Question 1

The correct answers are:

(a) (3)

(b) (1)

(c) (2)

(d) (1)

(e) (1)

(f) (3)

(g) (1)

(h) (2)

Question 2

(a) We construct the model on the canonical path space. Set

$$\Omega = \{\omega = (x_1, x_2) \,|\, x_1 \in \{1, 2, 3\}; x_2 \in \{1, 2\}\},$$

$\mathcal{F} = 2^\Omega$. Next, define P by setting $P[\{(x_1, x_2)\}] = p_{x_1} p_{x_1,x_2}$, where

$$p_1 = 0.5, \;\; p_2 = 0.3, \;\; p_3 = 0.2;$$
$$p_{1,1} = 0.25, \;\; p_{1,2} = 0.75, \;\; p_{2,1} = 0.6, \;\; p_{2,2} = 0.4 \text{ and } p_{3,1} = p_{3,2} = 0.5.$$

Finally, define random variables Y_1 by

$$\begin{aligned} Y_1\big((1,1)\big) &= Y_1\big((1,2)\big) = 1 - 0.1, \\ Y_1\big((2,1)\big) &= Y_1\big((2,2)\big) = 1 + 0.2, \\ Y_1\big((3,1)\big) &= Y_1\big((3,2)\big) = 1 + 0.25, \end{aligned}$$

and Y_2 by

$$\begin{aligned} Y_2\big((1,1)\big) &= Y_2\big((2,1)\big) = Y_2\big((3,1)\big) = 1 - 0.2, \\ Y_2\big((1,2)\big) &= Y_2\big((2,2)\big) = Y_2\big((3,2)\big) = 1 + 0.4. \end{aligned}$$

The filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ is given by

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_1 := \sigma(Y_1) \quad \text{and} \quad \mathcal{F}_2 := \sigma(Y_1, Y_2).$$

The bank account process $\widetilde{S}^0 = (\widetilde{S}^0_k)_{k=0,1,2}$ and the stock price process $\widetilde{S}^1 = (\widetilde{S}^1_k)_{k=0,1,2}$ are given by

$$\widetilde{S}^0_k = (1+r)^k \quad \text{and} \quad \widetilde{S}^1_k = 100 \prod_{i=1}^{k} Y_i \quad \text{for } k \in \{0, 1, 2\},$$

respectively.

(b) The first fundamental theorem of asset pricing says that our market is arbitrage-free if and only if there exists an EMM for S^1. The second fundamental theorem of asset pricing says that an arbitrage-free market is complete if and only if there exists a *unique* EMM for S^1.

Since the first period sub-market is a trinomial model about which we know that it is incomplete for any choice of $r > -1$, we also know that the entire market cannot be complete for any choice of $r > -1$. We therefore only look for $r > -1$ for which the set of EMMs for S^1 is not empty and this set will never be a singleton.

Any probability measure Q equivalent to P on $\mathcal{F}$ can be described by

$$Q[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2},$$

where the transition probabilities $q_{x_1}, q_{x_1,x_2} \in (0, 1)$ satisfy

$$\sum_{x_1=1}^{3} q_{x_1} = 1 \quad \text{and} \quad \sum_{x_2=1}^{2} q_{x_1,x_2} = 1 \quad \text{for every } x_1 \in \{0, 1, 2\}.$$

Next, S^1 is a $(Q, \mathbb{F})$-martingale if and only if we additionally have that

$$E_Q\left[\frac{Y_1}{1+r}\right] = 1 \quad \text{and} \quad E_Q\left[\frac{Y_2}{1+r}\,\middle|\,\mathcal{F}_1\right] = E_Q\left[\frac{Y_2}{1+r}\,\middle|\,\sigma(Y_1)\right] = 1.$$

But the corresponding transition probabilities q_{x_1,x_2} cannot depend on x_1 because the structure of the second period sub-markets is the same for all possible movements in the first period. So we can conclude that Y_1 and Y_2 are independent under Q and the above is equivalent to $q_{x_1}, q_{x_1,x_2} \in (0, 1)$,

$$\begin{aligned} 0.9q_1 + 1.2q_2 + 1.25q_3 &= 1 + r, \\ 0.8q_{1,1} + 1.4(1 - q_{1,1}) &= 1 + r, \end{aligned} \tag{1}$$

and $q_{1,2} = 1 - q_{1,1}$, $q_{2,1} = q_{3,1} = q_{1,1}$, $q_{2,2} = q_{3,2} = q_{1,2}$. The unique solution to the second equation in (1) is given by $q_{1,1} = \frac{2}{3} - \frac{5}{3}r$, which then implies that we must have $r \in \left(-\frac{1}{5}, \frac{2}{5}\right) = (-0.2, 0.4)$ in order for the second period sub-markets to be free of arbitrage. Parametrizing $q_1 = \lambda$ for $\lambda \in (0,1)$ and setting $q_2 = 1 - \lambda - q_3$, we obtain that

$$q_3 = 20r - 4 + 6\lambda, \quad \frac{2}{3} - \frac{10}{3}r < \lambda < \frac{5}{6} - \frac{10}{3}r,$$

where the second condition ensures that $q_3 \in (0,1)$. So if we want the set of EMMs to be non-empty, we must have that

$$0 < \frac{5}{6} - \frac{10}{3}r < 1 \quad \text{and} \quad \frac{2}{3} - \frac{10}{3}r < 1 \iff r \in \left(-\frac{1}{10}, \frac{1}{4}\right) = (-0.1, 0.25).$$

Because $q_3 = 20r - 4 + 6\lambda$, we get

$$q_2 = 1 - \lambda - q_3 = -20r + 5 - 7\lambda, \quad \frac{4}{7} - \frac{20}{7}r < \lambda < \frac{5}{7} - \frac{20}{7}r,$$

where the second condition ensures that $q_2 \in (0,1)$. So we must have that

$$0 < \frac{5}{7} - \frac{20}{7}r < 1 \quad \text{and} \quad \frac{4}{7} - \frac{20}{7}r < 1 \iff r \in \left(-\frac{1}{10}, \frac{1}{4}\right) = (-0.1, 0.25).$$

Bringing all the conditions on r together, we get that the given market is arbitrage-free if and only if $r \in \left(-\frac{1}{10}, \frac{1}{4}\right) = (-0.1, 0.25)$.

Alternatively, one could simply realize that the above model is a composition of a trinomial and a binomial model, for each of which we know a necessary condition for NA from the class. More specifically, the growth rate of $\widetilde{S}^0$ needs to be between the lowest and the largest possible growth rates of $\widetilde{S}^1$ for each of the two periods. For the first one, this means that $r \in \left(-\frac{1}{10}, \frac{1}{4}\right)$; for the second one, since the sub-markets are identical, $r \in \left(-\frac{1}{5}, \frac{2}{5}\right)$. Since both of these conditions need to hold true, we get that $r \in \left(-\frac{1}{10}, \frac{1}{4}\right)$, as before.

(c) An arbitrage opportunity is an admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth, with $V_2(\varphi) \geq 0$ P-a.s. and with $P[V_2(\varphi) > 0] > 0$.

Since the bank account grows faster than the stock in all states, the simplest arbitrage opportunity is to short, say, one stock and invest the proceedings into the bank account. Formally,

$$\vartheta_0 \equiv 0, \quad \vartheta_1 \equiv -1, \quad \vartheta_2 \equiv -1,$$

which clearly gives that $V_0(\varphi) = 0$. Moreover,

$$\widetilde{V}_2(\varphi) = 100 \times (1+r)^2 - \widetilde{S}^1_2 = 256 - \widetilde{S}^1_2 > 0 \ \ P\text{-a.s.}$$

shows that $V_2(\varphi) \geq 0$ P-a.s. and $P[V_2(\varphi) > 0] > 0$. So the above ϑ clearly defines an arbitrage opportunity.

(d) In order for the extended market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ to be arbitrage-free, we need $S^2 := \widetilde{S}^2 / \widetilde{S}^0$ to be a Q-martingale for some $Q \in \mathbb{P}_e(S^1)$. This will guarantee existence of an EMM for S^1 and S^2 at the same time. Since we know that the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free, the set of all EMMs is non-empty. Let us therefore fix a $Q \in \mathbb{P}_e(S^1)$ and define $S^2 = (S^2_k)_{k=0,1,2}$ by

$$S^2_2 = H \quad \text{and} \quad S^2_k = E_Q[H \,|\, \mathcal{F}_k] \quad \text{for } k \in \{0,1\},$$

and some arbitrary non-constant $H \in L^+_0(\mathcal{F}_2)$. This process is a Q-martingale by construction, and since S^0 and S^1 are Q-martingales because we picked $Q \in \mathbb{P}_e(S^1)$ and $S^0 \equiv 1$, the first fundamental theorem of asset pricing gives that $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is arbitrage-free.

Question 3

(a) In order to show that τ is a stopping time, it is enough to show that $\{\tau \leq k\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}_0$. We can write

$$\{\tau \leq k\} = \bigcup_{i=0}^{k} \{S_i = -a \text{ or } S_i = b\} = \bigcup_{i=0}^{k} \big(\{S_i = -a\} \cup \{S_i = b\}\big).$$

But S is adapted by the definition of $\mathbb{F}$, so $\{S_i = -a\} \in \mathcal{F}_i \subseteq \mathcal{F}_k$ and $\{S_i = b\} \in \mathcal{F}_i \subseteq \mathcal{F}_k$ for any $k \in \mathbb{N}_0$ and all $i \in \{0, 1, \dots, k\}$. Since σ-algebras are closed under countable (therefore also) finite unions, this implies that $\{\tau \leq k\} \in \mathcal{F}_k$ for all k.

(b) Clearly, S is adapted and integrable since for each $n \in \mathbb{N}_0$, S_n is a finite sum of $\mathcal{F}_n$-measurable and integrable random variables. For the martingale property, we compute

$$E\left[S_{n+1} - S_n \,|\, \mathcal{F}_n\right] = E\left[X_{n+1} \,|\, \mathcal{F}_n\right] = E\left[X_{n+1}\right] = 0.$$

In order to show that the stopped processes S^τ is an $\mathbb{F}$-martingale as well, we write

$$S_{n \wedge \tau} = \sum_{i=1}^{n} \mathbb{1}_{\{\tau \geq i\}} \Delta S_i.$$

Since $\{\tau \geq i\} = \{\tau \leq i-1\}^c$, $\mathbb{1}_{\{\tau \geq i\}}$ is predictable. It is also clearly bounded, so Theorem I.3.1 in the lecture notes tells us that this process is indeed an $\mathbb{F}$-martingale.

(c) We can argue as follows. Let

$$B_k = \{X_i = 1 \text{ for } (k-1)(a+b) \leq i < k(a+b)\}.$$

Because the X_i are independent, the B_k are independent, too. Moreover, $P\left[B_k\right] = \left(\frac{1}{2}\right)^{a+b}$ by the independence of X_i. Now let n be a large positive integer. If $\omega \in B_k$ for some $1 \leq k \leq n$, then $S_i(\omega)$ must reach $-a$ or b for some $1 \leq i \leq n(a+b)$, i.e.

$$\tau(\omega) \leq n(a+b).$$

Therefore

$$\{\tau > n(a+b)\} \subseteq \bigcap_{k=1}^{n} B_k^c.$$

Using the independence of the B_k, we obtain

$$P\left[\tau > n(a+b)\right] \leq P\left[\bigcap_{k=1}^{n} B_k^c\right] = \left(1 - 2^{-a-b}\right)^n \to 0.$$

So $P\left[\tau = \infty\right] = 0$.

(d) Since we know that τ is P-a.s. finite, $S_{n \wedge \tau} \to S_\tau$ P-a.s. as $n \to \infty$. Since we clearly have that $|S_\tau| \leq (a \vee b) \in L^1(P)$, we have by the dominated convergence theorem that

$$E\left[S_\tau\right] = E\left[\lim_{n \to \infty} S_{n \wedge \tau}\right] = \lim_{n \to \infty} E\left[S_{n \wedge \tau}\right] = \lim_{n \to \infty} S_0 = 0,$$

where the third equality uses that S^τ is a martingale.

(e) Note that S_τ only takes two values, more specifically $-a$ and b. We can therefore clearly write

$$0 = E\left[S_\tau\right] = -aP\left[S_\tau = -a\right] + bP\left[S_\tau = b\right].$$

Since we additionally have that

$$P\left[S_\tau = -a\right] + P\left[S_\tau = b\right] = 1$$

we obtain that $P\left[S_\tau = -a\right] = b/(a+b)$.

Question 4

(a) Set $Y_t^1 := e^{W_t + t/2}$ and $Y_t^2 := e^{W_t - t/2}$, $t \geq 0$, so that $X_t = Y_t^1 + Y_t^2$, $t \geq 0$. Applying Itô's formula to the continuous semimartingale $(W_t, t)_{t\geq 0}$ and the functions $f(x,t) = e^{x+t/2}$ and $g(x,t) = e^{x-t/2}$, respectively, gives that

$$dY_t^1 = Y_t^1 dW_t + \frac{1}{2} Y_t^1 dt + \frac{1}{2} Y_t^1 d[W]_t = Y_t^1 (dW_t + dt)$$

and, analogously,

$$dY_t^2 = Y_t^2 dW_t + \frac{1}{2} Y_t^2 dt - \frac{1}{2} Y_t^2 d[W]_t = Y_t^2 dW_t.$$

So

$$dX_t = (Y_t^1 + Y_t^2) dW_t + Y_t^1 dt = X_t dW_t + e^{W_t + t/2} dt.$$

We also clearly have that $X_0 = 2$.

(b) We first show that the process $\int W^2 dW$ is a $(P, \mathbb{F})$-martingale. Note that the integrand W^2 is adapted and continuous, therefore predictable and locally bounded, so $\int W^2 dW$ is at least a local $(P, \mathbb{F})$-martingale. We now use a result shown in exercise sheet 14, which says that if $W^2 \in L^2(W^T)$ for all $T \geq 0$, then $\int W^2 dW$ is even a true $(P, \mathbb{F})$-martingale. We compute

$$\begin{aligned} \|W^2\|^2_{L^2(W^T)} &= E\left[\int_0^\infty W_u^4 d(u \wedge T)\right] = E\left[\int_0^T W_u^4 du\right] = \int_0^T E\left[W_u^4\right] du \\ &= \int_0^T 3u^2 du = T^3 < \infty, \end{aligned}$$

which shows that $W^2 \in L^2(W^T)$ for all $T \geq 0$, and we can conclude that $\int W^2 dW$ is indeed a $(P, \mathbb{F})$-martingale.

Now, X is integrable by assumption, and it is clearly adapted because it is a continuous transformation of $\int W^2 dW$, which is adapted. Next, Jensen's inequality for conditional expectations gives us for all $0 \leq s \leq t$ that

$$\begin{aligned} E[X_t \,|\, \mathcal{F}_s] &= E\left[f\left(\int_0^t W_u^2 dW_u \right) \middle| \mathcal{F}_s \right] \leq f\left(E\left[\int_0^t W_u^2 dW_u \middle| \mathcal{F}_s \right] \right) \\ &= f\left(\int_0^s W_u^2 dW_u \right) = X_s, \end{aligned}$$

which shows the submartingale property of X. Since we know that equality in Jensen's inequality holds if and only if f is an affine function, we know that X is a $(P, \mathbb{F})$-martingale if and only if f is of the form $f : x \mapsto ax + b$ for some $a, b \in \mathbb{R}$. Taking any $a \neq 0$ thus gives us the desired example.

(c) Note that we have that

$$X_t = \mathcal{E}\left(\int \mu_s ds + \int \sigma_s dW_s \right)_t = \exp\left(\int_0^t \sigma_s dW_s + \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2 \right) ds \right).$$

This implies that

$$Y_t = \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right) = \mathcal{E}\left(\int \sigma_s dW_s \right)_t =: \mathcal{E}(M)_t$$

We have seen in the exercise sheets that $\mathcal{E}(M)$ is a local $(P, \mathbb{F})$-martingale if and only if M is a local $(P, \mathbb{F})$-martingale. But M is defined as a stochastic integral with respect to W (a $(P, \mathbb{F})$-martingale) of an adapted and continuous, thus predictable and locally bounded process σ, so M is a local $(P, \mathbb{F})$-martingale and we can conclude that so is Y.

(d) Applying Itô's formula to $f(x) = x^2$ and the continuous $(P, \mathbb{F})$-martingale W yields

$$W_T^2 = 2\int_0^T W_s dW_s + \int_0^T ds = T + \int_0^T 2W_s dW_s = E\left[W_T^2\right] + \int_0^T 2W_s dW_s.$$

This means that $c = E\left[W_T^2\right] = T$ and $\psi_t = 2W_t$ for $t \in [0, T]$.

Alternatively, analogously to the lecture,

$$E\left[W_T^2 \,\middle|\, \mathcal{F}_t\right] = E\left[(W_t + W_T - W_t)^2 \,\middle|\, \mathcal{F}_t\right] = W_t^2 + T - t =: g(W_t, t),$$

so

$$E\left[W_T^2 \,\middle|\, \mathcal{F}_t\right] = g(0,0) + \int_0^t \frac{\partial g}{\partial x}(W_s, s) dW_s.$$

Setting $t = T$ and using that W_T^2 is $\mathcal{F}_T$-measurable then leads to the same representation as above.

Question 5

(a) One way to show that S^1 satisfies the given SDE is by directly applying Itô's formula to the function $f(x,y) = \frac{x}{y}$ and the semimartingale $(\widetilde{S}^1_t, \widetilde{S}^0_t)_{t\in[0,T]}$. However, one can also use the product rule. Note that the first SDE is just an ODE whose unique solution is given by $\widetilde{S}^0 = e^{rt}$. Defining $X_t := e^{-rt}$, $Y_t := \widetilde{S}^1_t$, we get that $S^1_t = X_t Y_t$. Since e^{-rt} is continuous and of finite variation, we obtain by the product rule that

$$dS^1_t = e^{-rt} d\widetilde{S}^1_t + \widetilde{S}^1_t dY_t \qquad \Longleftrightarrow \qquad dS^1_t = S^1_t \mu dt + S^1_t \sigma dW_t - S^1_t r dt.$$

Grouping the "dt" terms together yields the desired result.

(b) The density process $\bar{Z} = (\bar{Z}_t)_{t\in[0,T]}$ is by definition given by

$$\begin{aligned}\bar{Z}_t = E_P\left[\frac{d\overline{Q}}{dP}\,\middle|\,\mathcal{F}_t\right] &= E_P\left[\exp\left(\frac{\mu - r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right)\,\middle|\,\mathcal{F}_t\right] \\ &= \exp\left(\frac{\mu - r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right) = \mathcal{E}\left(\frac{\mu - r}{\sigma}W\right)_t,\end{aligned}$$

where the third equality holds since we know from Proposition IV.2.2 from the lecture notes that $\left(\exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t\right)\right)_{t\in[0,T]}$ is a $(P,\mathbb{F})$-martingale.

In order to show that $\overline{Q} \approx P$, it is enough to show that $\frac{d\overline{Q}}{dP} > 0$. This is clear from the above, since $\bar{Z}$ is defined as a stochastic exponential of a continuous process and is thus strictly positive.

Since $\bar{Z} = \mathcal{E}(\overline{L})$ with $\overline{L} := \frac{\mu - r}{\sigma}W$, we have by Theorem VI.2.3 from the lecture notes that the process $\overline{W} = (\overline{W}_t)_{t\in[0,T]}$ defined by

$$\overline{W}_t = W_t - \left[\frac{\mu - r}{\sigma}W, W\right]_t = W_t - \frac{\mu - r}{\sigma}t$$

is a Brownian motion with respect to $\overline{Q}$ and $\mathbb{F}$. We can therefore rewrite the SDE for S^1 as

$$dS^1_t = S^1_t(\mu - r)dt + S^1_t \sigma d\overline{W}_t + S^1_t(\mu - r)dt \qquad \Longleftrightarrow \qquad dS^1_t = 2S^1_t(\mu - r)dt + S^1_t \sigma d\overline{W}_t,$$

or in integral form

$$S^1_t = S^1_0 + 2(\mu - r)\int_0^t S^1_s ds + \sigma \int_0^t S^1_s d\overline{W}_s.$$

In order for the above process to be a $(\overline{Q},\mathbb{F})$-martingale, we need the "ds" integral to vanish, which happens if and only if $\mu = r$.

(c) Let $Q^* \approx P$ on $\mathcal{F}_T$ denote the unique EMM for S^1 on $[0,T]$ in the Black–Scholes model. We have seen in the lecture that under Q^*, we have that

$$dS^1_t = S^1_t \sigma dW^*_t \quad \text{or} \quad S^1_t = S^1_0 \exp\left(\sigma W^*_t - \frac{1}{2}\sigma^2 t\right),$$

where W^* is a Brownian motion with respect to Q^* and $\mathbb{F}$. The unique arbitrage-free *discounted* price process for the binary call option is given by

$$V_t = E_{Q^*}\left[e^{-rT}\mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}}\,\middle|\,\mathcal{F}_t\right] = E_{Q^*}\left[e^{-rT}\mathbb{1}_{\{S^1_T \geq \widetilde{K}e^{-rT}\}}\,\middle|\,\mathcal{F}_t\right].$$

Denoting $K = \widetilde{K}e^{-rT}$, we can write

$$\begin{aligned}
V_t &= e^{-rT} E_{Q^*}\left[\mathbb{1}_{\{S_T^1 \geq K\}} \,\middle|\, \mathcal{F}_t\right] = e^{-rT} Q^*\left[x \exp\left(\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)\right) \geq K\right]\Bigg|_{x=S_t^1} \\
&= e^{-rT} Q^*\left[\log x + \sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t) \geq \log K\right]\Bigg|_{x=S_t^1} \\
&= e^{-rT} Q^*\left[\frac{W_T^* - W_t^*}{\sqrt{T-t}} \geq \frac{\log\frac{K}{x} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right]\Bigg|_{x=S_t^1} \\
&= e^{-rT}\left(1 - \Phi\left(\frac{\log\frac{K}{S_t^1} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) = e^{-rT}\Phi\left(\frac{\log\frac{S_t^1}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

For the *undiscounted* price process $\widetilde{V} = (\widetilde{V}_t)_{t\in[0,T]}$ we therefore have that

$$\widetilde{V}_t = e^{-r(T-t)}\Phi\left(\frac{\log\frac{S_t^1}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) = e^{-r(T-t)}\Phi\left(\frac{\log\frac{\widetilde{S}_t^1}{\widetilde{K}} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Denoting by ϕ the density of standard normal distribution $\mathcal{N}(0,1)$, the hedging strategy $\varphi \,\widehat{=}\, (V_0, \vartheta)$ with $\vartheta = (\vartheta_t)_{t\in[0,T]}$ is given by

$$\begin{aligned}
V_0 &= e^{-rT}\Phi\left(\frac{\log\frac{S_0^1}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) = e^{-rT}\Phi\left(\frac{\log\frac{\widetilde{S}_0^1}{\widetilde{K}} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right), \\
\vartheta_t &= \frac{e^{-rT}}{\sigma S_t^1\sqrt{T-t}}\phi\left(\frac{\log\frac{S_t^1}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{e^{-r(T-t)}}{\sigma \widetilde{S}_t^1\sqrt{T-t}}\phi\left(\frac{\log\frac{\widetilde{S}_t^1}{\widetilde{K}} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$