ETH Zürich, HS 2020
D-MATH
Prof. E. W. Farkas **August 2020**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐
Matriculation number	

Leave blank

Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as much as possible.
- ◇ Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The discounted market is denoted by (S^0, S^1).

(a) Let $\varphi = (\varphi^0, \vartheta)$ be a trading strategy. Which of the following does **not** hold?

(1) $V_k(\varphi) = C_k(\varphi) + G_k(\varphi)$

(2) $V_k(\varphi) = \varphi^0_0 + \sum_{j=1}^{k} \vartheta^{\text{tr}}_j (S^1_j - S^1_{j-1})$

(3) $\Delta \widetilde{G}_{k+1} = \varphi^0_{k+1}(\widetilde{S}^0_{k+1} - \widetilde{S}^0_k) + \vartheta_{k+1}(\widetilde{S}^1_{k+1} - \widetilde{S}^1_k)$.

(b) The set of attainable payoffs:

(1) is closed under addition.

(2) is closed under scalar multiplication.

(3) consists of integrable random variables.

(c) Which assumption about the market justifies the formula $\Delta C_{k+1} = \Delta \varphi^0_{k+1} + \Delta \vartheta_{k+1} S^1_{k+1}$?

(1) The investor is small.

(2) Trading strategies are unrestricted.

(3) None of the above.

(d) Suppose S^1 is a submartingale. Then:

(1) The market is arbitrage-free.

(2) For ϑ admissible, $\vartheta \cdot S^1$ is a submartingale.

(3) $(S^1 - 1)^+ - \frac{1}{2}(S^1 - 1)^-$ is a submartingale.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let X, Y be semimartingales. Then:

(1) XY is a martingale, if X and Y are martingales.

(2) XY is a martingale, if X is a martingale and Y has finite variation.

(3) XY has finite variation, if X and Y have finite variation.

(f) Which of the following is a stopping time?

(1) $\tau = \sup\{t \geq 0 : W_t \geq t\}$

(2) $\tau = \inf\{t \geq 1 : \int_1^t \operatorname{sgn}(W_{s-1}) dW_s > 2\}$, where $\operatorname{sgn}(z) = \mathbb{1}_{z>0} - \mathbb{1}_{z<0}$.

(3) $\tau = \inf\{t \geq 0 : W_{t^2} \geq 1\}$

(g) Let $Q \approx P$. Then:

(1) If X is a P-submartingale, it is a Q-submartingale.

(2) If X is a continuous P-martingale, then $X^2 - [X]$ is a continuous local Q-martingale.

(3) $[W]_t = t$ holds Q-almost surely.

(h) In the Black-Scholes model, let $f(\widetilde{S}_T)$ be a payoff, for a smooth function f. What is the PDE associated with its value process $\widetilde{v}(t, \widetilde{S}_t)$?

(1)

$$0 = \frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2\widetilde{x}^2\frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = f(\widetilde{x})$$

(2)

$$0 = -\frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2\widetilde{x}^2\frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = f(\widetilde{x})$$

(3)

$$f = -\frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2\widetilde{x}^2\frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = 0$$

Question 2 (8 **Points**)

The goal of this exercise is to study important properties of martingales, local martingales and discrete time stochastic integrals. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots}$.

(a) Let $(M_t)_{t=0,1,\ldots}$ be a discrete time martingale and let $(H_t)_{t=0,1,\ldots}$ be a bounded predictable process. Show that the discrete time stochastic integral process N defined by

$$N_t = \sum_{s=1}^{t} H_s \left(M_s - M_{s-1}\right)$$

is a martingale.

(b) Conclude that the stopped process M^τ is a martingale for any stopping time τ.

(c) Question (c) is **OPTIONAL**. You can gain extra **2 points** for solving it.
Let $(X_t)_{t=0,1,\ldots}$ be a discrete time local martingale and let $(K_t)_{t=0,1,\ldots}$ be a predictable process. Show that the discrete time stochastic integral process N defined by

$$N_t = \sum_{s=1}^{t} K_s \left(X_s - X_{s-1}\right)$$

is a local martingale.

(d) Let $(X_t)_{t=0,1,\ldots}$ be a discrete time local martingale, and let $(Y_t)_{t=0,1,\ldots}$ be a process such that $|X_s| \leq Y_t$ almost surely for all $0 \leq s \leq t$, and $Y_t \in L^1(P)$ for all $t = 0, 1, \ldots$. Show that X is a true martingale.

(e) Conclude that any integrable discrete time local martingale (i.e. $X_t \in L^1(P)$ for all $t = 0, 1, \ldots$) is a true martingale.
Remark: This result is only true in discrete time.

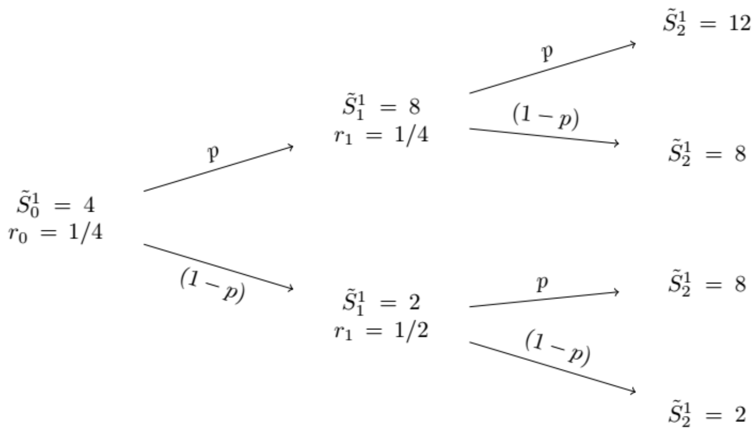
Question 3 (8 **Points**)

Consider a two step model $(\Omega, \mathcal{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ in which both the asset prices and the interest rate rate evolve randomly in time. More precisely, let the undiscounted price processes of the assets in our market be defined by $\tilde{S} = (\tilde{S}_0^0, \tilde{S}_0^1) = (1, 4)$, and

$$\tilde{S}_k^0 = \prod_{j=1}^{k} (1 + r_{j-1}) \quad \text{for } k = 1, \dots, T$$

$$\frac{\tilde{S}_{k+1}^1}{\tilde{S}_k^1} = Y_{k+1} \quad \text{for } k = 0, 1, \dots, T-1$$

where Y_k are i.i.d. random variables describing the returns of the risky asset at time k and r_k are i.i.d random variables describing the stochastic interest rates between time k and $k+1$. We endow our probability space with the natural filtration of Y, denoted by $\mathbb{F}$, and assume, for simplicity, that $\mathcal{F} = \sigma(Y_1, Y_2)$. The process Y is therefore adapted to $\mathbb{F}$, and r is supposed to be predictable with respect to $\mathbb{F}$. One dollar invested in or borrowed from the money market account $\tilde{S}^0$ at time k grows to an investment or debt of $1 + r_k$ at time $k+1$. We suppose that the distribution of Y_k and r_k under P is described by the following tree:



where $0 < p < 1$ is a fixed transition probability.

(a) Is the market arbitrage free? If yes, find the set of all EMMs for S^1. If not, construct an explicit arbitrage.

(b) Is the market complete?

(c) Compute the undiscounted price of a European Call option with maturity $T = 2$ and undiscounted strike $\tilde{K} = 7$.

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and W be a Brownian motion with respect to P and adapted to $\mathbb{F}$ and let

$$X_t = xe^{-at} + \mu(1 - e^{-at}) + b\int_0^t e^{-a(t-s)}dW_s \tag{1}$$

be an Ornstein–Uhlenbeck process with drift $\mu \in \mathbb{R}$, and parameters a, b, and $x \in \mathbb{R}$.

(a) Verify that $(X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = a(\mu - X_t)dt + bdW_t, \quad X_0 = x \tag{2}$$

(b) Show that

$$X_t \sim \mathcal{N}\left(xe^{-at} + \mu(1 - e^{-at}), \frac{b^2}{2a}(1 - e^{-2at})\right)$$

Hint: if you use results from the lecture/exercise sheets, it is enough to state them

(c) What is the distribution of the random variable $\int_0^T X_t dt$?

The following questions are **OPTIONAL**. You can gain a maximum of **5 bonus points** by correctly solving at least 3 out of the following 5 questions.

In the remaining part of the question we consider a simple stochastic interest rate model and derive an arbitrage-free price for the zero-coupon bond with maturity T and face value 1. This is a financial instrument that makes no periodic interest payment until its maturity, when it pays its face value. In particular one must have $\tilde{P}_T^{(T)} = 1$ for a zero coupon bond with maturity T and face value 1. Let $\hat{W} = (\hat{W}_t)_{t\geq 0}$ be a Brownian motion, adapted to $\mathbb{F}$, with respect to a fixed measure Q, which is equivalent to P. For simplicity, suppose that there is no risky asset in the market and the undiscounted bank account price process $\tilde{S}^0$ satisfies the following SDE:

$$d\widetilde{S}_t^0 = \widetilde{S}_t^0 r_t dt \quad \widetilde{S}_0^0 = 1$$

where $(r_t)_t$ is itself a stochastic process. In 1977, Vasicek proposed the following model for the stochastic short rate process r:

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma d\hat{W}_t \tag{3}$$

with a certain initial condition r_0. The parameters $\bar{r}$, λ and σ are given and assumed to be strictly positive.

(d) Give a financial interpretation of the parameters $\bar{r} > 0$, $\lambda > 0$ and $\sigma > 0$.

(e) Solve the ODE for the undiscounted bank account process, i.e find a closed form solution for $\tilde{S}_t^0$.

(f) Let $\tilde{P}_t^{(T)}$ denote the undiscounted time t price of a zero-coupon bond of maturity T and face value 1. Express $\tilde{P}_t^{(T)}$ in terms of a conditional expectation of a process involving r under the fixed measure Q.

(g) Show that the solution of the SDE (3) is given by

$$r_t = e^{-\lambda t}r_0 + (1 - e^{-\lambda t})\bar{r} + \int_0^t e^{-\lambda(t-s)}\sigma d\hat{W}_s. \tag{4}$$

Hint: Try to cast the dynamics (3) *into the general setting of Ornstein–Uhlenbeck processes* (2) *for which you know that the solution is given by* (1)

(h) Explain, how you could use the results of the previous questions to derive a closed form pricing formula for the initial (i.e. time $t = 0$) price $\tilde{P}_0^{(T)}$ of the zero coupon bond.

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \le s \le T)$. In the lectures we have studied the simple Black Scholes model where the volatility of the risky asset was assumed to be constant. In practice, one can observe so-called volatility-smiles in the empirical call surface contradicting the constant volatility assumption of the Black Scholes model. One possible approach to overcome this issue is to consider a non-random continuous function $\sigma : [0, +\infty) \to (0, +\infty)$ and assume that the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy

$$
\begin{aligned}
d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1, \\
d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu dt + \sigma(t) dW_t\right), & \widetilde{S}^1_0 &= S^1_0,
\end{aligned}
$$

with constants $\mu, r \in \mathbb{R}$. For simplicity you may assume $\widetilde{S}^1_0 = S^1_0 > 0$. As usual, let $S^0 = 1$ and $S^1 = (S^1_t)_{t\in[0,T]}$ where $S^1_t = \frac{\tilde{S}^1_t}{\tilde{S}^0_t}$ denote the discounted price processes.

(a) Derive an SDE for the risky stock's log price process $\log S^1_t$. Find an explicit solution for $\log S^1_t$ and conclude that the discounted risky stock price satisfies

$$
S^1_t = S^1_0 \exp\left(\int_0^t \left(\mu - r - \frac{\sigma^2(s)}{2} \right) ds + \int_0^t \sigma(s) dW_s \right).
$$

(b) What is the distribution of the random variable $\log S^1_t$ under the original measure $\mathbb{P}$?

(c) Find an EMM $\mathbb{Q}$ under which the dynamics of the discounted risky stock is given by

$$
dS^1_t = S^1_t \sigma(t) dW^*_t
$$

where W^*_t is a $\mathbb{Q}$- Brownian Motion.

(d) Express the initial (i.e time 0) undiscounted replication cost of a European call option with maturity T and undiscounted strike $\tilde{K}$ in terms of the function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$
F(v, m) := E\left[\left(e^{-v/2 + \sqrt{v}X} - m \right)^+ \right]
$$

where X is a standard Gaussian random variable.

(e) Question (e) is **OPTIONAL**. You can gain an extra **2 points** for a correct solution. Prove that the function F can be expressed explicitly in terms of the cumulative distribution function Φ of the standard gaussian distribution as follows:

$$
\begin{aligned}
F(v, m) &= E\left[\left(e^{-v/2 + \sqrt{v}X} - m \right)^+ \right] \\
&= \Phi\left(-\frac{\log m}{\sqrt{c}} + \frac{\sqrt{c}}{2} \right) - m\Phi\left(-\frac{\log m}{\sqrt{c}} - \frac{\sqrt{c}}{2} \right).
\end{aligned}
$$

(f) Assume, for simplicity, that the interest rate is zero: $r = 0$. Explain how you could use your answer to part (d) and quoted time 0 prices of at-the-money calls (i.e. call options whose strike is given by the initial price of the underlying: $\tilde{K} = \tilde{S}^1_0$) of different maturities T to estimate the function $\sigma(\cdot)$.

This model is often referred to as Hull-White extension of Black-Scholes