ETH Zürich HS 2015
D-MATH
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August 2016

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as far as possible.
- ◇ Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answers Sheet for Question 1

Please use this sheet to solve Question 1. Indicate the correct answer by a **cross ✗**. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following 8 subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get $-1/2$ points, and for no answer you get 0 points. You get at least zero points for the whole question 1. **Please use the printed form for your answers.** It is enough to put a cross; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with time horizon $T \in \mathbb{N}$, where $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, \ldots, T$ and some $r > -1$. Assume that $\widetilde{S}^1_0 := s^1_0 > 0$. Finally, the discounted market is denoted by (S^0, S^1) and the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0}^T$ is always generated by $\widetilde{S}^1$.

(a) Let σ, τ be two stopping times with respect to $\mathbb{F}$. Then it is always true that

- (1) $\tau \vee \sigma$ is not a stopping time.
- (2) $\sigma + \tau$ is a stopping time.
- (3) $(\tau - 1)^+$ is a stopping time.

(b) Let Q^* be an equivalent martingale measure for S^1. Then

- (1) the unique price process of an attainable payoff is a Q^*-martingale.
- (2) there is an admissible trading strategy whose gains process at time T is strictly positive with probability 1.
- (3) the market is complete.

(c) Let Q^* be the unique equivalent martingale measure for S^1. Then it is always true that

- (1) $Q^* \neq P$.
- (2) $\lambda Q^* + (1-\lambda)P \approx P$ for every $\lambda \in [0,1]$.
- (3) $\left(E\left[\frac{\mathrm{d}P}{\mathrm{d}Q^*}\middle|\mathcal{F}_k\right] S^1_k\right)_{k=0}^T$ is a $(P, \mathbb{F})$-martingale.

(d) Let $T = 5$ and suppose that $r = 1/2$, $\widetilde{S}^1_0 = 1$, and $\widetilde{S}^1_k = Y^k$, where $P[Y = 2] = 1/3$ and $P[Y = 1/2] = 2/3$. Consider the strategy $\varphi := (\varphi^0, \vartheta)$, where $\varphi^0 = (\varphi^0_k)_{k=0}^T$, $\vartheta = (\vartheta_k)_{k=0}^T$ with $\varphi^0_0 = \vartheta_0 = 0$ and

$$\varphi^0_k := -\mathbb{1}_{\{k \leq 3\}} \qquad \text{and} \qquad \vartheta_k := \mathbb{1}_{\{k \leq 3\}},$$

for all $k = 1, \ldots, T$. Which of the following assertions is true?

- (1) The market is arbitrage free and the strategy is not self-financing.
- (2) The market is not arbitrage free and the strategy is self-financing.
- (3) None of the previous answers is correct.

Throughout subquestions (e) to (h), W denotes a Brownian motion relative to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$.

(e) Which of the following assertions is true?

(1) the process $\left(W_t^{2016} + 2016t\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale.

(2) the process $\int \left(W_s^{2016} + 2016s\right) \mathrm{d}W_s$ is a $(P, \mathbb{F})$-martingale.

(3) the process $\left[\int \left(W_s^{2016} + 2016s\right) \mathrm{d}W_s\right]$ is a $(P, \mathbb{F})$-martingale.

(f) Let B be another $\mathbb{F}$-Brownian motion on $(\Omega, \mathcal{F}, P)$ such that W and B are independent. Then

(1) $\mathcal{E}(W + B) = \left(e^{B_t + W_t - t/2}\right)_{t\geq 0}$.

(2) $\mathcal{E}(W + B) = \mathcal{E}(W)\mathcal{E}(B)$.

(3) $\mathcal{E}(W + B)$ is a Brownian motion.

(g) Consider a square-integrable, adapted process X on $(\Omega, \mathcal{F}, P)$ with stationary independent increments and null at 0. Which of the following assertions is true?

(1) If $E[X_t] = 1$ for all $t \geq 0$, then X is a $(P, \mathbb{F})$-martingale.

(2) If X is a $(P, \mathbb{F})$-martingale, then $\left(X_t^2 - tE[X_1^2]\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-local martingale.

(3) If $E[X_t] \geq 0$ for all $t \geq 0$, then X is a $(P, \mathbb{F})$-supermartingale.

(h) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Which of the following conditions always implies that $\left(f(W_t)\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale?

(1) $f \equiv C$, for some $C \in \mathbb{R}$.

(2) $\int f'(W_s) \,\mathrm{d}W_s$ is a $(P, \mathbb{F})$-martingale.

(3) $E\left[\int_0^T \left(f''(W_s)\right)^2 \,\mathrm{d}s\right] < \infty$ for all $T \geq 0$.

Question 2 (8 **Points**)

Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^\Omega$, and P is the probability measure given by

$$p_1 := P[\{\omega_1\}] = \frac{1}{4}, \qquad p_2 := P[\{\omega_2\}] = \frac{1}{2}, \qquad p_3 := P[\{\omega_3\}] = \frac{1}{4}.$$

Consider the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$. Let $\big(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2\big)$ be a market on this space consisting of a bank account with price process $\widetilde{S}^0 := \big(\widetilde{S}^0_k\big)_{k=0,1}$ and two risky assets whose price processes are denoted by $\widetilde{S}^1 := \big(\widetilde{S}^1_k\big)_{k=0,1}$ and $\widetilde{S}^2 := \big(\widetilde{S}^2_k\big)_{k=0,1}$, where

$$\begin{aligned} \widetilde{S}^0_1 &= (1+r)\widetilde{S}^0_0 \qquad &&\text{with} \qquad \widetilde{S}^0_0 = 1, \\ \widetilde{S}^1_1 &= Y\widetilde{S}^1_0 \qquad &&\text{with} \qquad \widetilde{S}^1_0 = 101, \\ \widetilde{S}^2_1 &= \big(\widetilde{S}^1_1 - \widetilde{K}\big)^+ \qquad &&\text{with} \qquad \widetilde{S}^2_0 = s^2_0 > 0, \end{aligned}$$

for some $r > 0$, $\widetilde{K} \in \mathbb{R}$, and a random variable Y which takes positive values $1+d$,$1+m$ and $1+u$ on ω_1, ω_2 and ω_3, respectively. Finally denote by $S^i := \widetilde{S}^i/\widetilde{S}^0$, $i = 0, 1, 2$, the discounted price processes.

Suppose that $u = 0.04$, $m = r = 0.01$, and $d = -0.05$.

(a) Fix $\widetilde{K} = 102.01$ and $s^2_0 = 1$. Represent the discounted market as a tree diagram. Do not forget to indicate the transition probabilities as well as the values of the discounted price processes.

Hint: Note that $102.01 = 101 \cdot 1.01$.

(b) For each of the following two cases, determine for which values of s^2_0 the market is free of arbitrage, and for which values of s^2_0 it is complete.

(i) $\widetilde{K} = 98.98$.

(ii) $\widetilde{K} = 92.92$.

Hint: Note that $98.98 = 98 \cdot 1.01$ and $92.92 = 92 \cdot 1.01$.

(c) Fix $\widetilde{K} = 102.01$ and $s^2_0 = 1$. Then the probability measure Q^* given by

$$q^*_1 := Q^*[\{\omega_1\}] = \frac{1}{6}, \qquad q^*_2 := Q^*[\{\omega_2\}] = \frac{1}{2}, \qquad q^*_3 := Q^*[\{\omega_3\}] = \frac{1}{3}$$

is the unique equivalent martingale measure (EMM) for $\big(S^0, S^1, S^2\big)$. Recall that a put option on asset 1 with strike 99 is a contingent claim whose discounted payoff at time 1 is given by $H := \big(99 - S^1_1\big)^+$. Compute its unique (discounted) price process which admits no arbitrage and its replication strategy.

(d) Fix $\widetilde{K} = 102.01$ and $s^2_0 = 3$. Prove that the market is not free of arbitrage by explicitly constructing an arbitrage opportunity.

Question 3 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in \mathbb{N}$ the time horizon, and $\lambda > 0$. Consider a collection $(Y_j)_{j=1}^T$ of i.i.d., Poi(λ)-distributed random variables.
Define the discounted financial market (S^0, S^1) by

$$S^0 \equiv 1, \qquad S_k^1 := s_0^1 + \sum_{j=1}^k Y_j - k \quad \text{for } k = 1, \dots, T, \qquad \text{and} \qquad S_0^1 := s_0^1 \geq T + 1.$$

Finally, consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0}^T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for each $k = 1, \dots, T$.
Hint: $X \sim$ Poi(λ) means that X has a Poisson distribution with parameter λ and hence $P[X = n] = \frac{\lambda^n e^{-\lambda}}{n!}$ for all $n \in \mathbb{N} \cup \{0\}$. For this distribution, $E[X] = \mathrm{Var}[X] = \lambda$.

(a) Consider the probability measure Q^* given by $\frac{\mathrm{d}Q^*}{\mathrm{d}P} := e^{T(\lambda-1)} \prod_{j=1}^T \lambda^{-Y_j}$. Compute its density process with respect to P and prove that it is an equivalent martingale measure (EMM) for S^1.

Hint: You can use that $\sum_{n=0}^\infty \frac{1}{n!} = e$ and $\sum_{n=0}^\infty \frac{n}{n!} = e$.

(b) Prove that $(Y_j)_{j=1}^T$ is a collection of Poi(1)-distributed random variables under Q^*.

(c) Consider the (discounted) contingent claim $H := |S_T^1 - s_0^1|^2$ and compute its price process $V^{H,Q^*} := \big(E_{Q^*}[H|\mathcal{F}_k]\big)_{k=0}^T$ with respect to Q^*. Consider then the enlarged market (S^0, S^1, S^2), where $S_k^2 := V_k^{H,Q^*}$ for all $k = 0, \dots, T$. Is this market free of arbitrage?

Hint: One can show (and you can use) that $Y_1, \dots, Y_T$ are independent also under Q^*. Recall also that if $(X_n)_{n=1}^N$ is a collection of i.i.d. Poi(λ)-distributed random variables, then $(\sum_{n=1}^N X_n) \sim$Poi$(n\lambda)$. Finally, if you cannot prove point (b), you can assume it for this point.

(d) A power option on S^1 with power 2 and strike 1 is a contingent claim whose discounted payoff at time T is given by $H^P := \big(|S_T^1|^2 - 1\big)^+$. Prove that H^P is attainable in the enlargement of the market proposed in point (c), by explicitly constructing a replicating strategy.

Hint: Try to write H^P as a linear combination of S_T^1, S_T^2 and a constant term. If you did not find S_0^2 in the previous point, you can set it equal to s_0^2.

Question 4 (8 Points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to P and $\mathbb{F}$.

(a) Compute the covariance $\mathrm{Cov}(W_s, W_t)$ for $s, t \geq 0$.

(b) Define the process $X = (X_t)_{t\in[0,1]}$ by $X_t = (1+t)\exp(W_t)$. Use Itô's formula and Girsanov's theorem to construct a measure $Q \approx P$ on $\mathcal{F}_1$ such that X is a Q-martingale (on the time interval $[0, 1]$).
Hint: If $f : [0, 1] \to \mathbb{R}$ is a continuous function and the process $L = (L_t)_{t\in[0,1]}$ is given by $L_t = \int_0^t f(u)\,\mathrm{d}W_u$, then the stochastic exponential $\mathcal{E}(L)$ is a P-martingale.

(c) Fix $T \in (0, \infty)$. Find explicitly a number $c \in \mathbb{R}$ and a process $H \in L^2_{\mathrm{loc}}(W)$ such that

$$\exp(W_T) = c + \int_0^T H_t\,\mathrm{d}W_t \quad P\text{-a.s.}$$

and the stochastic integral process $\int H\,\mathrm{d}W$ is a $(P, \mathbb{F})$-martingale.

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy the SDEs

$$\begin{aligned} \mathrm{d}\widetilde{S}^0_t &= \widetilde{S}^0_t r \,\mathrm{d}t, & \widetilde{S}^0_0 &= 1, \\ \mathrm{d}\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\right), & \widetilde{S}^1_0 &= S^1_0, \end{aligned}$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $S^1_0 > 0$. Let Q^* denote the unique equivalent martingale measure for the discounted stock price process $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$.

(a) Show that the undiscounted stock price process satisfies the SDE

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t(r \,\mathrm{d}t + \sigma \,\mathrm{d}W^*_t)$$

for a Q^*-Brownian motion W^*.

(b) Derive the SDE satisfied by $\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1}$ under P (i.e., with a P-Brownian motion showing up in the SDE).

(c) Fix $T_0 \in (0, T)$ and consider the *forward-start log contract* with undiscounted payoff given by $\widetilde{H} = \log \frac{\widetilde{S}^1_T}{\widetilde{S}^1_{T_0}}$. Find a function $\tilde{v} : [0, T] \times (0, \infty) \times (0, \infty) \to \mathbb{R}$ such that the undiscounted value process $\widetilde{V}_t := \widetilde{S}^0_t E_{Q^*}\left[\widetilde{H}/\widetilde{S}^0_T \,\middle|\, \mathcal{F}_t\right]$, $t \in [0, T]$, is given by $\widetilde{V}_t = \tilde{v}(t, \widetilde{S}^1_t, \widetilde{S}^1_{t\wedge T_0})$ P-a.s.

Hint: Distinguish the two cases $t \leq T_0$ and $t > T_0$ for the computation of the conditional expectation.

(d) Let $\widetilde{H}^0 = \log \frac{\widetilde{S}^1_T}{\widetilde{S}^1_0}$. Construct a replicating strategy for H^0, i.e., find explicitly a number $V_0 \in \mathbb{R}$ and a predictable, locally bounded process ϑ such that

$$V_0 + \int_0^T \vartheta_t \,\mathrm{d}S^1_t = H^0 \quad P\text{-a.s.}$$

and the stochastic integral process $\int \vartheta \,\mathrm{d}S^1$ is a $(Q^*, \mathbb{F})$-martingale.

Hint: This part can be solved independently of part (c).