Question 1

The correct answers are:

(a) (2)

(b) (1)

(c) (1)

(d) (3)

(e) (3)

(f) (2)

(g) (3)

(h) (1)

Question 2

(a) Adaptedness follows from the adaptedness and predicatibility of the processes M and H respectively. Integrability is a consequence of the integrability of M and the boundedness of H. Remains to show the martingale property. For any $t = 1, 2, \ldots,$ we have

$$\begin{aligned}\mathbb{E}\left[N_t - N_{t-1} \mid \mathcal{F}_t\right] &= \mathbb{E}\left[H_t(M_t - M_{t-1}) \mid \mathcal{F}_t\right] \\ &= H_t \mathbb{E}\left[(M_t - M_{t-1}) \mid \mathcal{F}_t\right] \\ &= 0\end{aligned}$$

The first line uses the definition of the stochastic integral process, in the second line we used that H is a predictable process, and the final line is a consequence of the martingale property of M.

(b) Note that for any $t = 0, 1, \ldots,$ we have

$$\begin{aligned}M_t^\tau &= M_0 + \sum_{s=1}^{t\wedge\tau}(M_s - M_{s-1}) \\ &= M_0 + \sum_{s=1}^{t} \mathbb{1}_{s\le\tau}(M_s - M_{s-1}).\end{aligned}$$

Since the event $\{s \le \tau\} = \{\tau < s\}^c = \{\tau \le s-1\}^c$ is $\mathcal{F}_{s-1}$-measurable by definition of stopping time, we conclude that the process $H_s := \mathbb{1}_{s\le\tau}$ is predictable, and hence M^τ is a martingale as the stochastic integral of the bounded predictable process H with respect to the martingale M.

(c) Let $(\tau_n)_{n=0,1,\ldots}$ be a localizing sequence for X, i.e a sequence of increasing stopping times with $\tau_n \uparrow \infty$ such that X^{τ_n} is a martingale for all $n = 0, 1, \ldots.$ Define

$$\sigma_n := \inf\{t = 0, 1, \cdots : |K_{t+1}| > n\}$$

with the convention $\inf \emptyset = \infty$. Note that σ_n is a stopping time for all n, and $\sigma_n \uparrow \infty$. Indeed, we have

$$\{\sigma_n \le t-1\}^c = \{\sigma_n > t-1\} = \{\sigma_n \ge t\} = \bigcap_{0\le s\le t} \{|K_s| \le n\}.$$

Since K is predictable, we get $\{\sigma_n \le t-1\}^c \in \mathcal{F}_{t-1}$, and hence σ_n is a stopping time. It is clear that $\sigma_n \uparrow \infty$.
Let $\rho_n := \tau_n \wedge \sigma_n$. We claim that $(\rho_n)_{n=0,1,\ldots}$ is a localizing sequence for X and also for N. By definition, $\rho_n \uparrow \infty$ since both $\tau_n \uparrow \infty$ and $\sigma_n \uparrow \infty$. Moreover ρ_n is a stopping time for each $n = 0, 1, \ldots$ as the minimum of the two stopping times τ_n and σ_n. Finally, the stopped process $X^{\rho_n} = X^{\tau_n\wedge\sigma_n} = (X^{\tau_n})^{\sigma_n}$ is a stopped martingale, and hence a martingale by question b). The sequence $(\rho_n)_{n=0,1,\ldots}$ is therefore indeed a localizing sequence for X.
Note also that $(H_t)_{t=1,2,\ldots} := (K_t \mathbb{1}_{t\le\rho_n})_{t=1,2,\ldots}$ is predictable since K is predictable and ρ_n is a stopping time. Moreover, by definition of ρ_n and σ_n, the process H is bounded by n. Writing

$$\begin{aligned}N_t^{\rho_n} &= \sum_{s=1}^{t\wedge\rho_n} K_s\left(X_s - X_{s-1}\right) \\ &= \sum_{s=1}^{t} K_s \mathbb{1}_{s\le\rho_n}\left(X_s - X_{s-1}\right) \\ &= \sum_{s=1}^{t} K_s \mathbb{1}_{s\le\rho_n}\left(X_s^{\rho_n} - X_{s-1}^{\rho_n}\right) \\ &= \sum_{s=1}^{t} H_s\left(X_s^{\rho_n} - X_{s-1}^{\rho_n}\right),\end{aligned}$$

we see that the stopped process N^{ρ_n} is the stochastic integral of the bounded predictable process H with respect to the martingale X^{ρ_n}, and hence is a martingale by question (b). The sequence $(\rho_n)_{n=0,1,\dots}$ is therefore a localizing sequence for N, and N is thus indeed a local martingale.

(d) Adaptedness of X follows from the fact that X is a local martingale. Integrability is also clear since $|X_t| \leq Y_t \in L^1(P)$ by assumption. It therefore remains to show the martingale property.
Let $(\tau_n)_{n=0,1,\dots}$ be a localizing sequence for X. The assumption $|X_s| \leq Y_t$ almost surely for all $0 \leq s \leq t$, and $Y_t \in L^1(P)$ for all $t = 0, 1, \dots$ enables us to use Lebesgue's Dominated Convergence Theorem to write

$$\begin{aligned}\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}[\lim_{n\to\infty} X_{t\wedge\tau_n} \mid \mathcal{F}_s] \\ &= \lim_{n\to\infty} \mathbb{E}[X_{t\wedge\tau_n} \mid \mathcal{F}_s] \\ &= \lim_{n\to\infty} X_{s\wedge\tau_n} \\ &= X_s\end{aligned}$$

The process X is therefore a true martingale.

(e) Define $Y_t := \sum_{s=0}^{t} |X_s|$. The process Y is integrable by assumption and $|X_s| \leq Y_t$ for all $0 \leq s \leq t$. The conclusion follows from question (d).

Question 3

(a) Let us first introduce some notation. Let $T = 2$ be the time horizon of the market. We have $\Omega = \{u, d\}^T$. Let Q be a candidate EMM, and let us write

$$\begin{aligned} q_u &= Q[Y_1 = 2] \\ q_d &= Q[Y_1 = 1/2] \\ q_{u,u} &= Q[Y_2 = 3/2 | Y_1 = 2] \\ q_{u,d} &= Q[Y_2 = 1 | Y_1 = 2] \\ q_{d,u} &= Q[Y_2 = 4 | Y_1 = 1/2] \\ q_{d,d} &= Q[Y_2 = 1 | Y_1 = 1/2] \end{aligned}$$

The EMM conditions ($S_k^1 = \mathbb{E}[S_{k+1}^1 | \mathcal{F}_k]$, $Q \sim P$, Q probability measure) give

- for $k = 0$:

$$\begin{cases} q_u + q_d & = 1 \\ q_u, q_d & > 0 \\ \frac{8}{1+\frac{1}{4}} q_u + \frac{2}{1+\frac{1}{4}} q_d & = \frac{4}{1} \end{cases}$$

- for $k = 1$:

$$\begin{cases} q_{u,u} + q_{u,d} & = 1 \\ q_{u,u}, q_{u,d} & > 0 \\ \frac{12}{(1+\frac{1}{4})^2} q_{u,u} + \frac{8}{(1+\frac{1}{4})^2} q_{u,d} & = \frac{8}{1+\frac{1}{4}} \end{cases}$$

  and

$$\begin{cases} q_{d,u} + q_{d,d} & = 1 \\ q_{d,u}, q_{d,d} & > 0 \\ \frac{8}{(1+\frac{1}{4})(1+\frac{1}{2})} q_{d,u} + \frac{2}{(1+\frac{1}{4})(1+\frac{1}{2})} q_{d,d} & = \frac{2}{1+\frac{1}{4}} \end{cases}$$

The above system of equations have a unique solution given by

$$\begin{cases} q_u & = 1/2 \\ q_d & = 1/2 \\ q_{u,u} & = 1/2 \\ q_{u,d} & = 1/2 \\ q_{d,u} & = 1/6 \\ q_{d,d} & = 5/6 \end{cases}$$

The market is therefore arbitrage-free by the first fundamental theorem of asset pricing.

(b) Uniqueness of the EMM found in question (a), together with the assumption that $\mathcal{F} = \sigma(Y_1, Y_2)$, implies that the market is complete by the second fundamental theorem of asset pricing.

(c) To guarantee no-arbitrage in the enlarged market, the discounted initial process of the call option must a martingale under the risk neutral measure Q found in question (a). We thus have

$$\begin{aligned} \tilde{V}_0 &= \mathbb{E}_Q \left[\frac{(\tilde{S}_2^1 - \tilde{K})^+}{\tilde{S}_2^0} \right] \\ &= \frac{12-7}{(1+\frac{1}{4})^2} q_u q_{u,u} + \frac{8-7}{(1+\frac{1}{4})^2} q_u q_{u,d} + \frac{8-7}{(1+\frac{1}{4})(1+\frac{1}{2})} q_d q_{d,u} + 0 \\ &= \frac{226}{225} \approx 1.00444 \end{aligned}$$

Using the assumption $\tilde{S}_0^0 = 1$, we therefore conclude that the initial undiscounted price of the call option is $\frac{226}{225} \approx 1.00444$.

Question 4

(a) Note that

$$\begin{aligned} X_t &= xe^{-at} + \mu(1-e^{-at}) + be^{-at}\int_0^t e^{as}dW_s \\ &= f(t, M_t) \end{aligned}$$

where $f(t,m) := xe^{-at} + \mu(1-e^{-at}) + be^{-at}m$, and $M_t := \int_0^t e^{as}dW_s$. Applying Itô formula to the continuous semimartingale $(t, M_t)_{t\geq 0}$ and the C^2 function f, we get

$$\begin{aligned} dX_t &= -ae^{-at}\left(x - \mu + b\int_0^t e^{as}dW_s\right)dt + be^{-at}e^{at}dW_t \\ &= a(\mu - X_t)dt + bdW_t \end{aligned}$$

(b) From exercise sheet 13, we know that for all $f : \mathbb{R} \to \mathbb{R}$ deterministic continuous function, the random variable defined as the stochastic integral up to a fixed time with respect to a Brownian motion is Gaussian with parameters

$$\int_0^t f(s)dW_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)ds\right).$$

It follows that

$$\begin{aligned} X_t &\sim \mathcal{N}\left(xe^{-at} + \mu(1-e^{-at}), b^2\int_0^t (e^{-a(t-s)})^2 ds\right) \\ &= \mathcal{N}\left(xe^{-at} + \mu(1-e^{-at}), \frac{b^2}{2a}(1-e^{-2at})\right) \end{aligned}$$

(c) **First solution**

Rewriting the SDE $dX_t = a(\mu - X_t)dt + bdW_t$ in integral form, we get

$$\begin{aligned} X_T &= X_0 + \int_0^T a(\mu - X_t)dt + bW_T \\ &= x + a\mu T - a\int_0^T X_t dt + bW_T. \end{aligned}$$

We therefore have

$$\int_0^T X_t dt = -\frac{1}{a}\left(X_T - x - a\mu T - bW_T\right).$$

Similarly to question (b), we conclude that $\int_0^T X_t dt$ is normally distributed with mean

$$-\frac{1}{a}\left(xe^{-aT} + \mu(1-e^{-aT}) - x - a\mu T\right) = \frac{(x-\mu)(1-e^{-aT})}{a} + \mu T.$$

To compute the variance, note that

$$\begin{aligned} \mathrm{Var}\left(\int_0^T X_t dt\right) &= \mathrm{Var}\left(-\frac{1}{a}\left(X_T - x - a\mu T - bW_T\right)\right) \\ &= \frac{1}{a^2}\left(\mathrm{Var}(X_T) + b^2\mathrm{Var}(W_T) - 2b\mathrm{Cov}(X_T, W_T)\right) \\ &= \frac{1}{a^2}\left(\frac{b^2}{2a}(1-e^{-2aT}) + b^2 T - 2b\mathrm{Cov}(X_T, W_T)\right). \end{aligned}$$

To compute the covariance of X_T and W_T, we use Itô's isometry formula as follows:

$$\begin{aligned}
\mathrm{Cov}\,(X_T, W_T) &= \mathrm{Cov}\left(xe^{-aT} + (1-e^{-aT})\mu + \int_0^T e^{-a(T-s)} b dW_s, \int_0^T dW_s\right) \\
&= \mathrm{Cov}\left(\int_0^T e^{-a(T-s)} b dW_s, \int_0^T dW_s\right) \\
&= b\int_0^T e^{-a(T-s)} ds \\
&= \frac{b}{a}(1-e^{-aT}).
\end{aligned}$$

The variance of $\int_0^T X_t dt$ is thus given by

$$\begin{aligned}
\mathrm{Var}\left(\int_0^T X_t dt\right) &= \frac{1}{a^2}\left(\frac{b^2}{2a}(1-e^{-2aT}) + b^2 T - 2b\mathrm{Cov}(X_T, W_T)\right) \\
&= \frac{1}{a^2}\left(\frac{b^2}{2a}(1-e^{-2aT}) + b^2 T - \frac{2b^2}{a}(1-e^{-aT})\right)
\end{aligned}$$

The random variable $\int_0^T X_t dt$ is therefore Gaussian with parameters

$$\int_0^T X_t dt \sim \mathcal{N}\left(\frac{(x-\mu)(1-e^{-aT})}{a} + \mu T, \frac{1}{a^2}\left(\frac{b^2}{2a}(1-e^{-2aT}) + b^2 T - \frac{2b^2}{a}(1-e^{-aT})\right)\right).$$

Second solution
Alternatively, we can compute the integral explicitly using stochastic Fubini theorem

$$\begin{aligned}
\int_0^T X_t dt &= \int_0^T \left(xe^{-at} + (1-e^{-at})\mu + \int_0^t e^{-a(t-s)} b dW_s\right) dt \\
&= \mu T + \int_0^T (x-\mu)e^{-at} dt + b\int_0^T \int_0^t e^{-a(t-s)} dW_s dt \\
&= \mu T + \int_0^T (x-\mu)e^{-at} dt + b\int_0^T \int_s^T e^{-a(t-s)} dt dW_s \\
&= \frac{(x-\mu)(1-e^{-aT})}{a} + \mu T + \frac{b}{a}\int_0^T (1-e^{-a(T-s)}) dW_s.
\end{aligned}$$

Similarly to question (b), the integral $\frac{b}{a}\int_0^T (1-e^{-a(T-s)}) dW_s$ is Gaussian with mean 0 and variance $\frac{b^2}{a^2}\int_0^T (1-e^{-a(T-s)})^2 ds$, and hence the integral $\int_0^T X_t dt$ is also Gaussian with parameters

$$\begin{aligned}
\int_0^T X_t dt &\sim \mathcal{N}\left(\frac{(x-\mu)(1-e^{-aT})}{a} + \mu T, \frac{b^2}{a^2}\int_0^T (1-e^{-a(T-s)})^2 d_s\right) \\
&= \mathcal{N}\left(\frac{(x-\mu)(1-e^{-aT})}{a} + \mu T, \frac{1}{a^2}\left(\frac{b^2}{2a}(1-e^{-2aT}) + b^2 T - \frac{2b^2}{a}(1-e^{-aT})\right)\right).
\end{aligned}$$

(d) The parameter $\bar{r}$ is the mean of the limiting invariant distribution of $(r_t)_{t\geq 0}$, and can therefore be interpreted as long term mean. This is the mean level to which the process $(r_t)_{t\geq 0}$ reverts as $t \to \infty$. The speed of the mean reversion is characterised by the parameter $\lambda > 0$. Finally σ describes the volatility of the stochastic interest rate.

(e) Due to the similarity with the ordinary differential equation $\frac{y'}{y} = g \iff \log(y)' = g$, whose solution is given by $y(t) = C\exp\left(\int g(t)dt\right)$, one might try to apply Itô's formula to the function $f(x) = \log(x)$ and the positive continuous semimartingale $\widetilde{S}^0$. This yields

$$\begin{aligned}
\log(\widetilde{S}_t^0) &= \log(\widetilde{S}_0^0) + \int_0^t \frac{1}{\widetilde{S}_s^0} d\widetilde{S}_s^0 - \frac{1}{2}\int_0^t \frac{1}{(\widetilde{S}_t^0)^2} d[\widetilde{S}^0]_s \\
&= \int_0^t \frac{1}{\widetilde{S}_s^0}\widetilde{S}_s^0 r_s ds = \int_0^t r_s ds,
\end{aligned}$$

where we have used that $\widetilde{S}^0$ is of finite variation and therefore

$$[\widetilde{S}^0]_t = \left[\int \widetilde{S}^0 r ds\right]_t = \int_0^s (\widetilde{S}^0_s)^2 r_s^2 d[s]_s = 0,$$

Taking the exponential on both sides, we get

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(f) Since Q is an EMM, the discounted price process of the zero coupon bond must be a martingale under Q and therefore must satisfy

$$\frac{\tilde{P}_t^{(T)}}{e^{\int_0^t r_s ds}} = \mathbb{E}_Q\left[\frac{\tilde{P}_T^{(T)}}{e^{\int_0^T r_s ds}} \Big| \mathcal{F}_t\right]$$

Using that $\tilde{P}_T^{(T)} = 1$ we get

$$\tilde{P}_t^{(T)} = \mathbb{E}_Q\left[e^{-\int_t^T r_s ds} \big| \mathcal{F}_t\right]$$

(g) We note that r is an Ornstein–Uhlenbeck process with drift $\mu = \bar{r}$, and parameters $a = \lambda$ and $b = \sigma$. Using the result from question (a), we therefore have

$$r_t = e^{-\lambda t} r_0 + (1 - e^{-\lambda t})\bar{r} + \int_0^t e^{-\lambda(t-s)} \sigma d\hat{W}_s.$$

(h) By question (f), the initial undiscounted price of the zero coupon bond with maturity T is given by

$$\tilde{P}_0^{(T)} = \mathbb{E}_Q\left[e^{-\int_0^T r_s ds}\right].$$

Question (c) implies that, under the risk neutral measure Q, the integral appearing in the pricing formula is Gaussian with parameters

$$\begin{aligned}\int_0^T r_s ds &\sim \mathcal{N}\left(\frac{(r_0 - \bar{r})(1 - e^{-\lambda T})}{\lambda} + \bar{r}T, \frac{1}{\lambda^2}\left(\frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda T}) + \sigma^2 T - \frac{2\sigma^2}{\lambda}(1 - e^{-\lambda T})\right)\right) \\ &:= \mathcal{N}\left(\mu^*, (\sigma^*)^2\right),\end{aligned}$$

where we define

$$\mu^* := \frac{(r_0 - \bar{r})(1 - e^{-\lambda T})}{\lambda} + \bar{r}T,$$

and

$$(\sigma^*)^2 := \frac{1}{\lambda^2}\left(\frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda T}) + \sigma^2 T - \frac{2\sigma^2}{\lambda}(1 - e^{-\lambda T})\right).$$

The initial undiscounted price $\tilde{P}_0^{(T)}$ can therefore be computed using the moment generating function of the Gaussian random variable $\int_0^T r_s ds$:

$$\tilde{P}_0^{(T)} = \mathbb{E}_Q\left[e^{-\int_0^T r_s ds}\right] = e^{-\mu^* + \frac{(\sigma^*)^2}{2}}.$$

Question 5

(a) Applying Itô formula to the continuous semimartingale $\tilde{S} = (\tilde{S}^0, \tilde{S}^1)$ and the C^2 function $f(x, y) := x/y$, we derive the following SDE for the discounted stock price process S^1:

$$dS_t^1 = d\left(\frac{\tilde{S}_t^1}{\tilde{S}_t^0}\right) = \frac{d\tilde{S}_t^1}{\tilde{S}_t^0} - \frac{\tilde{S}_t^1}{(\tilde{S}_t^0)^2} d\tilde{S}_t^0$$
$$= S_t^1 \left((\mu - r)dt + \sigma(t)dW_t\right).$$

Applying Itô formula again to the continuous semimartingale S^1 and the C^2 function $g(x) := \log(x)$ leads to the following SDE for the log of the discounted stock price process:

$$d\log(S_t^1) = \frac{1}{S_t^1} dS_t^1 - \frac{1}{2}\frac{d < S^1 >_t}{(S_t^1)^2}$$
$$= \left(\mu - r - \frac{\sigma^2(t)}{2}\right) dt + \sigma(t)dW_t$$

The solution is given by

$$\log(S_t^1) = \log(S_0^1) + \int_0^t \left(\mu - r - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s)dW_s$$

or equivalently

$$S_t^1 = S_0^1 \exp\left(\int_0^t \left(\mu - r - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s)dW_s\right)$$

(b) From question (a), we know that

$$\log(S_t^1) = \log(S_0^1) + \int_0^t \left(\mu - r - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s)dW_s.$$

Since σ is a non-random continuous function, and W is a P-Brownian Motion, we have by a general result on stochastic integration (see Exercise sheet 13) that the stochastic integral $\int_0^t \sigma(s)dW_s$ is Gaussian with parameters

$$\int_0^t \sigma(s)dW_s \sim \mathcal{N}\left(0, \int_0^t \sigma^2(s)ds\right)$$

under the measure P. Hence $\log(S_t^1)$ is Gaussian under P with parameters

$$\log(S_t^1) \sim \mathcal{N}\left(\log(S_0^1) + \int_0^t \left(\mu - r - \frac{\sigma^2(s)}{2}\right) ds, \int_0^t \sigma^2(s)ds\right).$$

(c) Recall that Itô's lemma gave us in question (a)

$$dS_t^1 = S_t^1 \left((\mu - r)dt + \sigma(t)dW_t\right).$$

We can equivalently rewrite this as

$$dS_t^1 = S_t^1 \sigma(t) \left(\frac{\mu - r}{\sigma(t)} dt + dW_t\right)$$
$$= S_t^1 \sigma(t) dW_t^*$$

where we have defined $W_t^* = \int_0^t \frac{\mu - r}{\sigma(s)} ds + W_t$. Girsanov theorem tells us that

$$W_t^* = W_t - \int_0^t \frac{r - \mu}{\sigma(s)} ds$$
$$= W_t - \left\langle \int_0^t \frac{r - \mu}{\sigma(s)} dW_s, \int_0^t dW_s \right\rangle$$

is a Q-Brownian Motion under the measure $Q \sim P$ defined via the Radon Nykodym derivative

$$\frac{dQ_{|\mathcal{F}_T}}{dP_{|\mathcal{F}_T}} = \mathcal{E}\left(\int_0^T \frac{r-\mu}{\sigma(s)} dW_s\right) = \exp\left(\int_0^T \frac{r-\mu}{\sigma(s)} dW_s - \frac{1}{2}\int_0^T \left(\frac{r-\mu}{\sigma(s)}\right)^2 ds\right).$$

(d) It follows from question (a) that

$$\begin{aligned}
\tilde{S}_T^1 &= \tilde{S}_0^1 \exp\left(\int_0^T \left(\mu - \frac{\sigma^2(s)}{2}\right) ds + \int_0^T \sigma(s) dW_s\right) \\
&= \tilde{S}_0^1 \exp\left(\int_0^T \left(\mu - \frac{\sigma^2(s)}{2}\right) ds + \int_0^T (r-\mu) ds + \int_0^T \sigma(s) dW_s^*\right) \\
&= \tilde{S}_0^1 e^{rT} \exp\left(-\int_0^T \frac{\sigma^2(s)}{2} ds + \int_0^T \sigma(s) dW_s^*\right).
\end{aligned}$$

The initial (undiscounted) arbitrage free price of the call option is given by

$$\begin{aligned}
\tilde{V}_0 = V_0 &= \mathbb{E}_Q\left[\frac{(\tilde{S}_T^1 - \tilde{K})^+}{\tilde{S}_T^0}\right] \\
&= e^{-rT}\mathbb{E}_Q\left[(\tilde{S}_T^1 - \tilde{K})^+\right] \\
&= e^{-rT}\tilde{S}_0^1 e^{rT}\mathbb{E}_Q\left[\left(\exp\left(-\int_0^T \frac{\sigma^2(s)}{2} ds + \int_0^T \sigma(s) dW_s^*\right) - \frac{\tilde{K}}{\tilde{S}_0^1 e^{rT}}\right)^+\right] \\
&= \tilde{S}_0^1 F\left(\int_0^T \sigma^2(s) ds, \frac{\tilde{K}}{e^{rT}\tilde{S}_0^1}\right)
\end{aligned}$$

where in the last equation we have used that

$$\int_0^t \sigma(s) dW_s^* \sim \mathcal{N}\left(0, \int_0^t \sigma^2(s) ds\right)$$

under the measure Q (see exercise sheet 13).

(e) A simple computation using the density of the standard Gaussian distribution yields:

$$\begin{aligned}
F(v,m) &= \int_{-\infty}^{\infty} \left(e^{-v/2+\sqrt{v}x} - m\right)^+ \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{\frac{\log m}{\sqrt{v}}+\frac{\sqrt{v}}{2}}^{\infty} \left(e^{-v/2+\sqrt{v}x} - m\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{\frac{\log m}{\sqrt{v}}+\frac{\sqrt{v}}{2}}^{\infty} \frac{e^{-v/2+\sqrt{v}x-x^2/2}}{\sqrt{2\pi}} dx - m\int_{\frac{\log m}{\sqrt{v}}+\frac{\sqrt{v}}{2}}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{-\infty}^{-\frac{\log m}{\sqrt{v}}+\frac{\sqrt{v}}{2}} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds - m\int_{-\infty}^{-\frac{\log m}{\sqrt{v}}-\frac{\sqrt{v}}{2}} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds \\
&= \Phi\left(-\frac{\log m}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - m\Phi\left(-\frac{\log m}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right)
\end{aligned}$$

(f) Under the assumption $r = 0$, the initial price of the at the money call option (with strike

$\tilde{K} = \tilde{S}_0^1$) and maturity T is given by:

$$\begin{aligned}
\tilde{V}_0 &= \tilde{S}_0^1 F\left(\int_0^T \sigma^2(s)ds, e^{-rT}\right) \\
&= \tilde{S}_0^1 F\left(\int_0^T \sigma^2(s)ds, 1\right) \\
&= \tilde{S}_0^1 \left(\Phi\left(\frac{\sqrt{\int_0^T \sigma^2(s)ds}}{2}\right) - \Phi\left(-\frac{\sqrt{\int_0^T \sigma^2(s)ds}}{2}\right)\right) \\
&= \tilde{S}_0^1 \left(2\Phi\left(\frac{\sqrt{\int_0^T \sigma^2(s)ds}}{2}\right) - 1\right).
\end{aligned}$$

We therefore have

$$\int_0^T \sigma^2(s)ds = \left(2\Phi^{-1}\left(\frac{\tilde{V}_0}{2\tilde{S}_0^1} + \frac{1}{2}\right)\right)^2$$

Assuming that the quoted prices are arbitrage-free, we can therefore estimate the function $\sigma(\cdot)$ by a numerical approximation of $\frac{\partial}{\partial T}\left(2\Phi^{-1}\left(\frac{\tilde{V}_0}{2\tilde{S}_0^1} + \frac{1}{2}\right)\right)^2$.