Exercise 1

The correct answers are:

(a) (2)

(b) (1)

(c) (2)

(d) (3)

(e) (2)

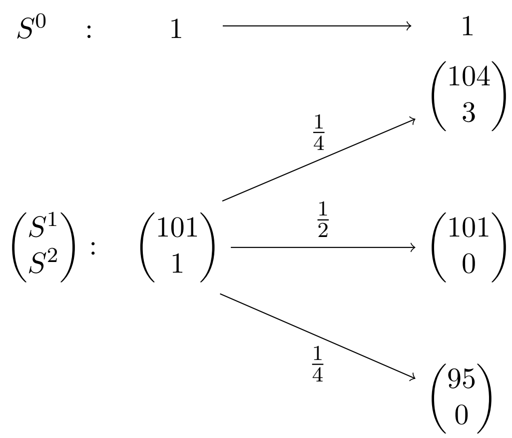
(f) (2)

(g) (2)

(h) (1)

Exercise 2

(a) (1 point) The tree diagram is the following.



(b) (3 points)We first compute the set of all equivalent martingale measures Q for S^1. Define

$$q_1 := Q[\{\omega_1\}], \qquad q_2 := Q[\{\omega_2\}], \qquad q_3 := Q[\{\omega_3\}].$$

Then Q is an EMM for S^1 if and only if $q_1, q_2, q_3 \in (0,1)$, $q_1 + q_2 + q_3 = 1$, and

$$101(1+d)q_1 + 101(1+m)q_2 + 101(1+u)q_3 = 101(1+r),$$

or equivalently

$$\begin{aligned}(q_1, q_2, q_3) &= \Big(\frac{u-r}{u-d}, 0, \frac{r-d}{u-d}\Big) + \lambda\Big(-\frac{u-r}{u-d}, 1, -\frac{r-d}{u-d}\Big) \\ &= \Big(\frac{1}{3}, 0, \frac{2}{3}\Big) + \lambda(-\frac{1}{3}, 1, -\frac{2}{3}\Big)\end{aligned}$$

for some $\lambda \in (0,1)$.
For the second part, recall that there are three possible cases

- If $s_0^2 = \frac{1}{1+r} E_Q[\widetilde{S}_1^2]$ for exactly one EMM Q for S^1, then the market is arbitrage-free and complete,
- If $s_0^2 = \frac{1}{1+r} E_Q[\widetilde{S}_1^2]$ for more than one EMM Q for S^1, then the market is arbitrage-free but not complete,
- If $s_0^2 \neq \frac{1}{1+r} E_Q[\widetilde{S}_1^2]$ for all the EMMs Q for S^1, then the market is even not arbitrage-free.

Hence we can do the following conclusions.

(i) For $\lambda \in (0,1)$, let Q^λ be the EMM for S^1 given by

$$(q_1^\lambda, q_2^\lambda, q_3^\lambda) = \Big(\frac{1}{3}, 0, \frac{2}{3}\Big) + \lambda(-\frac{1}{3}, 1, -\frac{2}{3}\Big).$$

Then, since in this case $\frac{\widetilde{K}}{1+r} = 98$ and thus

$$S_1^2(\omega_1) = 0, \qquad S_1^2(\omega_2) = 3, \qquad S_1^2(\omega_3) = 6,$$

we can conclude that $s_0^2 = \frac{1}{1+r} E_{Q^\lambda}[\widetilde{S}_1^2]$ if and only if

$$s_0^2 = 3\lambda + 6\Big(\frac{2}{3} - \lambda\frac{2}{3}\Big) = 4 - \lambda.$$

As a result, if $s_0^2 \in (3,4)$ the market is free of arbitrage and complete, and otherwise not even free of arbitrage.

(ii) For $\lambda \in (0,1)$, let again Q^λ be the EMM for S^1 given by

$$(q_1^\lambda, q_2^\lambda, q_3^\lambda) = \Big(\frac{1}{3}, 0, \frac{2}{3}\Big) + \lambda\big(-\frac{1}{3}, 1, -\frac{2}{3}\big).$$

Then, since in this case $\frac{\widetilde{K}}{1+r} = 92$ and thus

$$S_1^2(\omega_1) = 3, \qquad S_1^2(\omega_2) = 9, \qquad S_1^2(\omega_3) = 12,$$

we can conclude that $s_0^2 = \frac{1}{1+r} E_{Q^\lambda}[\widetilde{S}_1^2]$ if and only if

$$s_0^2 = 3\Big(\frac{1}{3} - \lambda\frac{1}{3}\Big) + 9\lambda + 12\Big(\frac{2}{3} - \lambda\frac{2}{3}\Big) = 9.$$

As a result, if $s_0^2 = 9$, the market is free of arbitrage but not complete, and otherwise not even free of arbitrage.

(c) (2 points) The unique (discounted) price process for H which admits no arbitrage is given by $V^H := \big(V_k^H\big)_{k=0,1}$, where

$$V_1^H = \frac{\widetilde{H}}{1+r} = \big(99 - S_1^1\big)^+ \text{ and } V_0^H = E_{Q^*}\Big[\frac{\widetilde{H}}{1+r}\Big] = \frac{1}{6}(99 - 95) = \frac{2}{3}.$$

A replication strategy for H is then an admissible, self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0^H, \vartheta^1, \vartheta^2)$ with $\vartheta^i = (\vartheta_k^i)_{k=0,1}$ for $i = 1,2$ such that $\vartheta_0^1 = \vartheta_0^2 = 0$ and

$$H = V_T(\varphi) = V_0^H + \vartheta_1^1 \Delta S_1^1 + \vartheta_1^2 \Delta S_1^2. \tag{1}$$

In our context, admissibility is automatically satisfied. By condition (1), we then have

$$\begin{aligned} H &= \frac{2}{3} + \vartheta_1^1(S_1^1 - S_0^1) + \vartheta_1^2(S_1^2 - S_0^2) \\ &\Leftrightarrow \begin{cases} 4 = 2/3 + \vartheta_1^1(95 - 101) + \vartheta_1^2(0-1) \\ 0 = 2/3 + \vartheta_1^1(101 - 101) + \vartheta_1^2(0-1) \\ 0 = 2/3 + \vartheta_1^1(104 - 101) + \vartheta_1^2\big((104-101) - 1\big) \end{cases} \\ &\Leftrightarrow \begin{cases} 4 = 2/3 - 6\vartheta_1^1 - \vartheta_1^2 \\ 0 = 2/3 - \vartheta_1^2 \\ 0 = 2/3 + 3\vartheta_1^1 + 2\vartheta_1^2 \end{cases} \end{aligned}$$

and hence $\vartheta_1^2 = \frac{2}{3}$ and $\vartheta_1^1 = -\frac{1}{3}\Big(\frac{2}{3} + 2\frac{2}{3}\Big) = -\frac{2}{3}$.

(d) (2 points) An arbitrage opportunity is an admissible, self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta^1, \vartheta^2)$ with $\vartheta^i = (\vartheta_k^i)_{k=0,1}$ for $i = 1,2$ such that $\vartheta_0^1 = \vartheta_0^2 = 0$ and

$$\vartheta_1^1 \Delta S_1^1 + \vartheta_1^2 \Delta S_1^2 \geq 0, \tag{2}$$

$$P\big[\vartheta_1^1 \Delta S_1^1 + \vartheta_1^2 \Delta S_1^2 > 0\big] > 0. \tag{3}$$

Again, admissibility is automatically satisfied; hence we only have to focus on conditions (2) and (3). For the first one, we have

$$
\begin{aligned}
&\vartheta_1^1(S_1^1 - S_0^1) + \vartheta_1^2(S_1^2 - S_0^2) \geq 0 \\
\Leftrightarrow &\begin{cases} 0 \leq \vartheta_1^1(95 - 101) + \vartheta_1^2(0 - 3) \\ 0 \leq \vartheta_1^1(101 - 101) + \vartheta_1^2(0 - 3) \\ 0 \leq \vartheta_1^1(104 - 101) + \vartheta_1^2\big((104 - 101) - 3\big) \end{cases} \\
\Leftrightarrow &\begin{cases} 0 \leq -6\vartheta_1^1 - 3\vartheta_1^2 \\ 0 \leq -3\vartheta_1^2 \\ 0 \leq 3\vartheta_1^1 \end{cases} \\
\Leftrightarrow &\ \vartheta_1^2 \leq 0 \quad \text{and} \quad \vartheta_1^1 \in \Big[0, -\frac{1}{2}\vartheta_1^2\Big].
\end{aligned}
$$

Observe that choosing $\vartheta_1^2 = 0$, condition (2) is satisfied if and only if $\vartheta_1^1 = 0$ and condition (3) cannot be satisfied. As a result, an arbitrary arbitrage opportunity has to be of the form $\varphi \mathrel{\widehat{=}} (0, \vartheta^1, \vartheta^2)$ with $\vartheta^i = (\vartheta_k^i)_{k=0,1}$ for $i = 1, 2$ such that $\vartheta_0^1 = \vartheta_0^2 = 0$, $\vartheta_1^2 < 0$, and $\vartheta_1^1 \in \Big[0, -\frac{1}{2}\vartheta_1^2\Big]$.

For instance one can choose $\vartheta_1^2 = -1$ and $\vartheta_1^1 = 0$.

Exercise 3

(a) (3 points) Start by computing the density process Z of Q^* with respect to P:

$$\begin{aligned}
Z_k = E\left[\frac{\mathrm{d}Q^*}{\mathrm{d}P}\middle|\mathcal{F}_k\right] &\overset{(*)}{=} e^{T(\lambda-1)}\Big(\prod_{j=1}^{k}\lambda^{-Y_j}\Big)E\Big[\lambda^{-Y_1}\Big]^{T-k} \\
&= e^{T(\lambda-1)}\Big(\prod_{j=1}^{k}\lambda^{-Y_j}\Big)\Big(\sum_{i=0}^{\infty}\lambda^{-i}\frac{\lambda^i e^{-\lambda}}{i!}\Big)^{T-k} \\
&= e^{T(\lambda-1)}\Big(\prod_{j=1}^{k}\lambda^{-Y_j}\Big)\Big(e^{-\lambda}\sum_{i=0}^{\infty}\frac{1}{i!}\Big)^{T-k} \\
&= e^{T(\lambda-1)}\Big(\prod_{j=1}^{k}\lambda^{-Y_j}\Big)\Big(e^{-(\lambda-1)}\Big)^{T-k} = e^{k(\lambda-1)}\Big(\prod_{j=1}^{k}\lambda^{-Y_j}\Big),
\end{aligned}$$

for $k = 1,\dots,T$ and $Z_0 = e^{T(\lambda-1)}E\Big[\lambda^{-Y_1}\Big]^T = 1$. In $(*)$, we used the i.i.d. property of $(Y_j)_{j=1}^T$ and the fact that Y_j is $\mathcal{F}_j$-measurable for each $j = 1,\dots,T$.

Since $\frac{\mathrm{d}Q^*}{\mathrm{d}P} > 0$ P-a.s., we already have that $Q^* \approx P$. One thus only has to show that S^1 is a Q^*-martingale.

- Adaptedness is clear.
- For the integrability, first note that

$$\begin{aligned}
E_{Q^*}[Y_j] = E\Big[e^{j(\lambda-1)}\Big(\prod_{i=1}^{j}\lambda^{-Y_i}\Big)Y_j\Big] &= e^{j(\lambda-1)}E\Big[\lambda^{-Y_1}\Big]^{j-1}E\Big[\lambda^{-Y_1}Y_1\Big] \\
&= e^{j(\lambda-1)}\Big(e^{-(\lambda-1)}\Big)^{j-1}\Big(\sum_{i=0}^{\infty}\lambda^{-i}i\frac{\lambda^i e^{-\lambda}}{i!}\Big) \\
&= e^{(\lambda-1)}\Big(e^{-\lambda}\sum_{i=0}^{\infty}i\frac{1}{i!}\Big) = e^{(\lambda-1)}\Big(e^{-(\lambda-1)}\Big) = 1.
\end{aligned}$$

  Hence for each $k = 1,\dots,T$, we can compute

$$E_{Q^*}[|S_k^1|] = E_{Q^*}[S_k^1] = s_0^1 - k + \sum_{j=1}^{k}E_{Q^*}[Y_j] = s_0^1 < \infty.$$

- It only remains to show the Q^*-martingale property of S^1. Fix $k \in \{0,\dots,T-1\}$; then we have

$$\begin{aligned}
E_{Q^*}[S_{k+1}^1 - S_k^1|\mathcal{F}_k] = E_{Q^*}[Y_{k+1}-1|\mathcal{F}_k] &\overset{Bayes}{=} E[e^{(\lambda-1)}\lambda^{-Y_{k+1}}(Y_{k+1}-1)|\mathcal{F}_k] \\
&\overset{(*)}{=} E[e^{(\lambda-1)}\lambda^{-Y_{k+1}}(Y_{k+1}-1)] \\
&= e^{(\lambda-1)}\Big(E[\lambda^{-Y_1}Y_1] - E[\lambda^{-Y_1}]\Big) \\
&= e^{(\lambda-1)}\Big(e^{-(\lambda-1)} - e^{-(\lambda-1)}\Big) = 0,
\end{aligned}$$

  where in $(*)$ we use that Y_{k+1} is independent of $\mathcal{F}_k$ under P.

(b) (1 point) For fixed $j \in \{1, \dots, T\}$ and $n \in \mathbb{N} \cup \{0\}$, we can use that $(Y_j)_{j=1}^T$ is a collection of i.i.d., Poi(λ)-distributed random variables under P to compute

$$\begin{aligned}
Q^*[Y_j = n] &= E\left[\mathbb{1}_{\{Y_j=n\}}\left(e^{T(\lambda-1)}\prod_{i=1}^T \lambda^{-Y_i}\right)\right] \\
&= e^{T(\lambda-1)}\left(\prod_{i\neq j} E\big[\lambda^{-Y_i}\big]\right)E\big[\mathbb{1}_{\{Y_j=n\}}(\lambda^{-Y_j})\big] \\
&= e^{T(\lambda-1)}\left(e^{-(\lambda-1)}\right)^{T-1}\left(\lambda^{-n}\frac{\lambda^n e^{-\lambda}}{n!}\right) = \frac{e^{-1}}{n!} = \frac{1^n e^{-1}}{n!},
\end{aligned}$$

proving that Y_j is Poi(1)-distributed under Q^*.

(c) (2 points) First note that since $(Y_j)_{j=1}^T$ is a collection of i.i.d. Poi(1)-distributed random variables under Q^*, by the hint we also have that $\sum_{j=1}^{\ell} Y_j$ is Poi(ℓ)-distributed under Q^*, for all $\ell = 1, \dots, T$. Recall that $H = |S_T^1 - s_0^1|^2 = |S_T^1 - S_k^1 + S_k^1 - s_0^1|^2$ and hence, since S^1 is a $(Q^*, \mathbb{F})$-martingale, we can compute

$$\begin{aligned}
E_{Q^*}[H|\mathcal{F}_k] &= E_{Q^*}\left[|S_T^1 - S_k^1|^2 \Big| \mathcal{F}_k\right] + |S_k^1 - s_0^1|^2 \\
&= E_{Q^*}\left[\left(\sum_{j=k+1}^T Y_j - (T-k)\right)^2\right] + |S_k^1 - s_0^1|^2 \\
&= \mathrm{Var}_{Q^*}\left[\sum_{j=k+1}^T Y_j\right] + |S_k^1 - s_0^1|^2 = (T-k) + |S_k^1 - s_0^1|^2.
\end{aligned}$$

As a result, the price process V^{H,Q^*} of H with respect to Q^* is given by

$$V_k^{H,Q^*} = |S_k^1 - s_0^1|^2 + (T-k)$$

for all $k = 0, \dots, T$.

Since Q^* is an equivalent martingale measure for (S^0, S^1, S^2), by the fundamental theorem of asset pricing, we can conclude that the proposed enlargement of the market is free of arbitrage.

(d) (2 points) Since $S_T^1 \geq s_0^1 - T \geq 1$, we have that $H^P = (S_T^1)^2 - 1$ P-a.s. Moreover,

$$(S_T^1)^2 - 1 = (S_T^1 - s_0^1)^2 + 2S_T^1 s_0^1 - (s_0^1)^2 - 1 = S_T^2 + 2S_T^1 s_0^1 - (s_0^1)^2 - 1.$$

Hence choosing $V_0 := -\big((s_0^1)^2 + 1\big) + 2(s_0^1)^2 + S_0^2$, $\vartheta_0^1 := \vartheta_0^2 := 0$, $\vartheta_k^1 := 2s_0^1$, and $\vartheta_k^2 := 1$ for each $k = 1, \dots, T$, we obtain that the self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta^1, \vartheta^2)$ replicates H^P. Indeed, for each $k = 1, \dots, T$, we can compute

$$\begin{aligned}
V_k(\varphi) &= V_0 + \sum_{j=1}^k \vartheta_j^1 \Delta S_j^1 + \sum_{j=1}^k \vartheta_j^2 \Delta S_j^2 \\
&= \Big(-\big((s_0^1)^2 + 1\big) + 2(s_0^1)^2 + S_0^2\Big) + 2s_0^1(S_k^1 - S_0^1) + (S_k^2 - S_0^2) \\
&= -\big((s_0^1)^2 + 1\big) + 2s_0^1 S_k^1 + S_k^2,
\end{aligned}$$

proving that φ is admissible (since $S_k^1, S_k^2 \geq 0$ for each $k = 1, \dots, T$) and $V_T(\varphi) = H^P$.

Exercise 4

(a) (2 points)Fix $t \geq s \geq 0$. Using that $E[W_s] = E[W_t] = 0$, the fact that the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and has mean zero (since W is a Brownian motion with respect to $\mathbb{F}$), and the fact that $E[W_s^2] = s$, we obtain

$$\operatorname{Cov}(W_s, W_t) = E[W_s W_t] = E[W_s(W_t - W_s) + W_s^2] = E[E[W_t - W_s \,|\, \mathcal{F}_s] W_s] + s = s.$$

If $s \geq t \geq 0$, then $\operatorname{Cov}(W_s, W_t) = t$ by symmetry. In summary, $\operatorname{Cov}(W_s, W_t) = s \wedge t$.

(b) (3 points) Note that $X_t = f(t, W_t)$ for the smooth function $f(t,x) = (1+t)\exp(x)$, $t \geq 0$, $x \in \mathbb{R}$. Hence, by Itô's formula,

$$\begin{aligned}
\mathrm{d}X_t &= \exp(W_t)\,\mathrm{d}t + (1+t)\exp(W_t)\,\mathrm{d}W_t + \frac{1}{2}(1+t)\exp(W_t)\,\mathrm{d}\langle W\rangle_t \\
&= X_t \frac{1}{1+t}\,\mathrm{d}t + X_t\,\mathrm{d}W_t + \frac{1}{2}X_t\,\mathrm{d}t \\
&= X_t\left(\left(\frac{1}{1+t} + \frac{1}{2}\right)\mathrm{d}t + \mathrm{d}W_t\right).
\end{aligned}$$

Define the process $L = (L_t)_{t\in[0,1]}$ by

$$L_t = -\int_0^t \left(\frac{1}{1+u} + \frac{1}{2}\right)\mathrm{d}W_u.$$

By the hint, the stochastic exponential $Z := \mathcal{E}(L)$ is a (true) P-martingale. Thus, we can define $Q \approx P$ on $\mathcal{F}_1$ by setting $\frac{\mathrm{d}Q}{\mathrm{d}P} = Z_1$. Then by Girsanov's theorem,

$$\widetilde{W}_t := W_t - \langle L, W\rangle_t = W_t + \int_0^t \left(\frac{1}{1+u} + \frac{1}{2}\right)\mathrm{d}u, \quad t \in [0,1],$$

defines a Q-Brownian motion $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,1]}$. Therefore,

$$\mathrm{d}X_t = X_t\,\mathrm{d}\widetilde{W}_t$$

and since $X_0 = 1$, we obtain $X = \mathcal{E}(\widetilde{W})$. We conclude that X is a Q-martingale.

(c) (3 points) Define the martingale $X = (X_t)_{t\in[0,T]}$ by $X_t = E[\exp(W_T) \,|\, \mathcal{F}_t]$. For each $t \in [0,T]$, using that $W_T - W_t$ is independent of $\mathcal{F}_t$ and $\mathcal{N}(0, T-t)$-distributed (since W is a Brownian motion with respect to P and $\mathcal{F}$) and that W_t is $\mathcal{F}_t$-measurable, we obtain

$$X_t = E[\exp(W_T - W_t)\exp(W_t) \,|\, \mathcal{F}_t] = \exp(W_t)\exp\left(\frac{1}{2}(T-t)\right) = f(t, W_t),$$

where $f(t,x) = \exp\left(x + \frac{1}{2}(T-t)\right)$, $t \geq 0$, $x \in \mathbb{R}$.

Applying Itô's formula, we find that for all $t \in [0,T]$

$$X_t = f(t, X_t) = X_0 + \int_0^t \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s \quad P\text{-a.s.};$$

note that the finite variation terms must vanish since X and W are continuous P-martingales by construction. In particular, the stochastic integral process $\int_0^\cdot \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s$ is a $(P, \mathbb{F})$-martingale and $\exp(W_T) = X_T = X_0 + \int_0^T \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s$ P-a.s. Hence, we can set

$$\begin{aligned}
c &:= X_0 = \exp\left(\frac{1}{2}T\right), \\
H_t &:= \frac{\partial f}{\partial x}(t, W_t) = \exp\left(W_t + \frac{1}{2}(T-t)\right).
\end{aligned}$$

As H is continuous and adapted, it is predictable and locally bounded and thus belongs to $L^2_{\mathrm{loc}}(W)$.

Exercise 5

(a) (2 points) It is known from the lecture notes that

$$W_t^* := W_t + \frac{\mu - r}{\sigma} t, \quad t \in [0, T],$$

defines a Q^*-Brownian motion W^* and that S^1 satisfies the SDE

$$\mathrm{d}S_t^1 = S_t^1 \sigma \,\mathrm{d}W_t^*.$$

By the product rule and the fact that $\widetilde{S}^0$ is continuous and of finite variation,

$$\mathrm{d}\widetilde{S}_t^1 = \mathrm{d}\left(S^1 \widetilde{S}^0\right)_t = S_t^1 \,\mathrm{d}\widetilde{S}_t^0 + \widetilde{S}_t^0 \,\mathrm{d}S_t^1 = \widetilde{S}_t^1 \left(r \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^*\right).$$

(b) (1 point) Itô's formula and the given dynamics of $\widetilde{S}^1$ under P yield

$$\mathrm{d}\left(\frac{1}{\widetilde{S}^1}\right)_t = -\frac{1}{(\widetilde{S}_t^1)^2} \,\mathrm{d}\widetilde{S}_t^1 + \frac{1}{(\widetilde{S}_t^1)^3} \,\mathrm{d}\langle \widetilde{S}^1 \rangle_t = \frac{1}{\widetilde{S}_t^1} \left(-\mu \,\mathrm{d}t - \sigma \,\mathrm{d}W_t + \sigma^2 \,\mathrm{d}t\right).$$

Using the product rule, the given dynamics of $\widetilde{S}^0$, and the fact that $\widetilde{S}^0$ is continuous and of finite variation, we then obtain

$$\mathrm{d}\widehat{S}_t^0 = \mathrm{d}\left(\frac{1}{\widetilde{S}^1} \widetilde{S}^0\right)_t = \widetilde{S}_t^0 \,\mathrm{d}\left(\frac{1}{\widetilde{S}^1}\right)_t + \frac{1}{\widetilde{S}_t^1} \,\mathrm{d}\widetilde{S}_t^0 = \widehat{S}_t^0 \left((\sigma^2 + r - \mu) \,\mathrm{d}t - \sigma \,\mathrm{d}W_t\right).$$

(c) (2.5 points) We first note that by part (a),

$$\widetilde{S}_t^1 = \widetilde{S}_0^1 \mathcal{E}\left(\int_0^\cdot r \,\mathrm{d}u + \sigma W^*\right)_t = S_0^1 \exp\left(\sigma W_t^* + \left(r - \frac{1}{2}\sigma^2\right)t\right), \quad t \in [0, T],$$

so that for $0 \le t \le u \le T$,

$$\frac{\widetilde{S}_u^1}{\widetilde{S}_t^1} = \exp\left(\sigma(W_u^* - W_t^*) + \left(r - \frac{1}{2}\sigma^2\right)(u - t)\right). \tag{4}$$

Using (4) for $u = T$ and $t = T_0$ gives

$$\widetilde{H} = \log \frac{\widetilde{S}_T^1}{\widetilde{S}_{T_0}^1} = \sigma(W_T^* - W_{T_0}^*) + \left(r - \frac{1}{2}\sigma^2\right)(T - T_0).$$

Suppose first that $t \in [0, T_0]$. As $W_T^* - W_{T_0}^*$ is independent of $\mathcal{F}_t$ (since W^* is a Q^*-Brownian motion and $t \le T_0$) and has expectation 0 under Q^*,

$$\begin{aligned} V_t = E_{Q^*}\left[\widetilde{H}/\widetilde{S}_T^0 \,\middle|\, \mathcal{F}_t\right] &= e^{-rT} E_{Q^*}\left[\sigma(W_T^* - W_{T_0}^*) + \left(r - \frac{1}{2}\sigma^2\right)(T - T_0) \,\middle|\, \mathcal{F}_t\right] \\ &= e^{-rT}\left(r - \frac{1}{2}\sigma^2\right)(T - T_0), \quad t \in [0, T_0]. \end{aligned} \tag{5}$$

Now, suppose that $t \in (T_0, T]$. Using (4), we obtain

$$\widetilde{H} = \log \frac{\widetilde{S}_T^1}{\widetilde{S}_{T_0}^1} = \log \frac{\widetilde{S}_T^1}{\widetilde{S}_t^1} + \log \frac{\widetilde{S}_t^1}{\widetilde{S}_{T_0}^1} = \sigma(W_T^* - W_t^*) + \left(r - \frac{1}{2}\sigma^2\right)(T - t) + \log \frac{\widetilde{S}_t^1}{\widetilde{S}_{t \wedge T_0}^1}.$$

With the same arguments as above,

$$\begin{aligned}
V_t &= E_{Q^*}\left[\widetilde{H}/\widetilde{S}^0_T \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-rT} E_{Q^*}\left[\sigma(W^*_T - W^*_t) + \big(r - \frac{1}{2}\sigma^2\big)(T-t) + \log\frac{\widetilde{S}^1_t}{\widetilde{S}^1_{t\wedge T_0}} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-rT}\left(\big(r - \frac{1}{2}\sigma^2\big)(T-t) + \log\frac{\widetilde{S}^1_t}{\widetilde{S}^1_{t\wedge T_0}}\right), \qquad t \in (T_0, T]. \qquad (6)
\end{aligned}$$

In view of (5) and (6), we find that $\widetilde{V}_t = V_t\widetilde{S}^0_t = V_t e^{rt} = \tilde{v}(t, \widetilde{S}^1_t, \widetilde{S}^1_{t\wedge T_0})$ where

$$\tilde{v}(t,x,y) := e^{-r(T-t)}\left(\big(r - \frac{1}{2}\sigma^2\big)(T - \max(t, T_0)) + \log\frac{x}{y}\right), \qquad t \in [0,T], x, y \in (0,\infty).$$

(d) (2.5 points) From the lecture notes we know that $S^1_t = S^1_0 \exp\left(\sigma W^*_t - \frac{1}{2}\sigma^2 t\right)$ and $\widetilde{S}^0_t = \exp(rt)$ for $t \in [0,T]$. As a result,

$$\widetilde{H}^0 = \log\frac{\widetilde{S}^1_T}{\widetilde{S}^1_0} = \log\frac{S^1_T\widetilde{S}^0_T}{S^1_0} = \sigma W^*_T + \left(r - \frac{1}{2}\sigma^2\right)T,$$

and hence

$$\begin{aligned}
V_t &= E_{Q^*}\left[\widetilde{H}^0/\widetilde{S}^0_T \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-rT} E_{Q^*}\left[\sigma W^*_T + \left(r - \frac{1}{2}\sigma^2\right)T \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-rT}\left(\sigma W^*_t + \left(r - \frac{1}{2}\sigma^2\right)T\right) \\
&= e^{-rT}\left(\log\frac{S^1_t}{S^1_0} + rT - \frac{1}{2}\sigma^2(T-t)\right), \qquad t \in [0,T].
\end{aligned}$$

We can thus write $V_t = v(t, S^1_t)$ where $v(t,x) = e^{-rT}\left(\log\frac{x}{S^1_0} + rT - \frac{1}{2}\sigma^2(T-t)\right)$, for $t \in [0,T]$ and $x > 0$.
By definition of V and Itô's formula, $H^0 = V_T = v(T, S^1_T)$ and for all $t \in [0,T]$

$$V_t = v(t, S^1_t) = v(0, S^1_0) + \int_0^t \frac{\partial v}{\partial x}(u, S^1_u)\, \mathrm{d}S^1_u \quad P\text{-a.s.}; \qquad (7)$$

note that the finite variation terms must vanish since V and S^1 are continuous $(Q^*, \mathbb{F})$-martingales by construction. In particular, the stochastic integral in (7) is a $(Q^*, \mathbb{F})$-martingale. We can thus set

$$\begin{aligned}
V_0 &:= v(0, S^1_0) = e^{-rT}\left(r - \frac{1}{2}\sigma^2\right)T, \\
\vartheta_t &:= \frac{\partial v}{\partial x}(t, S^1_t) = e^{-rT}\frac{1}{S^1_t}.
\end{aligned}$$

As ϑ is continuous and adapted, it is predictable and locally bounded.