arrow[d] \\
 & & \Omega^{\infty+d} \mathbb{L}^s(\mathbb{Z}[\pi_1 X])
\end{tikzcd}$$

connectivity of this map increases with d

fibre sequence

fibre sequence

K-theoretical accessible by other means

accessible ∞-loop spaces

<u>pseudo isotopy theory</u>

(Hatcher, Igusa, Waldhausen, Weiss-Williams)

<u>Surgery theory</u>

(Browder, Novikov, Sullivan, Wall)

Downside: Limited by the connectivity of d, the so-called pseudo-isotopy stable range

② Alternative approach? "compare M to htp. type of all its configuration spaces + maps between them".

$$B(\mathrm{Man}_d) \xrightarrow{E_{(-)}} B(\underset{\text{inverts weak equivalences}}{\mathcal{D}}(\mathrm{PSh}(\underbrace{\mathrm{Disc}_d}_{}))))^{\times}$$

homotopy invertible maps

$M \longmapsto E_M := \mathrm{Emb}(-, M)$

objects: $\coprod^k \mathbb{R}^d$ $k \geq 0$

morphism: codim 0 embeddings

objects: $\mathrm{Disc}_d^{op} \longrightarrow \mathrm{Top}$

morphisms: nat. trafos.

Note: $\mathrm{Emb}(\coprod^k \mathbb{R}^d, M) \simeq F_k(M)$ + framing data

ordered config space

$\hookrightarrow E_M$ records config. spaces + extra data..

On morphisms:

$$\begin{tikzcd}
\mathrm{Emb}(M^d, N^d) \arrow[r, "E_{(-)}"] \arrow[dr, "ev"'] & \mathrm{Map}^h(E_M, E_N)^{\times} \\
& T_\infty \mathrm{Emb}(M^d, N^d)^{\times} \arrow[u, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$\mathrm{Emb}(M^d,N^d)$ " $\mathrm{Diff}(M,N)$

$\cong$ Boavida de Brito - Weiss

$\hookrightarrow$ recovers embedding calculus in codim 0

Want to understand difference between $B\mathrm{Man}_d$ and $B\mathcal{D}(\mathrm{PSh}(\mathrm{Disc}_d))^{\times}$:

Def: For $X \in PSh(Disc_d)$, define the <u>Disc-structure space</u> of X as

$$S^{Disc}(X) = \mathrm{hofib}_X\left(BMan_d \longrightarrow BD(PSh(Disc_d))^\times\right)$$

$$\simeq \left(\coprod_{\{(M^d, E_M \xrightarrow{\simeq} X)\}} \mathrm{Aut}^h(E_M) \big/ \mathrm{Diff}(M) \right)_{id}$$

$$\Big/ \quad \begin{tikzcd} E_M \arrow[dr, "\simeq"] \arrow[d, "\Phi_*"'] & \\ E_N \arrow[r, "\simeq"'] & X \end{tikzcd} \quad \exists\, \Phi: M \cong N, \ \circlearrowright_h$$

path components of $S^{Disc}(X)$: <u>Disc-structure set</u>

Rem:

- $\exists$ version with boundary $S_\partial^{Disc}(X)$
- For $M \in Man_d$, write $S_\partial^{Disc}(M) := S_\partial^{Disc}(E_M) \ni (M, id_{E_M})$ based space
- $S_\partial^{Disc}(M)$ is functorial in codim 0 embeddings (when suitably modeled).

Task: Study $S^{Disc}(X)$.

Why?

- $D(PSh(Disc_d))$
  a) more amenable to htp. th. methods
  b) has filtration induced by bounding cardinalities of discs
  c) closely related to E_d-operad
  $\Rightarrow$ interesting in its own right.
- $S_\partial^{Disc}(X)$ might have a good description (not just in a range, more later)

- $\Omega_{id} S_\partial^{Disc}(M) \simeq \mathrm{hofib}\left(\mathrm{Emb}(M,M) \longrightarrow T_\infty \mathrm{Emb}(M,M)\right)$
  (captures quality of self-embedding calculus)
- $\mathrm{Man}_d \longrightarrow \mathcal{D}(\mathrm{PSh}(\mathrm{Disc}_d))$ is universal factorization homology invariant

③ Three properties of $S_\partial^{Disc}(M)$

Thm A (K.-Kupers) Let M^d and N^d compact d-mfds with $d \geq 5$. If M^d and N^d have the same 2-tangential type, then

$$S_\partial^{Disc}(M) \simeq S_\partial^{Disc}(N).$$

Rem: Extends (and relies on) work of Knudsen-Kupers.

- Relies on convergence of embedding calculus in codim ≥ 3 by Goodwillie-Klein.
- Part of the proof heavily inspired by literature on the space of psc-metrics on a manifold (especially work of Ebert-Wiemeler).

Def: M^d and N^d have the same tang. 2-type if

$\exists$

$$\begin{tikzcd} M \arrow[r, "\ell_M"] \arrow[rr, bend right, "T^s M"'] & B \arrow[r, "\Theta"] & BO \end{tikzcd}$$

for some map $\Theta : B \longrightarrow BO$

$$\begin{tikzcd} N \arrow[r, "\ell_N"] \arrow[rr, bend right, "T^s N"'] & B \arrow[r, "\Theta"] & BO \end{tikzcd}$$

such that ℓ_M and ℓ_N are 2-connected.

Ex: M^d and N^d are spin, then they have the same tang. 2-type iff $\pi_1 M \cong \pi_1 N$. (consider $B = B\mathrm{Spin} \times B\pi_1 M$)

$\underset{\text{Thm A}}{\Longrightarrow} S_\partial^{Disc}(M)$ only depends on d and $\pi_1 M$ if M is spin

Thm B (K.-Kupers) $S_\partial^{Disc}(M)_{id}$ is an ∞-loop space $\underbrace{\text{if } d \geq 10}_{}$.

often not necessary e.g. when $Wh(\pi_1 M) = 0$

not optimal, e.g. $d \geq 6$ if M is 1-cfd. and spin

Thm C (K.-Kupers) $S_\partial^{Disc}(M) \not\simeq *$ for $d \geq 6$ and M spin.

Rem: • $\Rightarrow$ self-emb. calc. does not convergence for any high-dim. spin manifold

• $S_\partial^{Disc}(D^3) \not\simeq *$ (KK)

• $S_\partial^{Disc}(M) \simeq *$ if $d \leq 2$ (KK)

• $\pi_0(S^{Disc}(M^4)) = \{\text{smooth structures on } M\}/_{\text{diffeo}}$ if M^4 is 1-cfd. (Knudsen-Kupers)

Cor: $\mathrm{hofib}_{id}(\mathrm{Emb}_\partial(M^d, M^d) \to T_\infty \mathrm{Emb}_\partial(M, M))$ is a nontrivial ∞-loopspace that only depends on d and $\pi_1 M$ if M is a compact spin-manifold of dimension $\underbrace{d \geq 10}_{}$.

$d \geq 6$ if $\pi_1 M = 0$.

④ On the proof of Thm C

M^d spin $d \geq 6$

Lemma: It suffices to prove this for D^d.

__Pf:__ Let M^d be a spin mfd and $d \geq 6$.

Choose a codimension 0-submanifold

$N \subseteq D^d$ with $\pi_1 N \cong \pi_1 M$

M spin $\Rightarrow$ N and M have the same tang. 2-type

$\underset{\text{Thm A}}{\Rightarrow} S_\partial^{\mathrm{Disc}}(M) \simeq S_\partial^{\mathrm{Disc}}(N)$. Choose $D^d \subseteq N$ and consider

$$\begin{tikzcd} S_\partial^{\mathrm{Disc}}(D^d) \arrow[r, "\text{extend}"] \arrow[r, phantom, shift right=4ex, "D^d \subseteq N"] \arrow[rr, bend right=40, "\mathrm{id}"'] \arrow[rr, phantom, bend right=20, "\circlearrowleft_h"] & S_\partial^{\mathrm{Disc}}(N) \arrow[r, "\text{extend}"] & S_\partial^{\mathrm{Disc}}(D^d) \end{tikzcd}$$

$\Rightarrow S_\partial^{\mathrm{Disc}}(D^d)$ is a htp. retract of $S_\partial^{\mathrm{Disc}}(M)$. $\square$

__Thm__ (Boavida de Brito - Weiss)

$$S_\partial^{\mathrm{Disc}}(D^d)_{\mathrm{id}} \simeq \Omega_0^{d+1} \frac{\mathrm{Aut}^h(E_d)}{\mathrm{Top}(d)} \quad \text{for } d \neq 4$$

Here: $\mathrm{Aut}^h(E_d)$ = derived automorphisms of E_d-operad

$\mathrm{Top}(d)$ = homeos of $\mathbb{R}^d$

(Combine this with Thm B to get:

__Cor:__ $\Omega_0^{d+1} \frac{\mathrm{Aut}^h(E_d)}{\mathrm{Top}(d)}$ is an ∞-loop space if $d \geq 6$.)

$\Rightarrow$ Thm C is closely related to:

$\underline{Q}$ (Dwyer-Hess, Weiss, Ayala-Francis-Tanaka) When is the map

$$BTop(d) \xrightarrow{(*)} BAut^{h}(E_d)$$

an equivalence?

Thm D (K.-Krannich) $(*)$ is an equivalence iff $d \leq 2$.

Rem: For $d=3$ and $d \geq 6$ we detect nontrivial htp groups in the fibre in degrees $>d+1$. This implies Thm C.

Strategy for Thm D

$$\underbrace{BTop(d)}_{\text{we know lots about, at least rationally (see previous two talks)}} \longrightarrow BAut^{h}(E_d) \xrightarrow{?} \underbrace{BAut^{h}(E_d^{\mathbb{Q}})}_{\text{has a complete description in terms of a graph complex (Fresse-Turchin-Willwacher)}}$$

rationalised operad

Thm (K.-Krannich) For $d \geq 3$ and $i \geq 2$ we have

i) $\pi_i(BAut^{h}(E_d)) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_i(BAut^{h}(E_d^{\mathbb{Q}}))$

or

ii) $\pi_*(BAut^{h}(E_d))$ is uncountable in degrees i or $i-1$.

Rem: Proof is quite technical.

$BTop(d)$ has countable htp groups (Milnor, Kirby-Siebenmann), so either ii) is the case and $\frac{Aut^h(E_d)}{Top(d)}$ has uncountable htp. groups, or i) is true and then we can play of knowledge about $BTop(d)_{\mathbb{Q}}$ against that about $BAut^h(E_d^{\mathbb{Q}})$.

↪ This leads to a proof of Thms. C and D