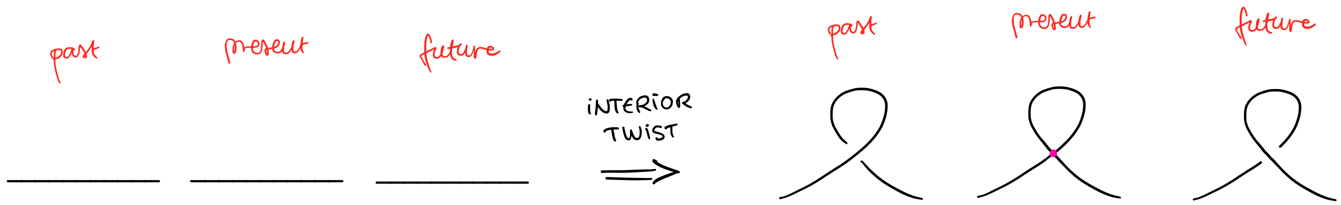
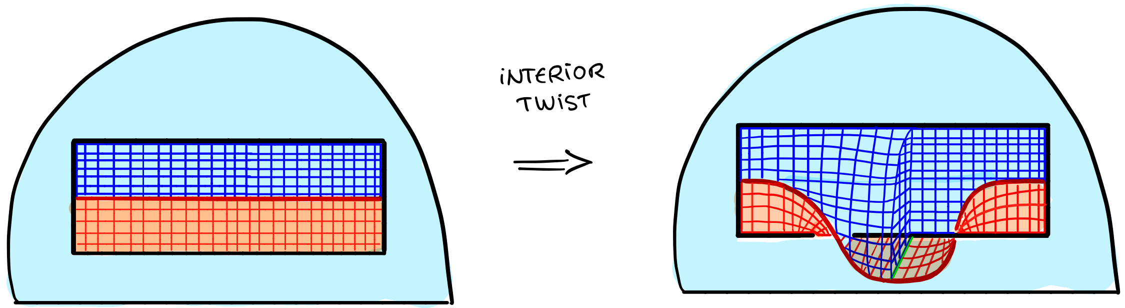
INTERIOR TWIST = THE CUSP HOMOTOPY = MOVE I

the non-regular move.

THE MOVIE:



3D PICTURE:



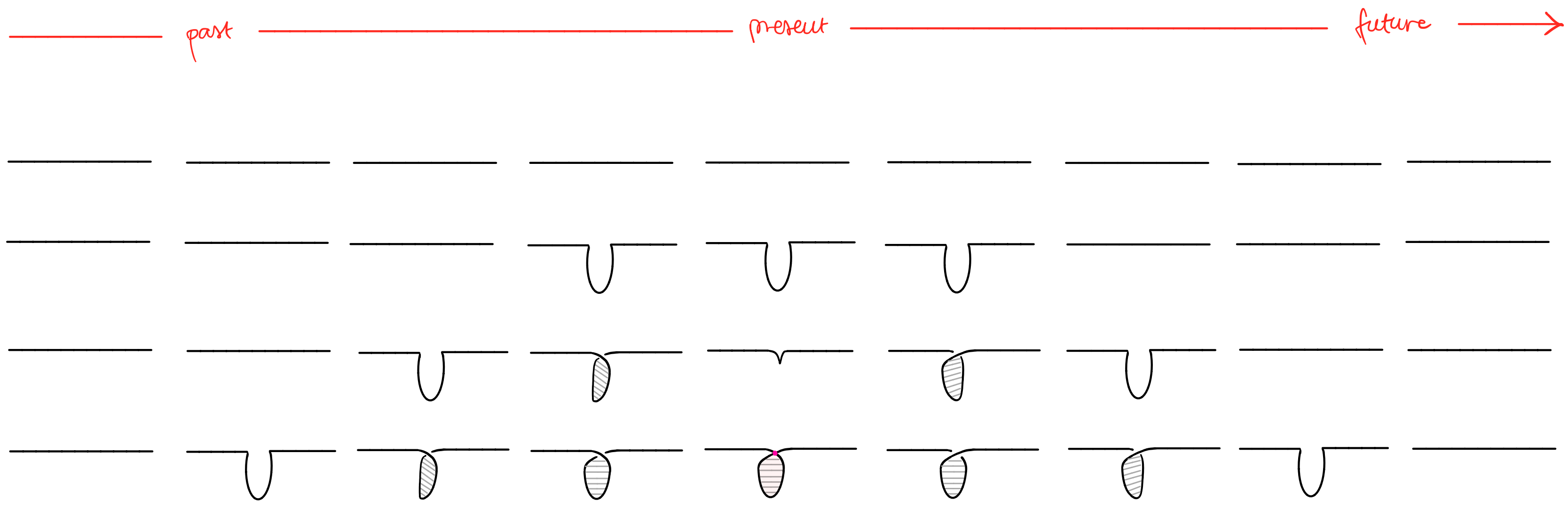
$$\mathbb{D}^2 \subseteq \mathbb{D}^3 \subseteq \mathbb{D}^4$$

have a line of selfintersection (green)
Perturb into past and future to get:

$$\mathbb{D}^2 \hookrightarrow \mathbb{D}^3 \subseteq \mathbb{D}^4$$

The cusp homotopy

Movie of movies:



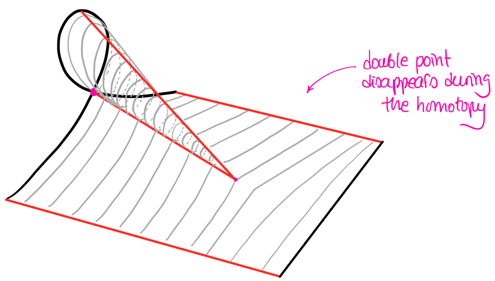
Standard $D : \mathbb{D}^2 \subseteq \mathbb{D}^4$

cusp homotopy

Twisted $\tilde{D} : \mathbb{D}^2 \looparrowright \mathbb{D}^4$ with $\mu = +1$.

the double point and the corresponding accessory disk

3D picture:



Note:

$$\lambda(\tilde{D}, \tilde{D}^{\uparrow}) = \overbrace{\mu(\tilde{D})}^{+1} + \overline{\mu(\tilde{D})} + e(\nu\tilde{D})$$

$\|$ since λ homotopy inv.

$$\lambda(D, D^{\uparrow}) = e(\nu D)$$ since D embedded.

$\Rightarrow e(\nu\tilde{D}) = e(\nu D) - 2$

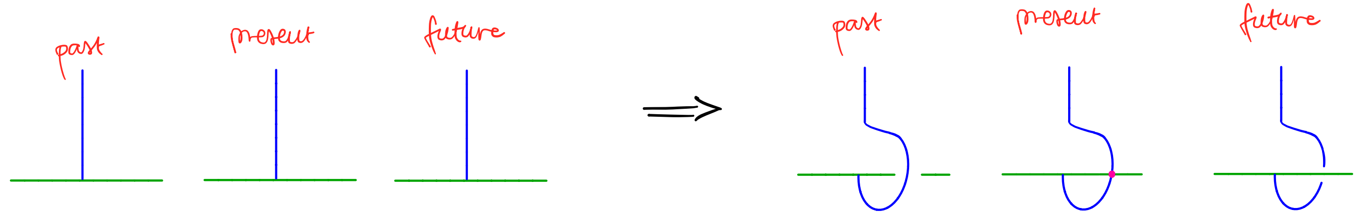
The framing of the twisted disk reduced by 2.

Note:

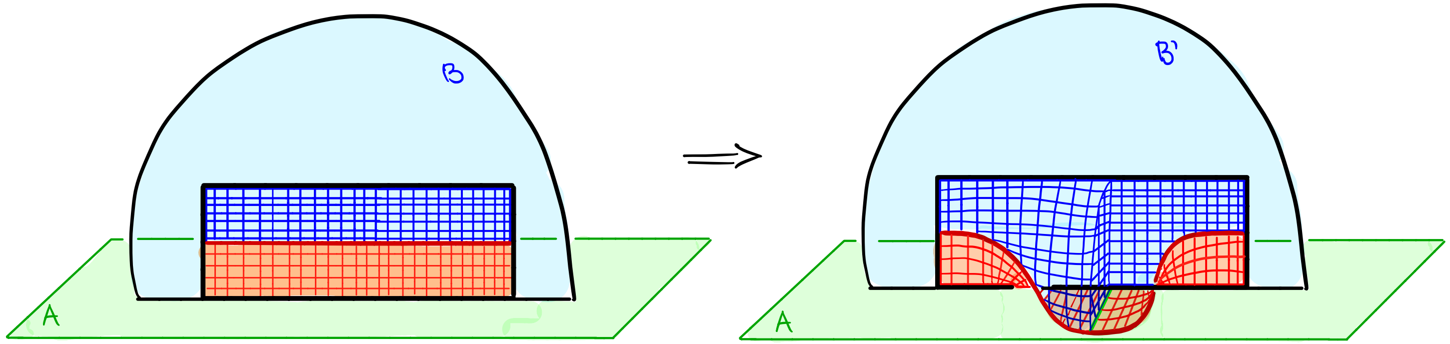
Reversing the movies wrt time we can get $\mu = -1$ and $\tilde{e} = e - 2$

Boundary Twist

THE MOVIE:



3D PICTURE:



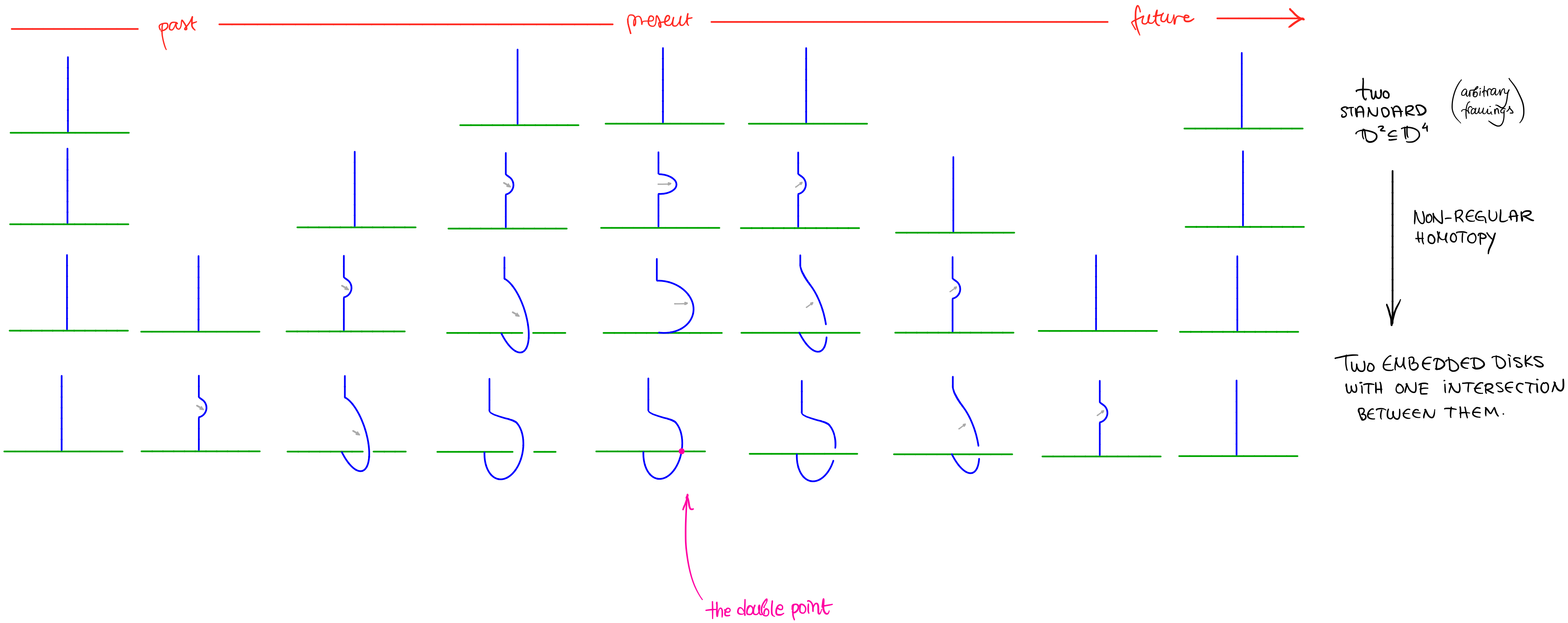
A, B : $\mathbb{D}^2 \subseteq \mathbb{D}^3 \subseteq \mathbb{D}^4$

B' is twisted.

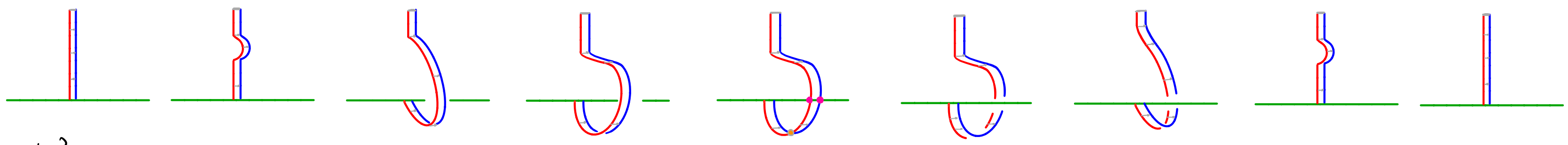
The homotopy

from embedded model to the Boundary Twist

Movie of movies:



Note: the parallel push-off $\tilde{B}^{\uparrow}$ now has one more intersection with $\tilde{B}$, than $B^{\uparrow}$ had with B.

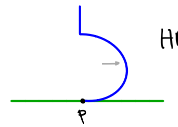


Therefore: $e(\nu\tilde{B}^{\uparrow}) = e(\nu B)$

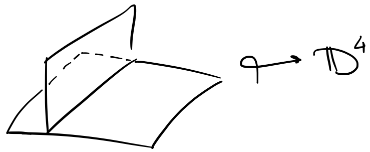
Remark. Both interior and boundary twist are produced using non-regular homotopies.

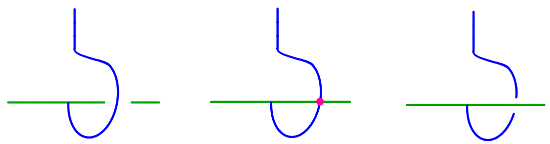
Recall: a regular homotopy is a homotopy through immersions.

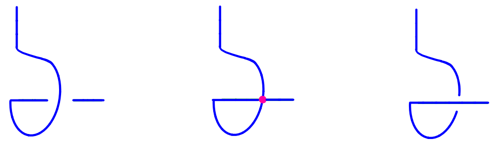
Namely: FOR THE INTERIOR TWIST WE HAD A MOMENT $t_0 \in [0,1]$ WHERE  HAPPENS WHICH IS NOT AN IMMERSION.

Similarly: FOR THE BOUNDARY TWIST WE HAD A MOMENT $t_0 \in [0,1]$ WHERE  HAPPENS.

HERE THE TWO DISKS SHARE THE SAME TANGENT VECTORS AT THE POINT $p \in \partial B \cap A$

This means that h_{t_0} is NOT a generic map 

Note: If in the boundary twist 

we take the "right green part" to be also part of B, and forget the rest of green:  is again just an interior twist!

And framing changes by ± 2 since we now see two intersections: