Question 1

The correct answers are:

(a) (2)

(b) (2)

(c) (1)

(d) (3)

(e) (3)

(f) (2)

(g) (1)

(h) (3)

Question 2

(a) We need to find real numbers q_u, q_m and q_d satisfying

$$\begin{aligned} q_u + q_m + q_d &= 1, \\ q_u, q_m, q_d &> 0, \\ 200q_u + 100q_m + 50q_d &= 100. \end{aligned} \tag{1}$$

Setting $q_u = 1 - q_m - q_d$ and plugging this into the third equation in (1), we obtain after straightforward simplifications that $q_m = 1 - \frac{3}{2}q_d$. Parametrising $q_d = \lambda$ where $\lambda \in (0, 1)$ so that we satisfy the first and the second equation in (1), we obtain $q_m = 1 - \frac{3}{2}\lambda$. This gives us the condition $\lambda \in \left(0, \frac{2}{3}\right)$ again so that the first and the second equation in (1) are satisfied. Plugging the expressions for q_m and q_d involving λ back into $q_u = 1 - q_m - q_d$, we obtain that $q_u = \frac{1}{2}\lambda$, which implies no additional restrictions on λ. Therefore, the set of all EMMs for S^1 is given by

$$\mathbb{P}_e(S^1) = \left\{ Q_\lambda \mathrel{\widehat{=}} (q_{u,\lambda}, q_{m,\lambda}, q_{d,\lambda}) = \left(\frac{1}{2}\lambda, 1 - \frac{3}{2}\lambda, \lambda\right) : \lambda \in \left(0, \frac{2}{3}\right) \right\}.$$

Two specific solutions are for instance given by the vectors $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ obtained for $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{5}$, respectively.

(b) Let $\|H\|_\infty = \max_{i=1,2,3} H(\omega_i)$ denote the sup-norm of H. The self-financing strategy $\vartheta = 0$ and $V_0 = \alpha\|H\|_\infty$ is clearly a superreplicating strategy for H for any $\alpha \geq 1$.

(c) Because $V_1(\varphi) = V_0 + \vartheta(S_1^1 - S_0^1)$, we need to solve

$$\begin{aligned} V_0 + 100\vartheta &\geq 110, \\ V_0 &\geq 10, \\ V_0 - 50\vartheta &\geq 0 \end{aligned}$$

for $V_0, \vartheta \in \mathbb{R}$. Adding the first equation to twice the last one gives $3V_0 \geq 110$, which implies the middle inequality. We thus need to solve

$$\begin{aligned} V_0 + 100\vartheta &= 110, \\ V_0 - 50\vartheta &= 0 \end{aligned}$$

to get $\vartheta = \frac{11}{15}$ and then $V_0 = 50\vartheta = \frac{110}{3}$. So $\pi(H) = \frac{110}{3}$.

(d) We need to find nonnegative numbers q_u^*, q_m^* and q_d^* satisfying

$$\begin{aligned} q_u^* + q_m^* + q_d^* &= 1, \\ 200q_u^* + 100q_m^* + 50q_d^* &= 100, \\ 110q_u^* + 10q_m^* &= \frac{110}{3}. \end{aligned}$$

Similarly to (a), the first two equations imply $q_m^* = 1 - 3q_u^*$ and $q_d^* = 2q_u^*$. Plugging back into the third equation, we get $q_u^* = \frac{1}{3}$, $q_m^* = 0$ and $q_d^* = \frac{2}{3}$. The measure Q^* is therefore not equivalent to P since $q_m^* = 0$.

Question 3

(a) Adaptedness of X follows from the fact that X is a local $(P, \mathbb{F})$-martingale. Integrability is also clear since $|X_k| \leq Y_k \in L^1(P)$ for all $k \in \mathbb{N}_0$ by assumption. It therefore remains to show the martingale property.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for X. The assumption $|X_j| \leq Y_k$ P-a.s. for all $0 \leq j \leq k$ and $Y_k \in L^1(P)$ for all $k \in \mathbb{N}$ enables us to use the dominated convergence theorem to write for all $k \in \mathbb{N}$ that

$$E\left[X_k \,|\, \mathcal{F}_{k-1}\right] = E\left[\lim_{n\to\infty} X_{k\wedge\tau_n} \,\middle|\, \mathcal{F}_{k-1}\right] = \lim_{n\to\infty} E\left[X_{k\wedge\tau_n} \,|\, \mathcal{F}_{k-1}\right] = \lim_{n\to\infty} X_{(k-1)\wedge\tau_n} = X_{k-1}.$$

The process X is therefore a true $(P, \mathbb{F})$-martingale.

(b) Define $Y = (Y_k)_{k\in\mathbb{N}_0}$ by $Y_k := \sum_{j=0}^{k} |X_j|$. The process Y is integrable by assumption and $|X_j| \leq Y_k$ for all $0 \leq j \leq k$. The result thus follows by a direct application of (a).

(c) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for X. First, we show that X is integrable. Fix a $k \geq 0$ and note that Fatou's lemma yields

$$E\left[X_k\right] = E\left[\lim_{n\to\infty} X_{k\wedge\tau_n}\right] \leq \liminf_{n\to\infty} E\left[X_{k\wedge\tau_n}\right] = \lim_{n\to\infty} E\left[X_0\right] = E\left[|X_0|\right] < \infty.$$

It follows from (b) that X is a true $(P, \mathbb{F})$-martingale.

(d) Define $\tau_n := \inf\{k \in \mathbb{N}_0 : |H_{k+1}| > n\}$ with $\inf \emptyset = +\infty$. Then τ_n is an $\mathbb{F}$-stopping time because H is $\mathbb{F}$-predictable; indeed,

$$\{\tau_n \leq \ell\} = \bigcup_{k=0}^{\ell} \{|H_{k+1}| > n\} = \bigcup_{k=1}^{\ell+1} \{|H_k| > n\} \in \mathcal{F}_\ell$$

because each $\{|H_k| > n\} \in \mathcal{F}_{k-1} \subseteq \mathcal{F}_\ell$ for $k \leq \ell + 1$. Moreover, $\tau_n \uparrow \infty$ as $n \to \infty$ because H is real-valued, and for each n and k,

$$|H_k^{\tau_n}| = |H_{k\wedge\tau_n}| = |H_k| \mathbb{1}_{\{\tau_n \geq k\}} + |H_{\tau_n}| \mathbb{1}_{\{\tau_n < k\}} \leq n$$

because $|H_j| \leq n$ for $j \leq \tau_n$ by the definition of τ_n. So H is locally bounded with the localising sequence $(\tau_n)_{n\in\mathbb{N}}$.

Question 4

(a) We compute

$$\begin{aligned}\|\widetilde{N}_-\|^2_{L^2(W^T)} &= E\left[\int_0^\infty \widetilde{N}^2_{t-} d\big[W^T\big]_t\right] = E\left[\int_0^T \widetilde{N}^2_{t-} dt\right] = E\left[\int_0^T \widetilde{N}^2_t dt\right] \\ &= \int_0^T E\big[\widetilde{N}^2_t\big] dt = \int_0^T \lambda t dt = \frac{\lambda}{2}T^2,\end{aligned} \tag{2}$$

where the third equality uses that $\widetilde{N}_t = \widetilde{N}_{t-}$ almost everywhere with respect to the Lebesgue measure on $[0,T]$, the fourth uses Fubini's theorem and the fifth that $E\big[\widetilde{N}^2_t\big] = \lambda t$ since the process $\big(\widetilde{N}^2_t - \lambda t\big)_{t\geq 0}$ is a $(P,\mathbb{F})$-martingale, as shown in Exercise 9.2 (b) in the exercise sheets.

(b) Since $\widetilde{N}_- \in L^2(W^T)$ by (a), the process $\int \widetilde{N}_- dW$ is a square-integrable martingale on $[0,T]$. So

$$E\left[\int_0^T \widetilde{N}_{t-} dW_t\right] = 0.$$

This gives

$$\operatorname{Var}\left[\int_0^T \widetilde{N}_{t-} dW_t\right] = E\left[\left(\int_0^T \widetilde{N}_{t-} dW_t\right)^2\right] = E\left[\int_0^T \widetilde{N}^2_{t-} dt\right] = \frac{\lambda}{2}T^2.$$

The second equality uses Itô's isometry and the third uses (2).

(c) We know from from Exercise 9.2 (a) of the exercise sheets that $\widetilde{N}$ is a $(P,\mathbb{F})$-martingale. So if $W \in L^2(\widetilde{N})$ on $[0,T]$, then M is a $(P,\mathbb{F})$-martingale. We estimate

$$E\left[\int_0^T W_t^2 d\big[\widetilde{N}\big]_t\right] = E\left[\int_0^T W_t^2 dN_t\right] \leq E\left[\int_0^T \Big(\sup_{t\in[0,T]} W_t^2\Big) dN_t\right] = E\left[N_T \sup_{t\in[0,T]} W_t^2\right],$$

where the first equality follows from Exercise 9.2 (b) in which it is shown that $\big[\widetilde{N}\big]_t = N_t$. By the Cauchy–Schwarz inequality, it then follows that

$$E\left[N_T \sup_{t\in[0,T]} W_t^2\right] \leq \left(E\left[N_T^2\right] E\left[\Big(\sup_{t\in[0,T]} W_t^2\Big)^2\right]\right)^{\frac{1}{2}} < \infty.$$

This is because all moments of the Poisson distribution are finite and because

$$\sup_{t\in[0,T]} W_t^2 = \sup_{t\in[0,T]} |W_t|^2 = \Big(\sup_{t\in[0,T]} |W_t|\Big)^2$$

and all moments of $\sup_{t\in[0,T]} |W_t|$ are finite as given in the hint. The second equality above follows from the fact that $\mathbb{R}_+ \ni x \mapsto x^2$ is an increasing function.

(d) Note that the process S is like N a pure jump process and of finite variation. Unlike N, the size of the k-th jump of S is given by W_k and is thus random. Therefore we know from the lecture that

$$[S]_t = \sum_{0<s\leq t} (\Delta S_s)^2 = \sum_{k=0}^{N_t} W_k^2.$$

Question 5

(a) Applying Itô's formula to the continuous semimartingale $\widetilde{S} = (\widetilde{S}^0, \widetilde{S}^1)$ and the C^2-function $f(x,y) := x/y$ on $\mathbb{R}^2_{++}$, we obtain that the P-dynamics of S^1 is given by

$$dS^1_t = d\left(\frac{\widetilde{S}^1_t}{\widetilde{S}^0_t}\right) = \frac{1}{\widetilde{S}^0_t} d\widetilde{S}^1_t - \frac{\widetilde{S}^1_t}{(\widetilde{S}^0_t)^2} d\widetilde{S}^0_t = S^1_t\left((\mu - r)dt + \sigma dW_t\right).$$

Applying again Itô's formula to the continuous semimartingale S^1 and the C^2-function $f(x) := 1/x$ on $\mathbb{R}_{++}$ leads to

$$d\left(1/S^1_t\right) = \frac{1}{S^1_t} dS^1_t \left(\left(r - \mu + \sigma^2\right) dt - \sigma dW_t\right) = -\frac{1}{S^1_t}\sigma\left(dW_t - \frac{r - \mu + \sigma^2}{\sigma} dt\right) = -\frac{1}{S^1_t} d\widehat{W}_t,$$

where $\widehat{W}_t = W_t - \frac{r-\mu+\sigma^2}{\sigma} t$. Girsanov's theorem gives that

$$\widehat{W} = W - \int \frac{r - \mu + \sigma^2}{\sigma} ds = W_t - \left\langle \int \frac{r - \mu + \sigma^2}{\sigma} dW, W \right\rangle_t$$

is a $(\widehat{Q}, \mathbb{F})$-Brownian motion, where $\widehat{Q} \approx P$ on $\mathcal{F}_T$ is given via the Radon–Nikodým derivative

$$\frac{d\widehat{Q}}{dP} = \mathcal{E}\left(\int \frac{r - \mu + \sigma^2}{\sigma} dW\right)_T = \exp\left(\left(\frac{r - \mu + \sigma^2}{\sigma}\right) W_T - \frac{1}{2}\left(\frac{r - \mu + \sigma^2}{\sigma}\right)^2 T\right).$$

(b) Expressing $dW_t = d\widehat{W}_t + \frac{r-\mu+\sigma^2}{\sigma} dt$ in terms of $d\widehat{W}_t$ and plugging back into the P-dynamics of $\widetilde{S}^1$, we get that the $\widehat{Q}$-dynamics of the undiscounted stock process $\widetilde{S}^1$ is given by

$$d\widetilde{S}^1 = \widetilde{S}^1\left((r + \sigma^2)dt + \sigma d\widehat{W}_t\right).$$

Applying Itô's formula with the C^2-function $g(x) = \log x$ on $\mathbb{R}_{++}$, we get

$$d \log \widetilde{S}^1_t = \left(r + \sigma^2/2\right) dt + \sigma d\widehat{W}_t,$$

and hence

$$\log \widetilde{S}^1_T = \log S^1_0 + \left(r + \sigma^2/2\right) T + \sigma \widehat{W}_T \sim \mathcal{N}\left(\log S^1_0 + \left(r + \sigma^2/2\right) T, \sigma^2 T\right).$$

(c) Bayes' formula gives

$$E_Q[H] = E_{\widehat{Q}}\left[\frac{dQ}{d\widehat{Q}} H\right].$$

To compute the Radon–Nikodým derivative $\frac{dQ}{d\widehat{Q}}$, we recall from (a) and from the lecture that the Radon–Nikodým derivative of $\widehat{Q}$ (respectively Q) with respect to P is given by

$$\frac{d\widehat{Q}}{dP} = \exp\left(\left(\frac{r - \mu + \sigma^2}{\sigma}\right) W_T - \frac{1}{2}\left(\frac{r - \mu + \sigma^2}{\sigma}\right)^2 T\right),$$

$$\frac{dQ}{dP} = \exp\left(\left(\frac{r - \mu}{\sigma}\right) W_T - \frac{1}{2}\left(\frac{r - \mu}{\sigma}\right)^2 T\right).$$

The Radon–Nikodým derivative $\frac{dQ}{d\widehat{Q}}$ is therefore given by

$$\frac{dQ}{d\widehat{Q}} = 1/\exp\left(\sigma W_T + \left(\mu - r - \sigma^2/2\right) T\right) = S^1_0/S^1_T,$$

and hence

$$E_Q[H] = E_{\widehat{Q}}\left[\frac{dQ}{d\widehat{Q}} H\right] = E_{\widehat{Q}}\left[\frac{\widetilde{S}^0_T}{\widetilde{S}^1_T} S^1_0 H\right] = \widetilde{S}^0_T S^1_0 E_{\widehat{Q}}\left[\frac{H}{\widetilde{S}^1_T}\right].$$

(d) The discounted initial price of the option is given by the expectation of the discounted payoff under the EMM for S^1, which in turn can be computed under the measure $\widehat{Q}$ as

$$\frac{\widetilde{V}_0}{\widetilde{S}_0^0} = E_Q\left[\frac{\widetilde{S}_T^1 \log \widetilde{S}_T^1}{\widetilde{S}_T^0}\right] = S_0^1 E_{\widehat{Q}}\left[\log \widetilde{S}_T^1\right] = S_0^1\big(\log S_0^1 + (r+\sigma^2/2)T\big),$$

where in the last equality we have used the result from (b).

(e) To avoid arbitrage opportunities, the pricing of any derivative must be done under the EMM Q under which

$$d\widetilde{S}_t^i = r\widetilde{S}_t^i dt + \sigma \widetilde{S}_t^i dW_t^{*,i}$$

for $i = 1, 2$, where $r \in \mathbb{R}$ is the risk-free rate of return. Despite having different drifts under P, the prices of the European call options on $\widetilde{S}^1$ and $\widetilde{S}^2$ therefore coincide. This is consistent with the observation that the Black–Scholes formula does not contain the drift parameter μ.