

Embedding calculus in codimension zero

(ongoing work with Alexander Kupers)

All manifolds are
compact and smooth,
(except \mathbb{R}^d)

Fix a closed manifold P^{d-1} .

The category Man_P has objects: d -mtds M with $\partial M \cong P$

morphisms: $\text{Diff}_2(M, N) = \text{Emb}_2(X, Y)$
(with topology)

Many questions in geom. topology are questions about $B\text{Man}_P$

$$\underline{\text{Ex.}}: \pi_0(B\text{Man}_P) = \{ (M, \partial M \cong P) \} / \text{diffeo}$$

↑
classifying space

$$\cdot \Omega_M B\text{Man}_P \cong \text{Diff}_2(M)$$

$$\cdot \text{Emb}(W^d, M^d) \in \text{hofib}_M \left(\begin{array}{c} B\text{Man}_{\partial W \sqcup \partial M} \\ \xrightarrow{(-) \cup_{\partial W} W} \\ B\text{Man}_{\partial M} \end{array} \right)$$

① The classical approach to $B\text{Man}_P$

Slogan: "compare M to its homotopy type ($+\epsilon$)"

For simplicity: $P = \emptyset$, write $\text{Man}_P = \text{Man}_d$.

$$B\text{Man}_d \longrightarrow B \text{Poinc}_d$$

obj: d -dim simple Poincaré complexes

morphisms: $\text{hAut}^s(X, Y)$ (with topology)

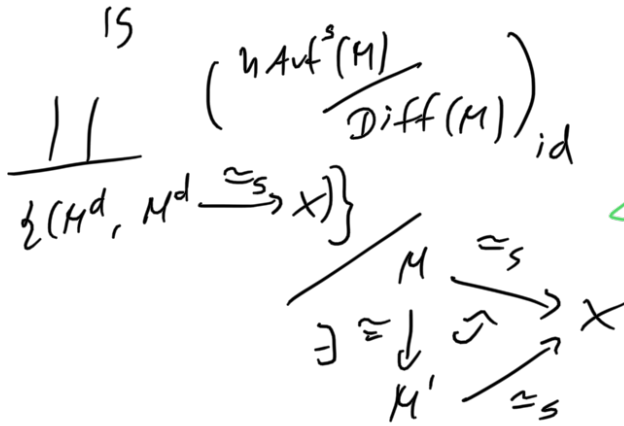
$B\text{Poinc}_d$ potentially accessible by unstable htp theory.

↳ study difference:

Def: For $X \in \text{Pointed}$, define

simple structure space:

$$S^S(M) = \text{hofib}_X (B\text{Mand} \rightarrow B\text{Pointed})$$



← path components of $S^S(M)$:
simple structure set.

To access $S^S(M)$, one considers the following factorisation

$$B\text{Mand} \rightarrow \widetilde{B\text{Mand}} \rightarrow B\text{Pointed}$$

objects: M^d mfd
morphisms: $\widetilde{\text{Diff}}(M, N)$
simplified version of diffeomorphism group

$$\begin{array}{ccccc}
 \text{hofib}(B\text{Mand} \rightarrow \widetilde{B\text{Mand}}) & & \text{hofib}_X(B\text{Mand} \rightarrow B\text{Pointed}) & & \text{hofib}_X(\widetilde{B\text{Mand}} \rightarrow B\text{Pointed}) \\
 M \parallel & & \parallel & & \parallel \\
 \text{isom} & & \text{isom} & & \text{isom} \\
 S(M) & \xrightarrow{\text{fibre sequence}} & S^S(X) & \xrightarrow{\text{fibre sequence}} & \widetilde{S^S(X)} \cong (M, M \xrightarrow{\cong_S} X)
 \end{array}$$

connectivity of this map increases with d

$$\downarrow (*)$$

$$\Omega^{\infty+1} \text{Wh}^{\text{Diff}(M)}$$

pseudo isotopy theory

(Hatcher, Igusa, Waldhausen, Weiss-Williams)

K -theoretical accessible by other means

fibre sequence

$$\downarrow$$

$$W(X) \approx \text{Map}(M, G/O)$$

$$\downarrow$$

$$\Omega^{\infty+d} L^S(\mathbb{Z}[t, t^{-1}])$$

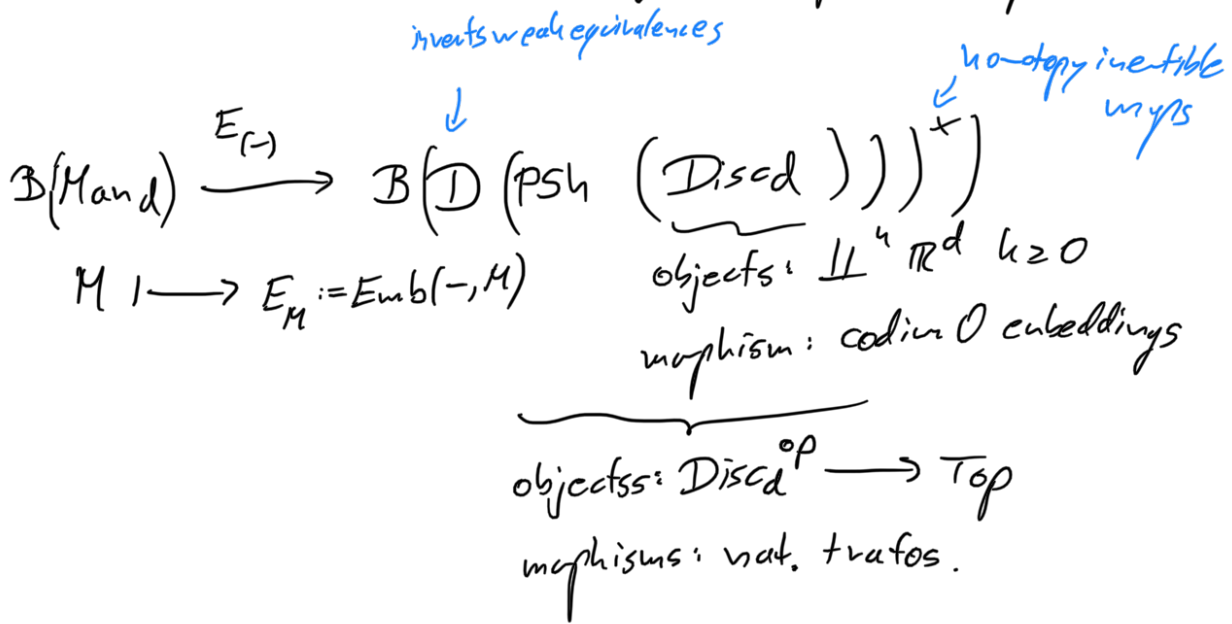
surgery theory

(Brouder, Novikov, Sullivan, Wall)

accessible co-loop spaces

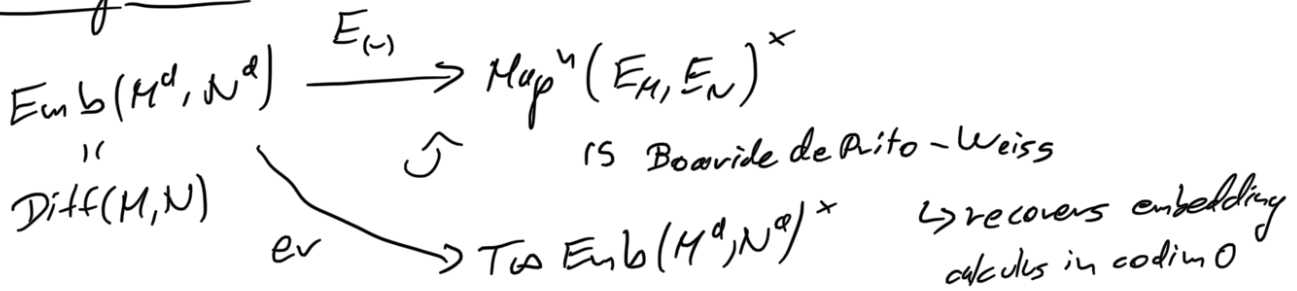
Downside: Limited by the connectivity of d , the so-called pseudo-isotopy stable range

② Alternative approach? "compare M to htp. type of all its configuration spaces + maps between them".



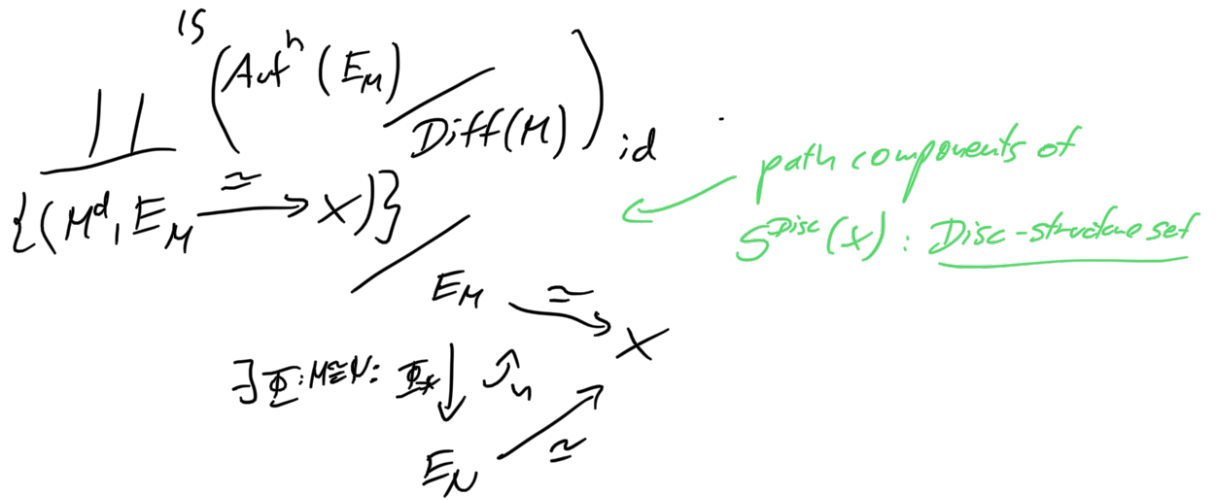
Note: $\text{Emb}(\mathbb{R}^n \times \mathbb{R}^d, M) \simeq \mathcal{F}_n(M) + \text{framing data}$
↑
ordered config space
 $\hookrightarrow E_M$ records config. spaces + extra data..

On morphisms:



Want to understand difference between $\mathcal{B}M \text{ and } d$ and $\mathcal{B}\mathcal{D}(\text{Push}(\text{Disc}^d))^*$:

Def: For $X \in \text{Psh}(\text{Discd})$, define the Disc-structure space of X as $S^{\text{Disc}}(X) = \text{hofib}_X(\mathcal{B}\text{Mand} \rightarrow \mathcal{B}\mathcal{D}(\text{Psh}(\text{Discd}))^*)$



- Rem:
- \exists version with boundary $S^{\text{Disc}}_{\partial}(X)$
 - For $M \in \text{Mand}$, write $S^{\text{Disc}}_{\partial}(M) := S^{\text{Disc}}_{\partial}(E_M)$
 - \cup
 - (M, id_{E_M}) based space
 - $S^{\text{Disc}}_{\partial}(M)$ is ^{functorial} V in codim 0 embeddings (when suitably modeled).

Task: Study $S^{\text{Disc}}(X)$.

- Why?
- $\mathcal{D}(\text{Psh}(\text{Discd}))$
 - a) more amenable to htp.th. methods
 - b) has filtration induced by bounding centralities of discs
 - c) closely related to E_d -operad \Rightarrow interesting in its own right.
 - $S^{\text{Disc}}_{\partial}(X)$ might have a good description (not just in a range, more later)

- $\Omega_{id} S_{\partial}^{Disc}(M) \cong \text{hofib}(\text{Emb}(M, M) \rightarrow \text{TopEmb}(M, M))$
(captures quality of self-embedding calculus)
- $\text{Man}_d \rightarrow \mathcal{D}(\text{PSb}(\text{Disc}))$ is universal factorization homology invariant

③ Three properties of $S_{\partial}^{Disc}(M)$

Thm A (U.-Myers) Let M^d and N^d compact d -mfds with $d \geq 5$.

If M^d and N^d have the same 2-tangential type, then

$$S_{\partial}^{Disc}(M) \cong S_{\partial}^{Disc}(N).$$

Rem: Extends (and relies on) work of Madsen-Myers.

- Relies on convergence of embedding calculus in codim ≥ 3 by Goodwillie-Klein.
- Part of the proof heavily inspired by literature on the space of psc-metrics on a manifold (especially work of Ebert-Wiemeler).

Def: M^d and N^d have the same tang. 2-type if

$$\exists \begin{array}{ccc} M & \xrightarrow{e_M} & B \xrightarrow{\Theta} BO \\ & \searrow \tau_M & \nearrow \\ N & \xrightarrow{e_N} & B \xrightarrow{\Theta} BO \end{array} \text{ for some map } \Theta: B \rightarrow BO$$

such that e_M and e_N are 2-connected.

Ex: M^d and N^d are spin, then they have the same tang. 2-type iff $\pi_1 M \cong \pi_1 N$. (consider $B = B\text{Spin} \times B\pi_1 M$)

$\Rightarrow S_{\partial}^{\text{Disc}}(M)$ only depends on d and $\pi_2 M$ if M is spin
Thm A

Thm B (U. Kupus) $S_{\partial}^{\text{Disc}}(M)$ is an ∞ -loop space if $d \geq 10$.
 often not necessary
 e.g. when $W(\pi_2 M) = 0$
 not optimal, e.g.
 $d \geq 6$ if M is 1-ctd.
 and spin

Thm C (U. Kupus) $S_{\partial}^{\text{Disc}}(M) \neq *$ for $d \geq 6$ and M spin.

Rem: \Rightarrow self-emb. calc. does not convergence for any high-dim.
 spin manifold

• $S_{\partial}^{\text{Disc}}(\mathbb{D}^3) \neq *$ (UK)

• $S_{\partial}^{\text{Disc}}(M) \neq *$ if $d \leq 2$ (UK)

• $\pi_0(S_{\partial}^{\text{Disc}}(M)) = \left\{ \begin{array}{l} \text{smooth structures} \\ \text{on } M \end{array} \right\} / \text{diffeo}$ if M^4 is 1-ctd.
 (U. Nielsen - Kupus)

Cor: $\text{hofib}_{\text{id}}(\text{Emb}_{\partial}^d(M, M) \rightarrow \text{TopEmb}_{\partial}^d(M, M))$ is a nontrivial
 ∞ -loop space that only depends on d and $\pi_2 M$ if M is
 a compact spin-manifold of dimension $d \geq 10$.
 $d \geq 6$ if $\pi_2 M = 0$.

(4) On the proof of Thm C

Lemma: It suffices to prove this for \mathbb{D}^d .

M^d spin
 $d \geq 6$

Pf: Let M^d be a spin mfd and $d \geq 6$.

Choose a codimension 0-submifold

$$N \subseteq D^d \text{ with } \pi_1 N \cong \pi_1 M$$

M spin $\Rightarrow N$ and M have the same tang. 2-type

$\Rightarrow S_2^{\text{Disc}}(M) \cong S_2^{\text{Disc}}(N)$. Choose $D^d \in N$ and consider

$$S_2^{\text{Disc}}(D^d) \xrightarrow{\text{extend}} S_2^{\text{Disc}}(N) \xrightarrow{\text{extend}} S_2^{\text{Disc}}(D^d)$$



$\Rightarrow S_2^{\text{Disc}}(D^d)$ is a htp. retract of $S_2^{\text{Disc}}(M)$. \square

Thm (Bourbaki de Brito - Weiss)

$$S_2^{\text{Disc}}(D^d) \xrightarrow{\text{id}} \Omega_0^{d+1} \frac{\text{Aut}^h(E_d)}{\text{Top}(d)} \text{ for } d \neq 4$$

Here: $\text{Aut}^h(E_d) =$ derived automorphisms of E_d -operad
 $\text{Top}(d) =$ homeos of \mathbb{R}^d

(Combine this with Thm B to get:

Cor: $\Omega_0^{d+1} \frac{\text{Aut}^h(E_d)}{\text{Top}(d)}$ is an ω -loop space if $d \geq 6$.)

\Rightarrow Thm C is closely related to:

Q (Dwyer-Hess
Weiss
Ayala-Francis-Tanaka) When is the map
 $B\text{Top}(d) \xrightarrow{(*)} B\text{Aut}^n(E_d)$
 an equivalence?

Thm D (U. Nupers) $(*)$ is an equivalence iff $d \leq 2$.

Rem: For $d=3$ and $d \geq 6$ we detect nontrivial ltp groups in the fibre in degrees $> d-1$. This implies Thm C.

Strategy for Thm D

$B\text{Top}(d) \rightarrow B\text{Aut}^n(E_d) \xrightarrow{?} B\text{Aut}^n(E_d^{\mathbb{Q}})$

rationalised operad
↓

we know lots about, at least rationally (see previous two talks)

has a complete description in terms of a graph complex (Fresse-Turchin-Wilkner)

Thm (U. Nupers) For $d \geq 3$ and $i \geq 2$ we have

$$2) \pi_i(B\text{Aut}^n(E_d)) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_i(B\text{Aut}^n(E_d^{\mathbb{Q}}))$$

or

ii) $\pi_*(B\text{Aut}^n(E_d))$ is countable in degrees i or $i-1$.

Rem: Proof is quite technical.

$B\text{Top}(d)$ has countable htp groups (Milnor, Kirby-Siebenmann), so
either i) is the case and $\frac{\text{Aut}^h(E_d)}{\text{Top}(d)}$ has uncountable htp. groups,
or ii) is true and then we can play off knowledge about
 $B\text{Top}(d)$ against that about $B\text{Aut}^h(E_d)$.

↳ This leads to a proof of Thms. C and D