ETH Zürich, HS 2017
D-MATH
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August 2018

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐
Matriculation number	

Leave blank

Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as much as possible.
- ◇ Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

Do not fill in

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following eight subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The discounted market is denoted by (S^0, S^1).

(a) Let (S^0, S^1) be an arbitrage-free incomplete market. Which of the following statements is **not** true?

(1) The set of all equivalent martingale measures for S^1 is uncountably infinite.

(2) There are multiple ways to assign a price process $V^H = (V^H_k)_{k=0,1,\dots,T}$ to an unattainable payoff $H \in L^0_+(\mathcal{F}_T)$, such that the market (S^0, S^1, V^H) is arbitrage-free.

(3) We cannot replicate any payoff $H \in L^0_+(\mathcal{F}_T)$ by trading in S^0 and S^1.

(b) Let Q be an equivalent martingale measure for (S^0, S^1). Then we always have that

(1) both $(\widetilde{S}^0, \widetilde{S}^1)$ and (S^0, S^1) are free of arbitrage.

(2) $E_Q\left[\widetilde{S}^1_k \,\middle|\, \mathcal{F}_{k-1}\right] = \widetilde{S}^1_{k-1}$ for all $k \in \{1, 2, \dots, T\}$.

(3) $(\widetilde{S}^0, \widetilde{S}^1)$ is a complete market.

(c) Let $M = (M_k)_{k=0,1,\dots,T}$ be a local $(P, \mathbb{F})$-martingale. Then

(1) since we are in finite discrete time, M is even a true $(P, \mathbb{F})$-martingale.

(2) there exists an $\mathbb{F}$-stopping time τ such that M^τ is a local $(P, \mathbb{F})$-martingale.

(3) M is not integrable.

(d) Let $Z = (Z_k)_{k=0,1,\dots,T}$ be the density process of $Q \approx P$ with respect to P. Then

(1) Z is s strictly positive $(P, \mathbb{F})$-martingale.

(2) $\frac{1}{Z}$ is a $(P, \mathbb{F})$-martingale.

(3) ZS^1 is a $(Q, \mathbb{F})$-martingale if and only if S^1 is $(P, \mathbb{F})$-martingale.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let $X = (X_t)_{t\geq 0}$ be a continuous $(P, \mathbb{F})$-semimartingale of finite variation. Then

(1) $\langle W, X\rangle_t = [X]_t$ P-a.s.

(2) $\left\langle 2W, W + \frac{1}{2}X\right\rangle_t = t$ P-a.s.

(3) $\langle W, W^2\rangle_t = \int_0^t W_s ds - t$ P-a.s.

(f) Which statement about a local $(P, \mathbb{F})$-martingale $M = (M_t)_{t\geq 0}$ null at 0 is **not** always true?

(1) If M is uniformly bounded from below, then M is a $(P, \mathbb{F})$-supermartingale.

(2) If M is bounded, then M is even a $(P, \mathbb{F})$-martingale.

(3) $\exp(M)$ is not integrable.

(g) Let N be a $(P, \mathbb{F})$-Poisson process with parameter $\lambda > 0$ and let $X = (X_t)_{t\geq 0}$ satisfy $dX_t = \sqrt{t}dW_t + dN_t$. Then for all $t > 0$

(1) $[X]_t = \frac{1}{2}t^2 + N_t$ P-a.s.

(2) $[X]_t = \frac{1}{2}t^2 + \sqrt{t}N_t$ P-a.s.

(3) $[X]_t = \frac{1}{2}t^2 + \lambda t$ P-a.s.

(h) Let $X = (X_t)$ be given by $X_t = \int_0^t H_s dW_s$ for some bounded predictable $H = (H_t)_{t\geq 0}$. Which of the following is true?

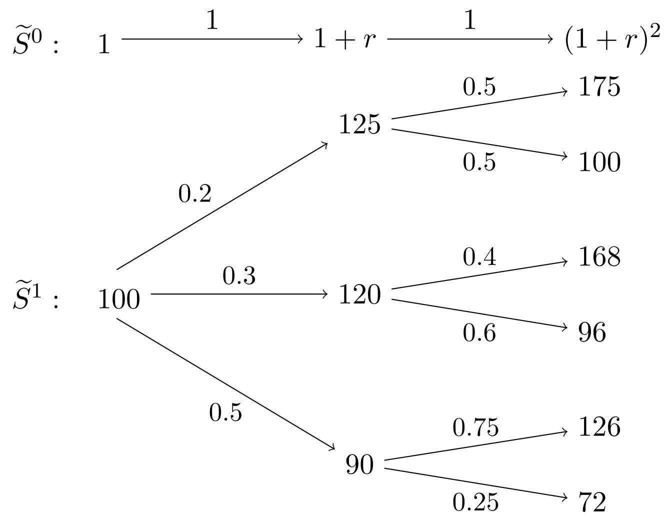
(1) X is a local $(P, \mathbb{F})$-martingale, but not necessarily a $(P, \mathbb{F})$-martingale.

(2) X is a $(P, \mathbb{F})$-martingale.

(3) X is not a local $(P, \mathbb{F})$-martingale, but still a $(P, \mathbb{F})$-semimartingale.

Question 2 (8 **Points**)

Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the bank account $\widetilde{S}^0$ and of the stock $\widetilde{S}^1$ are described by the trees below, where the numbers beside the branches denote transition probabilities and $r > -1$.



(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes $\widetilde{S}^0$ and $\widetilde{S}^1$ such that $\widetilde{S}^1_k = S^1_0 \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$. (2 pt)

(b) Explicitly describe the set of all equivalent martingale measures for $S^1 = \widetilde{S}^1 / \widetilde{S}^0$. For what values of $r > -1$ is the market free of arbitrage? For what values of $r > -1$ is the market free of arbitrage and complete? (3 pt)

(c) Give the definition of an arbitrage opportunity. Set $r = \frac{3}{5}$ and show that the market is not arbitrage-free by explicitly constructing an arbitrage opportunity $\varphi \mathrel{\widehat{=}} (0, \vartheta)$. (1 pt)

(d) Set $r = 0$, in which case the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free. Extend the market by a new *risky* (non-deterministic) asset $\widetilde{S}^2$ in such a way that the extended market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is again free of arbitrage. (2 pt)

Question 3 (8 **Points**)

Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P[X_i = 1] = P[X_i = -1] = 0.5$. Set $S_0 = 0$, and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be given by

$$\mathcal{F}_n = \sigma(S_k; 0 \leq k \leq n) \quad \text{for } n \in \mathbb{N}_0.$$

Let a, b be two positive integers. Define $\tau := \inf\{n > 0 : S_n = -a \text{ or } S_n = b\}$.

(a) Show that τ is an $\mathbb{F}$-stopping time. (1 pt)

(b) Show that both $S = (S_n)_{n\in\mathbb{N}_0}$ and $S^\tau = (S_{n\wedge\tau})_{n\in\mathbb{N}_0}$ are $\mathbb{F}$-martingales. (2 pt)

(c) Show that $P[\tau < \infty] = 1$.

Hint: Consider the sets

$$\begin{aligned} B_k &= \{X_i = 1 \textit{ for } (k-1)(a+b) \leq i < k(a+b)\} \\ &= \{\textit{the } k\textit{-th block of length } a+b \textit{ in the } X\textit{-sequence consists only of 1s}\} \end{aligned}$$

(2 pt)

(d) Using that τ is P-a.s. finite, compute $E[S_\tau]$. (2 pt)

(e) What is the probability that the random walk S reaches $-a$ before b? (1 pt)

Question 4 (8 Points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $W = (W_t)_{t\geq 0}$ a Brownian motion with respect to P and $\mathbb{F}$.

(a) Show that the process $X = (X_t)_{t\geq 0}$ given by

$$X_t = e^{W_t + t/2} + e^{W_t - t/2}, \quad t \geq 0,$$

satisfies the SDE

$$dX_t = X_t dW_t + e^{W_t + t/2} dt, \quad X_0 = 2.$$

(1 pt)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t = f\left(\int_0^t W_u^2 dW_u\right)$$

is a $(P, \mathbb{F})$-submartingale, provided that X is integrable. Give an example of a non-constant $f : \mathbb{R} \to \mathbb{R}$ such that this process is a $(P, \mathbb{F})$-martingale.
Hint: If $Z \sim \mathcal{N}(0, t)$, then $E\left[Z^4\right] = 3t^2$. (3 pt)

(c) Let $\mu = (\mu_t)_{t\geq 0}$ and $\sigma = (\sigma_t)_{t\geq 0}$ be two continuous adapted processes and $X = (X_t)_{t\geq 0}$ the unique semimartingale that solves

$$dX_t = X_t \left(\mu_t dt + \sigma_t dW_t\right), \quad X_0 = 1.$$

Show that the process $Y = (Y_t)_{t\geq 0}$ given by

$$Y_t = X_t \exp\left(-\int_0^t \mu_s ds\right)$$

is a local $(P, \mathbb{F})$-martingale. (2 pt)

(d) Find the representation of the random variable W_T^2 as

$$W_T^2 = c + \int_0^T \psi_s dW_s,$$

i.e. find the constant $c \in \mathbb{R}$ and the predictable process $\psi = (\psi_t)_{t\in[0,T]}$. (2 pt)

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), & \widetilde{S}^1_0 &= S^1_0, \end{aligned}$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $\widetilde{S}^1_0 = S^1_0 > 0$.

(a) Show that $S^1 := \widetilde{S}^1/\widetilde{S}^0$ satisfies

$$dS^1_t = S^1_t(\mu - r)dt + S^1_t \sigma dW_t.$$

(1 pt)

(b) Let the measure $\overline{Q}$ be given by

$$\frac{d\overline{Q}}{dP} = \mathcal{E}\left(\frac{\mu - r}{\sigma} W\right)_T.$$

Compute the density process $\overline{Z}$ of $\overline{Q}$ with respect to P and show that $\overline{Q}$ is equivalent to P. Using Girsanov's theorem, show that for $\mu \neq r$, S^1 is **not** a $(\overline{Q}, \mathbb{F})$-martingale. (3 pt)

(c) Compute the undiscounted value process $\widetilde{V} = (\widetilde{V}_t)_{t\in[0,T]}$ as well as the hedging strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ of a binary call option on $\widetilde{S}^1$ with the undiscounted payoff at time T given by

$$h(\widetilde{S}^1_T) = \mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}}.$$

(4 pt)