HOMOTOPY GROUPS OF SOME EMBEDDING SPACES

Danica Kosanović (ETH Zürich)

@ *La réunion annuelle du GDR Topologie algébrique*, Nantes, October, 2022

Based on the joint work with Peter Teichner (MPIM Bonn)
https://arxiv.org/abs/2105.13032

Table of contents

Motivation

Spaces of embeddings

- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \leq k \leq d$.
- **General goal.** Study the homotopy type of the space

$$\mathsf{Emb}_\partial(V, X)$$

of smooth neat embeddings $K\colon V \hookrightarrow X$ which near ∂V agree with a fixed basepoint $\mathrm{U}\colon V \hookrightarrow X$. We denote $\mathsf{s} := \mathrm{U}|_{\partial V}\colon \partial V \hookrightarrow \partial X$.

Spaces of embeddings

- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \leq k \leq d$.
- **General goal.** Study the homotopy type of the space

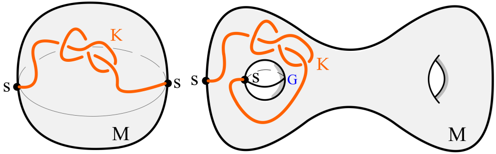
$$\mathrm{Emb}_\partial(V, X)$$

of smooth neat embeddings $K\colon V \hookrightarrow X$ which near ∂V agree with a fixed basepoint $\mathrm{U}\colon V \hookrightarrow X$. We denote $\mathsf{s} := \mathrm{U}|_{\partial V}\colon \partial V \hookrightarrow \partial X$.
- Recall that a smooth map K is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial X = K(\partial V)$.

- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \leq k \leq d$.
- **General goal.** Study the homotopy type of the space

$$\mathsf{Emb}_\partial(V, X)$$

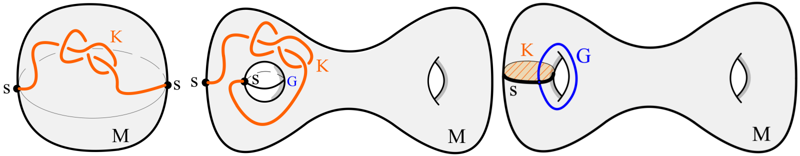
  of smooth neat embeddings $K \colon V \hookrightarrow X$ which near ∂V agree with a fixed basepoint $\mathrm{U} \colon V \hookrightarrow X$. We denote $\mathbf{s} := \mathrm{U}|_{\partial V} \colon \partial V \hookrightarrow \partial X$.
- Recall that a smooth map K is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial X = K(\partial V)$.
- For example, for $(k, d) = (1, 3)$



- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \leq k \leq d$.
- **General goal.** Study the homotopy type of the space

$$\mathrm{Emb}_\partial(V, X)$$

  of smooth neat embeddings $K\colon V \hookrightarrow X$ which near ∂V agree with a fixed basepoint $\mathrm{U}\colon V \hookrightarrow X$. We denote $\mathbf{s} := \mathrm{U}|_{\partial V}\colon \partial V \hookrightarrow \partial X$.
- Recall that a smooth map K is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial X = K(\partial V)$.
- For example, for $(k, d) = (1, 3)$ and $(2, 3)$:

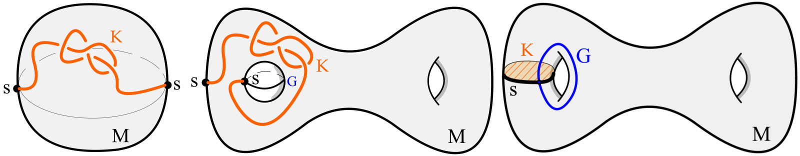


Spaces of embeddings

- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \leq k \leq d$.
- **General goal.** Study the homotopy type of the space

$$\mathsf{Emb}_\partial(V, X)$$

  of smooth neat embeddings $K\colon V \hookrightarrow X$ which near ∂V agree with a fixed basepoint $\mathrm{U}\colon V \hookrightarrow X$. We denote $\mathbf{s} := \mathrm{U}|_{\partial V}\colon \partial V \hookrightarrow \partial X$.
- Recall that a smooth map K is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial X = K(\partial V)$.
- For example, for $(k, d) = (1, 3)$ and $(2, 3)$:



- For $V = \mathbb{D}^k$, the setting with a dual: if there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial X$, such that G has trivial normal bundle and $G \pitchfork \mathbf{s} = \{pt\}$. Like pictures 2 and 3!

Spaces of embeddings

- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions.

Spaces of embeddings

- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions. E.g. use the fibration sequence

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M \setminus \mathbb{D}^d) \xrightarrow{-\cdot\nu_0} \mathrm{Emb}(\mathbb{S}^k, M) \xrightarrow{D_0} V_k(TM).$$

Spaces of embeddings

- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions. E.g. use the fibration sequence

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M \setminus \mathbb{D}^d) \xrightarrow{-\cdot\nu_0} \mathrm{Emb}(\mathbb{S}^k, M) \xrightarrow{D_0} V_k(TM).$$

- For example, (classical) knot theory studies isotopy classes of circles embedded into the 3-space: this is the set of connected components $\pi_0 \, \mathrm{Emb}_\partial(\mathbb{S}^1, \mathbb{R}^3)$.

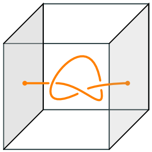
Spaces of embeddings

- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions. E.g. use the fibration sequence

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M \setminus \mathbb{D}^d) \xrightarrow{-\cdot\nu_0} \mathrm{Emb}(\mathbb{S}^k, M) \xrightarrow{D_0} V_k(TM).$$

- For example, (classical) knot theory studies isotopy classes of circles embedded into the 3-space: this is the set of connected components $\pi_0 \,\mathrm{Emb}_\partial(\mathbb{S}^1, \mathbb{R}^3)$. But we have

$$\{\text{knots}\}/_{\text{isotopy}} = \pi_0\,\mathrm{Emb}(\mathbb{S}^1, \mathbb{R}^3) \cong \pi_0\,\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^3) \cong \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^3)$$



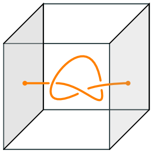
- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions. E.g. use the fibration sequence

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M \setminus \mathbb{D}^d) \xrightarrow{\quad - \cdot \nu_0 \quad} \mathrm{Emb}(\mathbb{S}^k, M) \xrightarrow{\quad D_0 \quad} V_k(TM).$$

- For example, (classical) knot theory studies isotopy classes of circles embedded into the 3-space: this is the set of connected components $\pi_0 \,\mathrm{Emb}_\partial(\mathbb{S}^1, \mathbb{R}^3)$. But we have

$$\{\text{knots}\}/_{\text{isotopy}} = \pi_0 \,\mathrm{Emb}(\mathbb{S}^1, \mathbb{R}^3) \cong \pi_0 \,\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^3) \cong \pi_0 \,\mathrm{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^3)$$

Note. Connected sum of knots is on arcs given by stacking the cubes horizontally – so well-defined on space-level!



Spaces of embeddings

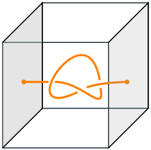
- **Remark.** Embeddings of closed manifolds can be reduced to the setting with boundary, modulo group extensions. E.g. use the fibration sequence

$$\mathsf{Emb}_\partial(\mathbb{D}^k, M \setminus \mathbb{D}^d) \xrightarrow{-\cdot\nu_0} \mathsf{Emb}(\mathbb{S}^k, M) \xrightarrow{D_0} V_k(TM).$$

- For example, (classical) knot theory studies isotopy classes of circles embedded into the 3-space: this is the set of connected components $\pi_0 \,\mathsf{Emb}_\partial(\mathbb{S}^1, \mathbb{R}^3)$. But we have

$$\{\text{knots}\}/\text{isotopy} = \pi_0 \,\mathsf{Emb}(\mathbb{S}^1, \mathbb{R}^3) \cong \pi_0 \,\mathsf{Emb}(\mathbb{S}^1, \mathbb{S}^3) \cong \pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^3)$$

Note. Connected sum of knots is on arcs given by stacking the cubes horizontally – so well-defined on space-level!



- Recently, intensively studied is the set of (long) 2-knots in a 4-manifold M:

$$\pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$$

This can be huge – for example, "spinning" a classical knot gives a 2-knot in $\pi_0 \,\mathsf{Emb}_\partial(\mathbb{S}^2, \mathbb{R}^4) \cong \pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$.

So wait, higher homotopy groups are irrelevant in low-dimensional topology?

Theorem (Space level light bulb trick [K–Teichner '21])

Theorem (Space level light bulb trick [K–Teichner '21])

For any $1 \leq k \leq d$, in a setting with a dual, any choice of $\mathrm{U}\colon \mathbb{D}^k \hookrightarrow M$ leads to an (explicit) homotopy equivalence

$$\mathsf{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\,\mathsf{Emb}_\partial^\varepsilon(\mathbb{D}^{k-1}, X).$$

where $X := M \cup_{\nu G} h^{d-k+1}$.

Recall that setting with a dual means: we have a d-manifold M and embedding $\mathbf{s} = \partial\mathrm{U}\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, such that there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$ with trivial normal bundle and such that $G \pitchfork \mathbf{s} = \{pt\}$.

So wait, higher homotopy groups are irrelevant in low-dimensional topology?

Theorem (Space level light bulb trick [K–Teichner '21])

For any $1 \leq k \leq d$, in a setting with a dual, any choice of $\mathbb{U}\colon \mathbb{D}^k \hookrightarrow M$ leads to an (explicit) homotopy equivalence

$$\mathsf{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, X).$$

where $X := M \cup_{\nu G} h^{d-k+1}$. In particular, if $d = 4$ we have

$$\pi_0\,\mathsf{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$$

Superscript ε means embedded disks are equipped with "push-offs"...

Theorem (Space level light bulb trick [K–Teichner '21])

For any $1 \leq k \leq d$, in a setting with a dual, any choice of $\mathrm{U}\colon \mathbb{D}^k \hookrightarrow M$ leads to an (explicit) homotopy equivalence

$$\mathsf{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\, \mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, X).$$

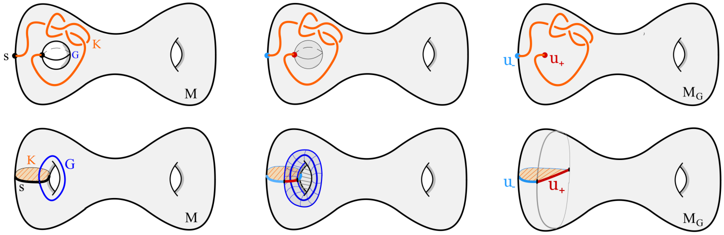
where $X := M \cup_{\nu G} h^{d-k+1}$. In particular, if $d = 4$ we have

$$\pi_0\, \mathsf{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\, \mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$$

Superscript ε means embedded disks are equipped with "push-offs"...

$k, d = 1, 3$

$k, d = 2, 3$



The main result today, and applications

How to compute homotopy groups?

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

How to compute homotopy groups?

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\implies$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

How to compute homotopy groups?

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\implies$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

Theorem [K–Teichner '22]

Fix ℓ, d such that $d \geq \ell + 3$ and $d - 2\ell \geq 1$. Let X be a d-dimensional smooth compact manifold with boundary, and fix $u \colon \mathbb{D}^\ell \hookrightarrow X$. Then

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\implies$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

Theorem [K–Teichner '22]

Fix ℓ, d such that $d \geq \ell + 3$ and $d - 2\ell \geq 1$. Let X be a d-dimensional smooth compact manifold with boundary, and fix $u\colon \mathbb{D}^\ell \hookrightarrow X$. Then

1. For $0 \leq n \leq d - 2\ell - 2$ we have $p_u\colon \pi_n(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell} X$.

How to compute homotopy groups?

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\Longrightarrow$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

Theorem [K–Teichner '22]

Fix ℓ, d such that $d \geq \ell + 3$ and $d - 2\ell \geq 1$. Let X be a d-dimensional smooth compact manifold with boundary, and fix $u\colon \mathbb{D}^\ell \hookrightarrow X$. Then

1. For $0 \leq n \leq d - 2\ell - 2$ we have $p_u\colon \pi_n(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell} X$.
2. There is a short exact sequence of groups (sets if $d - 2\ell - 1 = 0$):

$$\mathbb{Z}[\pi_1 X]\big/\langle 1\rangle \oplus rel_{\ell,d} \oplus \operatorname{dax}(\pi_{d-\ell}(X)) \underset{\text{Dax}}{\overset{\partial\mathfrak{r}}{\rightleftarrows}} \pi_{d-2\ell-1}(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \xrightarrow{p_u} \pi_{d-\ell-1} X.$$

where Dax is defined on the image of the realisation map $\partial\mathfrak{r}$ and is its explicit inverse, and $rel_{1,d} := \emptyset$ and $rel_{\ell,d} := \langle g - (-1)^{d-\ell} g : g \in \pi_1 X\rangle$ if $\ell \geq 2$

How to compute homotopy groups?

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\implies$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

Theorem [K–Teichner '22]

Fix ℓ, d such that $d \geq \ell + 3$ and $d - 2\ell \geq 1$. Let X be a d-dimensional smooth compact manifold with boundary, and fix $u \colon \mathbb{D}^\ell \hookrightarrow X$. Then

1. For $0 \leq n \leq d - 2\ell - 2$ we have $p_u \colon \pi_n(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell} X$.
2. There is a short exact sequence of groups (sets if $d - 2\ell - 1 = 0$):

$$\mathbb{Z}[\pi_1 X]/_{\langle 1 \rangle} \oplus rel_{\ell,d} \oplus \operatorname{dax}(\pi_{d-\ell}(X)) \underset{\text{Dax}}{\overset{\partial\mathfrak{r}}{\rightleftarrows}} \pi_{d-2\ell-1}(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \xrightarrow{p_u} \pi_{d-\ell-1} X.$$

where **Dax** is defined on the image of the realisation map $\partial\mathfrak{r}$ and is its explicit inverse, and $rel_{1,d} := \emptyset$ and $rel_{\ell,d} := \langle g - (-1)^{d-\ell} g : g \in \pi_1 X \rangle$ if $\ell \geq 2$

- Therefore, we have (after a bit more work to account for ε-augmentations) a (more or less) explicit description of $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$ for $n \leq d - 2k$ and $d \geq 4$, assuming there is a dual for the boundary condition $\mathbf{s} \colon \mathbb{S}^{k-1} \hookrightarrow \partial M$.

- Note. $\dim X - \dim \mathbb{D}^{k-1} < \dim M - \dim \mathbb{D}^k$

$\implies$ $\pi_{n+1} \operatorname{Emb}^{\varepsilon}_{\partial}(\mathbb{D}^{k-1}, X)$ is easier than $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$! We use the classical work of Dax to compute this in a range.

Theorem [K–Teichner '22]

Fix ℓ, d such that $d \geq \ell + 3$ and $d - 2\ell \geq 1$. Let X be a d-dimensional smooth compact manifold with boundary, and fix $u \colon \mathbb{D}^\ell \hookrightarrow X$. Then

1. For $0 \leq n \leq d - 2\ell - 2$ we have $p_u \colon \pi_n(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell} X$.
2. There is a short exact sequence of groups (sets if $d - 2\ell - 1 = 0$):

$$\mathbb{Z}[\pi_1 X]/\langle 1 \rangle \oplus rel_{\ell,d} \oplus \mathbf{dax}(\pi_{d-\ell}(X)) \underset{\text{Dax}}{\overset{\partial\mathfrak{r}}{\rightleftarrows}} \pi_{d-2\ell-1}(\operatorname{Emb}_{\partial}(\mathbb{D}^\ell, X), u) \xrightarrow{p_u} \pi_{d-\ell-1} X.$$

where Dax is defined on the image of the realisation map $\partial\mathfrak{r}$ and is its explicit inverse, and $rel_{1,d} := \emptyset$ and $rel_{\ell,d} := \langle g - (-1)^{d-\ell} g : g \in \pi_1 X \rangle$ if $\ell \geq 2$

- Therefore, we have (after a bit more work to account for ε-augmentations) a (more or less) explicit description of $\pi_n \operatorname{Emb}_{\partial}(\mathbb{D}^k, M)$ for $n \leq d - 2k$ and $d \geq 4$, assuming there is a dual for the boundary condition $\mathbf{s} \colon \mathbb{S}^{k-1} \hookrightarrow \partial M$.
- We make this more explicit, and compute many classes of examples in K' 21.

Applications of the two theorems

In this talk: After giving some applications, we discuss this theorem in detail.

Recall $X := M \cup_{\nu G} h^{d-1}$.

In this talk: After giving some applications, we discuss this theorem in detail.

Recall $X := M \cup_{\nu G} h^{d-1}$.

$$k = 1: \ \mathsf{Emb}_\partial(\mathbb{D}^1, M) \simeq \Omega\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^0, X) \simeq \Omega\mathbb{S}^{d-1} \times \Omega X$$

In this talk: After giving some applications, we discuss this theorem in detail.

Recall $X := M \cup_{\nu G} h^{d-1}$.

$k = 1:\ \mathsf{Emb}_\partial(\mathbb{D}^1, M) \simeq \Omega\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^0, X) \simeq \Omega\mathbb{S}^{d-1} \times \Omega X$

$d = 2:$ The map $\mathfrak{amb}$ is "point-pushing":
{arcs in a surface M, with ends fixed on two components of ∂M}/isotopy
$\cong \mathbb{Z} \oplus \pi_1(M \cup_G h^2)$.

Applications of the two theorems

In this talk: After giving some applications, we discuss this theorem in detail.

Recall $X := M \cup_{\nu G} h^{d-1}$.

$k = 1$: $\mathsf{Emb}_\partial(\mathbb{D}^1, M) \simeq \Omega\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^0, X) \simeq \Omega\mathbb{S}^{d-1} \times \Omega X$

$d = 2$: The map $\mathfrak{amb}$ is "point-pushing":
{arcs in a surface M, with ends fixed on two components of ∂M}/isotopy
$\cong \mathbb{Z} \oplus \pi_1(M \cup_G h^2)$.

$d = 3$: This recovers the classical LBT:
{arcs in a 3-manifold M with ends on two components of ∂M,
one of which is $\mathbb{S}^2$}/isotopy
$\cong \pi_1(M \cup_G h^3)$

Applications of the two theorems

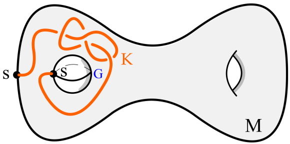
In this talk: After giving some applications, we discuss this theorem in detail.

Recall $X := M \cup_{\nu G} h^{d-1}$.

$k = 1:\ \mathrm{Emb}_\partial(\mathbb{D}^1, M) \simeq \Omega\,\mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^0, X) \simeq \Omega\mathbb{S}^{d-1} \times \Omega X$

$d = 2:$ The map $\mathfrak{amb}$ is "point-pushing":
{arcs in a surface M, with ends fixed on two components of ∂M}/isotopy
$\cong \mathbb{Z} \oplus \pi_1(M \cup_G h^2)$.

$d = 3:$ This recovers the classical LBT:
{arcs in a 3-manifold M with ends on two components of ∂M,
one of which is $\mathbb{S}^2$}/isotopy
$\cong \pi_1(M \cup_G h^3)$
$\Longrightarrow$ any knot in the chord to which a light bulb attaches can be unknotted!



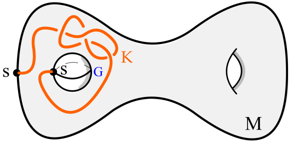
In this talk: After giving some applications, we discuss this theorem in detail.

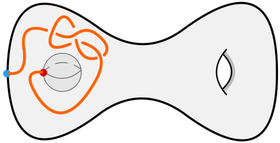
Recall $X := M \cup_{\nu G} h^{d-1}$.

$k = 1$: $\mathrm{Emb}_\partial(\mathbb{D}^1, M) \simeq \Omega\,\mathrm{Emb}_\partial^\varepsilon(\mathbb{D}^0, X) \simeq \Omega\mathbb{S}^{d-1} \times \Omega X$

$d = 2$: The map $\mathfrak{amb}$ is "point-pushing":
{arcs in a surface M, with ends fixed on two components of ∂M}/isotopy
$\cong \mathbb{Z} \oplus \pi_1(M \cup_G h^2)$.

$d = 3$: This recovers the classical LBT:
{arcs in a 3-manifold M with ends on two components of ∂M,
one of which is $\mathbb{S}^2$}/isotopy
$\cong \pi_1(M \cup_G h^3)$
$\implies$ any knot in the chord to which a light bulb attaches can be unknotted!





Applications of the theorem

$k = 2: \ \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$

Applications of the theorem

$k = 2: \ \mathsf{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$

$d = 4: \ \pi_0\,\mathsf{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\,\mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3).$

Applications of the theorem

$k = 2: \ \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\,\mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X)$.

$d = 4: \ \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\,\mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3)$.

$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

Applications of the theorem

$k = 2: \ \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X)$.

$d = 4: \ \pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3)$.

$\Longrightarrow$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

$\Longrightarrow$ We recover (and generalise) LBT for spheres of Gabai '20 and Schneiderman–Teichner '21.

Applications of the theorem

$k = 2: \ \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X)$.

$d = 4: \ \pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3)$.

$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

$\implies$ We recover (and generalise) LBT for spheres of Gabai '20 and Schneiderman–Teichner '21.

- Moreover, we get an (unexpected) group structure on $\pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^2, M)$!

Applications of the theorem

$k = 2:\ \operatorname{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega \operatorname{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$

$d = 4:\ \pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1 \operatorname{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3).$

$\Longrightarrow$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

$\Longrightarrow$ We recover (and generalise) LBT for spheres of Gabai '20 and Schneiderman–Teichner '21.

- Moreover, we get an (unexpected) group structure on $\pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, M)$!

$k = d{-}1:\ \operatorname{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{S}^1 \times \mathbb{D}^{d-1}) \simeq \Omega \operatorname{Emb}_\partial(\mathbb{D}^{d-2}, \mathbb{D}^d)$

$d = 4:\ \pi_0 \operatorname{Emb}_\partial(\mathbb{D}^3, \mathbb{S}^1 \times \mathbb{D}^3) \cong \pi_1 \operatorname{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$, cf. Budney–Gabai.

Applications of the theorem

$k = 2: \; \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X)$.

$d = 4: \; \pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3)$.

$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

$\implies$ We recover (and generalise) LBT for spheres of Gabai '20 and Schneiderman–Teichner '21.

- Moreover, we get an (unexpected) group structure on $\pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^2, M)$!

$k = d-1: \; \mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{S}^1 \times \mathbb{D}^{d-1}) \simeq \Omega\, \mathrm{Emb}_\partial(\mathbb{D}^{d-2}, \mathbb{D}^d)$

$d = 4: \; \pi_0\, \mathrm{Emb}_\partial(\mathbb{D}^3, \mathbb{S}^1 \times \mathbb{D}^3) \cong \pi_1\, \mathrm{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$, cf. Budney–Gabai.

$k = d: \;$ Recovers a theorem (and proof) of Cerf '68:

Theorem (Cerf '68)

There is a homotopy equivalence $\mathrm{Diff}^+_\partial(\mathbb{D}^d) \simeq \Omega\, \mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{D}^d)$. In particular,

$$\pi_0\, \mathrm{Diff}^+_\partial(\mathbb{D}^4) \cong \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^3, \mathbb{D}^4); \mathrm{U}).$$

Applications of the theorem

$k = 2: \ \mathrm{Emb}_\partial(\mathbb{D}^2, M) \simeq \Omega\,\mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, X).$

$d = 4: \ \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1\,\mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^1, M \cup_{\nu G} h^3).$

$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

$\implies$ We recover (and generalise) LBT for spheres of Gabai '20 and Schneiderman–Teichner '21.

- Moreover, we get an (unexpected) group structure on $\pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2, M)$!

$k = d-1: \ \mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{S}^1 \times \mathbb{D}^{d-1}) \simeq \Omega\,\mathrm{Emb}_\partial(\mathbb{D}^{d-2}, \mathbb{D}^d)$

$d = 4: \ \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^3, \mathbb{S}^1 \times \mathbb{D}^3) \cong \pi_1\,\mathrm{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$, cf. Budney–Gabai.

$k = d:$ Recovers a theorem (and proof) of Cerf '68:

Theorem (Cerf '68)

There is a homotopy equivalence $\mathrm{Diff}^+_\partial(\mathbb{D}^d) \simeq \Omega\,\mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{D}^d)$. In particular,

$$\pi_0\,\mathrm{Diff}^+_\partial(\mathbb{D}^4) \cong \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^3, \mathbb{D}^4); \mathrm{U}).$$

Open problem

Is $\pi_0\,\mathrm{Diff}^+_\partial(\mathbb{D}^4)$ trivial? Compute it.

See Budney-Gabai, Gay, Watanabe for some candidate diffeomorphisms.

Metastable homotopy groups

Stable, metastable, meta^{2}stable…(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \le \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies n < 2 \iff$ generically no double points.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies n < 2 \iff$ generically no double points.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies$ $n < 2 \iff$ generically no double points.

  - Can show: $\mathsf{Emb}(V, X) \hookrightarrow \mathsf{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

  $\implies n < 2 \iff$ generically no double points.
  - Can show: $\mathsf{Emb}(V, X) \hookrightarrow \mathsf{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.
- *Haefliger* '60s and *Dax* '70s: metastable range $\ell < \frac{2d}{3}$.

  $\implies n < 3 \iff$ generically no triple points.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

  $\implies$ $n < 2 \iff$ generically no double points.

  - Can show: $\mathrm{Emb}(V, X) \hookrightarrow \mathrm{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.

- *Haefliger* '60s and *Dax* '70s: metastable range $\ell < \frac{2d}{3}$.

  $\implies$ $n < 3 \iff$ generically no triple points.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies$ $n < 2 \iff$ generically no double points.

  - Can show: $\mathrm{Emb}(V, X) \hookrightarrow \mathrm{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.

- *Haefliger* '60s and *Dax* '70s: metastable range $\ell < \frac{2d}{3}$.

$\implies$ $n < 3 \iff$ generically no triple points.

  - Dax upgraded this to:

$$\mathrm{Emb}(V, X) \hookrightarrow \mathrm{P}_2(V, X) \text{ is } (2d - 3\ell - 3)\text{-connected},$$

  for a certain space $\mathrm{P}_2(V, X)$ built out of pairs of points in X.

Stable, metastable, meta^{2}stable…(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies n < 2 \iff$ generically no double points.

  - Can show: $\mathrm{Emb}(V, X) \hookrightarrow \mathrm{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.

- *Haefliger* '60s and *Dax* '70s: metastable range $\ell < \frac{2d}{3}$.

$\implies n < 3 \iff$ generically no triple points.

  - Dax upgraded this to:

$$\mathrm{Emb}(V, X) \hookrightarrow \mathrm{P}_2(V, X) \text{ is } (2d - 3\ell - 3)\text{-connected},$$

  for a certain space $\mathrm{P}_2(V, X)$ built out of pairs of points in X.

- *Goodwillie–Klein–Weiss* embedding calculus.

  - Construct a tower of spaces $\mathrm{P}_n(V, X)$, $n \geq 1$, with:
    $\mathrm{P}_1 = \mathrm{Imm}(V, X)$ and $\mathrm{P}_2(V, X) =$ the Haefliger–Dax space.

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

- *Whitney* '40s: stable range $\ell < \frac{d}{2}$.

$\implies$ $n < 2 \iff$ generically no double points.
  - Can show: $\mathsf{Emb}(V, X) \hookrightarrow \mathsf{Imm}(V, X)$ is $(d - 2\ell - 1)$-connected.
- *Haefliger* '60s and *Dax* '70s: metastable range $\ell < \frac{2d}{3}$.

$\implies$ $n < 3 \iff$ generically no triple points.
  - Dax upgraded this to:

$$\mathsf{Emb}(V, X) \hookrightarrow \mathrm{P}_2(V, X) \text{ is } (2d - 3\ell - 3)\text{-connected,}$$

  for a certain space $\mathrm{P}_2(V, X)$ built out of pairs of points in X.
- *Goodwillie–Klein–Weiss* embedding calculus.
  - Construct a tower of spaces $\mathrm{P}_n(V, X)$, $n \geq 1$, with: $\mathrm{P}_1 = \mathsf{Imm}(V, X)$ and $\mathrm{P}_2(V, X) =$ the Haefliger–Dax space.
  - $\mathsf{Emb}(V, X) \to \mathrm{P}_n(V, X)$ is $(nd - (n+1)\ell - (2n-1))$-connected (hard!).
  - Use homotopy theoretic tools to study $\mathrm{P}_n(V, X)$.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u \colon \pi_n(\mathrm{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathrm{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \leq n \leq d-2\ell-2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u \colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \le n \le d{-}2\ell{-}2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u\colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \le n \le d-2\ell-2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.

  - Firstly, study the relative homotopy group

    $$\pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X))$$

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $p_u\colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \leq n \leq d-2\ell-2.$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.

  - Firstly, study the relative homotopy group

    $$\pi_{d-2\ell-1}(\mathsf{Imm}(V,X), \mathsf{Emb}(V,X))$$

  - Then study the image of the map

    $$\delta_{\mathsf{Imm}}\colon \pi_{d-2\ell}\,\mathsf{Imm}(V,X) \to \pi_{d-2\ell-1}(\mathsf{Imm}(V,X), \mathsf{Emb}(V,X))$$

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u\colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \le n \le d{-}2\ell{-}2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.

  - Firstly, study the relative homotopy group

    $$\pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X))$$

  - Then study the image of the map

    $$\delta_{\mathsf{Imm}}\colon \pi_{d-2\ell}\,\mathsf{Imm}(V, X) \to \pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X))$$

  - The desired kernel is the cokernel of δ_{Imm}.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u\colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \leq n \leq d-2\ell-2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.
- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.
  - Firstly, study the relative homotopy group

    $$\pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X)) \cong {\mathbb{Z}[\pi_1 X]}/{rel_{\ell,d}}$$

  - Then study the image of the map

    $$\delta_{\mathsf{Imm}}\colon \pi_{d-2\ell}\,\mathsf{Imm}(V, X) \to \pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X))$$

  - The desired kernel is the cokernel of δ_{Imm}.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u\colon \pi_n(\mathsf{Emb}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_n(\mathsf{Imm}_\partial(\mathbb{D}^\ell, X), u) \cong \pi_{n+\ell}X, \qquad \text{for } 0 \leq n \leq d{-}2\ell{-}2.$$

  is just the well-known computation of the homotopy groups of immersions, using Smale–Hirsch theory.

- For $n = d - 2\ell - 1$ we still have a surjection

  $$\pi_{d-2\ell-1}\,\mathsf{Emb}_\partial(\mathbb{D}^\ell, X) \twoheadrightarrow \pi_{d-2\ell-1}\,\mathsf{Imm}_\partial(\mathbb{D}^\ell, X) \cong \pi_{d-\ell-1}X.$$

  Dax tells us how to compute its kernel.

  - Firstly, study the relative homotopy group

    $$\pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X)) \cong {}^{\mathbb{Z}[\pi_1 X]}\!/_{rel_{\ell,d}}$$

  - Then study the image of the map

    $$\delta_{\mathsf{Imm}}\colon \pi_{d-2\ell}\,\mathsf{Imm}(V, X) \to \pi_{d-2\ell-1}(\mathsf{Imm}(V, X), \mathsf{Emb}(V, X))$$

    It turns out this is given as the image of a certain homomorphism $\mathsf{dax}\colon \pi_{d-\ell}X \to \mathbb{Z}[\pi_1 X \setminus 1]$.

  - The desired kernel is the cokernel of δ_{Imm}.

About the lowest degree in the metastable range

Theorem [Dax '72]

There is an isomorphism $\pi_{d-2\ell-1}(\mathrm{Imm}(V,X),\mathrm{Emb}(V,X),u)\cong\Omega_0(\mathcal{C}_u;\theta_u)$, the degree 0 normal bordism group of a certain space $\mathcal{C}_u$ with a stable normal bundle θ_u over it.

About the lowest degree in the metastable range

Theorem [Dax '72]

There is an isomorphism $\pi_{d-2\ell-1}(\mathsf{Imm}(V,X),\mathsf{Emb}(V,X),u) \cong \Omega_0(\mathcal{C}_u;\theta_u)$, the degree 0 normal bordism group of a certain space $\mathcal{C}_u$ with a stable normal bundle θ_u over it.

Theorem [K–Teichner '22]

There is an isomorphism $\mathsf{Dax}\colon \pi_{d-2\ell-1}(\mathsf{Imm}(V,X),\mathsf{Emb}(V,X),u) \to \mathbb{Z}[\pi_1 X]/rel_{\ell,d}$ given as follows: represent a relative class by a "perfect" map

$$F\colon (\mathbb{I}^{d-2\ell-1},\mathbb{I}^{d-2\ell-2}\times\{0\},\mathbb{I}^{d-2\ell-2}\times\{1\}\cup\partial\mathbb{I}^{d-2\ell-2}\times\mathbb{I}) \to (\mathsf{Imm},\mathsf{Emb},u)$$

i.e. F is smooth and its track

$$\widetilde{F}\colon \mathbb{I}^{d-2\ell-1}\times V \to \mathbb{I}^{d-2\ell-1}\times X, \quad (\vec{t},v)\mapsto(\vec{t},F(\vec{t},v)),$$

has no triple points and double points $(\vec{t}_i,x_i)\in\mathbb{I}^{d-2\ell-1}\times V$ for $i=1,\dots,r$ are isolated and transverse.

Theorem [K–Teichner '22]

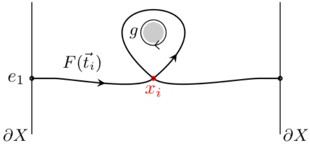
There is an isomorphism $\mathbf{Dax}\colon \pi_{d-2\ell-1}(\mathrm{Imm}(V,X), \mathrm{Emb}(V,X), u) \to \mathbb{Z}[\pi_1 X]/rel_{\ell,d}$ given as follows: represent a relative class by a "perfect" map

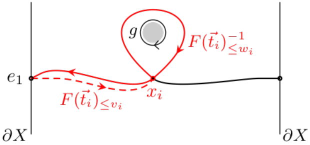
$$F\colon (\mathbb{I}^{d-2\ell-1}, \mathbb{I}^{d-2\ell-2}\times\{0\}, \mathbb{I}^{d-2\ell-2}\times\{1\}\cup\partial\mathbb{I}^{d-2\ell-2}\times\mathbb{I}) \to (\mathrm{Imm}, \mathrm{Emb}, u)$$

i.e. F is smooth and its track

$$\widetilde{F}\colon \mathbb{I}^{d-2\ell-1}\times V \to \mathbb{I}^{d-2\ell-1}\times X, \quad (\vec{t}, v)\mapsto(\vec{t}, F(\vec{t}, v)),$$

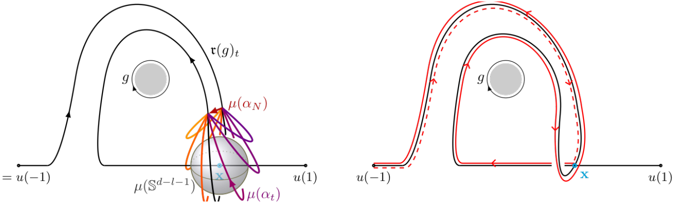
has no triple points and double points $(\vec{t}_i, x_i)\in\mathbb{I}^{d-2\ell-1}\times V$ for $i=1,\dots,r$ are isolated and transverse. Then $\mathbf{Dax}([F]) = \sum_{i=1}^r \varepsilon_{(\vec{t}_i,x_i)} g_{(\vec{t}_i,x_i)}$ is the sum of signed double point loops of $\widetilde{F}$.





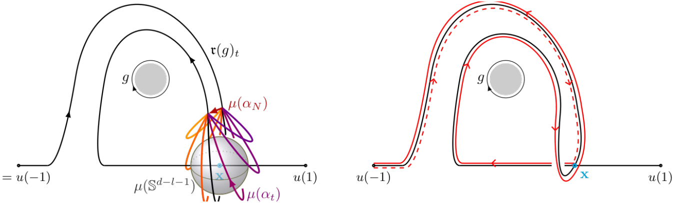
Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial\mathfrak{r}(g)$ is given by

Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial\mathfrak{r}(g)$ is given by



The realisation map and the Dax invariant

Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial\mathfrak{r}(g)$ is given by



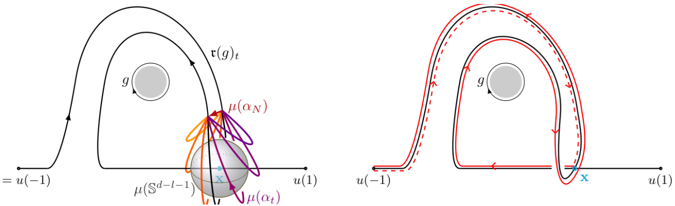
Finally, for $V = \mathbb{D}^\ell$ we can describe $\mathrm{im}(\delta_{\mathsf{Imm}})$ as $\langle 1 \rangle \oplus \mathrm{im}(\mathsf{dax})$ where

$$\mathsf{dax}\colon \pi_{d-\ell} X \to \mathbb{Z}[\pi_1 X \setminus 1], \quad \mathsf{dax}(a) = \mathsf{Dax}(\widetilde{A}),$$

where we represent $a \in \pi_{d-\ell} X$ by a map $A\colon \mathbb{I}^{d-2\ell} \times \mathbb{D}^\ell \to X$.

The realisation map and the Dax invariant

Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial\mathfrak{r}(g)$ is given by



Finally, for $V = \mathbb{D}^\ell$ we can describe $\mathsf{im}(\delta_{\mathsf{Imm}})$ as $\langle 1\rangle \oplus \mathsf{im}(\mathsf{dax})$ where

$$\mathsf{dax}\colon \pi_{d-\ell}X \to \mathbb{Z}[\pi_1 X \setminus 1], \quad \mathsf{dax}(a) = \mathsf{Dax}(\widetilde{A}),$$

where we represent $a \in \pi_{d-\ell}X$ by a map $A\colon \mathbb{I}^{d-2\ell} \times \mathbb{D}^\ell \to X$.

We can compute this in many classes of examples! See [K '21].

Thank you!