Long-term returns in stochastic interest rate models: Convergence in law.

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Abstract

Using an extension of the Cox-Ingersoll-Ross [1] stochastic model of the short interest rate r, we study the convergence in law of the long-term return in order to make some approximations. We use the theory of Bessel processes and observe the convergence in law of the sequence $\left(\sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt} (X_u + \frac{\delta_u}{2\beta}) du\right)_{t \geq 0}$ with the X a generalized Besselsquare process with drift with stochastic reversion level. By Aldous' criterion, we are able to prove that this sequence converges in law to a Brownian motion.

Key Words: Generalized Besselsquare processes, convergence in law, Aldous' criterion, stochastic interest rates.

AMS subject classification: 60Gxx, 60H10, 60H30.

1 Introduction.

Controlling the risk induced by interest rate fluctuations is of crucial importance for banks and insurance companies. Interest rate models can be used to obtain explicit formulae for pricing interest rate derivative securities and to construct a hedging strategy. They are also a necessary tool in managing long-term life insurance contracts.

As in "Long-term returns in stochastic interest rate models." [2], we study the long-term return using an extension of the Cox, Ingersoll and Ross [1] stochastic model of the short interest rate r. Cox, Ingersoll and Ross express the short interest rate dynamics as

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t.$$

with $(B_t)_{t\geq 0}$ a Brownian motion and κ, γ and σ positive constants. This is a fairly good model since r cannot become negative and the randomly moving interest rate is elastically pulled towards a reversion level. However, this is a constant level, namely the long-term value γ. It is more reasonable to conjecture that the market will influence this level. Schaefer and Schwartz [3], Hull and White [4] and Longstaff and Schwartz [5], proposed time-dependent parameters. We extend the CIR model by assuming a stochastic reversion level. In this way, we can treat more factor models.

If we define X by a transformation of the CIR square root process r, namely $X = \frac{4}{\sigma^2}r$, then X is a Besselsquare process with drift $-\kappa/2$ and dimension $\frac{4\kappa\gamma}{\sigma^2}$. The many results obtained by Yor [6,7], convinced us that these processes are very tractable. Therefore, we consider a family of stochastic processes X, which contains the Besselsquare processes with drift.

More precisely, we study processes X satisfying the stochastic differential equation:

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s \quad \forall s \in I\!R^+$$

with δ a non-negative adapted stochastic process and $\beta < 0$. In "Existence of Solutions of Stochastic Differential Equations related to the Bessel process" [8], we have shown that this stochastic differential equation has a unique (non-negative) strong solution as soon as $\int_0^t \delta_u \, du < \infty$ a.e. for all $t \in I\!R^+$.
In this paper, we will assume that $\frac{1}{t}\int_0^t \delta_u du \overset{a.e.}{\to} \overline{\delta}$ with $\overline{\delta} > 0$.

The following generalized CIR two-factor model is an element of this family:

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t \qquad (1) \\ d\gamma_t &= \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t \end{aligned}$$

with κ, $\tilde{\kappa} > 0$; γ^*, σ and $\tilde{\sigma}$ positive constants and $(B'_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ two Brownian motions. These Brownian motions are correlated in an arbitrarily way. Most authors suppose for technical reasons that the Brownian motions are uncorrelated or have a constant correlation. We do not need this assumption. The Brownian motions even may have a random correlation!

In "Long-term returns in stochastic interest rate models." [2], we found a convergence theorem. Under some conditions such as $\frac{1}{s}\int_0^s \delta_u du \overset{\text{a.e.}}{\to} \overline{\delta}$ with $\overline{\delta} : \Omega \to I\!R^+$, the following convergence almost everywhere holds:

$$\frac{1}{s}\int_0^s X_u du \overset{a.e.}{\longrightarrow} \frac{-\overline{\delta}}{2\beta}.$$

In this paper, we study the convergence in law. By convergence in law or in distribution we mean weak convergence of probability measures on the space $\mathcal{C}(I\!R^+, I\!R)$ of continuous sample paths. This space endowed with the topology of uniform convergence on compact sets of $I\!R^+$, is a Polish space. (See Revuz–Yor [7] p. 472).

We are interested in the convergence in law since it is always useful to know how the long-term return is distributed in the limit so that one can find approximations. In some earlier papers, authors have modeled interest rates by Wiener models. On long-term, the Central Limit Theorems are indeed applicable.

We suppose the same hypothesis as in [2] so that we can apply the previous convergence theorem. Using the theory of Bessel processes, we prove the following theorem:

Theorem 1 *Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ is given and that a stochastic process $X : \Omega \times I\!R^+ \to I\!R^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s \quad \forall s \in I\!R^+$$

with $(B_s)_{s\geq 0}$ a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\beta < 0$.
Let us make the following assumptions about the adapted and measurable process $\delta : \Omega \times I\!R^+ \to I\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \overset{\text{a.e.}}{\to} \overline{\delta}$ *where $\overline{\delta}$ is a real number,* $\overline{\delta} > 0$
- *There is a constant k such that* $\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!E\left[\delta_u^2\right] du \leq k$
- *For all $a \in I\!R^+$* $\quad \lim_{t\to\infty} \frac{1}{t}\int_{t-a}^t I\!E[\delta_u^2]du = 0.$

Under these conditions, the following convergence in distribution holds:

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \overset{\mathcal{L}}{\longrightarrow} (B'_t)_{t\geq 0}$$

where $(B'_t)_{t\geq 0}$ is a Brownian motion and where '$\overset{\mathcal{L}}{\longrightarrow}$' denotes convergence in law.

Remark that there is no assumption about the correlation between the process X and the process δ.

The organisation of the paper is as follows: Section 2 contains some technical lemmas. We show that, under the conditions of theorem 1, the second moment of X_t is bounded by a constant and that for a given $\varepsilon > 0$, there exists a constant c such that $\mathbb{P}\left[\sup_{u \leq t} \frac{X_u}{\sqrt{t}} > c\right] \leq \varepsilon$. In section 3, the main theorem is proved. We also state an alternative version of theorem 1, whose conditions are measure-invariant. Since in Finance the measure is often transformed, measure-invariance is an important property. Section 4 gives some applications of theorem 1. We explain the usefulness of theorem 1 by the two-factor model (1).

2 Technical lemmas.

In this section, we prove a technical lemma and its corollary, both needed in the proof of the convergence result. We assume without further notice that B is a continuous Brownian motion with respect to the filtration $(\mathcal{F}_s)_{s \geq 0}$.We consider a family of stochastic processes X, which contains the Besselsquare processes with drift.

Lemma 1 *Suppose the stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)\, ds + g(X_s) dB_s \quad \forall s \in \mathbb{R}^+ \tag{2}$$

with

- $\beta < 0$
- *$g : \mathbb{R} \to \mathbb{R}^+$ is a function, vanishing at zero and such that there is a constant b with $|g(x) - g(y)| \leq b\sqrt{|x - y|}$.*
- *$\delta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is an adapted and measurable process such that $\frac{1}{t}\int_0^t \mathbb{E}\left[\delta_u^2\right] du \leq k$ for all $t \geq 1$ with k a constant.*

Then for all $t \geq 1$

$$\mathbb{E}\left[\sup_{u \leq t} X_u^2\right] \leq K\, t$$

with K a constant independent of t and in fact only depending on β, k and b.

Remark: Before proving the lemma, we recall from [8] that a (continuous) adapted process X, given by the stochastic differential equation (2) has a unique strong non-negative solution as soon as $\int_0^t \delta_u \, du < \infty$ a.e. for all $t \in I\!R^+$. In [2], we obtained the expectation of X_s, namely:

$$I\!E[X_s] = e^{2\beta s} X_0 + e^{2\beta s} \int_0^s I\!E\left[\delta_u\right] e^{-2\beta u} \, du. \tag{3}$$

Proof. By Itô's formula,

$$X_t^2 = X_0^2 + 4\beta \int_0^t X_u^2 du + 2 \int_0^t X_u \delta_u du + \int_0^t g^2(X_u) du + 2 \int_0^t X_u g(X_u) dB_u.$$

Let us denote the continuous local martingale $2 \int_0^t X_u g(X_u) dB_u$ by M_t. We recall the notation $M_t^* = \sup_{u \leq t} M_u$. For all $u \leq t$, the positivity of X and δ imply the following inequality:

$$X_u^2 - 4\beta \int_0^u X_s^2 ds \leq X_0^2 + 2 \int_0^t X_u \delta_u du + \int_0^t g^2(X_u) du + M_t^*.$$

Since this is true for all $u \leq t$, we conclude that the supremum of the left hand side is smaller than the right hand side.

$$\sup_{u \leq t} \left(X_u^2 - 4\beta \int_0^u X_s^2 ds \right) \leq X_0^2 + 2 \int_0^t X_u \delta_u du + \int_0^t g^2(X_u) du + M_t^*.$$

Since $\beta < 0$,

$$\sup_{u \leq t} X_u^2 - 4\beta \int_0^t X_s^2 ds \leq 2 \sup_{u \leq t} \left(X_u^2 - 4\beta \int_0^u X_s^2 ds \right).$$

Consequently,

$$\begin{aligned} I\!E\left[\sup_{u \leq t} X_u^2\right] \;\leq\; & 2X_0^2 + 4\beta I\!E\left[\int_0^t X_u^2 du\right] + 4 I\!E\left[\int_0^t X_u \delta_u du\right] \\ & + 2 I\!E\left[\int_0^t g^2(X_u) du\right] + 2 I\!E\left[M_t^*\right]. \end{aligned}$$

Applying Cauchy–Schwarz' inequality in the third term and the inequality $|g(x)| \leq b\sqrt{x}$ in the fourth term, we obtain:

$$\begin{aligned} I\!E\left[\sup_{u \leq t} X_u^2\right] \leq & \; 2X_0^2 + 4\beta I\!E\left[\int_0^t X_u^2 du\right] \\ & + 4 I\!E\left[\int_0^t X_u^2 du\right]^{1/2} I\!E\left[\int_0^t \delta_u^2 du\right]^{1/2} + 2b^2 I\!E\left[\int_0^t X_u du\right] + 2 I\!E\left[M_t^*\right]. \end{aligned}$$

We remark that $X_u \le \frac{b^2}{-4\beta} + \frac{-\beta}{b^2} X_u^2$ since $0 \le \frac{-\beta}{b^2}\left(X_u + \frac{b^2}{2\beta}\right)^2$. Substituting this inequality, we find:

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t} X_u^2\right] - 2\beta \mathbb{E}\left[\int_0^t X_u^2 du\right] &\le 2X_0^2 + \frac{b^4}{-2\beta} t \\ &+ 4\mathbb{E}\left[\int_0^t X_u^2 du\right]^{1/2} \mathbb{E}\left[\int_0^t \delta_u^2 du\right]^{1/2} + 2\mathbb{E}\left[M_t^*\right].\end{aligned}$$

By the Burkholder–Davis–Gundy inequality (see Revuz–Yor [7] p. 151), we can rewrite the last term:

$$\begin{aligned}\mathbb{E}\left[M_t^*\right] &\le c\mathbb{E}\left[< M, M >_t^{1/2}\right] \\ &\le c\, b\, 2\mathbb{E}\left[\left(\int_0^t X_u^2 X_u du\right)^{1/2}\right] \\ &\le c\, b\, 2\mathbb{E}\left[\left(\int_0^t X_u du\right)^{1/2} \left(\sup_{u\le t} X_u^2\right)^{1/2}\right] \\ &\le c\, b\, 2\mathbb{E}\left[\int_0^t X_u du\right]^{1/2} \mathbb{E}\left[\sup_{u\le t} X_u^2\right]^{1/2} \\ &\le c\, b\, 2\, t^{1/4}\mathbb{E}\left[\int_0^t X_u^2 du\right]^{1/4} \mathbb{E}\left[\sup_{u\le t} X_u^2\right]^{1/2}\end{aligned}$$

where we applied the inequality of Cauchy–Schwarz in the last inequalities.
Summarising, we conclude that:

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t} X_u^2\right] - 2\beta \mathbb{E}\left[\int_0^t X_u^2 du\right] &\le 2X_0^2 + \frac{b^4}{-2\beta} t \\ + 4\mathbb{E}\left[\int_0^t X_u^2 du\right]^{1/2} \mathbb{E}\left[\int_0^t \delta_u^2 du\right]^{1/2} &+ cb4t^{1/4}\mathbb{E}\left[\int_0^t X_u^2 du\right]^{1/4} \mathbb{E}\left[\sup_{u\le t} X_u^2\right]^{1/2}.\end{aligned}$$

Remark that by hypothesis $\mathbb{E}\left[\int_0^t \delta_u^2 du\right] \le k\, t$ for $t \ge 1$.
If we denote $\mathbb{E}\left[\sup_{u\le t} X_u^2\right]^{1/2}$ by A, $\sqrt{-2\beta}\mathbb{E}\left[\int_0^t X_u^2 du\right]^{1/2}$ by B and the constants by C_i $(i \ge 1)$, then we have to find the solution of the inequality:

$$A^2 + B^2 \le C_1 + C_2 t + C_3 t^{1/2} B + C_4 t^{1/4} A B^{1/2}.$$

We remark that $B^{1/2} \le \frac{1}{C_4 t^{1/4}} B + \frac{C_4 t^{1/4}}{4}$ since $\frac{1}{C_4 t^{1/4}}\left(B^{1/2} - \frac{C_4 t^{1/4}}{2}\right)^2 \ge 0$.
Substitution yields:

$$A^2 + B^2 \le C_1 + C_2 t + C_3 t^{1/2} B + AB + C_5 t^{1/2} A.$$

Since $AB \leq \frac{1}{2}(A^2 + B^2)$, we obtain:

$$A^2 + B^2 \leq 2C_1 + 2C_2 t + 2C_3 t^{1/2} B + 2C_5 t^{1/2} A.$$

This is equivalent with:

$$\left(A - C_5 t^{1/2}\right)^2 + \left(B - C_3 t^{1/2}\right)^2 \leq 2C_1 + 2C_2 t + C_3^2 t + C_5^2 t.$$

Thus $\left(A - C_5 t^{1/2}\right)^2$ as well as $\left(B - C_5 t^{1/2}\right)^2$ are smaller than the right hand side. Substituting the expression for A, we find for all $t \geq 1$

$$\mathbb{E}\left[\sup_{u \leq t} X_u^2\right] \leq K\, t$$

with K a constant, only depending on k, β and b.

q.e.d.

Corollary 1. *Suppose the stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in \mathbb{R}^+$$

with

- $\beta < 0$
- *$g : \mathbb{R} \to \mathbb{R}^+$ is a function, vanishing at zero and such that there is a constant b with $|g(x) - g(y)| \leq b\sqrt{|x - y|}$.*
- *$\delta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is an adapted and measurable process such that $\frac{1}{t}\int_0^t \mathbb{E}\left[\delta_u^2\right] du \leq k$ for all $t \geq 1$ with k a constant.*

Then, for $\varepsilon > 0$, there exists a real number c such that for all $t \geq 1$

$$\mathbb{P}\left[\sup_{u \leq t} \frac{X_u}{\sqrt{t}} > c\right] \leq \varepsilon.$$

Proof. By Chebyshev's inequality

$$\mathbb{P}\left[\sup_{u \leq t} \frac{X_u}{\sqrt{t}} > c\right] \leq \frac{1}{c^2 t}\mathbb{E}\left[\sup_{u \leq t} X_u^2\right].$$

Since by lemma 1, $\mathbb{E}\left[\sup_{u \leq t} X_u^2\right] \leq K\, t$ with K a constant, we obtain for all $c > 0$ and all $t \geq 1$

$$\mathbb{P}\left[\sup_{u \leq t} \frac{X_u}{\sqrt{t}} > c\right] \leq \frac{K}{c^2}.$$

Thus, we conclude that for $\varepsilon > 0$, there exists a real number c such that for all $t \geq 1$

$$\mathbb{P}\left[\sup_{u \leq t} \frac{X_u}{\sqrt{t}} > c\right] \leq \varepsilon.$$

q.e.d.

3 Convergence in law to a Brownian motion.

In this section, we give the proof of the main result of this paper, namely the convergence in law to a Brownian motion. We suppose the same hypothesis as in [2] so that we can apply the convergence a.e. result in our proof of the convergence in law.

Theorem 1 *Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ is given and that a stochastic process $X : \Omega \times I\!\!R^+ \to I\!\!R^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s \quad \forall s \in I\!\!R^+$$

with $(B_s)_{s\geq 0}$ a Brownian motion and $\beta < 0$. Let us make the following assumptions about the adapted and measurable process $\delta : \Omega \times I\!\!R^+ \to I\!\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \overset{\text{a.e.}}{\to} \overline{\delta}$ *where $\overline{\delta}$ is a real number, $\overline{\delta} > 0$*
- $\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!\!E\left[\delta_u^2\right] du \leq k$ *with k a constant independent of t.*
- *For all $a \in I\!\!R^+$ $\lim_{t\to\infty} \frac{1}{t}\int_{t-a}^t I\!\!E\left[\delta_u^2\right] du = 0$.*

Under these conditions, the following convergence in distribution is true:

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \overset{\mathcal{L}}{\longrightarrow} (B'_t)_{t\geq 0}$$

with $(B'_t)_{t\geq 0}$ a Brownian motion.

Remark: The statements of theorem 1 imply the conditions of the theorem in the previous paper [2] and hence, we may conclude that the long-term return converges a.e.:

$$\frac{1}{s}\int_0^s X_u du \overset{\text{a.e.}}{\longrightarrow} \frac{-\overline{\delta}}{2\beta}. \tag{4}$$

Proof: First, we will check that the sequence (Y^n), defined by

$$Y_t^n = \sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du \qquad\qquad n \geq 1$$

converges to the Brownian motion $B_t^{'}$ in the sense of finite distributions. Afterwards, we will show that the sequence is weakly relatively compact by using Aldous' criterion for tightness.

To prove the convergence in the sense of finite distributions, we show that for any finite collection $(t_1, \cdots, t_k)$ of times, the random variables $(Y^n_{t_1}, \cdots, Y^n_{t_k})$ converge in law to $(B'_{t_1}, \cdots, B'_{t_k})$. Thus, we have to check that:

$$\left(\sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt_1} \left(X_u + \frac{\delta_u}{2\beta}\right) du, \cdots, \sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt_k} \left(X_u + \frac{\delta_u}{2\beta}\right) du\right) \xrightarrow{\mathcal{L}} \left(B'_{t_1}, \cdots, B'_{t_k}\right).$$

We recall the following well known theorem of probability theory (See Feller [9] p. 247):

Theorem A *Suppose that a sequence $Z_n : \Omega \longrightarrow \mathbb{R}^k$ is given. If $Z_n = U_n + V_n$ where*

- $U_n \xrightarrow{\mathcal{L}} U$ *with $U \sim \mu$ a probability measure*
- $V_n \xrightarrow{\mathbb{P}} 0$

then

$$Z_n \xrightarrow{\mathcal{L}} U.$$

From the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s$$

we arrive at the following formula:

$$\sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt} \left(X_u + \frac{\delta_u}{2\beta}\right) du = \sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \frac{X_{nt} - X_0}{2\beta} - \sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt} \frac{\sqrt{X_s}}{\beta} dB_s.$$

Consequently, we can rewrite the random variables:

$$\begin{aligned} &(Y^n_{t_1}, \cdots, Y^n_{t_k}) \\ = \quad & \left(\sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \frac{X_{nt_1} - X_0}{2\beta}, \cdots, \sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \frac{X_{nt_k} - X_0}{2\beta}\right) \\ & - \left(\sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt_1} \frac{\sqrt{X_s}}{\beta} dB_s, \cdots, \sqrt{\frac{-2\beta^3}{\bar{\delta} n}} \int_0^{nt_k} \frac{\sqrt{X_s}}{\beta} dB_s\right). \end{aligned}$$

The first vector converges in distribution to zero because each component converges to zero in probability. Namely, for all $\varepsilon > 0$, there exists a t_0 such that for all $t \geq t_0$ $\frac{\mathbb{E}[X_t]}{\sqrt{t+1}} < \varepsilon$. Indeed,

$$\frac{\mathbb{E}[X_t]}{\sqrt{t+1}} = \frac{e^{2\beta t} X_0}{\sqrt{t+1}} + \frac{1}{\sqrt{t+1}} \mathbb{E}\left[e^{2\beta t} \int_0^t e^{-2\beta u} \delta_u du\right]$$

by remark (3) in section 2.

Trivially, the first term converges to zero. Let us treat the second term:

$$\begin{aligned}
&\frac{1}{\sqrt{t+1}}\mathbb{E}\left[e^{2\beta t}\int_0^t e^{-2\beta u}\delta_u du\right] \\
= \;& \frac{1}{\sqrt{t+1}}\mathbb{E}\left[e^{2\beta t}\int_0^{t-a} e^{-2\beta u}\delta_u du\right] + \frac{1}{\sqrt{t+1}}\mathbb{E}\left[e^{2\beta t}\int_{t-a}^{t} e^{-2\beta u}\delta_u du\right] \\
\leq \;& \mathbb{E}\left[\left(\int_0^{t-a} e^{4\beta(t-u)}du\right)^{1/2}\left(\frac{1}{t+1}\int_0^{t-a}\delta_u^2 du\right)^{1/2}\right] \\
& + \mathbb{E}\left[\left(\int_{t-a}^{t} e^{4\beta(t-u)}du\right)^{1/2}\left(\frac{1}{t+1}\int_{t-a}^{t}\delta_u^2 du\right)^{1/2}\right] \\
\leq \;& \frac{e^{2\beta a}}{2\sqrt{-\beta}}\left(\frac{1}{t+1}\int_0^{t-a}\mathbb{E}[\delta_u^2]du\right)^{1/2} + \frac{1}{2\sqrt{-\beta}}\left(\frac{1}{t+1}\int_{t-a}^{t}\mathbb{E}[\delta_u^2]du\right)^{1/2}.
\end{aligned}$$

Since by hypothesis $\sup_{t\geq 1}\frac{1}{t}\int_0^t \mathbb{E}\left[\delta_u^2\right]du \leq k$ with k a constant, the first term is smaller than $Ke^{2\beta a}$ with K a constant. We can choose a such that $Ke^{2\beta a} < \varepsilon/2$. Since by hypothesis for all $a \in \mathbb{R}^+$, $\lim_{t\to\infty}\frac{1}{t}\int_{t-a}^t \mathbb{E}[\delta_u^2]du = 0$, there exists a $t_0 \geq a$ such that for all $t \geq t_0$: $\frac{1}{t+1}\int_{t-a}^t \mathbb{E}[\delta_u^2]du < -\beta\varepsilon^2$. Thus, we proved that $\frac{\mathbb{E}[X_t]}{\sqrt{t+1}} \longrightarrow 0$. Consequently,

$$\frac{X_{nt_i}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0 \qquad\qquad i = 1,\cdots,k.$$

By theorem A, it remains to be shown that

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt_1}\frac{\sqrt{X_s}}{\beta}dB_s,\cdots,\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt_k}\frac{\sqrt{X_s}}{\beta}dB_s\right)$$

converges in distribution to a k-dimensional Brownian motion.

Let us define the stopping times

$$\tau_t = \inf\left\{s \mid \int_0^s X_u du \geq \frac{-\overline{\delta}t}{2\beta}\right\}.$$

Let us remark that two properties hold for these stopping times:

$$\lim_{t\to\infty}\tau_t = \infty$$

and

$$\lim_{t\to\infty}\frac{\tau_t}{t} = 1. \tag{5}$$

Let us give a short proof of the second statement:
By theorem 1, $\frac{1}{u}\int_0^u X_s ds \xrightarrow{\text{a.e.}} \frac{-\overline{\delta}}{2\beta}$. Thus, for an arbitrary $\varepsilon > 0$, we have for all u large enough that:

$$(1-\varepsilon)\frac{-\overline{\delta}u}{2\beta} \leq \int_0^u X_s ds \leq (1+\varepsilon)\frac{-\overline{\delta}u}{2\beta}.$$

This implies that $\lim_{t\to\infty} \tau_t = \infty$ and hence, we have for all ε and u large enough that:

$$(1-\varepsilon)\frac{-\overline{\delta}\tau_u}{2\beta} \leq \int_0^{\tau_u} X_s ds \leq (1+\varepsilon)\frac{-\overline{\delta}\tau_u}{2\beta}.$$

But by definition of τ_u, $\int_0^{\tau_u} X_s ds = \frac{-\overline{\delta}u}{2\beta}$. After a simplification, we obtain for u large enough:

$$\frac{1}{1+\varepsilon} \leq \frac{\tau_u}{u} \leq \frac{1}{1-\varepsilon}.$$

Since this is true for any ε, we have proved that $\lim_{t\to\infty} \frac{\tau_t}{t} = 1$.
With the help of the theorem of Dambis, Dubins and Schwarz (see Revuz–Yor [7], p. 170), it is easy to verify that $\left(\sqrt{\frac{-2\beta}{\overline{\delta}n}}\int_0^{\tau_{nt}} \sqrt{X_u} dB_u\right)_{t\geq 0}$ is a Brownian motion $B^{'}$.

Since trivially,

$$\begin{aligned}
&\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt_1} \frac{\sqrt{X_s}}{\beta} dB_s, \cdots, \sqrt{\frac{-2\beta^3}{\overline{line\delta}n}}\int_0^{nt_k} \frac{\sqrt{X_s}}{\beta} dB_s\right) \\
&= \left(\sqrt{\frac{-2\beta}{\overline{\delta}n}}\int_0^{\tau_{nt_1}} \sqrt{X_u} dB_u, \cdots, \sqrt{\frac{-2\beta}{\overline{\delta}n}}\int_0^{\tau_{nt_k}} \sqrt{X_u} dB_u\right) \\
&\quad + \left(\sqrt{\frac{-2\beta}{\overline{\delta}n}}\int \left(1\!\!1_{[\![0,nt_1]\!]} - 1\!\!1_{[\![0,\tau_{nt_1}]\!]}\right)\sqrt{X_u} dB_u, \cdots\right. \\
&\qquad\qquad \left.\cdots, \sqrt{\frac{-2\beta}{\overline{\delta}n}}\int \left(1\!\!1_{[\![0,nt_k]\!]} - 1\!\!1_{[\![0,\tau_{nt_k}]\!]}\right)\sqrt{X_u} dB_u\right)
\end{aligned}$$

By theorem A, we only need to show the convergence in distribution or in probability to zero of the second vector. We will show that each component converges in probability to zero. This result follows from stochastic integration theory. We cite the theorem that we will use, see e.g. Karatzas-Shreve [10] p. 147, proposition 2.26 which remains valid for $T \equiv \infty$:

Theorem B *Let us denote* $L^{2,0} = \{\, H \mid H$ *predictable with* $\int_0^\infty H_u^2 du < \infty \,\}$ *and let us define*

$$L^{2,0} \longrightarrow L^0(\Omega, \mathcal{F}_\infty, I\!\!P) \qquad H \longmapsto (H\cdot B)_\infty$$

If $\int_0^\infty (H_u^n)^2 du \xrightarrow{I\!\!P} 0$, *then* $(H^n \cdot B)_\infty \xrightarrow{I\!\!P} 0$.

We apply this theorem for $i = 1, \cdots, k$ and for the processes:

$$H^{n,i}_{\cdot} \equiv \left(1\!\!1_{[\![0,nt_i]\!]} - 1\!\!1_{[\![0,\tau_{nt_i}]\!]} \right) \frac{\sqrt{X_{\cdot}}}{\sqrt{nt_i}} \qquad i = 1, \cdots, k.$$

In this case, the condition of theorem B is fulfilled:

$$\int_0^\infty (H^{n,i}_u)^2 du = \frac{1}{nt_i} \int \left(1\!\!1_{[\![0,nt_i]\!]} - 1\!\!1_{[\![0,\tau_{nt_i}]\!]} \right) X_u du \xrightarrow{I\!\!P} 0.$$

Indeed

$$\begin{aligned} &\frac{1}{nt_i} \int \left(1\!\!1_{[\![0,nt_i]\!]} - 1\!\!1_{[\![0,\tau_{nt_i}]\!]} \right) X_u du \\ &\quad = \frac{1}{nt_i} \left(\int_0^{nt_i} X_u du \right) - \frac{\tau_{nt_i}}{nt_i} \left(\int_0^{\tau_{nt_i}} \frac{X_u}{\tau_{nt_i}} du \right) \xrightarrow{\text{a.e.}} 0 \end{aligned}$$

because

- $\frac{1}{s} \int_0^s X_u du \xrightarrow{\text{a.e.}} \frac{-\overline{\delta}}{2\beta}$ by remark (4)
- $\frac{\tau_{nt_i}}{nt_i} \xrightarrow{\text{a.e.}} 1$ by remark (5).

Thus, we conclude that for all $i = 1, \cdots, k$

$$(H^{n,i} \cdot B)_\infty = \frac{1}{\sqrt{nt_i}} \int \left(1\!\!1_{[\![0,nt_i]\!]} - 1\!\!1_{[\![0,\tau_{nt_i}]\!]} \right) \sqrt{X_u} dB_u \xrightarrow{I\!\!P} 0$$

or equivalently that for all $i = 1, \cdots, k$

$$\sqrt{\frac{-2\beta t_i}{\overline{\delta} n}} (H^{n,i} \cdot B)_\infty = \sqrt{\frac{-2\beta}{\overline{\delta} n}} \int \left(1\!\!1_{[\![0,nt_i]\!]} - 1\!\!1_{[\![0,\tau_{nt_i}]\!]} \right) \sqrt{X_u} dB_u \xrightarrow{I\!\!P} 0.$$

Thus, we have shown that for any finite collection $(t_1, \cdots, t_k)$ of times, the random variables $\left(\sqrt{\frac{-2\beta^3}{\overline{\delta} n}} \int_0^{nt_1} \left(X_u + \frac{\delta_u}{2\beta} \right) du, \cdots, \sqrt{\frac{-2\beta^3}{\overline{\delta} n}} \int_0^{nt_k} \left(X_u + \frac{\delta_u}{2\beta} \right) du \right)$ converge in law to $(B'_{t_1}, \cdots, B'_{t_k})$.

We will now prove that the laws of (Y^n) form a weakly relatively compact sequence. We have to check two conditions (see e.g. Jacod–Shiryaev [11], p. 320):

$$\begin{aligned} &\text{For all } N \in I\!\!N^*, \varepsilon > 0, \text{ there exists } n_0 \in I\!\!N^* \text{ and } K \in I\!\!R^+ \\ &\qquad \text{such that for all } n \geq n_0 : \qquad I\!\!P \left(\sup_{t \leq N} |Y^n_t| > K \right) \leq \varepsilon. \end{aligned}$$

and Aldous' criterion, namely

$$\begin{aligned} &\text{For all } N \in I\!\!N^*, \varepsilon > 0 : \\ &\qquad \lim_{\theta \to 0} \limsup_n \sup_{S \leq T \leq S+\theta} I\!\!P(|Y^n_T - Y^n_S| \geq \varepsilon) = 0 \end{aligned}$$

where S and T are restricted to the set of $\mathcal{F}^n$-stopping times that are bounded by N.

Let us start with Aldous' criterion. This is equivalent to:

$$\begin{aligned}&\text{For all } N \in I\!N^*,\ \varepsilon > 0 :\\ &\lim_{\theta\to 0} \limsup_n \sup_S I\!P\big(\sup_{0\le u\le\theta} \left|Y^n_{S+u} - Y^n_S\right| \ge \varepsilon\big) = 0\end{aligned}$$

where S is restricted to the set of $\mathcal{F}^n$-stopping times bounded by N. We will therefore search a bound for

$$I\!P\left[\sup_{0\le u\le\theta} \left|Y^n_{S+u} - Y^n_S\right| \ge 2\varepsilon\right]$$

or if we substitute the expression of Y^n, for:

$$I\!P\left[\sup_{0\le u\le\theta}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\left(X_t + \frac{\delta_t}{2\beta}\right)dt\right| \ge 2\varepsilon\right].$$

From the stochastic differential equation, we easily find

$$\int \frac{dX_u}{2\beta} = \int (X_u + \frac{\delta_u}{2\beta})du + \int \frac{\sqrt{X_u}}{\beta}dB_u.$$

We replace $\int (X_u + \frac{\delta_u}{2\beta})du$ and obtain:

$$I\!P\left[\sup_{0\le u\le\theta}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\frac{\sqrt{X_t}}{\beta}dB_t + \sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\frac{dX_t}{2\beta}\right| \ge 2\varepsilon\right].$$

Trivially, this probability is smaller than

$$\begin{aligned}I\!P&\left[\sup_{0\le u\le\theta}\left|-\sqrt{\frac{-2\beta}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\sqrt{X_t}dB_t\right| \ge \varepsilon\right] \\ &+ I\!P\left[\sup_{0\le u\le\theta}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\frac{dX_t}{2\beta}\right| \ge \varepsilon\right].\end{aligned} \tag{6}$$

Let us concentrate on the first term. We know that $\left(\int_0^s \sqrt{X_t}dB_t\right)_{s\ge 0}$ is a martingale. If S is a $\mathcal{F}^n$-stopping time, then nS and $n(S+u)$ are $\mathcal{F}$-stopping times. Consequently $\left(\int_{Sn}^{(S+u)n}\sqrt{X_t}dB_t\right)_{u\ge 0}$ is a $(\mathcal{F}_{Sn+nu})_{u\ge 0}$-martingale.

Therefore, we can apply the martingale inequality

$$\begin{aligned}&I\!P\left[\sup_{0\le u\le\theta}\left|-\sqrt{\frac{-2\beta}{\overline{\delta}n}}\int_{Sn}^{(S+u)n}\sqrt{X_t}dB_t\right| \ge \varepsilon\right]\\ &\le \quad \frac{-2\beta}{\overline{\delta}n\varepsilon^2}\left\|\int_{Sn}^{(S+\theta)n}\sqrt{X_t}dB_t\right\|_2^2.\end{aligned}$$

By stochastic calculus, this is equal to

$$\frac{-2\beta}{\overline{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} X_u du\right].$$

We can replace $\int_{Sn}^{(S+\theta)n} X_u du$, using the integrated stochastic differential equation:

$$\int_{Sn}^{(S+\theta)n} dX_u = 2\beta\int_{Sn}^{(S+\theta)n} X_u\, du + \int_{Sn}^{(S+\theta)n} \delta_u\, du + 2\int_{Sn}^{(S+\theta)n} \sqrt{X_u} dB_u.$$

Since $\int_{Sn}^{(S+\theta)n} \sqrt{X_u} dB_u$ is a martingale, its expected value is equal to zero and we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\le u\le\theta}\left|-\sqrt{\frac{-2\beta}{\overline{\delta} n}}\int_{Sn}^{(S+u)n} \sqrt{X_t} dB_t\right| \ge \varepsilon\right] \\
&\le \frac{1}{\overline{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} \delta_u du\right] - \frac{1}{\overline{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} dX_u\right]. \qquad (7)
\end{aligned}$$

We will handle both terms in the right hand side separately. Let us start with the first term. By an application of the Cauchy-Schwarz inequality, we obtain for all $N \in \mathbb{N}^*$ and $\varepsilon > 0$:

$$\begin{aligned}
&\lim_{\theta\to 0}\ \limsup_n\ \sup_S \frac{1}{\overline{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} \delta_u du\right] \\
&\le \lim_{\theta\to 0}\limsup_n\ \sup_S \frac{1}{\overline{\delta}\varepsilon^2}\mathbb{E}\left[(\theta n)^{1/2}\left(\frac{1}{n^2}\int_{Sn}^{(S+\theta)n} \delta_u^2 du\right)^{1/2}\right] \\
&\le \lim_{\theta\to 0}\limsup_n \frac{1}{\overline{\delta}\varepsilon^2}(\theta n)^{1/2}\left(\frac{1}{n^2}\int_0^{(N+\theta)n} \mathbb{E}\left[\delta_u^2\right] du\right)^{1/2} \\
&\le \lim_{\theta\to 0}\limsup_n \frac{1}{\overline{\delta}\varepsilon^2}\sqrt{\theta}\left((N+1)\frac{1}{(N+1)n}\int_0^{(N+1)n} \mathbb{E}\left[\delta_u^2\right] du\right)^{1/2} \\
&\le \lim_{\theta\to 0}\sqrt{\theta}\frac{\sqrt{k(N+1)}}{\overline{\delta}\varepsilon^2} = 0
\end{aligned}$$

where we have used the hypothesis that $\sup_{t\ge 1}\frac{1}{t}\int_0^t \mathbb{E}\left[\delta_u^2\right] du \le k$ with$k$ a constant.In the case of the second term of inequality (7), we remark that

$$\lim_{\theta\to 0}\ \limsup_n\ \sup_S \frac{-1}{\overline{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} dX_u\right]$$

$$
\begin{aligned}
&= \lim_{\theta\to 0}\ \limsup_n\ \sup_S \frac{\mathbb{E}[X_{Sn}] - \mathbb{E}[X_{(S+\theta)n}]}{\bar{\delta} n\varepsilon^2}\\
&\leq \lim_{\theta\to 0}\ \limsup_n\ \sup_S \frac{\mathbb{E}[X_{Sn}]}{\bar{\delta} n\varepsilon^2}\\
&\leq \lim_{\theta\to 0}\ \limsup_n \frac{\mathbb{E}\left[\sup_{u\leq Nn} X_u\right]}{nN}\frac{N}{\bar{\delta}\varepsilon^2}.
\end{aligned}
$$

If we use Jensen's inequality, we obtain:

$$
\frac{\mathbb{E}\left[\sup_{u\leq nN}\ X_u\right]}{nN} \leq \frac{\mathbb{E}\left[\sup_{u\leq nN}\ X_u^2\right]^{1/2}}{nN}
$$

By corollary 1, we find

$$
\frac{\mathbb{E}\left[\sup_{u\leq nN}\ X_u\right]}{nN} \leq \frac{K\sqrt{nN}}{nN}.
$$

Thus,

$$
\frac{\sup_{u\leq nN} X_u}{nN} \xrightarrow{L^1} 0.
$$

We conclude that the second term in inequality (7) also converges:

$$
\lim_{\theta\to 0}\ \limsup_n\ \sup_S \frac{-1}{\bar{\delta} n\varepsilon^2}\mathbb{E}\left[\int_{Sn}^{(S+\theta)n} dX_u\right] = 0.
$$

Let us now look at the second probability in inequality (6) and let us check if for all $N \in \mathbb{N}^*$ and $\varepsilon > 0$:

$$
\lim_{\theta\to 0}\ \limsup_n\ \sup_S \mathbb{P}\left[\sup_{0\leq u\leq\theta}\left|\sqrt{\frac{-2\beta^3}{\bar{\delta} n}}\int_{Sn}^{(S+u)n}\frac{dX_t}{2\beta}\right| \geq \varepsilon\right] = 0
$$

where S is restricted to the set of $\mathcal{F}^n$-stopping times bounded by N. We start the calculations:

$$
\begin{aligned}
&\sup_S \mathbb{P}\left[\sup_{0\leq u\leq\theta}\left|\int_{Sn}^{(S+u)n}\frac{dX_t}{\sqrt{n}}\right| \geq 2\varepsilon\right]\\
&\quad\leq \sup_S \mathbb{P}\left[\left(\sup_{0\leq u\leq\theta}\frac{X_{(S+u)n}}{\sqrt{n}} + \frac{X_{Sn}}{\sqrt{n}}\right) \geq 2\varepsilon\right]\\
&\quad\leq \sup_S \mathbb{P}\left[\sup_{0\leq u\leq\theta}\frac{X_{(S+u)n}}{\sqrt{n}} \geq \varepsilon\right] + \sup_S \mathbb{P}\left[\frac{X_{Sn}}{\sqrt{n}} \geq \varepsilon\right]\\
&\quad\leq \mathbb{P}\left[\sup_{0\leq v\leq (N+\theta)n}\frac{X_v}{\sqrt{n(N+\theta)}} \geq \varepsilon\sqrt{N+\theta}\right] + \mathbb{P}\left[\sup_{0\leq v\leq Nn}\frac{X_v}{\sqrt{nN}} \geq \varepsilon\sqrt{N}\right].
\end{aligned}
$$

And by lemma 2, Aldous' criterion is fulfilled.

To obtain tightness, we also have to check the remaining condition:

$$\text{For all } N \in \mathbb{N}^*, \varepsilon > 0, \text{ there exists a } n_0 \in \mathbb{N}^* \text{ and a } K \in \mathbb{R}^+$$
$$\text{such that for all } n \geq n_0 \;\; \mathbb{P}\left(\sup_{t\leq N} |Y_t^n| > K\right) \leq \varepsilon.$$

By an analogous reasoning:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\leq N}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\left(\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)\right| > 2K\right)\\
&\quad\leq \; \mathbb{P}\left(\left(\sup_{t\leq N}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\int_0^{nt}\frac{\sqrt{X_u}}{\beta}dB_u\right| + \sup_{t\leq N}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\frac{X_{tn}-X_0}{2\beta}\right|\right) > 2K\right)\\
&\quad\leq \; \mathbb{P}\left(\sup_{t\leq N}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\int_0^{nt}\frac{\sqrt{X_u}}{\beta}dB_u\right| > K\right) &(8)\\
&\qquad + \mathbb{P}\left(\sup_{t\leq N}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\frac{X_{tn}-X_0}{2\beta}\right| > K\right). &(9)
\end{aligned}$$

For the first term (8), we can apply the martingale inequality:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\leq N}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\int_0^{nt}\frac{\sqrt{X_u}}{\beta}dB_u\right| > K\right)\\
&\quad\leq \; \frac{-2\beta}{\overline{\delta} n K^2}\left\|\int_0^{Nn}\sqrt{X_u}dB_u\right\|_2^2\\
&\quad\leq \; \frac{-2\beta}{\overline{\delta} n K^2}\int_0^{Nn}\mathbb{E}[X_u]du\\
&\quad\leq \; \frac{-2\beta}{\overline{\delta} n K^2}\int_0^{Nn}e^{2\beta s}X_0 ds - \frac{2\beta}{\overline{\delta} n K^2}\int_0^{Nn}e^{2\beta s}\int_0^s e^{-2\beta u}\mathbb{E}[\delta_u]du
\end{aligned}$$

where we have applied remark (4) of section 2. By Fubini's theorem, we find:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\leq N}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta} n}}\int_0^{nt}\frac{\sqrt{X_u}}{\beta}dB_u\right| > K\right)\\
&\quad\leq \; \frac{-2\beta N X_0}{\overline{\delta} K^2} - \frac{2\beta}{\overline{\delta} n K^2}\int_0^{Nn}e^{-2\beta u}\mathbb{E}[\delta_u]du\int_s^{nN}e^{2\beta s}ds\\
&\quad\leq \; \frac{-2\beta N X_0}{\overline{\delta} K^2} - \frac{2\beta}{\overline{\delta} n K^2}\int_0^{Nn}\mathbb{E}[\delta_u]du\\
&\quad\leq \; \frac{-2\beta N X_0}{\overline{\delta} K^2} - \frac{2\beta N}{\overline{\delta} K^2}\left(\frac{1}{nN}\int_0^{Nn}\mathbb{E}[\delta_u^2]du\right)^{1/2}
\end{aligned}$$

$$\leq \quad \frac{-2\beta N(X_0+\sqrt{k})}{\overline{\delta}K^2}$$

since by hypothesis $\sup_{t\geq 1}\frac{1}{t}\int_0^t I\!E\left[\delta_u^2\right]du \leq k$ with k a constant.

We conclude that for all $N \in I\!N^*$, $\varepsilon > 0$, there exists a $n_0 \in I\!N^*$ and a $K \in I\!R^+$ such that for all $n \geq n_0$:

$$I\!P\left(\sup_{t\leq N}\left|-\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\frac{\sqrt{X_u}}{\beta}dB_u\right| > K\right) \leq \varepsilon/2.$$

As regards the second term (9), it remains to be shown that for all $N \in I\!N^*$, $\varepsilon > 0$, there exists a $n_0 \in I\!N^*$ and a $K \in I\!R^+$ such that for all $n \geq n_0$:

$$I\!P\left(\sup_{t\leq N}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\frac{X_{tn}-X_0}{2\beta}\right| > K\right) < \varepsilon/2.$$

We will transform this condition to a previous result:

$$\begin{aligned}
&I\!P\left(\sup_{t\leq N}\left|\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\frac{X_{tn}-X_0}{2\beta}\right| > K\right) \\
&\leq \quad I\!P\left(\sup_{t\leq nN}\frac{X_t}{\sqrt{nN}} > K'\right) + I\!P\left(\frac{X_0}{\sqrt{n}} > K\text{"}\right).
\end{aligned}$$

Again by lemma 2, the first term is smaller than $\varepsilon/4$ for K' large enough. The convergence of the second term is a triviality.

q.e.d.

This theorem is very useful in deducing results about the limit-distribution of the long-term return. In the following section, we will give some applications.

First, we want to improve theorem 1 in the sense that we search for conditions which are measure-invariant. Since in Finance the measure is often transformed to obtain risk-neutral measures, measure-invariant hypothesis are important. Therefore, we will give an alternative version of theorem 1, in which the assumptions are not expressed in function of moments but in which the boundedness in L^0 of a convex hull is needed. As boundedness in L^0 is a measure-invariant property, this is an improvement.

Before stating theorem 2, we give a direct corollary of theorem 1, which is needed in the proof of theorem 2:

Corollary 2 *Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ is given and that a stochastic process $X : \Omega \times I\!R^+ \to I\!R^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s \quad \forall s \in I\!R^+$$

with $(B_s)_{s\geq 0}$ a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\beta < 0$.
Let us make the following assumptions about the adapted and measurable process $\delta : \Omega \times I\!R^+ \to I\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \overset{a.e.}{\to} \overline{\delta}$ *with* $\overline{\delta} \in I\!R_0^+$
- *There exists a function ψ such that $\lim_{x\to\infty} \psi(x)/x^2 = \infty$ and such that there exists a constant K so that*

$$\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!E\left[\psi(\delta_u)\right] du \leq K. \tag{10}$$

Under these conditions, the following convergence in distribution holds:

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta} n}} \int_0^{nt} \left(X_u + \frac{\delta_u}{2\beta}\right) du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B'_t)_{t\geq 0}$$

with $(B'_t)_{t\geq 0}$ a Brownian motion.

Proof: We will check that the conditions of theorem 1 are fulfilled.

First, we prove that $\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!E\left[\delta_u^2\right] du \leq k$ with k a constant independent of t. Since $\lim_{x\to\infty} \psi(x)/x^2 = \infty$, there exists a c large enough such that $x^2 \leq \psi(x)\varepsilon_c$ holds for all $x \geq c$, with ε_c a constant. Consequently:

$$\begin{aligned}
\frac{1}{t}\int_0^t I\!E\left[\delta_u^2\right] du &= \frac{1}{t}\int_0^t I\!E\left[\delta_u^2 1\!\!1_{(\delta_u < c)}\right] du + \frac{1}{t}\int_0^t I\!E\left[\delta_u^2 1\!\!1_{(\delta_u \geq c)}\right] du \\
&\leq c^2 + \frac{1}{t}\int_0^t I\!E\left[\psi(\delta_u)\varepsilon_c\right] du \\
&\leq c^2 + \varepsilon_c K = k
\end{aligned}$$

where k is a constant independent of t.

Analogously, we prove that for all $a \in I\!R^+$ $\quad \lim_{t\to\infty} \frac{1}{t}\int_{t-a}^t I\!E\left[\delta_u^2\right] du = 0$. Indeed,

$$\begin{aligned}
\frac{1}{t}\int_{t-a}^t I\!E\left[\delta_u^2\right] du &= \frac{1}{t}\int_{t-a}^t I\!E\left[\delta_u^2 1\!\!1_{(\delta_u < c)}\right] du + \frac{1}{t}\int_{t-a}^t I\!E\left[\delta_u^2 1\!\!1_{(\delta_u \geq c)}\right] du \\
&\leq \frac{c^2 a}{t} + \frac{1}{t}\int_{t-a}^t I\!E\left[\psi(\delta_u)\varepsilon_c\right] du
\end{aligned}$$

For a given ε, we choose c such that $\varepsilon_c \frac{1}{t}\int_0^t I\!E\left[\psi(\delta_u)\right] du \leq \varepsilon/2$ and for this fixed c, we can choose t large enough such that $\frac{c^2 a}{t} \leq \varepsilon/2$.

q.e.d.

By using techniques from functional analysis and by stopping and localisation, we find theorem 2 which has measure-invariant conditions.

Theorem 2 *Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ is given and that all martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ are continuous. Assume that a stochastic process $X : \Omega \times I\!R^+ \to I\!R^+$ is defined by the stochastic differential equation on $I\!R^+$*

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X_s}dB_s$$

with $(B_s)_{s\geq 0}$ a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\beta < 0$.
Let us make the following assumptions about the adapted and measurable process $\delta : \Omega \times I\!R^+ \to I\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \overset{\text{a.e.}}{\to} \overline{\delta}$ *with* $\overline{\delta} \in I\!R_0^+$
- *there exists a function ψ such that*

$$\lim_{x\to\infty} \frac{\psi(x)}{x^2} = \infty$$

and such that the convex hull of the set $\left\{\frac{1}{t}\int_0^t \psi(\delta_u)du \mid t \geq 1\right\}$, namely $conv\left(\frac{1}{t}\int_0^t \psi(\delta_u)du \mid t \geq 1\right)$, is bounded in L^0.

Under these conditions, the following convergence in distribution holds:

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}} \int_0^{nt} \left(X_u + \frac{\delta_u}{2\beta}\right) du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B'_t)_{t\geq 0} \tag{11}$$

with $(B'_t)_{t\geq 0}$ a Brownian motion.

Proof. Since $\mathrm{conv}\left(\frac{1}{t}\int_0^t \psi(\delta_u)du \mid t \geq 1\right)$ is bounded in L^0, there exist a $\alpha > 0$ and a $\mathcal{F}_\infty$-measurable function h with $0 < h \leq 1$ such that for all t:

$$I\!E\left[\frac{1}{t}\int_0^t \psi(\delta_u)\, du\, h\right] \leq \alpha.$$

Let us fix this function h and let us define the stopping times T_m by

$$T_m = \inf\left\{t \mid h_t = I\!E[h \mid \mathcal{F}_t] \leq \frac{1}{m}\right\}.$$

Since $h > 0$, the probabilities $I\!P[T_m = \infty]$ are increasing to 1. Let us define the stochastic process

$$\delta_u^m = \begin{cases} \delta_u & u \leq T_m \\ \overline{\delta} & u > T_m. \end{cases}$$

For the stochastic process X^m, defined by the stochastic differential equation

$$dX_s^m = (2\beta X_s^m + \delta_s^m)ds + 2\sqrt{X_s^m}dB_s,$$

the hypotheses of corollary 2 are fulfilled. Indeed, there is a constant k such that for all $t \geq 0$

$$\frac{1}{t}\int_0^t \mathbb{E}\left[\psi\left(\delta_u^m\right) du\right] du \leq k.$$

This is shown as follows:

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u^m\right) du\right] \\
&= \mathbb{E}\left[\frac{1}{t}\int_0^{T_m} \psi\left(\delta_u\right) du\, \mathbb{1}_{(T_m<t)}\right] + \mathbb{E}\left[\frac{1}{t}\int_{T_m}^t \psi\left(\overline{\delta}\right) du\, \mathbb{1}_{(T_m<t)}\right] \\
&+ \mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u\right) du\, \mathbb{1}_{(T_m\geq qt)}\right].
\end{aligned}$$

Since $mh_{T_m} \geq 1$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u^m\right) du\right] \\
&\leq \mathbb{E}\left[\frac{1}{t}\int_0^{T_m} \psi\left(\delta_u\right) du\, mh_{T_m}\, \mathbb{1}_{(T_m<t)}\right] + \mathbb{E}\left[\frac{1}{t}\int_{T_m}^t \psi\left(\overline{\delta}\right) du\, \mathbb{1}_{(T_m<t)}\right] \\
&+ \mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u\right) du\, mh_{T_m}\, \mathbb{1}_{(T_m\geq t)}\right].
\end{aligned}$$

If we take the conditional expectation and recall that $h_{T_m} = \mathbb{E}[h \mid \mathcal{F}_{T_m}]$, we find

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u^m\right) du\right] \\
&\leq \psi\left(\overline{\delta}\right) \mathbb{P}[T_m < t] + \mathbb{E}\left[mh\, \frac{1}{t}\int_0^{T_m\wedge t} \psi\left(\delta_u\right) du\right].
\end{aligned}$$

Since also $\mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u\right) du\, h\right] \leq \alpha$, we may conclude that for all $t \geq 1$:

$$\mathbb{E}\left[\frac{1}{t}\int_0^t \psi\left(\delta_u^m\right) du\right] \leq \psi\left(\overline{\delta}\right) \mathbb{P}[T_m < t] + m\alpha \leq K(m)$$

where K is a constant depending on m but not on t. Since the hypotheses of corollary 2 are fulfilled in case of the process X^m, we can apply corollary 2:

$$(Y_t^{n,m})_{t\geq 0} = \left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u^m + \frac{\delta_u^m}{2\beta}\right) du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B'_t)_{t\geq 0}$$

with $(B'_t)_{t\geq 0}$ a Brownian motion. Thus, for every continuous and bounded function $f : C(\mathbb{R}^+, \mathbb{R}) \longrightarrow \mathbb{R}$:

$$\mathbb{E}\left[f(Y^{n,m})\right] \longrightarrow \int f d\mu$$

where μ denotes the Wiener measure.

In order to prove that

$$(Y_t^n)_{t\geq 0} = \left(\sqrt{\frac{-2\beta^3}{\bar{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B'_t)_{t\geq 0}$$

with $(B'_t)_{t\geq 0}$ a Brownian motion, we show that for every bounded and continuous function $f : C(\mathbb{R}^+, \mathbb{R}) \longrightarrow \mathbb{R}$:

$$\mathbb{E}\left[f(Y^n)\right] \longrightarrow \int f d\mu.$$

This is quite standard:

$$\begin{aligned}
&\left|\mathbb{E}\left[f(Y^n)\right] - \int f d\mu\right| \\
&\leq \left|\mathbb{E}\left[f(Y^{n,m})\right] - \int f d\mu\right| + \left|\mathbb{E}\left[f(Y^n)\right] - \mathbb{E}\left[f(Y^{n,m})\right]\right| \\
&\leq \left|\mathbb{E}\left[f(Y^{n,m})\right] - \int f d\mu\right| + \mathbb{E}\left[\left|f(Y^n) - f(Y^{n,m})\right|\right] \\
&\leq \left|\mathbb{E}\left[f(Y^{n,m})\right] - \int f d\mu\right| + 2\|f\|_\infty \mathbb{P}[\exists u \; Y_u^{n,m} \neq Y_u^n]
\end{aligned}$$

Since $\mathbb{P}\left[T_m = \infty\right] \uparrow 1$ for m going to infinity, also $\mathbb{P}[\exists u \; X_u^{n,m} \neq X_u^n] \to 0$ and consequently $\mathbb{P}[\exists u \; Y_u^{n,m} \neq Y_u^n] \to 0$. Thus, for every $\varepsilon > 0$, we can fix m such that the second term is smaller than $\varepsilon/2$. Since $(Y_t^{n,m})_{t\geq 0}$ converges to a Brownian motion, there exist n large enough that the first term is bounded by $\varepsilon/2$.

q.e.d.

An immediate consequence of this corollary is that the theorem holds if there exists $\varepsilon > 0$ such that $\mathrm{conv}\left(\frac{1}{t}\int_0^t \delta_u^{2+\varepsilon} du \mid t \geq 1\right)$ is bounded in L^0.

4 Approximations.

In this section, we propose some applications of theorem 1. We consider the possibility of approximating $\int_0^t r_u du$. It is very interesting to know an approximation of $\int_0^t r_u du$ since this integral appears in bond prices, accumulation and discounting factors, etcetera.

As an example, we apply theorem 1 to the generalized Cox–Ingersoll–Ross two-factor model (1) and we approximate $\int_0^t r_u du$ by the sum of two terms: the constant reversion level γ^* multiplied by t and a scaled Brownian motion which represents random changes. Afterwards, we evaluate the approximation in case of the Cox–Ingersoll–Ross model.

For a stochastic process $(X_u)_{u\geq 0}$ given by $dX_u = (2\beta X_u + \delta_u)du + 2\sqrt{X_u}d\tilde{B}_u$ and satisfying the assumptions of theorem 1, the following convergence in law to a Brownian motion $(B_t)_{t\geq 0}$ was shown:

$$\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B_t)_{t\geq 0}$$

This result inspires us to approximate

$$\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du$$

by B_t for n large enough.

Using the scaling property of Brownian motion, namely $\sqrt{n}B_{t/n} \stackrel{d}{=} B_t$, we can estimate

$$\int_0^t X_u du \qquad \text{by} \qquad -\int_0^t \frac{\delta_u}{2\beta}du + \sqrt{\frac{\overline{\delta}}{-2\beta^3}}B_t.$$

As $\frac{1}{t}\int_0^t \delta_u du \xrightarrow{\text{a.e.}} \overline{\delta}$, we obtain for t large enough

$$\int_0^t X_u du \qquad \approx \qquad -\frac{\overline{\delta}}{2\beta}t + \sqrt{\frac{\overline{\delta}}{-2\beta^3}}B_t.$$

In "Long-term returns in stochastic interest rate models." [2], it is proved that under the hypothesis of theorem 1

$$\frac{1}{t}\int_0^t X_u du \xrightarrow{a.e.} -\frac{\overline{\delta}}{2\beta}.$$

Therefore, we approximate $\int_0^t X_u du$ by the constant convergence a.e. limit of the long-term return times t plus a scaled Brownian motion.

Let us now study the two-factor model (1), which is a generalisation of the Cox–Ingersoll–Ross model (see also [2]):

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB'_t \\ d\gamma_t &= \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t \end{aligned}$$

with κ, $\tilde{\kappa} > 0$; γ^*, σ and $\tilde{\sigma}$ positive constants and $(B'_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ two Brownian motions. These Brownian motions may be correlated in an arbitrarily way, they even may have a random correlation. As mentioned in the introduction, this is in contrast with the assumptions of most papers: most of the authors suppose for technical reasons that the Brownian motions are uncorrelated or have a constant correlation.

Since we are interested in the convergence of the long-term return $\frac{1}{t}\int_0^t r_u du$, we verify if theorem 1 is applicable:

If we make the transformation $X_u = \frac{4}{\sigma^2} r_u$, then X_u satisfies the stochastic differential equation

$$dX_u = \left(\frac{4\kappa\gamma_u}{\sigma^2} + 2\left(-\frac{\kappa}{2}\right) X_u\right)du + 2\sqrt{X_u}dB_u^{'}.$$

In terms of theorem 1:

$$dX_u = (\delta_u + 2\beta X_u)du + 2\sqrt{X_u}dB_u^{'}$$

with

- $\beta = -\frac{\kappa}{2} < 0$
- $\delta_u = \frac{4\kappa\gamma_u}{\sigma^2} \quad \forall u \in I\!R^+$, which is measurable and adapted since γ is a CIR square root process.

We check the hypothesis of theorem 1:

It is easy to show that (see e.g. [2]):

$$\frac{1}{s}\int_0^s \delta_u du \xrightarrow{\text{a.e.}} \frac{4\kappa\gamma^*}{\sigma^2} \text{ with } \frac{4\kappa\gamma^*}{\sigma^2} > 0.$$

Since γ follows a CIR square root process, its second moment is given by:

$$I\!E\left[\gamma_s^2\right] = (2\tilde{\kappa}\tilde{\gamma} + \tilde{\sigma}^2)\left[\frac{\gamma_0 - \tilde{\gamma}}{\tilde{\kappa}}e^{-\tilde{\kappa}s} + \frac{\tilde{\gamma}}{2\tilde{\kappa}}\right] + e^{-2\tilde{\kappa}s}\left[\gamma_0^2 + \frac{\tilde{\gamma} - \gamma_0^2}{2\tilde{\kappa}}(2\tilde{\kappa}\tilde{\gamma} + \tilde{\sigma}^2)\right].$$

As δ_s equals $\frac{4\kappa\gamma_s}{\sigma^2}$, a technical calculation leads to the results that for all $a \in I\!R^+$

$$\lim_{t\to\infty} \frac{1}{t}\int_{t-a}^t I\!E\left[\delta_s^2\right] ds = 0$$

and that

$$\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!E\left[\delta_s^2\right] ds \leq k$$

with k a constant independent of t. Consequently, theorem 1 is applicable and one finds that

$$\left(\frac{\kappa}{\sigma\sqrt{\gamma^* n}}\int_0^{nt}(r_u - \gamma_u)du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B_t)_{t\geq 0}.$$

This is not a trivial result since r and γ may have an arbitrary, random correlation.

Repeating the reasoning above, we approximate

$$\int_0^t r_u du \approx \int_0^t \gamma_u du + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}} B_t$$

or

$$\int_0^t r_u du \approx \gamma^* t + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}} B_t$$

for t large enough.

This is a very easy approximation. We approximate the surface under the instantaneous interest rate curve by the rectangle with height the constant reversion level γ^* plus a factor which represents random changes by using a scaled Brownian motion.

In order to evaluate this estimator, we have a look at the moments of the estimator. They do not equal those of $\int_0^t r_u du$, but they are the same asymptotically. Moreover, if we assume that r_t follows the CIR square root process

$$dr_t = \kappa(\gamma^* - r_t)dt + \sigma\sqrt{r_t} dB_t^{'}$$

with γ^* as well as κ and σ constants and if we assume that r_0 is distributed according to its stationary distribution $\Gamma\left(\frac{2\kappa\gamma^*}{\sigma^2}, \frac{2\kappa}{\sigma^2}\right)$, we find that the expectation values are equal for all t.

Indeed, by Fubini and the expectation value $\mathbb{E}[r_u]$ we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^t r_u du\right] &= \int_0^\infty \mathbb{E}_x\left[\int_0^t r_u du\right] f_{R_0}(x)dx \\ &= \int_0^\infty \left[\int_0^t \left(xe^{-\kappa s} + \gamma^*(1-e^{-\kappa s})\right) ds\right] f_{R_0}(x)dx \\ &= \frac{1-e^{-\kappa t}}{\kappa}\int_0^\infty x f_{R_0}(x)dx + \gamma^* t - \gamma^* \frac{1-e^{-\kappa t}}{\kappa} \end{aligned}$$

where we have used that $f_{R_0} = \Gamma\left(\frac{2\kappa\gamma^*}{\sigma^2}, \frac{2\kappa}{\sigma^2}\right)$ is a density function. If we recall that $\int_0^\infty x f_{R_0}(x)dx = \mathbb{E}[R_0] = \frac{2\kappa\gamma^*/\sigma^2}{2\kappa/\sigma^2} = \gamma^*$, the result follows:

$$\mathbb{E}\left[\int_0^t r_u du\right] = \gamma^* t = \mathbb{E}\left[\gamma^* t + \frac{\sigma\sqrt{\gamma^*}}{\kappa} B_t\right].$$

In case of the CIR square root process, an explicit formula for the bond price is given by Pitman–Yor [6] and Cox–Ingersoll–Ross [1]. From Pitman-Yor [6], we recall that the bond price has been given by

$$\begin{aligned} &\mathbb{E}_x\left[\exp\left(-\int_0^t r_u du\right)\right] \\ &= \frac{\exp\left\{-\frac{x}{\sigma^2} w \frac{1+\kappa/w \coth(wt/2)}{\coth(wt/2)+\kappa/w}\right\} e^{\kappa x/\sigma^2} e^{\kappa^2\gamma^* t/\sigma^2}}{\left(\cosh(wt/2) + \kappa/w \sinh(wt/2)\right)^{\frac{2\kappa\gamma^*}{\sigma^2}}} \end{aligned}$$

with $w = \sqrt{\kappa^2 + 2\sigma^2}$ and $x = r_0$.

By our approximation, we obtain

$$\mathbb{E}_0\left[\exp\left(-\gamma^* t - \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t\right)\right] = \exp\left(-\gamma^* t + \frac{\sigma^2\gamma^*}{2\kappa^2}t\right)$$

If we make the quotient of the two results to the power $1/t$, for t going to infinity, we should obtain the value 1. But:

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\mathbb{E}_0\left[\exp\left(-\gamma^* t - \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t\right)\right]^{1/t}}{\mathbb{E}_x\left[\exp\left(-\int_0^t r_u du\right)\right]^{1/t}} \\
&= \lim_{t\to\infty} \frac{\exp\left\{\gamma^*\left(\frac{\sigma^2}{2\kappa^2} - 1\right)\right\}}{\exp\left(\frac{\kappa^2\gamma^*}{\sigma^2}\right)/2\cosh\left(\frac{\kappa\gamma^*\omega}{\sigma^2}\right)} \neq 1
\end{aligned}$$

The approximation to a Brownian motion is too slow.

In practice however, the period of interest usually is shorter than 40 or 50 years. Using the parameters estimated within the empirical work of Chan, Karolyi, Longstaff and Sanders [12], namely $\kappa = 0.2339$, $\gamma^* = 0.0808$ and $\sigma = 0.0854$, we have calculated the exact bond prices and the proposed approximation.

As the approximation does not depend on the present interest rate r_0, we do not need the knowledge of r_0 to calculate them. The approximations for the prices of bonds with duration t year can be found in the first column of table 1. The other columns collect the quotients of the exact bond prices by the approximations for different values of the present interest rate r_0.

Clearly, three situations are possible: for $r_0 < \gamma^*$, the approximation underestimates the bond price; for $r_0 > \gamma^*$ there is an overestimation and for $r_0 \approx \gamma^*$, the fit is fairly satisfying. In general, the quality of the approximation of a long-term zero coupon depends on the value of the parameters.

However, if the objective is to approximate the distribution of the long-term return of an investment made at time 0, it is appropriate to approximate $\int_0^t r_u du$ by a scaled Brownian motion with drift for t going to infinity. A lot of authors previously proposed Wiener models. The argumentation of using Wiener models is that Central Limit Theorems are applicable on long-term.

As an illustration, we have simulated $\gamma^* + \sqrt{\frac{\kappa^2\gamma^*}{40\sigma^2}}B_1$ and $\frac{1}{40}\int_0^{40} r_u du$ where $(r_u)_{u\geq 0}$ specifies the CIR process from above with the parameters taken from [12]. For both the long-term return and the approximation, we have calculated the probabilities of being in the intervals $[\ln(1+(i-1)/100), \ln(1+i/100)]$ for $1 \leq i \leq 18$. Figure 1 shows the histogram of the approximation (__) which is independent of r_0 and the histograms of the long-term return for $r_0 = 0.07$ (- - -), for $r_0 = 0.04$ (....) and for $r_0 = 0.1$ (_._.).

Table 1: Bond prices: The exact values and the approximations.

t	Approximation	Exact/Approximation		
		$r_0 = 0.07$	$r_0 = 0.04$	$r_0 = 0.1$
1	0.92735916	1.00423575	1.03152406	0.97784472
5	0.68586677	1.00988853	1.10162997	0.92578721
10	0.47041327	1.00885379	1.12789547	0.90237617
15	0.32264084	1.00619245	1.13247085	0.89399487
20	0.22128864	1.00320458	1.13114345	0.88973624
25	0.15177453	1.00014389	1.12823772	0.88659304
30	0.10409711	0.99707115	1.12491739	0.88375461
35	0.07139675	0.99400234	1.12149405	0.88100392
40	0.04896866	0.99094135	1.11805093	0.87828267

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