Question 1

The correct answers are:

(a) (2)

(b) (3)

(c) (1)

(d) (2)

(e) (3)

(f) (2)

(g) (3)

(h) (1)

Question 2

(a) Any probability measure $Q \approx P$ can be specified by four parameters $q_{\uparrow\uparrow}, q_{\downarrow\uparrow}, q_{\uparrow\downarrow}, q_{\downarrow\downarrow}$ (with the obvious interpretation), which must all be strictly between 0 and 1. Moreover, for this probability measure to be an EMM for S^1 and S^2 the following conditions must be satisfied

$$\begin{aligned} &q_{\uparrow\uparrow} + q_{\downarrow\uparrow} + q_{\uparrow\downarrow} + q_{\downarrow\downarrow} = 1, \\ &(1+u_1)q_{\uparrow *} + (1+d_1)q_{\downarrow *} = 1+r, \\ &(1+u_2)q_{*\uparrow} + (1+d_2)q_{*\downarrow} = 1+r, \end{aligned}$$

where $q_{\uparrow *} = q_{\uparrow\uparrow} + q_{\uparrow\downarrow}$, $q_{\downarrow *} = q_{\downarrow\uparrow} + q_{\downarrow\downarrow}$, $q_{*\uparrow} = q_{\uparrow\uparrow} + q_{\downarrow\uparrow}$ and $q_{*\downarrow} = q_{\uparrow\downarrow} + q_{\downarrow\downarrow}$. The first and the second equation let us solve explicitly

$$q_{\uparrow *} = \frac{r-d_1}{u_1-d_1}, \quad q_{\downarrow *} = \frac{u_1-r}{u_1-d_1}$$

and similarly the first with the third give

$$q_{*\uparrow} = \frac{r-d_2}{u_2-d_2}, \quad q_{*\downarrow} = \frac{u_2-r}{u_2-d_2}.$$

For these to be positive we already require $u_1 > r > d_1, u_2 > r > d_2$. Using a parameter $q_{\uparrow\uparrow} =: a$, we get

$$q_{\downarrow\uparrow} = \frac{r-d_2}{u_2-d_2} - a, \quad q_{\uparrow\downarrow} = \frac{r-d_1}{u_1-d_1} - a \quad \text{and} \quad q_{\downarrow\downarrow} = 1 - \frac{r-d_2}{u_2-d_2} - \frac{r-d_1}{u_1-d_1} + a.$$

Given the conditions we already have, it is clear that this leads to a well-defined equivalent measure only if

$$a \in \left(\max\left(0, \frac{r-d_2}{u_2-d_2} + \frac{r-d_1}{u_1-d_1} - 1\right), \min\left(\frac{r-d_1}{u_1-d_1}, \frac{r-d_2}{u_2-d_2}\right)\right).$$

So we have established that for no arbitrage to occur, $u_1 > r > d_1, u_2 > r > d_2$ must hold and a must belong to the given interval, and moreover the entire measure is then specified by the formulas above. Conversely, we can check that any such a generates an EMM. This interval thus describes all EMMs for S^1 and S^2. Moreover, the interval is non-empty if and only if the four inequalities hold

$$\begin{aligned} &\frac{r-d_2}{u_2-d_2} > 0, \quad \frac{r-d_1}{u_1-d_1} > 0 \\ &\frac{r-d_2}{u_2-d_2} > \frac{r-d_2}{u_2-d_2} + \frac{r-d_1}{u_1-d_1} - 1 \\ &\frac{r-d_1}{u_1-d_1} > \frac{r-d_2}{u_2-d_2} + \frac{r-d_1}{u_1-d_1} - 1 \end{aligned}$$

These are all true as long as $0 < \frac{r-d_i}{u_2-d_i} < 1$, which in turn is implied by $d_i < r < u_i$. Therefore these conditions are also sufficient for absence of arbitrage.

(b) Since the market is arbitrage-free by assumption, there exists an EMM for (S^0, S^1, S^2) by the fundamental theorem of asset pricing. The same EMM is therefore an EMM for both (S^0, S^1) and (S^0, S^2), which means again by the fundamental theorem of asset pricing that both of these markets are arbitrage-free. But both of these markets are simple binomial markets so they are also complete. We can thus hedge $f(\widetilde{S}_1^1)$ by trading only in $\widetilde{S}^1$. We do this by finding an initial value c_1 and a trading strategy θ_1 such that

$$c_1 + \theta_1(\widetilde{S}_1^1 - \widetilde{S}_0^1) = f(\widetilde{S}_1^1) \quad P\text{-a.s.},$$

since we assume $r = 0$. This gives equations

$$\begin{aligned} c_1 + \theta_1 u_1 &= f(1+u_1), \\ c_1 + \theta_1 d_1 &= f(1+d_1), \end{aligned}$$

which can indeed be solved with

$$\theta_1 = \frac{f(1+u_1) - f(1+d_1)}{u_1 - d_1},$$
$$c_1 = \frac{u_1 f(1+d_1) - d_1 f(1+u_1)}{u_1 - d_1}.$$

Analogously, we can hedge $g(\widetilde{S}_1^2)$ by trading in $\widetilde{S}^2$ only, and then add up the hedging strategies. In the specific case given, we have that

$$\begin{aligned}
\theta_1 &= \frac{f(1+u_1) - f(1+d_1)}{u_1 - d_1} = \frac{1.2^2 - 0.8^2}{0.4} = 2, \\
\theta_2 &= \frac{g(1+u_2) - g(1+d_2)}{u_2 - d_2} = \frac{0 - (-0.3)}{0.9} = \frac{1}{3}, \\
c_1 + c_2 = V_0(\widetilde{H}) &= \frac{u_1 f(1+d_1) - d_1 f(1+u_1)}{u_1 - d_1} + \frac{u_2 g(1+d_2) - d_2 g(1+u_2)}{u_2 - d_2} \\
&= \frac{0.2 \times 1.2^2 + 0.2 \times 0.8^2}{0.4} + \frac{0.6 \times (-0.3) + 0.3 \times 0}{0.9} = 1.04 - 0.2 = 0.84.
\end{aligned}$$

(c) Using that $r = 0$, we obtain

$$\begin{aligned}
E_Q[\widetilde{H}] &= q_{*\uparrow}(1.6) + q_{\uparrow\downarrow}(1.2) + q_{\downarrow\downarrow}(0.8) \\
&= \frac{0.3}{0.9} \times 1.6 + \left(\frac{0.2}{0.4} - a\right) \times 1.2 + (1 - \frac{0.3}{0.9} - \frac{0.2}{0.4} + a) \times 0.8 \\
&= \frac{19}{15} - \frac{2}{5}a.
\end{aligned}$$

One can see that the interval for a in this setting is given by $(0, \frac{1}{3})$ and so the expectations under EMMs form the set

$$\Pi(\widetilde{H}) = \left(\frac{17}{15}, \frac{19}{15}\right).$$

In particular, this set is not a singleton, so according to Theorem 1.2 on page 49 in the lecture notes, $\widetilde{H}$ is not attainable.

Question 3

(a) Since $Q \approx P$, we know that $Z > 0$ P-a.s. by Radon–Nikodým theorem. Therefore also $\frac{1}{Z} > 0$ P-a.s. As a continuous transformation of an adapted process Z, $\frac{1}{Z}$ is adapted. We also have by Lemma 3.1 in chapter 3 of the lecture notes and the adaptedness of $\frac{1}{Z}$ that

$$E_Q\left[\left|\frac{1}{Z_k}\right|\right] = E_Q\left[\frac{1}{Z_k}\right] = E_P\left[Z_k \frac{1}{Z_k}\right] = 1$$

for all $k \in \{0, 1, \ldots, T\}$. This gives the Q-integrability of $\frac{1}{Z}$ as well as the fact that

$$E_Q\left[\frac{1}{Z_0}\right] = 1.$$

Additionally, we have by the Bayes formula from Lemma 3.1 in chapter 3 of the lecture notes and again by the adaptedness of $\frac{1}{Z}$ that

$$E_Q\left[\frac{1}{Z_k}\middle|\mathcal{F}_{k-1}\right] = E_P\left[\frac{Z_k}{Z_{k-1}}\frac{1}{Z_k}\middle|\mathcal{F}_{k-1}\right] = E_P\left[\frac{1}{Z_{k-1}}\middle|\mathcal{F}_{k-1}\right] = \frac{1}{Z_{k-1}}$$

for all $k \in \{0, 1, \ldots, T\}$. So $\frac{1}{Z}$ is a strictly positive Q-martingale with $E_Q\left[\frac{1}{Z_0}\right] = 1$. Since $A \in \mathcal{F}_k$ and because $\frac{1}{Z}$ is adapted, the random variable $\frac{1}{Z_k}\mathbb{1}_A$ is $\mathcal{F}_k$-measurable and we have again by the Lemma 3.1 in chapter 3 of the lecture notes

$$E_Q\left[\frac{1}{Z_k}\mathbb{1}_A\right] = E_P\left[Z_k\frac{1}{Z_k}\mathbb{1}_A\right] = E_P\left[\mathbb{1}_A\right] = P\left[A\right].$$

(b) Let $\eta := \tau \wedge \sigma = \min\{\tau, \sigma\}$. Then we have for all $k \in \{0, 1, \ldots, T\}$

$$\begin{aligned}\{\eta \leq k\} &= \{\min\{\tau, \sigma\} \leq k\} = \{\omega \in \Omega \text{ such that } \tau(\omega) \leq k \text{ or } \sigma(\omega) \leq k\} \\ &= \{\tau \leq k\} \cup \{\sigma \leq k\} \in \mathcal{F}_k,\end{aligned}$$

since τ and σ are $\mathbb{F}$-stopping times by assumption and σ-algebras are closed under countable unions.

(c) We know from (b) that if σ_n is an $\mathbb{F}$-stopping time for all $n \in \mathbb{N}$ then $\tau_n \wedge \sigma_n$ is an $\mathbb{F}$-stopping time as well. Since $\tau_n \uparrow T$ P-a.s. by assumption, it is first enough to show that σ_n is an $\mathbb{F}$-stopping time for all $n \in \mathbb{N}$ and that $\sigma_n \uparrow T$ P-a.s. The latter is clear from the very definition of σ_n. In order to show that σ_n is an $\mathbb{F}$-stopping time, we have

$$\begin{aligned}\{\sigma_n \leq k\} &= \{\omega \in \Omega \text{ such that } |\vartheta_l| > n \text{ for some } l \in \{0, 1, \ldots, k+1\}\} \\ &= \bigcup_{l=1}^{k+1} \{|\vartheta_l| > n\} \in \mathcal{F}_k\end{aligned}$$

by the fact that ϑ is $\mathbb{F}$-predictable and that σ-algebras are closed under countable unions. In order to show that $X^{\tau_n \wedge \sigma_n}$ is a $(P, \mathbb{F})$-martingale for all $n \in \mathbb{N}$ we use the fact that a stopped martingale is a martingale (which has been proved in one of the exercise sheets) and that X^{τ_n} is a $(P, \mathbb{F})$-martingale since $(\tau_n)_{n \in \mathbb{N}}$ is a localizing sequence for X. Denoting $Y = X^{\tau_n}$ we indeed we have for $k \in \{0, 1, \ldots, T\}$ that

$$X_k^{\tau_n \wedge \sigma_n} = X_{\tau_n \wedge \sigma_n \wedge k} = X^{\tau_n}_{\sigma_n \wedge k} = Y_{\sigma_n \wedge k} = Y_k^{\sigma_n}.$$

(d) Let us denote $\rho_n := \tau_n \wedge \sigma_n$. Then we have for all $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, T\}$ that

$$(\vartheta \bullet X)_{\rho_n \wedge k} = \sum_{j=1}^{\rho_n \wedge k} \vartheta_j (X_j - X_{j-1}) = \sum_{j=1}^{k} \vartheta_j (X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)}) = (\vartheta \bullet X^{\rho_n})_k.$$

But X^{ρ_n} is a $(P, \mathbb{F})$-martingale for all $n \in \mathbb{N}$ and $|\vartheta_j| \leq n$ for all $j \leq \rho_n$ and

$$X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)} = 0 \text{ for all } j > \rho_n,$$

so we have for all $k \in \{0, 1, \ldots, T\}$ that

$$\begin{aligned}
E\left[|(\vartheta \bullet X)_{\rho_n \wedge k}|\right] &= E\left[\left|\sum_{j=1}^{k} \vartheta_j (X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)})\right|\right] \leq \sum_{j=1}^{k} E\left[|\vartheta_j||X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)}|\right] \\
&= \sum_{j=1}^{k} E\left[|\vartheta_j||X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)}|\mathbb{1}_{\{j \leq \rho_n\}}\right] \\
&\leq \sum_{j=1}^{k} nE\left[|X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)}|\mathbb{1}_{\{j \leq \rho_n\}}\right] \leq \sum_{j=1}^{k} nE\left[|X_{\rho_n \wedge j} - X_{\rho_n \wedge (j-1)}|\right] \\
&\leq \sum_{j=1}^{k} n(E\left[|X_{\rho_n \wedge j}|\right] + E\left[|X_{\rho_n \wedge (j-1)}|\right]) < \infty,
\end{aligned}$$

which is the integrability of $(\vartheta \bullet X)^{\rho_n}$. We also have for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, T\}$ that

$$\begin{aligned}
&E\left[(\vartheta \bullet X)_{\rho_n \wedge k} - (\vartheta \bullet X)_{\rho_n \wedge (k-1)} \,\middle|\, \mathcal{F}_{k-1}\right] = E\left[\vartheta_{k-1}(X_{\rho_n \wedge k} - X_{\rho_n \wedge (k-1)}) \,\middle|\, \mathcal{F}_{k-1}\right] \\
&\quad = E\left[\vartheta_{k-1}(X_{\rho_n \wedge k} - X_{\rho_n \wedge (k-1)})\mathbb{1}_{\{k \leq \rho_n\}} \,\middle|\, \mathcal{F}_{k-1}\right] \\
&\quad = \mathbb{1}_{\{k \leq \rho_n\}}\vartheta_{k-1} E\left[X_{\rho_n \wedge k} - X_{\rho_n \wedge (k-1)} \,\middle|\, \mathcal{F}_{k-1}\right] = 0,
\end{aligned}$$

where the third equality follows from the fact that $|\vartheta_k \mathbb{1}_{\{k \leq \rho_n\}}| \leq n$ is bounded and

$$\{\rho_n \geq k\} = \{\rho_n > k-1\} = \{\rho_n \leq k-1\}^C \in \mathcal{F}_{k-1}.$$

Adaptedness is clear since $(\vartheta \bullet X)_k^{\rho_n}$ is a sum of products of $\mathcal{F}_k$-measurable random variables so we conclude that $(\vartheta \bullet X)^{\rho_n}$ is a $(P, \mathbb{F})$-martingale for all $n \in \mathbb{N}$, which shows that $(\rho_n)_{n \in \mathbb{N}}$ is a localizing sequence for $(\vartheta \bullet X)$.

Question 4

(a) BW is adapted since B and W are. By independence, $E[|B_t W_t|] = E[|B_t|]E[|W_t|] < \infty$ since B, W are integrable. Finally,

$$\begin{aligned} E[B_t W_t \mid \mathcal{F}_s] &= E[(B_t - B_s + B_s)(W_t - W_s + W_s) \mid \mathcal{F}_s] \\ &= E[(B_t - B_s)(W_t - W_s)] + B_s W_s \\ &= B_s W_s, \end{aligned}$$

where the last equality follows from the independence of the increments of W and B. Thus BW is a $(P, \mathbb{F})$-martingale. Alternatively, this can also be shown using the product rule and arguing that the stochastic integrals that show up are indeed martingales.

(b) $\mathcal{M}_0^2$ is the space of martingales M null at 0 with $\sup_{t\geq 0} E[M_t^2] < \infty$.

Note that we clearly have for all $t \geq 0$ that

$$X_t^T = \left(\int_0^t s dW_s\right)^T + B_t^T$$

Since every deterministic time is also an $\mathbb{F}$-stopping time, we can apply the *behavior under stopping* property of stochastic integrals from page 89 in the lecture notes to obtain that

$$X_t^T = \int_0^t s dW_s^T + B_t^T$$

For each $T > 0$, $B^T \in \mathcal{M}_0^2$ since $E[(B_t^T)^2] = t \wedge T \leq T < \infty$. On the other hand, by the properties of stochastic integrals $\left(\int_0^\cdot s dW_s^T\right) \in \mathcal{M}_0^2$ since the process $H = (H_s)_{s\geq 0}$ defined by $H_s := s$ is in $L^2(W^T)$, because it is clearly predictable since it is $\mathbb{F}$-adapted and continuous and

$$E\left[\int_0^\infty s^2 d\left[W^T\right]_s\right] = E\left[\int_0^\infty s^2 d(s \wedge T)\right] = E\left[\int_0^T s^2 ds\right] = \frac{T^3}{3} < \infty.$$

Clearly, both summands are null at 0, thus $X^T \in \mathcal{M}_0^2$. Finally, applying Itô's formula to the C^2-function $f(x) = x^2$ with $f'(x) = 2x$, $f''(x) = 2$ and the $(P, \mathbb{F})$-semimartingale X, together with the fact that

$$\begin{aligned} \langle X\rangle_t &= \left\langle \int_0^\cdot s dW_s + B \right\rangle_t = \left\langle \int_0^\cdot s dW_s \right\rangle_t + 2\left\langle \int_0^\cdot s dW_s, B \right\rangle_t + \langle B\rangle_t \\ &= \int_0^t s^2 d\langle W\rangle_s + 2\int_0^t s d\langle W, B\rangle_t + \langle B\rangle_t = \frac{t^3}{3} + t \end{aligned}$$

since $\langle W\rangle_t = \langle B\rangle_t = t$, $\langle W, B\rangle_t = 0$ (by the independence of W and B), gives that

$$X_t^2 = 2\int_0^t X_s dX_s + \langle X\rangle_t = 2\int_0^t s X_s dW_s + 2\int_0^t X_s dB_s + \frac{t^3}{3} + t.$$

(c) Since $X^T \in \mathcal{M}_0^2$, $(X^T)^2 - [X]^T$ is a $(P, \mathbb{F})$-martingale. Since X is continuous, we have that $[X]_t = \langle X\rangle_t$ so by (b) we get that

$$\begin{aligned} E[(X_T)^2 \mid \mathcal{F}_t] &= X_t^2 + E[[X]_T - [X]_t \mid \mathcal{F}_t] = X_t^2 + \frac{T^3 - t^3}{3} + (T - t) \\ &= T + \frac{T^3}{3} + 2\int_0^t s X_s dW_s + 2\int_0^t X_s dB_s. \end{aligned}$$

(d) The measure Q with $\frac{dQ}{dP} = Z_T$ is a probability measure equivalent to P on $\mathcal{F}_T$ since

$$\frac{dQ}{dP} = \mathcal{E}(\alpha B + \beta W)_T = \exp\left(\alpha B_T + \beta W_T - \frac{\alpha^2 + \beta^2}{2} T\right)$$

is finite and strictly positive P-a.s., and $E_P[Z_T] = E_P[Z_0] = 1$ since Z is a $(P, \mathbb{F})$-martingale. By Girsanov's theorem, since W and B are two independent $(P, \mathbb{F})$-Brownian motions and noting that

$$[B, \alpha B + \beta W]_t = \alpha [B, B]_t + [B, W]_t = \alpha [B]_t = \alpha t,$$
$$[W, \alpha B + \beta W]_t = [W, B]_t + \beta [W, W]_t = \beta [W]_t = \beta t,$$

we get that $\widetilde{B}$ and $\widetilde{W}$ are $(Q, \mathbb{F})$-Brownian motions.

Finally, we can rewrite X^2 as

$$\begin{aligned} X_t^2 &= 2\int_0^t sX_s dW_s + 2\int_0^t X_s dB_s + \frac{t^3}{3} + t \\ &= 2\int_0^t sX_s d\widetilde{W}_s + 2\int_0^t X_s d\widetilde{B}_s + 2\int_0^t (\beta s + \alpha) X_s ds + \frac{t^3}{3} + t. \end{aligned}$$

(e) No. Notice that $X_0^2 = 0$ P-a.s., while $X_t^2 \geq 0$ P-a.s. and $P[X_t^2 > 0] > 0$ for all $t > 0$. These conditions must hold under any equivalent measure Q^*, and if we additionally want X^2 to be a $(Q^*, \mathbb{F})$-martingale, then we also need to have that $E_{Q^*}[X_t^2] = E_{Q^*}[X_0^2] = 0$. These two conditions are, however, not compatible, since if $E_{Q^*}[X_t^2] = 0$, then $X_t^2 = 0$ Q^*-a.s.

Question 5

(a) By Fubini's theorem,

$$E\left[\int_0^t W_s ds\right] = \int_0^t E[W_s]ds = 0$$

and

$$\begin{aligned}
E\left[\left(\int_0^t W_s ds\right)^2\right] &= E\left[\left(\int_0^t W_s ds\right)\left(\int_0^t W_u du\right)\right] = E\left[\int_0^t \int_0^t W_s W_u ds du\right] \\
&= \int_0^t \int_0^t E[W_s W_u] ds du = \int_0^t \int_0^t (s \wedge u) ds du \\
&= \int_0^t \left(\frac{u^2}{2} + (t-u)u\right) du = \int_0^t \left(-\frac{u^2}{2} + tu\right) du = -\frac{t^3}{6} + \frac{t^3}{2} = \frac{t^3}{3},
\end{aligned}$$

where the fourth equality follows from the fact that for $s \geq u$, we have

$$\begin{aligned}
E\left[W_s W_u\right] &= E\left[(W_s - W_u + W_u)W_u\right] = E\left[(W_s - W_u)W_u\right] + E\left[W_u^2\right] \\
&= E\left[W_s - W_u\right] E\left[W_u\right] + u = u
\end{aligned}$$

and similarly for $s < u$, $E\left[W_s W_u\right] = s$.

For the conditional distribution we can rewrite

$$\int_0^T W_s ds = \int_0^t W_s ds + (T-t)W_t + \int_t^T (W_s - W_t) ds.$$

The first two summands are $\mathcal{F}_t$-measurable, while $(W_s - W_t)_{s\in[0,T]}$ is by the Markov property a new $(P, \mathbb{F})$-Brownian motion independent of $\mathcal{F}_t$. Therefore we can use the earlier calculations to compute the distribution as

$$\int_0^T W_s ds \sim \mathcal{N}\left(\int_0^t W_s ds + (T-t)W_t, \frac{(T-t)^3}{3}\right)$$

conditionally on $\mathcal{F}_t$.

(b) We compute

$$\begin{aligned}
V_t = E[H \mid \mathcal{F}_t] &= P[A_t > K \mid \mathcal{F}_t] \\
&= P\left[T\log(x) + \int_0^T \left(\sigma W_s - \frac{\sigma^2}{2}s\right) ds > T\log(K) \,\middle|\, \mathcal{F}_t\right] \\
&= P\left[\int_0^T W_s ds > \frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} \,\middle|\, \mathcal{F}_t\right].
\end{aligned}$$

By (a) we know the conditional distribution of $\int_0^T W_s ds$, and we can write

$$\int_0^T W_s ds \overset{(d)}{=} \int_0^t W_s ds + (T-t)W_t + \frac{(T-t)^{\frac{3}{2}}}{\sqrt{3}} Z,$$

for $Z \sim \mathcal{N}(0,1)$ and independent of $\mathcal{F}_t$. Therefore we can write the above as

$$\begin{aligned}
&P\left[\int_0^T W_s ds > \frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} \,\middle|\, \mathcal{F}_t\right] = \\
&= P\left[\int_0^t W_s ds + (T-t)W_t + \frac{(T-t)^{\frac{3}{2}}}{\sqrt{3}} Z > \frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} \,\middle|\, \mathcal{F}_t\right] \\
&= P\left[Z > \frac{\sqrt{3}}{(T-t)^{\frac{3}{2}}}\left(\frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} - \int_0^t W_s ds - (T-t)W_t\right) \,\middle|\, \mathcal{F}_t\right] \\
&= 1 - \Phi(X_t),
\end{aligned}$$

as we wanted, using the aforementioned independence.

If instead we use that the conditional distribution of $\int_0^T W_s ds$ given $\mathcal{F}_t$ is $\mathcal{N}(m_t, v_t^2)$ we still get $V_t = 1 - \Phi(X_t)$ with X given in the generic form by

$$X_t = \frac{1}{v_t}\left(\frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} - m_t\right).$$

(c) We can find the dynamics of $\Phi(X)$ using Itô's formula. Note that X_t is a smooth function of t, $\int_0^t W_s ds$ and W_t. We know that V_t must be a $(P, \mathbb{F})$-martingale (by the tower law), and because all the terms are continuous, any continuous finite variation part must vanish. Note that t and $\int_0^t W_s ds$ are finite variation processes, and so we only need to think about the derivative with respect to W_t (i.e., we know a priori that all the other terms must cancel out). Therefore that simply yields

$$\begin{aligned} V_t = V_0 + \int_0^t dV_s &= V_0 + \int_0^t -\Phi'(X_s)\left(-\frac{\sqrt{3}}{(T-s)^{\frac{1}{2}}}\right) dW_s \\ &= V_0 + \int_0^t \left(\frac{\sqrt{3}e^{\frac{-X_s^2}{2}}}{\sigma S_s\sqrt{2\pi(T-s)}}\right) dS_s \end{aligned}$$

giving

$$\theta_t = \left(\frac{\sqrt{3}e^{\frac{-X_t^2}{2}}}{\sigma S_t\sqrt{2\pi(T-t)}}\right)$$

and

$$V_0 = 1 - \Phi(X_0) = 1 - \Phi\left(\sqrt{3}\left(\frac{\log(K/x)}{\sigma\sqrt{T}} + \frac{\sqrt{T}\sigma}{4}\right)\right).$$