Question 1

The correct answers are:

(a) (2)

(b) (1)

(c) (2)

(d) (3)

(e) (2)

(f) (3)

(g) (1)

(h) (2)

Question 2

(a) By the fundamental theorem of asset pricing (Theorem II.2.1 in the lecture notes), the market (S^0, S^1) is arbitrage-free if and only if there exists an EMM Q for the discounted stock price process S^1.

Any probability measure $Q \approx P$ on $\mathcal{F}$ can be described by

$$Q[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2},$$

where $q_{x_1}, q_{x_1,x_2} \in (0,1)$ satisfy that $\sum_{x_1 \in \{-1,1\}} q_{x_1} = 1$ and $\sum_{x_2 \in \{-1,1\}} q_{x_1,x_2} = 1$ for all $x_1 \in \{-1, 1\}$. Next, since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 only takes two values, S^1 is a $(Q, \mathbb{F})$-martingale if and only if $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$E_Q\left[\frac{Y_1}{1+r}\right] = 1, \quad E_Q\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = 1+u\right] = 1 \quad \text{and} \quad E_Q\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = 1+d\right] = 1.$$

This is equivalent to $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$\begin{aligned} q_1(1+u) + (1-q_1)(1+d) &= 1+r &\iff&& q_1 &= \frac{r-d}{u-d}, \\ q_{1,1}(1+2u) + (1-q_{1,1})(1+2d) &= 1+2r &\iff&& q_{1,1} &= \frac{r-d}{u-d}, \\ q_{-1,1}(1+u) + (1-q_{-1,1})(1+d) &= 1+2r &\iff&& q_{-1,1} &= \frac{2r-d}{u-d}. \end{aligned}$$

In conclusion, the market (S^0, S^1) is arbitrage-free if and only if

$$\frac{r-d}{u-d} \in (0,1) \quad \text{and} \quad \frac{2r-d}{u-d} \in (0,1) \qquad \iff \qquad d < r < u \quad \text{and} \quad d < 2r < u.$$

(b) We know from the second fundamental theorem of asset pricing that an arbitrage-free market is complete is and only if $\mathbb{P}_e(S^1)$ is a singleton. However, we know from (a) that if (S^0, S^1) is arbitrage-free, then the EMM for (S^0, S^1) is unique. So we know that (S^0, S^1) is arbitrage-free and complete if and only if $d < r < u$ and $d < 2r < u$.

Even if one did not find the conditions on u, r and d in (a), we can still argue each of the single-period submarkets of our market (S^0, S^1) is a binomial market and it thus admits a unique EMM if it is arbitrage-free. So the whole market admits at most one EMM and the conditions for this market to be arbitrage-free and complete are the same as for it to be just arbitrage-free.

(c) Since any replicating strategy is self-financing by definition, we have

$$H = V_2(\varphi) = V_0 + (\vartheta \bullet S^1)_2 \quad P\text{-a.s.}$$

Because the market (S^0, S^1) is arbitrage-free by assumption, we must have

$$V_k^H = V_k(\varphi) \quad P\text{-a.s. for all } k \in \{0, 1, 2\}. \tag{1}$$

Again because (S^0, S^1) is arbitrage-free, $\mathbb{P}_e(S^1)$ is non-empty by the first fundamental theorem of asset pricing, and S^1 is a $(Q, \mathbb{F})$-martingale for any $Q \in \mathbb{P}_e(S^1)$. Additionally, ϑ is admissible by the definition of a replicating strategy, so $(\vartheta \bullet S^1)$ is also a $(Q, \mathbb{F})$-martingale for any $Q \in \mathbb{P}_e(S^1)$. We therefore have by the martingale property of $V(\varphi)$ (which follows trivially from the martingale property of the stochastic integral process $(\vartheta \bullet S^1)$) and (1) that

$$V_k^H = V_k(\varphi) = E_Q\left[V_2(\varphi) \,|\, \mathcal{F}_k\right] = E_Q\left[H \,|\, \mathcal{F}_k\right] \quad P\text{-a.s. for all } k \in \{0, 1, 2\}$$

for any $Q \in \mathbb{P}_e(S^1)$.

(d) Note that V^H and V^K are necessarily $(Q, \mathbb{F})$-martingales. We only show this for V^H since one can proceed exactly in the same way for V^K. V^H is $\mathbb{F}$-adapted by the definition of conditional expectation,

$$E_Q\left[V_{k+1}^H \,\middle|\, \mathcal{F}_k\right] = E_Q\left[E_Q\left[H \,|\, \mathcal{F}_{k+1}\right] \,|\, \mathcal{F}_k\right] = E_Q\left[H \,|\, \mathcal{F}_k\right] = V_k^H$$

for all $k \in \{0, 1\}$ and

$$E_Q\left[|V_k^H|\right] = E_Q\left[|E_Q\left[H \,|\, \mathcal{F}_k\right]|\right] \leq E_Q\left[E_Q\left[|H| \,|\, \mathcal{F}_k\right]\right] = E_Q\left[|H|\right] < \infty$$

for all $k \in \{0, 1, 2\}$, where we have used Jensen's inequality.

Since Q is an EMM for (S^0, S^1) by assumption, Q is therefore an EMM for (S^0, S^1, V^H, V^K). But (S^0, S^1) is complete by assumption, so the second fundamental theorem of asset pricing gives that Q is the only EMM for (S^0, S^1). Since the set of EMMs for (S^0, S^1, V^H, V^K) is the intersection of set of EMMs for (S^0, S^1) and (V^H, V^K), the market (S^0, S^1, V^H, V^K) is complete even if V^H and V^K admit additional EMMs.

Question 3

(a) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for X and define $M := \max_{k\in\{0,1,\dots,T\}} Y_k$. Then we have for all $k \in \{0,1,\dots,T\}$ and all $n \in \mathbb{N}$ that

$$\begin{aligned} |X_{\tau_n\wedge k}| &= \left|\sum_{i=1}^{T} X_{\tau_n\wedge k}\mathbb{1}_{\{\tau_n=i\}}\right| \leq \sum_{i=1}^{T} |X_{\tau_n\wedge k}|\mathbb{1}_{\{\tau_n=i\}} \\ &= \sum_{i=1}^{k} |X_i|\mathbb{1}_{\{\tau_n=i\}} + \sum_{i=k+1}^{T} |X_k|\mathbb{1}_{\{\tau_n=i\}} \\ &\leq \sum_{i=1}^{k} M\mathbb{1}_{\{\tau_n=i\}} + \sum_{i=k+1}^{T} M\mathbb{1}_{\{\tau_n=i\}} = M \leq \sum_{i=1}^{T} Y_i. \end{aligned}$$

But since $Y_i \in L^1(P)$ for all $i \in \{0,1,\dots,T\}$ by assumption, $\sum_{i=1}^{T} Y_i \in L^1(P)$ as well and dominated convergence theorem gives

$$E\left[|X_k|\right] = E\left[\lim_{n\to\infty} |X_{\tau_n\wedge k}|\right] = \lim_{n\to\infty} E\left[|X_{\tau_n\wedge k}|\right] \leq \lim_{n\to\infty} \sum_{i=1}^{T} E\left[Y_i\right] = \sum_{i=1}^{T} E\left[Y_i\right] < \infty,$$

for all $k \in \{0,1,\dots,T\}$, which is the integrability of X. The same bound and dominated convergence theorem for conditional expectations can be used to show the martingale property of X. Indeed

$$\begin{aligned} E\left[X_k \,|\, \mathcal{F}_{k-1}\right] &= E\left[\lim_{n\to\infty} X_{\tau_n\wedge k} \,\middle|\, \mathcal{F}_{k-1}\right] = \lim_{n\to\infty} E\left[X_{\tau_n\wedge k} \,|\, \mathcal{F}_{k-1}\right] \\ &= \lim_{n\to\infty} X_{\tau_n\wedge(k-1)} = X_{k-1} \end{aligned}$$

for all $k \in \{1,\dots,T\}$. Since adaptedness of X is clear, we get that X is indeed a true $(P,\mathbb{F})$-martingale.

(b) Let $X = (X_k)_{k=0,1,\dots,T}$ be an integrable local $(P,\mathbb{F})$-martingale. Since we clearly have for all $k \in \{0,1,\dots,T\}$ that $|X_k| \leq |X_k|$ and $E\left[|X_k|\right] < \infty$ by assumption the result from (a) applies.

(c) Let $\mathcal{F}_n = \sigma(Y_1,\dots,Y_n)$. By the definition of a stopping time, we only need to show that

$$\{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

But

$$\{\tau \leq n\} = \{Y_k > 1 \text{ for some } k \leq n\} = \bigcup_{k=1}^{n} \{Y_k > 1\} \in \mathcal{F}_n$$

as $\{Y_k > 1\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$ by the very definition of $\mathcal{F}_k$. So τ is a stopping time with respect to the filtration generated by Y.

(d) We have that

$$\begin{aligned} P[\tau = +\infty] &= P[Y_n \leq 1 \text{ for all } n \in \mathbb{N}] = P\left[\bigcap_{n=1}^{\infty} \{Y_n \leq 1\}\right] = \prod_{n=2}^{\infty} P[Y_n \leq 1] \\ &= \prod_{n=2}^{\infty} \left(P[Y_n = 0] + P[Y_n = 1]\right) = \prod_{i=2}^{\infty} \left(\left(1 - \frac{1}{n}\right)^n + \left(1 - \frac{1}{n}\right)^{n-1}\right) \\ &= \prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{n}{n-1}\right) \leq 3\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^n. \end{aligned}$$

But $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} < 1$, so there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $(1 - \frac{1}{n})^n < C$, where $C \in (e^{-1}, 1)$. So

$$P[\tau = +\infty] \leq 3\prod_{n=2}^{N} \left(1 - \frac{1}{n}\right)^n \prod_{n=N+1}^{\infty} C^n = 0.$$

Question 4

(a) Since W is a $(P, \mathbb{F})$-Brownian motion, in particular, $W_t - W_0 = W_t$ P-a.s. is independent of $\mathcal{F}_0$, and in particular independent of Z. Therefore, as Z and W_t are independent random variables, we obtain that

$$X_t = Z + f(t)W_t \sim \mathcal{N}(0, tf^2(t) + 1).$$

Moreover, we can calculate the covariance using the independence of increments of W and their independence from Z: assuming that $t \geq s$,

$$\begin{aligned} E[X_t X_s] &= E[(Z + f(t)W_t)(Z + f(s)W_s)] \\ &= E[Z^2] + f(s)E[ZW_s] + f(t)E[ZW_t] + f(t)f(s)E[W_t W_s] \\ &= 1 + f(t)f(s)E[(W_t - W_s)W_s] + f(t)f(s)E[W_s^2] \\ &= 1 + sf(t)f(s). \end{aligned}$$

If we assume that $t < s$, we analogously obtain that

$$E[X_t X_s] = 1 + tf(t)f(s).$$

Putting the two cases together gives

$$E[X_t X_s] = 1 + (s \wedge t)f(t)f(s).$$

(b) Continuous version of Itô's formula says that if $X = (X_t)_{t \geq 0}$ is a continuous $\mathbb{R}^d$-valued $(P, \mathbb{F})$-semimartingale and $g : \mathbb{R}^d \to \mathbb{R}$ is in C^2, then the process $g(X) = (g(X_t))_{t \geq 0}$ is a real-valued $(P, \mathbb{F})$-semimartingale and we explicitly have P-a.s. for all $t \geq 0$

$$g(X_t) = g(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial g}{\partial x^i}(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 g}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j \rangle_s.$$

Applying Itô's formula to the $(P, \mathbb{F})$-semimartingale $(t, W_t)_{t \geq 0}$ and the C^2-function $g(x, y) = f(x)y$, using that t is a continuous finite variation process and hence has trivial quadratic variation and covariations, and since

$$g_x(x, y) = f'(x)y,\ g_y(x, y) = f(x),\ g_{xy}(x, y) = f'(x),\ g_{xx}(x, y) = f''(x)y,\ g_{yy}(x, y) = 0,$$

we obtain

$$X_t - Z = g(t, W_t) = \int_0^t f'(s)W_s ds + \int_0^t f(s)dW_s = \int_0^t f'(s)W_s ds + Y_s,$$

since $X_0 = 0$ as $W_0 = 0$ P-a.s. Finally, since $\langle M + N \rangle_t = \langle M \rangle_t + 2\langle M, N \rangle_t + \langle N \rangle_t$ and since the process $(\int_0^t f'(s)W_s ds)_t$ is of finite variation (as has been shown in the exercise sheet), we get that

$$\langle X \rangle_t = \langle Y \rangle_t = \int_0^t f^2(s) d\langle W \rangle_s = \int_0^t f^2(s) ds.$$

(c) Note that we can rewrite

$$Y_T = \int_0^T f(s)dW_s = Y_t + \int_t^T f(s)d(W_s - W_{t \wedge s}). \tag{2}$$

By the Markov property of Brownian motion, $W_s - W_{t \wedge s}$ is a new Brownian motion (with respect to the translated filtration $\mathcal{G}_s = \mathcal{F}_{s+t}$) independent of $\mathcal{F}_t$. Moreover Y_t is of course measurable with respect to $\sigma(Y_t) \subseteq \mathcal{F}_t$, and therefore we get that

$$Z_t = E[\exp(uY_T) \mid \mathcal{F}_t] = E[\exp(uY_T) \mid Y_t].$$

Then clearly Z is adapted and since $\exp(uY_T)$ is non-negative and integrable, it follows by the tower property of conditional expectations that Z is integrable and has the martingale property. Now from Itô's formula applied to $Z_t = F(Y_t, t)$, noting again that t has trivial quadratic variation and covariations and using the quadratic variation from (b), we obtain that

$$Z_t = Z_0 + \int_0^t \frac{\partial F}{\partial t}(Y_s, s)ds + \int_0^t \frac{\partial F}{\partial y}(Y_s, s)dY_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial y^2}(Y_s, s)f^2(s)ds.$$

Thus collecting the finite variation terms we deduce that the equation

$$\frac{\partial F}{\partial t}(Y_t, t) + \frac{\partial^2 F}{\partial y^2}(Y_t, t)f^2(t) = 0 \quad P\text{-a.s. for all } t \in (0, T)$$

must hold for the finite variation part to vanish and for Z to be a $(P, \mathbb{F})$-martingale. The terminal condition is clear from (2) since conditionally on $Y_T = y$, $\exp(uY_T) = \exp(uy)$.

(d) We can check that

$$F_t(y, t) = \frac{u^2}{2}\frac{d}{dt}\left(\int_t^T f^2(s)ds\right)F(y, t) = -\frac{u^2 f^2(t)}{2}F(y, t)$$

and

$$F_{yy}(y, t) = u^2 F(y, t),$$

which indeed satisfies the equation. Moreover the terminal condition is satisfied since the integral term vanishes when $t = T$. Now plugging in $y = t = 0$ we get

$$F(0, 0) = \exp\left(\frac{u^2}{2}\int_0^T f^2(s)ds\right) = E[\exp(uY_T) \mid Y_0 = 0] = E[\exp(uY_T)],$$

where the last equality follows from the fact that $Y_0 = 0$ already holds P-a.s. By varying u this gives us the moment generating function, or Laplace transform, of Y_T. By uniqueness of the Laplace transform, since for $N \sim \mathcal{N}(\mu, \sigma^2)$ we have that

$$E[\exp(uN)] = \exp\left(\mu u + \frac{u^2\sigma^2}{2}\right),$$

we get that

$$Y_T \sim \mathcal{N}\left(0, \int_0^T f^2(s)ds\right).$$

Question 5

(a) We can rewrite the SDE as $d\widetilde{S}^1_t = \widetilde{S}^1_t dX_t$, where $X = (X_t)_{t\in[0,T]}$ is given by $X_t = \mu t + \sigma W_t$. We then know that since X is clearly a continuous $(P, \mathbb{F})$-semimartingale, the solution to this new SDE is given by

$$\widetilde{S}^1_0 \mathcal{E}(X)_t = \widetilde{S}^1_0 \exp\left(X_t - \frac{1}{2}[X]_t\right) = \widetilde{S}^1_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right).$$

The process is clearly positive and since it is in fact a C^2 transformation of the $(P, \mathbb{F})$-semimartingale $(W_t, t)_{t\geq 0}$, it is itself a $(P, \mathbb{F})$-semimartingale by Itô's lemma.

(b) We do so by applying Itô's lemma to the $(P, \mathbb{F})$-semimartingale $(\widetilde{S}^0, \widetilde{S}^1)$ and the C^2 function $f(x, y) = x/y$. Since we have that

$$f_x(x, y) = \frac{1}{y}, \quad f_y(x, y) = -\frac{x}{y^2}, \quad f_{yy}(x, y) = \frac{2x}{y^3}$$

and since $\widetilde{S}^0_t = \exp(rt)$ (which is the solution the SDE for $\widetilde{S}^0$) defines a continuous process of finite variation and also

$$\langle \widetilde{S}^1 \rangle_t = (\widetilde{S}^1_t)^2 \sigma^2 t,$$

by the rules for computing quadratic variations of stochastic integrals from page 89 in the lecture notes, we get

$$d\widehat{S}^0_t = \frac{1}{\widetilde{S}^1_t} d\widetilde{S}^0_t - \frac{\widetilde{S}^0_t}{(\widetilde{S}^1_t)^2} d\widetilde{S}^1_t + \frac{\widetilde{S}^0_t}{(\widetilde{S}^1_t)^3} d\langle \widetilde{S}^1 \rangle_t = \widehat{S}^0_t \big((r - \mu + \sigma^2)t - \sigma dW_t\big).$$

(c) Since the process $Z = (Z_t)_{t\geq 0}$ defined by

$$Z_t = \mathcal{E}(\alpha W)_t = \exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t\right)$$

is a $(P, \mathbb{F})$-martingale for all $\alpha \in \mathbb{R}$ (for instance by Proposition 2.2 on page 70 in the lecture notes) and $Z_0 = 1$ P-a.s., Girsanov's theorem says that the process $\widehat{W} = (\widehat{W}_t)_{t\geq 0}$ defined by

$$\widehat{W}_t := W_t - [\alpha W, W]_t = W_t - \alpha t$$

is a $(\widehat{Q}, \mathbb{F})$-Brownian motion, where $\widehat{Q}$ is probability which is locally equivalent to P and whose density process is Z defined as above. Rewriting the SDE from (b) using $\widehat{W}$, we get

$$d\widehat{S}^0_t = \widehat{S}^0_t \big((r - \mu + \sigma^2 - \sigma\alpha)t - \sigma d\widehat{W}_t\big).$$

If we set $\alpha = (r - \mu + \sigma^2)/\sigma$, the finite variation term disappears and the SDE reduces to

$$d\widehat{S}^0_t = \widehat{S}^0_t d(-\sigma \widehat{W}_t).$$

Since $-\sigma\widehat{W}$ is a continuous semimartingale, the solution to this SDE is given by

$$\widehat{S}^0_t = \mathcal{E}(-\sigma\widehat{W}) = \exp\left(-\sigma\widehat{W}_t - \frac{1}{2}\sigma^2 t\right),$$

which is a $(\widehat{Q}, \mathbb{F})$-martingale again by Proposition 2.2 on page 70 in the lecture notes. This means that the probability measure $\widehat{Q}$ whose density process with respect to P is given by $\mathcal{E}\big((r - \mu + \sigma^2)/\sigma W\big)$ is an EMM for $\widehat{S}^0$.

(d) Since the Black–Scholes model is complete and because $\widehat{S}^1$ is a $(\widehat{Q}, \mathbb{F})$-martingale, the unique discounted value process $\widehat{V} := \widetilde{V}/\widetilde{S}^1$ of the claim whose payoff in undiscounted terms is given by $\widetilde{H}$ is

$$\begin{aligned}
\widehat{V}_t &= E_{\widehat{Q}}\left[\widehat{H} \,\middle|\, \mathcal{F}_t\right] = E_{\widehat{Q}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T} \,\middle|\, \mathcal{F}_t\right] = E_{\widehat{Q}}\left[\mathbb{1}_{\{\widetilde{S}^1_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] = \widehat{Q}\left[\widetilde{S}^1_T \geq K \,\middle|\, \mathcal{F}_t\right] \\
&= \widehat{Q}\left[\widetilde{S}^1_t \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma W_{T-t}\right) \geq K \,\middle|\, \mathcal{F}_t\right] \\
&= \widehat{Q}\left[y \exp\left(\left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma \widehat{W}_{T-t}\right) \geq K\right]\Bigg|_{y=\widetilde{S}^1_t} \\
&= \widehat{Q}\left[\frac{\widehat{W}_{T-t}}{\sqrt{T-t}} \geq \frac{\log\frac{K}{y} - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]\Bigg|_{y=\widetilde{S}^1_t}.
\end{aligned}$$

Because $\widehat{W}$ is a $(\widehat{Q}, \mathbb{F})$-Brownian motion, $\widehat{W}_{T-t}/\sqrt{T-t} \sim \mathcal{N}(0,1)$ under $\widehat{Q}$, so

$$\widehat{V}_t = 1 - \Phi\left(\frac{\log\frac{K}{\widetilde{S}^1_t} - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\log\frac{\widetilde{S}^1_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

This means that the undiscounted value process $\widetilde{V}$ of the payoff $\widetilde{H}$ is given by

$$\widetilde{V}_t = \widetilde{S}^1_t \Phi\left(\frac{\log\frac{\widetilde{S}^1_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$