ETH Zürich HS 2016
D-MATH
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August 2017

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as far as possible.
- ◇ Most of the subquestions can be solved independently of each other.

$\star\star\star$ Good luck! $\star\star\star$

Answers Sheet for Question 1

Please use this sheet to solve Question 1. Indicate the correct answer by a **cross** ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (**8 points**)

For each of the following 8 subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get $-1/2$ points, and for no answer you get 0 points. You get at least zero points for the whole question 1. **Please use the printed form for your answers.** It is enough to put a cross; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with time horizon $T \in \mathbb{N}$, where $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, \dots, T$ and some $r > -1$. Assume that $\widetilde{S}^1_0 := s^1_0 > 0$. Finally, the discounted market is denoted by (S^0, S^1) and the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0}^T$ is always generated by $\widetilde{S}^1$.

(a) Let σ, τ be two stopping times with respect to $\mathbb{F}$. Then it is always true that

(1) the product $\sigma\tau$ is a stopping time.

(2) the sum $\sigma + \tau$ is a stopping time.

(3) the difference $\sigma - \tau$ is a stopping time.

(b) Assume the market is arbitrage-free and complete. Then

(1) for every contingent claim H, the cost of replicating H is unique.

(2) every self-financing portfolio is admissible.

(3) every admissible portfolio is self-financing.

(c) Suppose that P is a probability measure and S^1 is a $(P, \mathbb{F})$-submartingale. Then S^1 is a $(P, \mathbb{F})$-martingale **if**

(1) $S^1_0 \leq E_P[S^1_T]$.

(2) $S^1_\sigma \geq E_P[S^1_\tau \mid \mathcal{F}_\sigma]$ for every pair of stopping times σ, τ satisfying $\sigma \leq \tau$.

(3) the stochastic integral process $\varphi \cdot S^1$ is a $(P, \mathbb{F})$-supermartingale for some predictable process φ.

(d) Assume that the market is free of arbitrage. Then it is always true that

(1) the set of equivalent martingale measures contains at most one element.

(2) the set of equivalent martingale measures contains exactly one element.

(3) the set of equivalent martingale measures contains at least one element.

Throughout subquestions (e) to (h), W denotes a Brownian motion relative to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$.

(e) Which of the following is *not* a Brownian motion under P?

(1) $(W_{t+c} - W_c)_{t\geq 0}$ for a fixed $c > 0$.
(2) $(t^2 W_{1/t^2})_{t\geq 0}$ with the convention $0W_{1/0} = 0$.
(3) $(tW_{1/t})_{t\geq 0}$ with the convention $0W_{1/0} = 0$.

(f) Let $H \in L^2_{loc}(W)$. Which of the following is *not* sufficient to guarantee that the stochastic integral process $H \cdot W$ is a martingale?

(1) $(H \cdot W)_T \geq 0$ for each $T < \infty$.
(2) $E\left[\int_0^T H_s^2 ds\right] < \infty$ for each $T < \infty$.
(3) $(H \cdot W)_T$ is dominated in absolute value by an integrable random variable for each $T < \infty$.

(g) Which of the following is *not* true for the process $Y = (Y_t)_{t\geq 0}$ defined by $Y_t := \exp(W_t - \frac{1}{2}t)$?

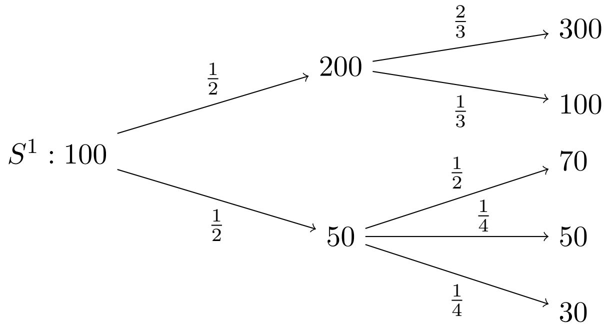
(1) $\lim_{t\to\infty} Y_t = 0$ P-a.s.
(2) Y has the Markov property, but not the strong Markov property.
(3) $Y = \mathcal{E}(W)$.

(h) Which of the following is *not* true for the process $Z = (Z_t)_{t\geq 0}$ defined by $Z_t := \int_0^t W_s dW_s$?

(1) Z is a square-integrable martingale in $\mathcal{M}_0^2$.
(2) Z is a semimartingale.
(3) $Z_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$ P-a.s. for all $t \geq 0$.

Question 2 (8 **points)**

Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the *discounted* stock price S^1 are described by the following tree, where the numbers beside the branches denote transition probabilities.



More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with $\Omega := \{(1,1), (1,2), (1,3), (2,1), (2,2)\}$, $\mathcal{F} = 2^\Omega$ and the probability measure $\mathbb{P}$ defined by $\mathbb{P}[\{(x_1, x_2)\}] := p_{x_1} p_{x_1,x_2}$, where

$$p_1 = p_2 = \frac{1}{2}, \quad p_{1,1} = p_{1,2} = \frac{1}{4}, \quad p_{1,3} = \frac{1}{2} \quad \text{and} \quad p_{2,1} = \frac{1}{3}, \quad p_{2,2} = \frac{2}{3}.$$

The discounted bank account process $(S_k^0)_{k=0,1,2}$ is given by $S_k^0 = 1$, $k = 0,1,2$, and the discounted stock price process $(S_k^1)_{k=0,1,2}$ by

$$\begin{array}{lllll} S_0^1 = 100, & S_1^1((1,j)) = 50, & S_1^1((2,\ell)) = 200, & j = 1,2,3, \ \ell = 1,2. & \\ S_2^1((1,1)) = 30, & S_2^1((1,2)) = 50, & S_2^1((1,3)) = 70, & S_2^1((2,1)) = 100, & S_2^1((2,2)) = 300. \end{array}$$

The filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ is given by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(S_1^1)$ and $\mathcal{F}_2 := \sigma(S_1^1, S_2^1) = \mathcal{F}$.

(a) Show that the market is free of arbitrage and explicitly describe the set $\mathbb{P}_e(S^1)$ of all equivalent martingale measures for S^1.

(b) Show that the discounted payoff of the European call option $C^K = (S_2^1 - K)^+$ is attainable for the strike price $70 \leq K < 300$ by explicitly constructing a hedging portfolio for C^K.

(c) Show that the discounted payoff of the European call option $C^K = (S_2^1 - K)^+$ is **not** attainable for the strike price $50 \leq K < 70$.

(d) For which values of the strike price $50 \leq K < 300$ is the payoff of the European **put** option $P^K = (K - S_2^1)^+$ attainable? Justify your answer.

Question 3 (8 points)

Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a financial market in discrete time with time horizon T and assume that the bank account satisfies $\widetilde{S}^0_k = (1+r)^k$ for $k = 0, \ldots, T$ with $r \geq 0$. Let $\mathbb{P}_e(S^1)$ denote the set of all equivalent martingale measures for the discounted stock price process S^1 on a filtered measurable space $\big(\Omega, (\mathcal{F}_k)_{k=0}^T\big)$ with $\mathcal{F}_0$ trivial and $\mathcal{F}_k = \sigma(S^1_j : j \leq k)$ for $k \geq 1$. We assume $\mathbb{P}_e(S^1) \neq \emptyset$ and take $Q \in \mathbb{P}_e(S^1)$ fixed. Fix $\widetilde{K} > 0$ and $k \in \{1, \ldots, T\}$. The undiscounted payoff of a *European call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity k is denoted by $\widetilde{C}^{Eu}_k$ and given by

$$\widetilde{C}^{Eu}_k := \big(\widetilde{S}^1_k - \widetilde{K}\big)^+ ,$$

whereas the undiscounted payoff of an *Asian call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity k is denoted by $\widetilde{C}^{As}_k$ and given by

$$\widetilde{C}^{As}_k := \left(\frac{1}{k}\sum_{j=1}^{k} \widetilde{S}^1_j - \widetilde{K}\right)^+ ,$$

and the undiscounted payoff of a *lookback call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity k is denoted by $\widetilde{C}^{lb}_k$ and given by

$$\widetilde{C}^{lb}_k := \left(\max_{1\leq j\leq k} \widetilde{S}^1_j - \widetilde{K}\right)^+ .$$

(a) Show that the function

$$\{1, \ldots, T\} \ni k \mapsto E_Q\left[\frac{\widetilde{C}^{Eu}_k}{\widetilde{S}^0_k}\right] \in \mathbb{R}_+$$

is increasing.

(b) Show that for all $k = 1, \ldots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^{As}_k}{\widetilde{S}^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{\widetilde{C}^{Eu}_j}{\widetilde{S}^0_j}\right].$$

(c) Deduce that for all $k = 1, \ldots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^{As}_k}{\widetilde{S}^0_k}\right] \leq E_Q\left[\frac{\widetilde{C}^{Eu}_k}{\widetilde{S}^0_k}\right].$$

(d) Show that $\left(\widetilde{C}^{lb}_k / \widetilde{S}^0_k\right)_{k=1}^T$ is a Q-submartingale on $\big(\Omega, (\mathcal{F}_k)_{k=1}^T\big)$.

(e) Fix $M > 0$ and by considering the stopping time

$$\tau := \inf\left\{k \in \{1, \ldots, T\} : \left(\widetilde{S}^1_k - \widetilde{K}\right)^+ \geq M\right\} \wedge T,$$

show that

$$Q\left[\widetilde{C}^{lb}_T \geq M\right] \leq \frac{1}{M} E_Q\left[\widetilde{C}^{Eu}_T\right].$$

(f) Assume that the underlying space Ω is finite, i.e., $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ for some $N \in \mathbb{N}$. Determine first the family of all non-negative contingent claims $H \in L^0_+(\Omega, \mathcal{F}_T; \mathbb{R}_+)$, where $\mathcal{F}_T = \sigma(S^1_j : j \leq T)$, satisfying

$$H^2 = H$$

and then their prices under the measure Q in the case $r = 0$.

Question 4 (8 **points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $W = (W_t)_{t\geq 0}$ a Brownian motion with respect to P and $\mathbb{F}$.

(a) Express the value of the stochastic integral $\int_0^t W_s e^{W_s} dW_s$ in the form $f(W_t) + \int_0^t g(W_s)ds$, where $f, g : \mathbb{R} \to \mathbb{R}$.

(b) Assume that S satisfies the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \;\; \mu, \sigma \in \mathbb{R}.$$

Compute the dynamics of the process $Y = |S|^3$.

(c) Find a measure Q (by giving its Radon–Nikodým derivative) such that the process

$$X_t := W_t + t - t^3, \;\; t \in [0, T],$$

is a Q-Brownian motion on $[0, T]$, for a fixed $T > 0$.

Question 5 (8 points)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t \in [0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{B} = (\widetilde{B}_t)_{t \in [0,T]}$ and the undiscounted stock price process $\widetilde{S} = (\widetilde{S}_t)_{t \in [0,T]}$ satisfy the SDEs

$$
\begin{aligned}
\mathrm{d}\widetilde{B}_t &= \widetilde{B}_t r \,\mathrm{d}t, & \widetilde{B}_0 &= 1, \\
\mathrm{d}\widetilde{S}_t &= \widetilde{S}_t \left(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\right), & \widetilde{S}_0 &= S_0,
\end{aligned}
$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $\widetilde{S}_0 = S_0 > 0$. Denote by $S = (S_t)_{0 \leq t \leq T}$ the discounted stock price process. Recall that the unique equivalent martingale measure Q for S is given by the Radon–Nikodým derivative

$$
\frac{dQ}{dP} = \mathcal{E}\left(-\int \frac{\mu - r}{\sigma} dW\right)_T .
$$

(a) Find the dynamics of the process $1/S$ under the measure Q and express them in terms of a Q-Brownian motion.

(b) Show that $1/S$ is a Q-submartingale.

(c) Find the hedging strategy for the *log-contract* g with the discounted payoff given by

$$
g(S_T) = \log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T.
$$