**Question 1**

The correct answers are:

(a) (3)

(b) (1)

(c) (3)

(d) (2)

(e) (2)

(f) (3)

(g) (1)

(h) (3)

## Question 2

(a) **2 points**
Let $Y$ be a martingale deflator (note that in particular $Y_T S_T$ is $P$-integrable). We want to construct an EMM $Q$. Therefore we define a new measure $Q$ with Radon-Nikodym density

$$\frac{dQ}{dP} = \frac{Y_T \tilde{S}_T^0}{E_P[Y_T \tilde{S}_T^0]}$$

Observe that

- $Q$ is a probability measure since $Q[\Omega] = E_Q[\mathbb{1}_\Omega] = E_P\left[\frac{dQ}{dP}\right] = 1$
- $Q$ is equivalent to $P$ since $\frac{dQ}{dP} > 0$ by the positiveness of the martingale deflator $Y$.

Remains to show that $Q$ is a martingale measure. By Bayes formula,

$$E_Q\left[\frac{\tilde{S}_T^1}{\tilde{S}_T^0}|\mathcal{F}_t\right] = \frac{E_P[\tilde{S}_T^1 Y_T|\mathcal{F}_t]}{E_P[\tilde{S}_T^0 Y_T|\mathcal{F}_t]}$$

Since $Y$ is a martingale deflator, the numerator is equal to $E_P[\tilde{S}_T^1 Y_T|\mathcal{F}_t] = \tilde{S}_t^1 Y_t$ and the denominator is equal to $E_P[\tilde{S}_T^0 Y_T|\mathcal{F}_t] = \tilde{S}_t^0 Y_t$. Simplifying by the non-negative $Y_t$ gives

$$E_Q\left[\frac{\tilde{S}_T^1}{\tilde{S}_T^0}|\mathcal{F}_t\right] = \frac{\tilde{S}_t^1}{\tilde{S}_t^0}$$

and hence $Q$ is a martingale measure (adaptedness of the discounted price process does not depend on the probability measure and integrability of $\frac{\tilde{S}_T^1}{\tilde{S}_T^0}$ under $\mathbb{Q}$ follows easily from the integrability of $Y_T \tilde{S}_T^1$ under $\mathbb{P}$).
Conversely, suppose that $Q$ is an EMM. Let

$$Z_t = E_P\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$$

Note that by definition $Z$ is a $P$-martingale. Moreover since $Q$ is equivalent to $P$, the process $Z$ is positive. Define

$$Y_t = \frac{Z_t}{\tilde{S}_t^0}$$

We now show that $Y$ is a martingale deflator. First,note that $Y$ is positive since $Z$ and $\tilde{S}^0$ are both positive. Moreover the process $Y$ satisfies

$$\begin{aligned} E_P\left[\tilde{S}_T^0 Y_T|\mathcal{F}_t\right] &= E_P\left[Z_T|\mathcal{F}_t\right] \\ &= Z_t \\ &= \tilde{S}_t^0 Y_t \end{aligned}$$

Furthermore, $\tilde{S}_T^1/\tilde{S}_T^0$ is $Q$-integrable (by the definition of martingale) and hence $\tilde{S}_T^1 Y_T$ is $P$-integrable. We can thus conclude using Bayes formula that

$$\begin{aligned} E_P\left[\tilde{S}_T^1 Y_T|\mathcal{F}_t\right] &= E_Q\left[\frac{\tilde{S}_T^1}{\tilde{S}_T^0}|\mathcal{F}_t\right] E_P\left[\tilde{S}_T^0 Y_T|\mathcal{F}_t\right] \\ &= \frac{\tilde{S}_t^1}{\tilde{S}_t^0}\tilde{S}_t^0 Y^t \\ &= \tilde{S}_t^1 Y_t \end{aligned}$$

so $Y$ is a martingale deflator.

(b) **2 points**
Since the market is complete, there is no problem with integrability because $Y_t$ is bounded for all $t \geq 0$. Adaptedness is clear by assumption. Remains to prove the supermartingale property.

Using our assumption that $\tilde{S}^0_{t+1} \geq \tilde{S}^0_t$ for all $t \geq 0$, we have

$$Y_t \leq \frac{Y_t \tilde{S}^0_t}{\tilde{S}^0_s}$$

and hence using that $Y$ is a martingale deflator, we get

$$\begin{aligned} E_P[Y_t|\mathcal{F}_s] &\leq E_P\left[\frac{Y_t\tilde{S}^0_t}{\tilde{S}^0_s}|\mathcal{F}_s\right] \\ &= \frac{1}{\tilde{S}^0_s} E_P[Y_t \tilde{S}^0_t|\mathcal{F}_s] \\ &= \frac{Y_s \tilde{S}^0_s}{\tilde{S}^0_s} = Y_s \end{aligned}$$

Hence $Y$ is a $P$-supermartingale.

(c) **2 points**
Jensen's inequality and the martingale property of $Y\tilde{S}^1$ together imply

$$\begin{aligned} E_P[(Y_t\tilde{S}^1_t - Y_t\tilde{K})^+|\mathcal{F}_s] &\geq (E_P[Y_t\tilde{S}^1_t - Y_t\tilde{K}|\mathcal{F}_s])^+ \\ &= (Y_s S^1_s - \tilde{K} E_P[Y_t|\mathcal{F}_s])^+ \\ &\geq (Y_s S^1_s - Y_s\tilde{K})^+ \end{aligned}$$

where the supermartingale property of $Y$ has been used in the last line.

(d) **1 point**
By no arbitrage, we know form lecture that the discounted initial price of the European Call option is

$$C(T,\tilde{K}) = E_Q\left[\frac{(\tilde{S}^1_T - \tilde{K})^+}{\tilde{S}^0_T}\right]$$

Using the one-to-one correspondence between EMMs and martingale deflators given by

$$\frac{dQ}{dP} = \frac{Y_T\tilde{S}^0_T}{E_P[Y_T\tilde{S}^0_T]}$$

we conclude using Bayes formula that

$$\begin{aligned} C(T,\tilde{K}) &= \frac{E_P\left[Y_T(\tilde{S}^1_T - \tilde{K})^+\right]}{E_P\left[Y_T\tilde{S}^T_0\right]} \\ &= \frac{E_P\left[Y_T(\tilde{S}^1_T - \tilde{K})^+\right]}{Y_0\tilde{S}^0_0} \\ &= \frac{E_P\left[Y_T(\tilde{S}^1_T - \tilde{K})^+\right]}{\tilde{S}^0_0} \end{aligned}$$

The undiscounted initial price of the option is therefore

$$\tilde{C}(T,\tilde{K}) = \tilde{S}^0_0 C(T,\tilde{K}) = E_P\left[Y_T(\tilde{S}^1_T - \tilde{K})^+\right]$$

(e) **1 point**
That $\tilde{K} \to \tilde{C}(T,\tilde{K})$ is decreasing and convex is immediate from the same properties of $K \to (\tilde{S}^1_T - \tilde{K})^+$. That $T \to \tilde{C}(T,\tilde{K})$ is increasing is a consequence of the submartingale property of $Y(\tilde{S}^1 - K)^+$.

## Question 3

(a) **3 points**
We use $\Omega = \{u, m, d\}^T$, and define the random variables $Y_k(\omega) = 1 + \omega_k$.

Begin by introducing the notation $I_k = \{u, m, d\}^k$ for the set of outcomes until time $k$ and $J_k = \{u, m, d\}^{T-k}$ for the set of future outcomes. Then set $S^1 := \tilde{S}^1/\tilde{S}^0$. By rewriting the martingale condition $S^1_k = E_Q[S^1_{k+1}|\mathcal{F}_k]$, we obtain

$$1 + r = E_Q\left[Y_{k+1}|\mathcal{F}_k\right] = \sum_{\omega^k \in I_k} E_Q[Y_{k+1}|\{\omega^k\} \times J_k] 1_{\{\omega^k\}\times J_k},$$

for $k = 0, 1, \ldots, T-1$. With the notation $Q[Y_{k+1} = 1+v|\{\omega^k\} \times J_k] = q^v_{\omega^k}$, for $v \in \{u, m, d\}$ and $\omega^k \in I_k$, this condition reduces to

$$q^u_{\omega^k} u + q^m_{\omega^k} m + q^d_{\omega^k} d = r, \quad \forall \omega^k \in I_k,$$

because $q^u_{\omega^k} + q^m_{\omega^k} + q^d_{\omega^k} = 1$. In the case $k = 0$, we have

$$q^u_{\omega^0} u + q^m_{\omega^0} m + q^d_{\omega^0} d = r,$$

where $q^v_{\omega^0} = Q[Y_1 = 1 + v]$. The solution for all $k = 0, \ldots, T$ are analogous to the one period case handled below.

When $T = 1$, we omit the dependence of $q$ on $k$ and therefore need to find $q_i$ for $i \in \{u, m, d\}$ such that

$$\begin{aligned}
&1 + r = (1 + u)q_u + (1 + m)q_m + (1 + d)q_d, && \text{(Martingale property)}\\
&1 = q_u + q_m + q_d, && (Q[\Omega] = 1)\\
&q_i \in (0, 1), \quad i \in \{u, m, d\}. && (Q \approx P)
\end{aligned}$$

We parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$q_u = \frac{(r - d) - (m - d)\lambda}{u - d},$$
$$q_d = \frac{(u - r) - (u - m)\lambda}{u - d}.$$

Now we just have to restrict $\lambda$ according to the third condition. This amounts to choosing $\lambda$ such that

$$\begin{aligned}
q_m \in (0, 1) &\Leftrightarrow \lambda \in (0, 1),\\
q_u \in (0, 1) &\Leftrightarrow \lambda \in \left(\frac{r-u}{m-d}, \frac{r-d}{m-d}\right),\\
q_d \in (0, 1) &\Leftrightarrow \lambda \in \left(\frac{d-r}{u-m}, \frac{u-r}{u-m}\right).
\end{aligned}$$

Since $u > m > d$ and $u > r > d$ this reduces to

$$\lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

For the general case ($T \geq 1$) the same argument can be used to write the set of solutions, with the parameter $\lambda_{\omega^k}$, as

$$(q^u_{\omega^k}, q^m_{\omega^k}, q^d_{\omega^k}) = \left(\frac{(r-d) - (m-d)\lambda_{\omega^k}}{u-d}, \lambda_{\omega^k}, \frac{(u-r) - (u-m)\lambda_{\omega^k}}{u-d}\right),$$

where

$$\lambda_{\omega^k} \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

For any sequence of $\lambda_{\omega^k}$, $\omega^k \in I_k$ for $k = 0, \dots, T-1$ as above, we get an (equivalent) martingale measure $Q$, namely

$$Q[\{\omega\}] = \prod_{k=1}^{T} q_{\omega^{k-1}}^{\omega_k},$$

where $\omega = (\omega_1, \dots, \omega_k, \dots, \omega_T) \in \Omega$ and $\omega^{k-1} = (\omega_1, \dots, \omega_{k-1}) \in I_{k-1}$ and $q_{\omega^{k-1}}^{\omega_k}$, as defined earlier, is the conditional probability under $Q$ that $Y_k$ takes the value $1 + \omega_k$, given that we are in the node $\omega^{k-1}$ at time $k-1$, for $k = 1, \dots, T$.

(b) **2 points**
Since $\tilde{S}_0^1 = 1$ is deterministic, $\mathcal{F}_0$ must be trivial, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Therefore $\tilde{H} \in L_+^0(\mathcal{F}_T)$ can be replicated if and only if there exists an admissible self-financing trading strategy $\phi = (\tilde{V}_0, \vartheta_1) \in \mathbb{R}^2$ such that

$$\tilde{V}_0(1+r) + \vartheta_1 \tilde{S}_1^1 = \tilde{H}$$

hold in every state of the world, which, using $\tilde{S}_0^1 = 1$, is equivalent to

$$\tilde{V}_0(1+r) + \vartheta_1 Y_1 = \tilde{H}$$

Writing $\tilde{H}^u$, $\tilde{H}^m$ and $\tilde{H}^d$ for the outcomes of the random variable $\tilde{H}$ on the different market scenarios "up", "middle" and "down" respectively, the above vector equation leads to the following system of equations

$$\begin{cases} \tilde{V}_0(1+r) + \vartheta_1(1+u) = \tilde{H}^u \\ \tilde{V}_0(1+r) + \vartheta_1(1+m) = \tilde{H}^m \\ \tilde{V}_0(1+r) + \vartheta_1(1+d) = \tilde{H}^d \end{cases}$$

This system admits non-trivial solutions if and only if

$$\det \begin{bmatrix} 1+r & 1+u & \tilde{H}^u \\ 1+r & 1+m & \tilde{H}^m \\ 1+r & 1+d & \tilde{H}^d \end{bmatrix} = 0.$$

(c) **2 points**
Since $\tilde{S}_0^1 = 1$ is deterministic, $\mathcal{F}_0$ must be trivial, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and thus our optimisation for $T = 1$ simplifies to an optimisation over $\vartheta \in \mathbb{R}$:

$$\min_{\vartheta \in \mathbb{R}} E\left[\left((\tilde{S}_1^1 - \tilde{K})^+ - (\vartheta \bullet \tilde{S}^1)_T\right)^2\right]$$

which using $\tilde{S}_0^1 = 1$, is equivalent to

$$\min_{\vartheta \in \mathbb{R}} E\left[\left((Y_1 - 1)^+ - \vartheta(Y_1 - 1)\right)^2\right]$$

Using the given values for $u$, $m$, $d$, $p_u$, $p_m$ and $p_d$, the problem becomes

$$\min_{\vartheta \in \mathbb{R}} \frac{2}{3}(0.5 - 0.5\vartheta)^2 + 0 + \frac{1}{6}(0.5\vartheta)^2$$

whose solution is given by $\vartheta^* = \frac{4}{5}$ and the minimal mean squared hedging error is $\frac{1}{30}$.

**Question 4**

(a) **1 point**
The parameter $\bar{r}$ is the mean of the limiting ivariant distribution of $(r_t)_{t\geq 0}$ and can therefore be interpreted as long term mean. This is the mean level to which the process $(r_t)_{t\geq 0}$ reverts as $t \to \infty$. The speed of the mean revesion is characterised by the parameter $\lambda > 0$. FInally $\sigma$ describes the volatility of the stochastic interest rate.

(b) **2 points**
Due to the similarity with the ordinary differential equation $\frac{y'}{y} = g \iff \log(y)' = g$, whose solution is given by $y(t) = C \exp\left(\int g(t)dt\right)$, one might try to apply Itô's formula to the function $f(x) = \log(x)$ and the positive continuous semimartingale $\widetilde{S}^0$. This yields

$$\begin{aligned}\log(\widetilde{S}^0_t) &= \log(\widetilde{S}^0_0) + \int_0^t \frac{1}{\widetilde{S}^0_s} d\widetilde{S}^0_s - \frac{1}{2}\int_0^t \frac{1}{(\widetilde{S}^0_t)^2} d[\widetilde{S}^0]_s \\ &= \int_0^t \frac{1}{\widetilde{S}^0_s} \widetilde{S}^0_s r_s ds = \int_0^t r_s ds,\end{aligned}$$

where we have used that $\widetilde{S}^0$ is of finite variation and therefore

$$[\widetilde{S}^0]_t = \left[\int \widetilde{S}^0 r ds\right]_t = \int_0^s (\widetilde{S}^0_s)^2 r_s^2 d[s]_s = 0,$$

Taking the exponential on both sides, we get

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(c) **2 points**
Since $Q$ is an EMM, the discounted price process of the zero coupon bond must be a martingale under $Q$ and therefore must satisfy

$$\frac{\tilde{P}_t^{(T)}}{e^{\int_0^t r_s ds}} = \mathbb{E}_Q\left[\frac{\tilde{P}_T^{(T)}}{e^{\int_0^T r_s ds}} | \mathcal{F}_t\right]$$

Using that $\tilde{P}_T^{(T)} = 1$ we get

$$\tilde{P}_t^{(T)} = \mathbb{E}_Q\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right]$$

(d) **3 points**
Using Ito's lemma, we get that the undicsounted price dynamics satisfy

$$\begin{aligned}d\tilde{V}(t, r_t) &= \frac{\partial v}{\partial t}(t, r_t)dt + \frac{\partial \tilde{V}}{\partial r}(t, r_t)dr_t + \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial r_t^2}(t, r_t)d < r >_t \\ &= \left(\frac{\partial \tilde{V}}{\partial t}(t, r_t) + \lambda(\bar{r} - r_t)\frac{\partial \tilde{V}}{\partial r_t} + \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{V}}{\partial r_t^2}(t, r_t)\right)dt + \frac{\partial \tilde{V}}{\partial r_t}(t, r_t)\sigma d\hat{W}_t\end{aligned}$$

Applying Ito once more with the semimartingale $(\tilde{V}, \tilde{S}^0)$ and the $C^2$ function $f(x, y) = \frac{x}{y}$, we get the dynamics of the discounted price process $V = \frac{\tilde{V}}{\tilde{S}^0}$:

$$\begin{aligned}dV(t, r_t) &= \frac{1}{\tilde{S}_0^t} d\tilde{V} - \frac{\tilde{V}}{\tilde{S}_t^0} r_t dt \\ &= \frac{1}{\tilde{S}_0^t}\left[d\tilde{V} - r_t \tilde{V} dt\right] \\ &= \frac{1}{\tilde{S}_0^t}\left[\left(\frac{\partial \tilde{V}}{\partial t}(t, r_t) + \lambda(\bar{r} - r_t)\frac{\partial \tilde{V}}{\partial r_t} + \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{V}}{\partial r_t^2}(t, r_t) - r_t\tilde{V}(t, r_t)\right)dt + \frac{\partial \tilde{V}}{\partial r_t}(t, r_t)\sigma d\hat{W}_t\right]\end{aligned}$$

The LHS is a $Q$-martingale by definition and the stochastic integral on the RHS defines a local $Q$-martingale. Therefore the finite variational part on the RHS is a local martingale starting at 0 and of finite variation and hence it must vanish. This gives

$$\int_0^t \left( \frac{\partial \tilde{V}}{\partial t}(s, r_s) + \lambda(\bar{r} - r_t)\frac{\partial \tilde{V}}{\partial r_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial r_t^2}(s, r_s) - r_s \tilde{V}(s, r_s) \right) ds = 0 \quad \text{for all } t \geq 0$$

which is equivalent to

$$\frac{\partial \tilde{V}}{\partial t}(t, r_t) + \lambda(\bar{r} - r_t)\frac{\partial \tilde{V}}{\partial r}(t, r_t) + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial r_t^2}(t, r_t) = r_t \tilde{V}$$

on $(0, \infty) \times (-\infty, \infty)$ since the integral up to time $t$ can vanish for all $t \geq 0$ if and only if the integrand vanishes on it's support.

Finally the undiscounted value process must satisfy the boundary condition

$$\tilde{V}(T, r_T) = \tilde{P}_T^{(T)} = 1$$

(e) **3 BONUS points**
The idea is to take the partial derivatives of the ansatz and plug them back into the PDE obtained in the last question. To simplify the notations, we will write $\dot{R} := \frac{dR}{dt}$ and $\dot{Q} := \frac{dQ}{dt}$ for the derivatives of the functions $R$ and $Q$ with respect to $t$. An easy computation gives:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} = \tilde{V}(-r_t \dot{R} - \dot{Q}) \\ \frac{\partial \tilde{V}}{\partial r} = \tilde{V} R \\ \frac{\partial^2 \tilde{V}}{\partial r^2} = \tilde{V} R^2 \end{cases}$$

Plugging back the above expressions into the pricing PDE yields

$$\tilde{V}\left(-r_t \dot{R} - \dot{Q}\right) + \lambda\left(\bar{r} - r_t\right)\tilde{V} R + \frac{1}{2}\sigma^2 \tilde{V} R^2 = r_t \tilde{V}$$

which is equivalent to

$$r_t \tilde{V}\left(\dot{R} - \lambda R\right) + \tilde{V}\left(-\dot{Q} + \lambda \bar{r} R + \frac{1}{2}\sigma^2 R^2\right) = r_t \tilde{V} \tag{1}$$

Equation (1) has to hold on the full support of the function $\tilde{V}$, which is only possible if the coefficients of the terms in $r_t$ on the LHS equals the coefficients in $r_t$ on the RHS and similarly for the coefficients for the terms without $r_t$. This leads to the following system of ODEs:

$$\begin{cases} -\dot{R} - \lambda R = 1 \\ \dot{Q} = \lambda \bar{r} R + \frac{1}{2}\sigma^2 R^2 \end{cases}$$

The initial conditions are $R(0) = Q(0) = 0$ so that $\tilde{V}(T, r_T) = \exp\left(r_T R(T - T) + Q(T - T)\right) = 1$.

The equation for $R$ can easily be solved via variation of constants method. Finally $Q$ can be simply obtained by integrating the solution for $R$.

(a) **Solving homogeneous equation for** $R$ Consider the homogeneous equation

$$-\dot{R} - \lambda R = 0$$

which is clearly equivalent to

$$\frac{\dot{R}}{R} = -\lambda$$

Integrating on both sides gives

$$\ln R(t) = -\lambda t + c_1$$

for some constant $c_1$ Taking the exponential, we find

$$R(t) = K \exp(-\lambda t)$$

for some constant $K$.

(b) **Solving inhomogeneous equation for $R$ (variation of constants)** Suppose that $K$ is actually a function of time $t$ to solve the inhomegeneous case. Using the solution for the homogeneous case, the inhomogeneous equation

$$-\dot{R} - \lambda R = 1$$

leads to the following ODE for $K$:

$$\dot{K} \exp(-\lambda t) = -1$$

whose solution is

$$K(t) = \frac{-\exp(\lambda t)}{\lambda} + c_2$$

for some constant $c_2$. The solution to the inhomogeneous equation $-\dot{R} - \lambda R = 1$ is therefore

$$R(t) = \left(c_2 - \frac{\exp(\lambda t)}{\lambda}\right) \exp(-\lambda t)$$

Using the initial condition $R(0) = 0$ we found $c_2 = \frac{1}{\lambda}$ and therefore

$$R(t) = \frac{\exp(-\lambda t) - 1}{\lambda}$$

which is equivalent to

$$R(T-t) = \frac{\exp(-\lambda(T-t)) - 1}{\lambda}$$

(c) **Solving ODE for $Q$** The solution of $\dot{Q} = \lambda \bar{r} R + \frac{1}{2}\sigma^2 R^2$ is obtained by integrating the RHS:

$$\begin{aligned} Q(T-t) &= Q(0) + \int_0^{T-t} \left(\lambda \bar{r} R(s) + \frac{1}{2}\sigma^2 (R(s))^2\right) ds \\ &= \int_0^{T-t} \left(\lambda \bar{r} R(s) + \frac{1}{2}\sigma^2 (R(s))^2\right) ds \\ &= \bar{r} \frac{1 - e^{-\lambda(T-t)} - \lambda(T-t)}{\lambda} + \frac{\sigma^2 \left(4e^{-\lambda(T-t)} - e^{-2\lambda(T-t)} + 2\lambda(T-t) - 3\right)}{4\lambda^3} \end{aligned}$$

where in the second equality we have used the initial condition $Q(0) = 0$ and in the last equality we have used the expression for $R(z)$ found previously and have evaluated the integral explicitly.

Finally plugging back the solutions for $R(T-t)$ and $Q(T-t)$ into our initial ansatz, we get the the solution to the pricing PDE is

$$\begin{aligned} \tilde{V}(t, r_t) &= \exp\left(r_t R(T-t) + Q(T-t)\right) \\ &= \exp\left(r_t \frac{e^{-\lambda(T-t)} - 1}{\lambda} + \bar{r} \frac{1 - e^{-\lambda(T-t)} - \lambda(T-t)}{\lambda} + \frac{\sigma^2 \left(4e^{-\lambda(T-t)} - e^{-2\lambda(T-t)} + 2\lambda(T-t) - 3\right)}{4\lambda^3}\right) \end{aligned}$$

**Question 5**

To price the contingent claims, we first need to find an EMM $Q$. To get find this measure, we first need to derive the dynamics of the discounted price process $S^1$ under $P$ and then try to find a candidate $Q$ under which $S^1$ can be expressed as a stochastic integral with respect to a $Q$-Brownian Motion and therefore is a $Q$-local martingale. To show that $S^1$ is actually a true $Q$-martingale, we solve the SDE for $S^1$ under $Q$ and show explicitly that the solution defines a $Q$-martingale and therefore our candidate $Q$ is indeed an EMM. Uniqueness of the EMM $Q$ follows from Ito's representation theorem

Using Ito's formula with the semimartingale $(\tilde{S}^1, \tilde{S}^0)$ and the $C^2$ function $f(x, y) = \frac{x}{y}$ we get the $P$-dynamics of the discounted price process $S^1$:

$$dS_t^1 = S_t^1\left((\mu - r)dt + \sigma dW_t\right) \tag{2}$$

To get a candidate EMM $Q$ we rewrite (2) as

$$\begin{aligned} dS_t^1 &= S_t^1\left((\mu - r)dt + \sigma dW_t\right) \\ &= S_t^1\sigma\left(\frac{\mu - r}{\sigma}dt + dW_t\right) \\ &= S_t^1\sigma dW_t^* \end{aligned}$$

with $W^* = (W_t)_{0\le t\le T}$ defined by

$$W_t^* = W_t + \frac{\mu - r}{\sigma}t = W_t + \int_0^t \lambda ds \qquad \text{for } 0 \le t \le T$$

where $\lambda = \frac{\mu - r}{\sigma}$ is the instantaneous market price of risk of $S^1$. Girsanov theorem tells us that $W^*$ is a Brownian motion under the probability measure $Q^*$ given by

$$\frac{dQ^*}{dP} = \mathcal{E}\left(-\int \lambda dW\right)_T = \exp(-\lambda W_T - \frac{1}{2}\lambda^2 T)$$

By a general result on stochastic integration, the stochastic integral process

$$S_t^1 = S_0^1 + \int_0^t S_s^1 \sigma dW_s^*$$

is then a continuous $Q$-local martingale. It is even a $Q$-martingale since we get using a log transformation that the solution of $dS_t^1 = S_t^1\sigma dW_t^*$ is given by

$$S_t^1 = S_0^1 \exp\left(\sigma W_t^* - \frac{1}{2}\sigma^2 t\right)$$

which indeed defines a $Q$-martingale as $W^*$ is a $Q$-Brownian motion.

**Remark:** the proof of existence and uniqueness of the EMM $Q$ is not required, students are allowed to state the results from the lecture.

Let $H = \frac{\tilde{H}}{\tilde{S}_T^0}$ denote the discounted payoff. The no arbitrage discounted price of $H$ is given by the $Q$-martingale $V^* = (V_t^*)_{0\le t\le T}$ defined as

$$V_t^* = \mathbb{E}_Q\left[H|\mathcal{F}_t\right] \text{ for } 0 \le t \le T \tag{3}$$

To find the initial replication price and the hedging strategy, we need to compute the above conditional expectation (3) for the particular value of $H$ and express it as a function of current time $t$ and current underlying price $S_t^1$, i.e. find a function $v$ such that $V_t^* = v(t, S_t^1)$ for all $0 \le t \le T$. If the function $v$ is smooth enough (which is the case for the given payoffs), we can apply Ito's formula to get

$$v(t, S_t^1) = v(0, S_0^1) + \int_0^t \frac{\partial}{\partial x}v(t, S_t^1)dS_t^1 + \text{ continuous FV process.}$$

Since the left-hand side and the stochastic integral on the right-hand side are local $Q$-martingales, the "continuous FV process" is a local $Q$-martingale as well and since it apparently is null at 0, it must be identically equal to 0. We thus immediately obtain that the hedging strategy as

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1)$$

i.e. a s the spatial derivative of $v$, evaluated along the trajectories of $S^1$. Note that $\vartheta$ represents the holdings in the risky asset $\tilde{S}^1$. To find the holding $\phi_t$ in the numeraire $\tilde{S}^0$, one simply solves the budget equation

$$\phi_t + \vartheta_t S_t^1 = V_t^* = v(t, S_t^1)$$

which gives

$$\phi_t = v(t, S_t^1) - \vartheta_t S_t^1$$

Finally, the initial discounted replication cost is given by $v(0, S_0^1)$.

It thus remains to find the function $v$ for the four different payoffs of the exercise. For all subquestions, we will use that the undiscounted terminal price $\tilde{S}_T^1$ can be expressed as

$$\begin{aligned}\tilde{S}_T^1 &= e^{rT} S_T^1 \\ &= e^{rT} S_t^1 \exp\left(\sigma(W_T^* - W_t^*) - \frac{\sigma^2}{2}(T-t)\right)\end{aligned}$$

in terms of the $Q$-Brownian motion $W^*$.

(a) **4 points**

$$\begin{aligned}V_t^* &= \mathbb{E}_Q\left[\frac{(\tilde{S}_T^1)^p}{\tilde{S}_T^0} \big| \mathcal{F}_t\right] \\ &= \mathbb{E}_Q\left[\frac{\left(e^{rT} S_t^1 \exp\left(\sigma(W_T^* - W_t^*) - \frac{\sigma^2}{2}(T-t)\right)\right)^p}{\tilde{S}_T^0} \big| \mathcal{F}_t\right] \\ &= e^{rT(p-1)} (S_t^1)^p \exp\left(-\frac{p\sigma^2(T-t)}{2}\right) \mathbb{E}_Q\left[e^{p\sigma(W_T^* - W_t^*)}\right] \\ &= (S_t^1)^p \exp\left((p-1)(r + p\frac{\sigma^2}{2})(T-t)\right) \exp\left((p-1)t\right) \\ &= (S_t^1)^p \exp\left((p-1)\left(rT + p\frac{\sigma^2}{2}(T-t)\right)\right) \\ &= (\tilde{S}_t^1)^p e^{-rtp} \exp\left((p-1)\left(rT + p\frac{\sigma^2}{2}(T-t)\right)\right) \\ &= v(t, S_t^1)\end{aligned}$$

with

$$v(t, x) = x^p \exp\left((p-1)\left(rT + p\frac{\sigma^2}{2}(T-t)\right)\right)$$

By the above arguments, the hedging strategy is fully characterised by

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1) = p(S_t^1)^{p-1} \exp\left((p-1)\left(rT + p\frac{\sigma^2}{2}(T-t)\right)\right)$$

and $V_0^* = v(0, S_0^1) = (S_0^1)^p \exp\left((p-1)\left(rT + p\frac{\sigma^2}{2}T\right)\right)$.

(b) **4 points**
Similarly we get

$$\begin{aligned}
V_t^* &= \mathbb{E}_Q\left[\frac{\left(\log(\tilde{S}_T^1)\right)^2}{\tilde{S}_T^0}\big|\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[\frac{\left(\log\left(e^{rT}S_t^1\exp\left(\sigma(W_T^*-W_t^*)-\frac{\sigma^2}{2}(T-t)\right)\right)\right)^2}{\tilde{S}_T^0}\big|\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[\frac{\left(rT+\log S_t^1-\frac{\sigma^2}{2}(T-t)\right)^2+2\left(rT+\log S_t^1-\frac{\sigma^2}{2}(T-t)\right)\sigma\left(W_T^*-W_t^*\right)+\sigma^2\left(W_T^*-W_t^*\right)^2}{e^{rT}}\big|\mathcal{F}_t\right] \\
&= e^{-rT}\left(\left(\log S_t^1+rT-\frac{\sigma^2}{2}(T-t)\right)^2+\sigma^2(T-t)\right) \\
&= v(t,S_t^1)
\end{aligned}$$

with

$$v(t,x)=e^{-rT}\left(\left(\log x+rT-\frac{\sigma^2}{2}(T-t)\right)^2+\sigma^2(T-t)\right)$$

By the above arguments, the hedging strategy is fully characterised by

$$\vartheta_t=\frac{\partial}{\partial x}v(t,S_t^1)=\frac{2e^{-rT}\left(\log S_t^1+rT-\frac{\sigma^2}{2}(T-t)\right)}{S_t^1}$$

and $V_0^*=v(0,S_0^1)=e^{-rT}\left(\left(\log S_0^1+rT-\frac{\sigma^2}{2}T\right)^2+\sigma^2 T\right)$.

(c) **4 points**
We start by showing the hint. Let $F(c,m):=E\left[\left(e^{-c/2+\sqrt{c}X}-m\right)^+\right]$ where $X\sim\mathcal{N}(0,1)$ is a standard normal random variable, and $c$ and $m$ are positive constants. A simple computation using the density of $X$ yields

$$\begin{aligned}
F(c,m) &= \int_{-\infty}^{\infty}\left(e^{-c/2+\sqrt{c}x}-m\right)^+\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{\frac{\log m}{\sqrt{c}}+\frac{\sqrt{c}}{2}}^{\infty}\left(e^{-c/2+\sqrt{c}x}-m\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{\frac{\log m}{\sqrt{c}}+\frac{\sqrt{c}}{2}}^{\infty}\frac{e^{-c/2+\sqrt{c}x-x^2/2}}{\sqrt{2\pi}}dx-m\int_{\frac{\log m}{\sqrt{c}}+\frac{\sqrt{c}}{2}}^{\infty}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{-\infty}^{-\frac{\log m}{\sqrt{c}}+\frac{\sqrt{c}}{2}}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds-m\int_{-\infty}^{-\frac{\log m}{\sqrt{c}}-\frac{\sqrt{c}}{2}}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds \\
&= \Phi\left(-\frac{\log m}{\sqrt{c}}+\frac{\sqrt{c}}{2}\right)-m\Phi\left(-\frac{\log m}{\sqrt{c}}-\frac{\sqrt{c}}{2}\right)
\end{aligned}$$

The discounted time $t$ value of the European call option is

$$\begin{aligned}
V_t^* &= \mathbb{E}_Q\left[\frac{\left(\tilde{S}_T^1 - \tilde{K}\right)^+}{\tilde{S}_T^0}|\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[\frac{\left(e^{rT}S_t^1\exp\left(\sigma(W_T^* - W_t^*) - \frac{\sigma^2}{2}(T-t)\right) - \tilde{K}\right)^+}{\tilde{S}_T^0}|\mathcal{F}_t\right] \\
&= S_t^1\mathbb{E}_Q\left[\left(\exp\left(\sigma(W_T^* - W_t^*) - \frac{\sigma^2}{2}(T-t)\right) - \frac{\tilde{K}}{e^{rT}S_t^1}\right)^+|\mathcal{F}_t\right] \\
&= S_t^1F(c,m) \\
&= v(t,S_t^1)
\end{aligned}$$

with $c = \sigma^2(T-t)$ and $m = \frac{\tilde{K}}{e^{rT}S_t^1} = \frac{K}{S_t^1}$ where we defined the discounted strike $K = \frac{\tilde{K}}{e^{rT}}$ and the function $v$ by

$$\begin{aligned}
v(t,S_t^1) &= S_t^1F(c,m) \\
&= S_t^1\left[\Phi\left(-\frac{\log\left(\frac{K}{S_t^1}\right)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right) - \frac{K}{S_t^1}\Phi\left(-\frac{\log\left(\frac{K}{S_t^1}\right)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)\right] \\
&= S_t^1\Phi\left(\frac{\log\left(\frac{S_t^1}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log\left(\frac{S_t^1}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

By the above arguments, the hedging strategy is fully characterised by

$$\vartheta_t = \frac{\partial}{\partial x}v(t,S_t^1) = \Phi\left(\frac{\log\left(\frac{S_t^1}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

and

$$V_0^* = v(0,S_0^1) = S_0^1\Phi\left(\frac{\log\left(\frac{S_0^1}{K}\right) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{S_0^1}{K}\right) - \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)$$

We have used the hint to compute the partial derivative $\frac{\partial}{\partial x}v(t,S_t^1)$.

(d) **4 points**
As suggested by the hint, we derive the dynamics of $(S_t^1)^2$ and relate the bonus question to pricing of a standard European call option (part c)). Note that under $P$, the undiscounted price process is given by

$$\tilde{S}_t^1 = \tilde{S}_0^1\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Therefore the squared undiscounted price satisfies

$$\begin{aligned}
(\tilde{S}_t^1)^2 &= (\tilde{S}_0^1)^2\exp\left(\left(2\mu - \sigma^2\right)t + 2\sigma W_t\right) \\
&= (\tilde{S}_0^1)^2\exp\left(\left(\bar{\mu} - \frac{\bar{\sigma}^2}{2}\right)t + \bar{\sigma}W_t\right)
\end{aligned}$$

where $\bar{\mu} = 2\mu - \sigma^2$ and $\bar{\sigma} = 2\sigma$. Therefore $(S_t^1)^2$ is again a geometric Brownian motion but with new drift $\bar{\mu} = 2\mu + \sigma^2$ and new diffusion $\bar{\sigma} = 2\sigma$. This is very helpful as it guarantees that we can use a similar argument to question c).

Using the relation between $W$ and $W^*$, we directly get that the undiscounted squared asset price statisfies

$$\begin{aligned}(\tilde{S}_T^1)^2 &= (\tilde{S}_0^1)^2 \exp\left(\left(2r-\sigma^2\right)T + 2\sigma W_T^*\right)\\ &= e^{2rT}(S_t^1)^2 \exp\left(-\sigma^2(T-t) + 2\sigma(W_T^* - W_t^*)\right)\end{aligned}$$

The discounted time $t$ value of the European call option is

$$\begin{aligned}V_t^* &= \mathbb{E}_Q\left[\frac{\left((\tilde{S}_T^1)^2 - \tilde{K}\right)^+}{\tilde{S}_T^0}\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}_Q\left[\frac{\left(e^{2rT}(S_t^1)^2 \exp\left(-\sigma^2(T-t) + 2\sigma(W_T^* - W_t^*)\right) - \tilde{K}\right)^+}{\tilde{S}_T^0}\Big|\mathcal{F}_t\right]\\ &= e^{rT+\sigma^2(T-t)}(S_t^1)^2 F(c,m)\\ &= v(t, S_t^1)\end{aligned}$$

with $c = 4\sigma^2(T-t)$ and $m = \frac{\tilde{K}}{e^{2rT}(S_t^1)^2 e^{\sigma^2(T-t)}} = \frac{K}{e^{rT+\sigma^2(T-t)}(S_t^1)^2}$ where we defined the discounted strike $K = \frac{\tilde{K}}{e^{rT}}$ and the function $v$ by

$$\begin{aligned}v(t, S_t^1) &= e^{rT+\sigma^2(T-t)}(S_t^1)^2 F(c,m)\\ &= e^{rT+\sigma^2(T-t)}(S_t^1)^2 \Phi\left(\frac{rT + 3\sigma^2(T-t) + \log\left(\frac{(S_t^1)^2}{K}\right)}{4\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{rT - \sigma^2(T-t) + \log\left(\frac{(S_t^1)^2}{K}\right)}{4\sigma\sqrt{T-t}}\right)\end{aligned}$$

We therefore have that the hedging strategy is fully characterized by

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1)$$

and

$$V_0^* = v(0, S_0^1)$$

Due to time constraints, the students are not required to explicitly evaluate the delta hedge.