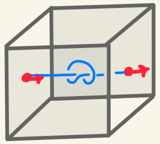
11.11.2020 Danica

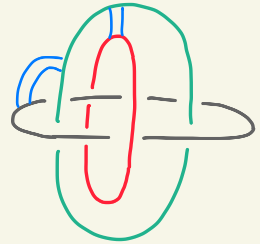
$d=3$



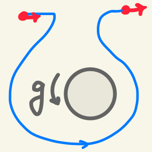
$\mathrm{Emb}_\partial(I, I^3)$ $\begin{cases} \pi_0 \mathrm{Emb}_\partial(I, I^3) \cong \pi_0 \mathrm{Emb}_\partial(S^1, S^3) \\ \text{but can stack} \rightsquigarrow \text{conn. sum on space level.} \\ \text{but } \mathrm{Emb}_\partial(I, I^3)_u \simeq * \ (\text{here it's } S^3 \times S^2) \end{cases}$

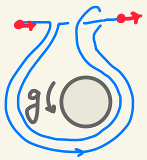
$\mathrm{Emb}_\partial(S^{2n-1}, S^{3n})$

$\pi_0 = \begin{matrix} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2 & n \text{ odd} > 1 \end{matrix}$



... π_0 knotting in smooth cat, none in PL / TOP.





$g \in \pi_1 M$

$\mathrm{Emb}_\partial(I, M)$

$\mathrm{Emb}_\partial(I, Y \setminus D^3) \simeq \mathrm{Emb}_{\vec{*}}(S^1, Y)$

Away from π_0: for $d \geq 4$

$\pi_i \mathrm{Emb}_\partial(I, M) \cong \pi_i \mathrm{Imm}_\partial(I, M)$

for $0 \leq i \leq d-4$ and (see the next slide)

$\frac{\mathbb{Z}[\pi_1 M]}{\mathbb{Z} \oplus \delta(\pi_{d-1} M)} \hookrightarrow \pi_{d-3} \mathrm{Emb}_\partial(I, M) \twoheadrightarrow \pi_{d-3} \mathrm{Imm}_\partial(I, M)$

$\boxed{\mathrm{Emb}_\partial(X, M)}$

X, M smooth compact

dim: k d

with $u^\varepsilon: \partial X \times [0, \varepsilon] \hookrightarrow M$

$\partial X \times \{0\} \overset{u}{\hookrightarrow} \partial M$

and any $K: X \hookrightarrow M$

$K|_{\partial X \times [0,\varepsilon]} = u^\varepsilon$

$\mathrm{Emb}_\partial(D^2, M)$

$\mathrm{Emb}_\partial(\Sigma_g, M)$

Given $h \in H_2(M; \mathbb{Z})$
what is minimal genus
realized by embedded surface?

recent examples of smooth $\Sigma_g \hookrightarrow D^4$
top. isotopic
but smoothly NOT

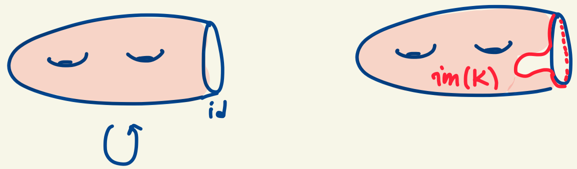
for $g \geq 2$
$g = 0$
$g = 1$

$\mathrm{Diff}_\partial M = \mathrm{Emb}_\partial(M, M)$

exciting $\mathrm{Diff}_\partial D^d$

Weiss fibre sequence

$\mathrm{Diff}_\partial D^d \longrightarrow \mathrm{Diff}_\partial M \longrightarrow \mathrm{Emb}_{\partial/2}(M, M)$



Why care about $\pi_{i>0}$?

For example: joint work w/ P. Teichner

THM. [basically known to Cerf]

$$\mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\, \mathrm{Emb}_\partial^{\uparrow}(I, M^+)$$

$M^+ = M \cup_{G \times D^2} h^3$

$u: \partial \mathbb{D}^2 \hookrightarrow \partial M$ has geometric dual $G: \mathbb{S}^2 \hookrightarrow \partial M$, $u \pitchfork s = \{pt\}$

dim = 4

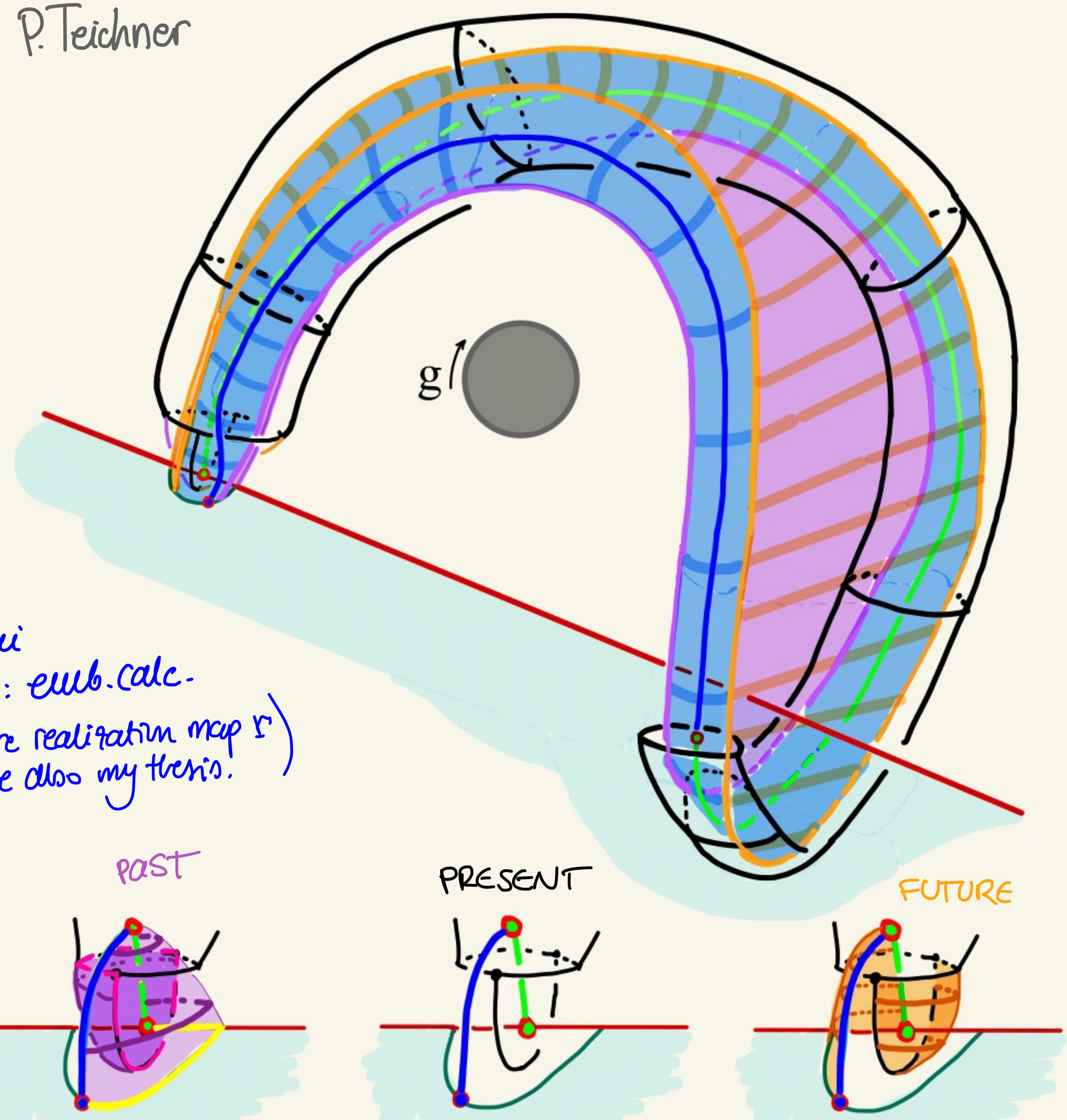
THM.

$$\mathbb{Z}[\pi_1 M] / \delta(\pi_3 M) \xrightarrow{r} \pi_1 \mathrm{Emb}_\partial^{\uparrow}(I, M^+) \to \pi_2 M^+$$

given on $g \in \pi_1 M$ by the picture

Dax
Gabai
also: emb. calc.
(for the realization map r see also my thesis.)

Cor. $\pi_0 \mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong$ (and iso explicit)



Thm. $\ker(\pi_0 \mathrm{Emb}_\partial(\mathbb{D}^2, M) \to \pi_2 M) \cong \dfrac{\mathbb{Z}[\pi_1 M]^{\mathbb{Z}/2}}{\delta(\pi_3 M)} \ni \eta = \bar{\eta}$

subset of disks htpic to the given one.

Cor. Light Bulb Theorems of Gabai & Schneiderman–Teichner

$\boxed{\text{embedding calculus}}$

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_n(X,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_2(X,M) \arrow[d] \\
\mathrm{Emb}_\partial(X,M) \arrow[uuur, "ev_n"] \arrow[ur, "ev_2"] \arrow[r, hook, "ev_1"', "\underset{\mathrm{inc}}{\shortparallel}"] & T_1(X,M) = \mathrm{Imm}_\partial(X,M)
\end{tikzcd}
$$

HOPE: ev_n as $n \to \infty$ "become better and better" approximations.

INDEED:

> **THM.** [Goodwillie–Klein '15]
> If $(k,d) := (\dim X, \dim M) \neq (1,3)$ then
> ev_n is $\big(1-k+n(d-k-2)\big)$-connected.

Cor. For $d-k>2$ $\underset{\lim ev_n}{\overset{\shortparallel}{ev_\infty}}$ is a weak equivalence.

Def. $$T_n(X,M) := \operatorname*{holim}_{V \in \mathcal{O}_{\leq n}(X)} \mathrm{Emb}_\partial(V,M)$$

where

$\mathcal{O}_{\leq n}(X)$ is the poset of open subsets in X which are diffeomorphic to $\bigsqcup_{\leq n} D^k$

Observe: $\mathrm{Emb}_\partial(V,M) \simeq \mathrm{Conf}^{fr}_{\leq n}(M)$

MOTTO: $f \in T_n(X,M)$ is the data of consistent choices of configurations in M parametrized by X.

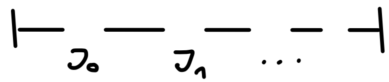
so ev_n will natural: restrictions.

$$\boxed{\mathrm{Emb}_\partial(I,M)}$$

turns out $T_n(I,M) \simeq \operatorname*{holim}_{\emptyset\neq S\subseteq\{0,1,\dots,n\}} \mathrm{Emb}(I\setminus J_S, M)$

where $J_S := \bigsqcup_{i\in S} J_i$

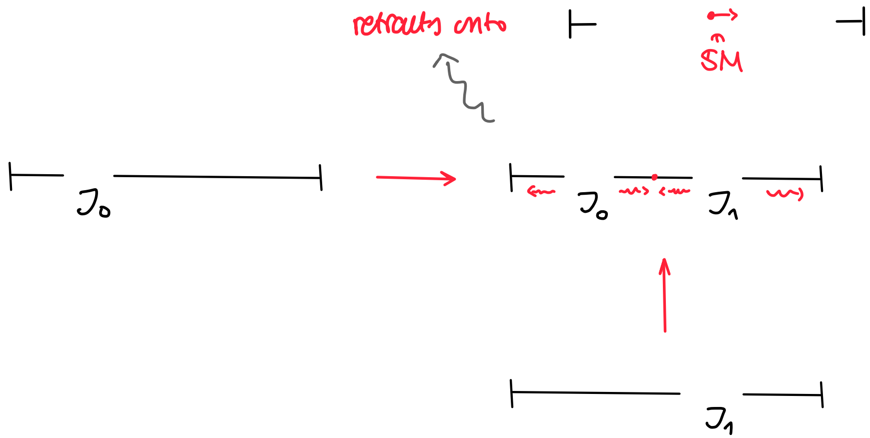
where $J_i \subseteq I$ fixed ascending disjoint



$\boxed{n=0}$ $T_0(I,M) = \mathrm{Emb}(I\setminus J_0, M) \simeq *$

$\boxed{n=1}$

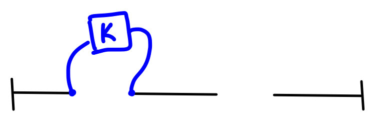
$$\begin{tikzcd}
 & * \arrow[d, phantom, "\cong" rotate=90] & & SM \text{ unit sphere bundle} \arrow[d, phantom, "\cong" rotate=90] \\
 & \mathrm{Emb}_\partial(I\setminus J_1, M) \arrow[rr, "r_1^0"] & & \mathrm{Emb}_\partial(I\setminus J_{01}, M) \\
 & T_1 \arrow[u] \arrow[rr] \arrow[d, phantom, "\cong" rotate=90] & & \mathrm{Emb}_\partial(I\setminus J_0, M) \arrow[u] \arrow[d, phantom, "="] \\
\text{by def of holim (homotopy pullback)} \Longrightarrow & \Omega SM & & T_0 \simeq *
\end{tikzcd}$$



THM [Smale] $\mathrm{Imm}_\partial(I,M) \simeq \Omega SM$.

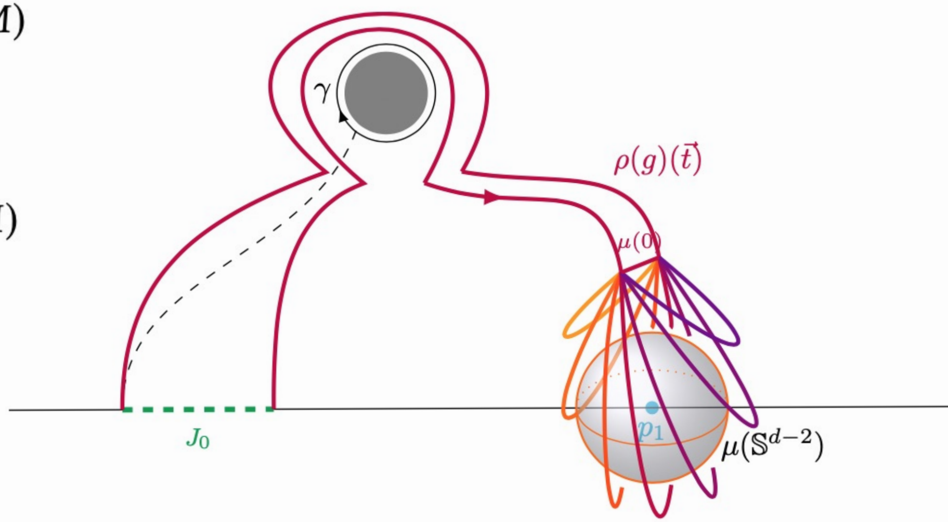
Note: $\operatorname{hofib}_{U|_{I\setminus J_0}}(T_1 \to T_0) \underset{\text{Fact}}{\simeq} \operatorname{hofib}(r_1^0) \underset{\substack{\text{since}\\ r_1^0 \text{ fibration}\\ \text{by Palais-Cerf}}}{\simeq} \operatorname{fib}(r_1^0) := \mathrm{Emb}_\partial(J_0, M\setminus U|_{I\setminus J_{01}})$

$\cong$ $\mathrm{Imm}_\partial(I,M)$



$\boxed{n=3}$

$$\begin{tikzcd}[column sep=small]
 & \underset{\cong\, SM}{\mathcal{E}mb_\partial(I\setminus J_{12},M)} \arrow[rr] & & \mathcal{E}mb_\partial(I\setminus J_{012},M) \simeq \mathrm{Conf}_2'(M) \\
\underset{\cong\, *}{\mathcal{E}mb_\partial(I\setminus J_2,M)} \arrow[ur] \arrow[rr] & & \underset{\cong\, SM}{\mathcal{E}mb_\partial(I\setminus J_{02},M)} \arrow[ur] & \\
 & \underset{\cong\, *}{\mathcal{E}mb_\partial(I\setminus J_1,M)} \arrow[uu] \arrow[rr] & & \underset{\cong\, SM}{\mathcal{E}mb_\partial(I\setminus J_{01},M)} \arrow[uu] \\
 & & \underset{\cong\, *}{\mathcal{E}mb_\partial(I\setminus J_0,M)} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$



$$\begin{tikzcd}
 & F_2 \simeq \Omega^2\Sigma^{d-1}(\Omega M)_+ \arrow[d] \\
\mathcal{E}mb_\partial(I,M) \arrow[r] \arrow[dr, "\text{isomorphism below } d-3"'] & T_2 \arrow[d] \\
 & T_1 \simeq \mathcal{I}mm_\partial(I,M)
\end{tikzcd}$$

isomorphism below $2(d-3)$.
so isomorphism on $d-3$.

Cor.

$$\begin{tikzcd}
\pi_{d-2}(\Omega SM) = \mathbb{Z}\oplus\pi_{d-1}M \arrow[d, "\delta"] \\
\pi_{d-3}F_2 \cong \mathbb{Z}[\pi_1 M] \arrow[d] \\
\pi_{d-3}\mathcal{E}mb_\partial(I,M) \arrow[d] \\
\pi_{d-3}(\Omega SM) \cong \pi_{d-2}M
\end{tikzcd}$$

corresp. generator.

Now use long exact seq. of homotopy gps. $\longrightarrow$