HIGHER HOMOTOPY GROUPS IN LOW DIMENSIONAL TOPOLOGY

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joint with Peter Teichner (MPIM Bonn)

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Talk based on: https://arxiv.org/abs/2105.13032

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Introduction

Spaces of embeddings

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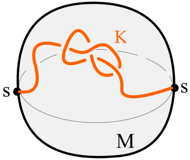

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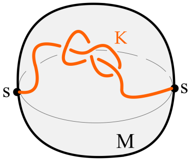

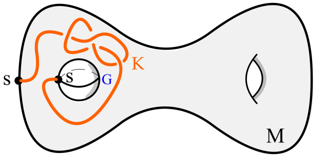

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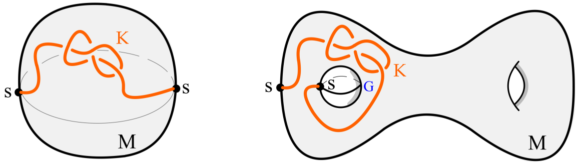

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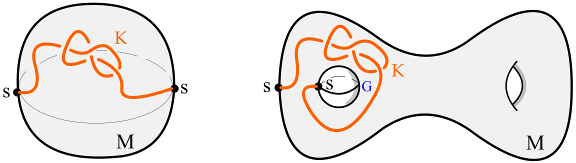

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- More recently, intensively studied is the set of 2-knots $\pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, M)$ for a 4-manifold M. This can be huge – for example, "spinning" a classical knot gives a 2-knot in $\pi_0 \operatorname{Emb}_\partial(\mathbb{S}^2, \mathbb{R}^4) \cong \pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$.

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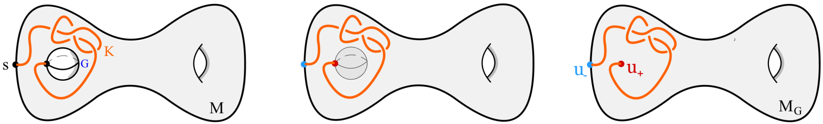
- Although usually only the sets of components are considered, we will see that higher homotopy groups of embedding spaces are also useful.

Space Level Light Bulb Theorem

Theorem [K-Teichner]

In the setting with a dual, if we denote $M_G := M \cup_{\nu G} h^{d-k+1}$, then



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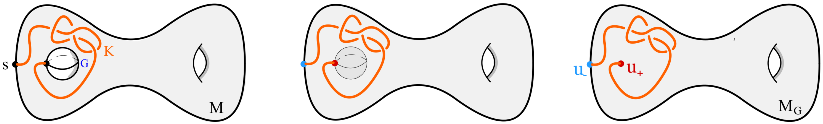

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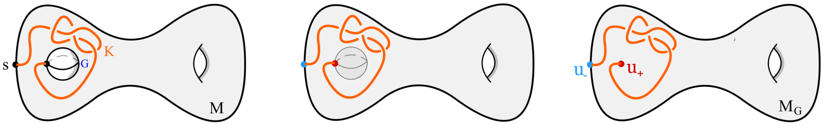
- Supscript ε means each embedded disk is equipped with a “push-off”(...).



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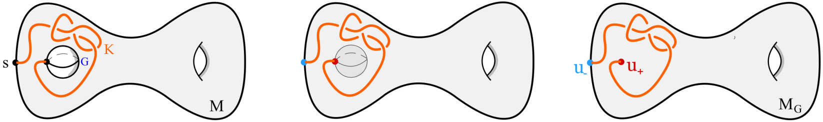

Example: $k = 1, d = 3$

This recovers the classical LBT: isotopy classes of arcs in a 3-manifold M with ends on two components of ∂M, one of which is $\mathbb{S}^2$, are in bijection with $\pi_1(M \cup_G h^3)$. $\implies$ a knot in the chord for a light bulb can be unknotted!

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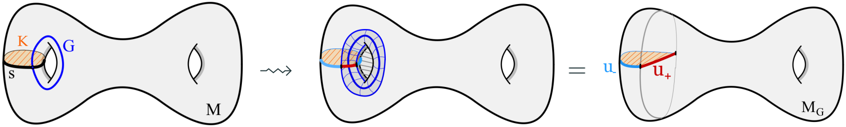
$k = d$: Recovers a theorem (and proof) of Cerf '68:

$$\mathrm{Diff}^+_\partial(\mathbb{D}^d) = \mathrm{Emb}_\partial(\mathbb{D}^d, \mathbb{D}^d) \simeq \Omega\,\mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{D}^d).$$

In particular, $\pi_0\,\mathrm{Diff}^+_\partial(\mathbb{D}^4) \cong \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^3, \mathbb{D}^4); \mathrm{U})$. Open: is this nontrivial?

Cerf’s trick

Cerf’s trick: Proof of Space Level LBT



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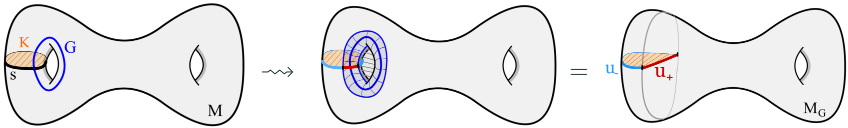

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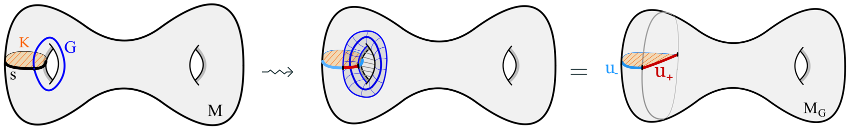

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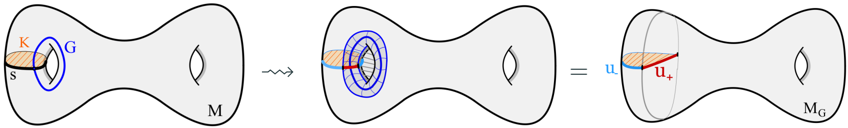

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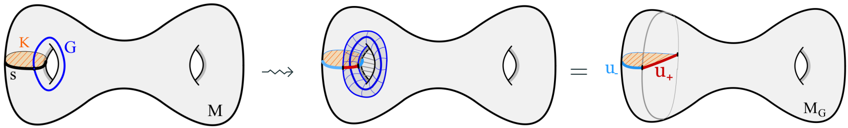

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where: $\mathfrak{amb}_{\mathrm{U}}$ is the connecting map (use the family ambient isotopy theorem to extend loops), $\mathfrak{fol}^\varepsilon_{\mathrm{U}}(K)$ is the loop of ε-augmented $(k-1)$-disks foliating the sphere $-\mathrm{U} \cup K$. $\square$

LBT for 2-disks in 4-manifolds

The 4D setting with a dual

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- a knot $\mathsf{s}\colon \mathbb{S}^1 \hookrightarrow \partial M$,
- an embedded sphere $G\colon \mathbb{S}^2 \hookrightarrow \partial M$,

so that s and G intersect transversely and positively in a single point. Recall that we study the set of isotopy classes $\mathsf{Emb}_\partial[\mathbb{D}^2, M] := \pi_0 \, \mathsf{Emb}_\partial(\mathbb{D}^2, M)$ of neat smooth embeddings $K\colon \mathbb{D}^2 \hookrightarrow M$ which on $\partial\mathbb{D}^2$ agree with s.

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By Space Level LBT we have $\mathsf{Emb}_\partial[\mathbb{D}^2, M] := \pi_1 \mathsf{Emb}_\partial^\varepsilon(\mathbb{D}^1, M \cup_{\nu G} h^3)$ and we can compute the latter group! Moreover, we can interpret the resulting group structure on the original set, as follows.

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LBT for 2-disks

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$\Longleftrightarrow$ the relative Dax invariant, given by a clever count of double point loops in a homotopy to K, detects the action:

$$\mathrm{Dax}(K + \mathrm{fm}(r)^G, K) = [r].$$

Picture of LBT for 2-disks

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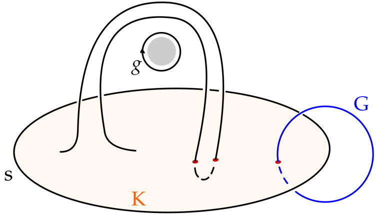

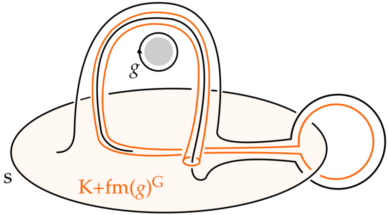

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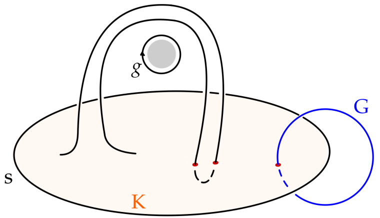

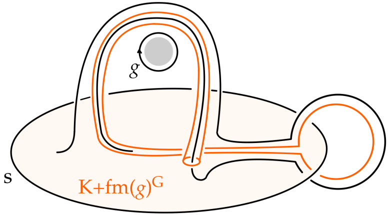

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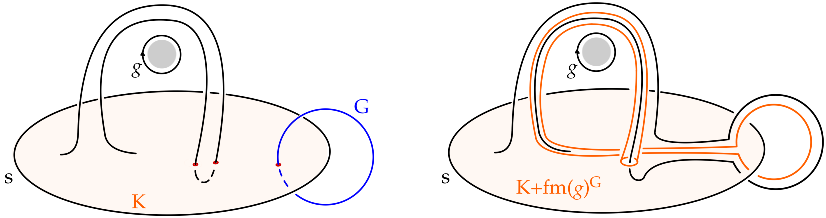

- A similar construction by Gabai ('21).
- We recover LBT for spheres of Gabai ('20) and Schneiderman–Teichner ('21).

Thank you!