Exercise 1

The correct answers are:

(a) (2)

(b) (3)

(c) (2)

(d) (1)

(e) (2)

(f) (2)

(g) (3)

(h) (2)

Exercise 2

(a) An arbitrage opportunity is an admissible, self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $\vartheta = (\vartheta_k)_{k=0,1}$ such that $\vartheta_0 = 0$ and

$$V_1(\varphi) = \vartheta_1 \Delta S_1^1 \geq 0, \tag{1}$$
$$P\big[\vartheta_1 \Delta S_1^1 > 0\big] > 0 \tag{2}$$

In this context, admissibility is automatically satisfied; hence we only have to focus on conditions (??) and (??). For the first one, we have

$$\begin{aligned}
&\vartheta_1(S_1^1 - S_0^1) \geq 0 \\
\Leftrightarrow &\begin{cases} 0 \leq \vartheta_1(101 - 101) \\ 0 \leq \vartheta_1(102 - 101) \\ 0 \leq \vartheta_1(105 - 101) \end{cases} \\
\Leftrightarrow &\ \vartheta_1 \geq 0.
\end{aligned}$$

Moreover, at least one of the three inequalities is strict if and only if $\vartheta_1 > 0$. As a result, an arbitrary arbitrage opportunity has to be of the form $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $\vartheta = (\vartheta_k)_{k=0,1}$ such that $\vartheta_0 = 0$ and $\vartheta_1 > 0$.

For example, the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $\vartheta = (\vartheta_k)_{k=0,1}$ such that $\vartheta_0 = 0$ and $\vartheta_1 = 1$ is an arbitrage opportunity.

For an arbitrary d, a self-financing trading strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ is an arbitrage opportunity if and only if

$$\begin{cases} 0 \leq \vartheta_1(100(1+d) - 101) \\ 0 \leq \vartheta_1(102 - 101) \\ 0 \leq \vartheta_1(105 - 101) \end{cases},$$

where at least one of the listed inequalities has to be strict. The second and the third inequalities are satisfied if and only if $\vartheta_1 \geq 0$. As a result, the market is free of arbitrage if and only if $100(1+d) - 101 < 0$ which is equivalent to the condition $d < 0.01$.

(b) We first compute the set of all equivalent martingale measures Q for S^1. Define

$$q_d := Q[\{\omega_d\}], \qquad q_m := Q[\{\omega_m\}], \qquad q_u := Q[\{\omega_u\}].$$

Then Q is an EMM for S^1 if and only if $q_d, q_m, q_u \in (0,1)$, $q_d + q_m + q_u = 1$, and

$$101(1+d)q_d + 101(1+m)q_m + 101(1+u)q_u = 101(1+r),$$

or equivalently

$$\begin{aligned}
(q_d, q_m, q_u) &= \Big(\frac{u-r}{u-d}, 0, \frac{r-d}{u-d}\Big) + \lambda\Big(-\frac{u-m}{u-d}, 1, -\frac{m-d}{u-d}\Big) \\
&= \Big(\frac{2}{3}, 0, \frac{1}{3}\Big) + \lambda(-\frac{1}{2}, 1, -\frac{1}{2}\Big)
\end{aligned}$$

for some $\lambda \in (0, 2/3)$.

For the second part, recall that a payoff H is attainable if and only if H has the same and finite expectation under all EMMs Q for S^1.

For $\lambda \in (0, 2/3)$, let now Q^λ be the EMM for S^1 given by

$$(q_d^\lambda, q_m^\lambda, q_u^\lambda) = \Big(\frac{2}{3}, 0, \frac{1}{3}\Big) + \lambda(-\frac{1}{2}, 1, -\frac{1}{2}\Big).$$

Computing

$$E_{Q^\lambda}[H^{Put}] = 3\Big(\frac{2}{3} - \lambda\frac{1}{2}\Big) + 0\lambda + 0\Big(\frac{1}{3} - \lambda\frac{1}{2}\Big) = 2 - \frac{3}{2}\lambda,$$

we can thus conclude that H^{Put} is not attainable.

(c) (i) Since S^1 and $S^2 := \frac{\widetilde{S}^2}{\widetilde{S}^0}$ are both martingales with respect to Q^*, by the fundamental theorem of asset pricing, the proposed enlargement of the market is free of arbitrage.

(ii) For this sub-point, there are two possible answers:

- One can see from the calculations of point (b) that S^2 is not a Q-martingale for any $Q \in \mathbb{P}_e(S^1)\setminus\{Q^*\}$. As a result, Q^* is the unique EMM for this market, which is thus complete.
- First note that

$$S_0^2 = \widetilde{S}_0^2 = E_{Q^*}\left[H^{Put}\right] = 3q_d^* + 0q_m^* + 0q_u^* = \frac{3}{2}. \tag{3}$$

One can then show that the system of linear equations given by

$$\begin{cases} h_1 = V_0 + \vartheta_1^1 \Delta S_1^1(\omega_d) + \vartheta_1^2 \Delta S_1^2(\omega_d) \\ h_2 = V_0 + \vartheta_1^1 \Delta S_1^1(\omega_m) + \vartheta_1^2 \Delta S_1^2(\omega_m) \\ h_3 = V_0 + \vartheta_1^1 \Delta S_1^1(\omega_u) + \vartheta_1^2 \Delta S_1^2(\omega_u) \end{cases}$$

has a solution $(V_0, \vartheta_1^1, \vartheta_1^2) \in \mathbb{R}^3$ for every triple (h_1, h_2, h_3) such that $h_i \geq 0$ for all $i = 1, 2, 3$. For instance, this can be proved by showing that the determinant of the coefficients matrix is different from 0:

$$\left| \begin{pmatrix} 1 & \Delta S_1^1(\omega_d) & \Delta S_1^2(\omega_d) \\ 1 & \Delta S_1^1(\omega_m) & \Delta S_1^2(\omega_m) \\ 1 & \Delta S_1^1(\omega_u) & \Delta S_1^2(\omega_u) \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & -2 & 3/2 \\ 1 & 1 & -3/2 \\ 1 & 4 & -3/2 \end{pmatrix} \right| = 9 \neq 0.$$

(d) A replication strategy for $H^{Call} := \frac{1}{1+r}\widetilde{H}^{Call} = (S_1^1 - 101)^+$ is an admissible, self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0^{H^{Call}}, \vartheta^1, \vartheta^2)$ with $\vartheta^i = (\vartheta_k^i)_{k=0,1}$ for $i = 1, 2$ such that $\vartheta_0^1 = \vartheta_0^2 = 0$ and

$$H^{Call} = V_T(\varphi) = V_0^{H^{Call}} + \vartheta_1^1 \Delta S_1^1 + \vartheta_1^2 \Delta S_1^2 \qquad P\text{-a.s.} \tag{4}$$

In our context, admissibility is automatically satisfied. Note that $S_0^2 = 3/2$, as showed in equation (**??**). By condition (**??**), we then have

$$\begin{aligned} H^{Call} =& V_0^{H^{Call}} + \vartheta_1^1(S_1^1 - S_0^1) + \vartheta_1^2(S_1^2 - S_0^2) \\ \Leftrightarrow & \begin{cases} 0 = V_0^{H^{Call}} + \vartheta_1^1(99 - 101) + \vartheta_1^2(3 - 3/2) \\ 1 = V_0^{H^{Call}} + \vartheta_1^1(102 - 101) + \vartheta_1^2(0 - 3/2) \\ 4 = V_0^{H^{Call}} + \vartheta_1^1(105 - 101) + \vartheta_1^2(0 - 3/2) \end{cases} \\ \Leftrightarrow & \begin{cases} 0 = V_0^{H^{Call}} - 2\vartheta_1^1 + \frac{3}{2}\vartheta_1^2 \\ 1 = V_0^{H^{Call}} + \vartheta_1^1 - \frac{3}{2}\vartheta_1^2 \\ 4 = V_0^{H^{Call}} + 4\vartheta_1^1 - \frac{3}{2}\vartheta_1^2 \end{cases} \end{aligned}$$

end hence $\vartheta_1^1 = 1$, $\vartheta_1^2 = \frac{2}{3}$, and $V_0^{H^{Call}} = 1$.

Exercise 3

(a) Start computing the density process Z of Q^* with respect to P.

$$Z_k = E\Big[\frac{\mathrm{d}Q^*}{\mathrm{d}P}\Big|\mathcal{F}_k\Big] = E\Big[\Big(\frac{4}{3}\Big)^T S_T^1\Big|\mathcal{F}_k\Big] \overset{(*)}{=} \Big(\frac{4}{3}\Big)^T\Big(\prod_{j=1}^k Y_j\Big)E[Y_1]^{T-k}$$
$$= \Big(\frac{4}{3}\Big)^T\Big(\prod_{j=1}^k Y_j\Big)\Big(\frac{3}{4}\Big)^{T-k} = \Big(\frac{4}{3}\Big)^k S_k^1,$$

for $k = 1,\dots,T$ and $Z_0 = \big(\frac{4}{3}\big)^T E[Y_1]^T = 1$. In $(*)$, we used the i.i.d. property of $(Y_j)_{j=1,\dots,T}$ and the fact that Y_j is $\mathcal{F}_j$-measurable for each $j = 1,\dots,T$.

Since $\frac{\mathrm{d}Q^*}{\mathrm{d}P} > 0$ P-a.s., we already have that $Q^* \approx P$. One thus only has to show that S^1 is a $(Q^*,\mathbb{F})$-martingale.

- Adaptedness is clear.
- For the integrability, note that $|S_k^1| \le (3/2)^k$ P-a.s. for each $k = 1,\dots,T$, since $|Y_j| \le 3/2$ P-a.s. for each $j = 1,\dots,T$.
- It only remains to show the $(Q^*,\mathbb{F})$-martingale property of S^1. Fix $k \in \{0,\dots,T-1\}$, then we have

$$E_{Q^*}[S_{k+1}^1/S_k^1|\mathcal{F}_k] = E_{Q^*}[Y_{k+1}|\mathcal{F}_k] \overset{Bayes}{=} \frac{1}{Z_k}E[Z_{k+1}Y_{k+1}|\mathcal{F}_k]$$
$$= E\Big[\frac{4}{3}\,Y_{k+1}^2\Big|\mathcal{F}_k\Big] \overset{i.i.d.}{=} E\Big[\frac{4}{3}\,Y_1^2\Big] = 1.$$

Alternatively, by the lecture we know that S^1 is a $(Q^*,\mathbb{F})$-martingale is and only if ZS^1 is a $(P,\mathbb{F})$-martingale. But $Z_kS_k^1 = \big(\frac{4}{3}\big)^k(S_k^1)^2 = \prod_{j=1}^k \frac{4}{3}Y_j^2$ is a product of P-i.i.d. random variables, hence a $(P,\mathbb{F})$-martingale if and only if each factor has expectation 1 with respect to P. Noting that $E\big[\frac{4}{3}Y_j^2\big] = 1$ by the remark, we can thus conclude that S^1 is a $(Q^*,\mathbb{F})$-martingale.

(b) We start proving that τ is a stopping time. Fix $k \in \{1,\dots,T-1\}$; then we can compute

$$\{\tau \le k\} = \bigcup_{j=1}^k \underbrace{\{Y_j > 1\}}_{\in\mathcal{F}_j\subseteq\mathcal{F}_k} \in \mathcal{F}_k,$$

since σ-algebras are closed under countable unions. Moreover, since $\tau \ge 1$, we have that $\{\tau \le 0\} = \emptyset \in \mathcal{F}_0$, and since $\tau \le T$, we have that $\{\tau \le T\} = \Omega \in \mathcal{F}_T$.

For the second part, we need to show that φ^0 is adapted, $\vartheta_0 = 0$, and ϑ is predictable. By construction, $\vartheta_0 = 0$. For the predictability of ϑ, we only need to show that $\{k \le \tau\} \in \mathcal{F}_{k-1}$ for each $k = 1,\dots,T$. Fixing $k \in \{1,\dots,T\}$ we have

$$\{k \le \tau\} = \{k > \tau\}^c = \{k-1 \ge \tau\}^c \in \mathcal{F}_{k-1},$$

since $\{k-1 \ge \tau\} \in \mathcal{F}_{k-1}$ by the definition of a stopping time and since σ-algebras are closed under taking complements. The proof of the adaptedness of φ^0 is analogous; in fact, $\varphi^0 = -\vartheta$ is even predictable like ϑ.

(c) The strategy $\overline{\varphi}$ is self-financing if and only if

$$\overline{\varphi}_{k+1}^0 - \overline{\varphi}_k^0 + (\vartheta_{k+1} - \vartheta_k)S_k^1 = \Delta C_{k+1}(\overline{\varphi}) = 0 \qquad P\text{-a.s.},$$

for all $k = 0, \dots, T-1$. Hence, $\overline{\varphi}$ is self-financing if and only if

$$\overline{\varphi}^0_{k+1} - \overline{\varphi}^0_k = -(\vartheta_{k+1} - \vartheta_k) S^1_k = \begin{cases} 1 & \text{if } k = 0 \\ -S^1_k & \text{if } k = \tau \\ 0 & \text{else} \end{cases} .$$

Thus, using that $\overline{\varphi}^0_0 = 0$, for each $k = 1, \dots, T$ we must have that

$$\overline{\varphi}^0_k = \overline{\varphi}^0_0 + \sum_{j=0}^{k-1} (\overline{\varphi}^0_{j+1} - \overline{\varphi}^0_j) = 0 + \mathbb{1}_{\{k-1 \geq 0\}} - S^1_\tau \mathbb{1}_{\{k-1 \geq \tau\}} = 1 - S^1_\tau \mathbb{1}_{\{k > \tau\}}.$$

The value process of $\overline{\varphi}$ is then given by $V_0(\overline{\varphi}) = \overline{\varphi}^0_0 = 0$ and

$$V_k(\overline{\varphi}) = V_0(\overline{\varphi}) + G_k(\vartheta) = 0 + \sum_{j=1}^{k} \vartheta_j \Delta S^1_j = -\sum_{j=1}^{\tau \wedge k} \Delta S^1_j + \sum_{j=(\tau \wedge k)+1}^{k} 0 = 1 - S^1_{\tau \wedge k},$$

for all $k = 1, \dots, T$.

Alternatively, one can also deduce $\overline{\varphi}^0$ from the value process of $\overline{\varphi}$. Indeed for all $k = 1, \dots, T$ we can compute

$$\overline{\varphi}^0_k = V_k(\overline{\varphi}) - \vartheta_k S^1_k = 1 - S^1_{\tau \wedge k} + \mathbb{1}_{\{k \leq \tau\}} S^1_k = 1 - S^1_\tau \mathbb{1}_{\{k > \tau\}}.$$

Finally, since $S^1_k = \prod_{j=1}^k Y_j \leq (3/2)^k \leq (3/2)^T$, we can conclude that for each $k = 1, \dots, T$

$$1 - S^1_{\tau \wedge k} \geq 1 - (3/2)^T \quad P\text{-a.s.},$$

and thus that $\overline{\varphi}$ is an admissible trading strategy.

(d) There are different possible argumentations:

- S^1 is a $(Q^*, \mathbb{F})$-martingale; hence by the stopping theorem, the process

$$V(\overline{\varphi}) = \left(1 - S^1_{\tau \wedge k}\right)_{k=0,\dots,T}$$

  is a $(Q^*, \mathbb{F})$-martingale as well (Corollary 1.3.2).
- S^1 is a $(Q^*, \mathbb{F})$-martingale and ϑ is bounded, hence $V(\overline{\varphi}) = \vartheta \cdot S^1$ is a $(Q^*, \mathbb{F})$-martingale as well (Theorem 1.3.1).

In general, it is always true that the value process $V(\varphi)$ of an admissible self-financing strategy φ is a $(Q^*, \mathbb{F})$-martingale. Indeed, first note that the a-admissibility of φ gives us that the gains process can be written as a stochastic integral process bounded from below:

$$G(\vartheta) = \vartheta \cdot S^1 = V(\varphi) - V_0 \geq -a - |V_0| \quad P\text{-a.s.}$$

Since S^1 is a $(Q^*, \mathbb{F})$-martingale, we can then directly conclude that $G(\vartheta)$, and thus $V(\varphi)$, is a $(Q^*, \mathbb{F})$-martingale as well (Theorem 1.3.3).

Exercise 4

(a) By the product rule, using that the process $(2T - t)_{t\in[0,T]}$ is continuous and of finite variation,

$$\mathrm{d}X_t = (2T-t)\,\mathrm{d}I_t - I_t\,\mathrm{d}t = (2T-t)\frac{1}{2T-t}\,\mathrm{d}W_t - I_t\,\mathrm{d}t = \mathrm{d}W_t - I_t\,\mathrm{d}t.$$

Thus, the quadratic variation of X is $[X]_t = [W]_t = t$, $t \in [0,T]$. Since I is not P-a.s. zero, X is not a local martingale and hence cannot be a Brownian motion.

(b) Fix $a, b \in \mathbb{R}$ and note that $M_t = f(t, W_t)$ for the smooth function $f(t,x) = atx + bx^3$, $t \geq 0$, $x \in \mathbb{R}$. Hence, by Itô's formula,

$$\mathrm{d}M_t = aW_t\,\mathrm{d}t + (at + 3bW_t^2)\,\mathrm{d}W_t + \frac{1}{2}(6bW_t)\,\mathrm{d}\langle W\rangle_t = (a+3b)W_t\,\mathrm{d}t + (at + 3bW_t^2)\,\mathrm{d}W_t,$$

where we use that $\langle W\rangle_t = t$. Therefore, M is a local martingale (with respect to P and $\mathbb{F}$) if and only if its finite variation part is zero, i.e. if and only if $a + 3b = 0$. We claim that in this case, M is even a (true) martingale. Indeed, for $a + 3b = 0$, we have

$$M_t = M_0 + a\int_0^t u\,\mathrm{d}W_u + 3b\int_0^t W_u^2\,\mathrm{d}W_u,$$

and both stochastic integrals are martingales by Exercise 12-3 (c) and (d).

(c) First, assume that $\alpha \neq 0$. Then

$$\lim_{t\to\infty}(W_t + \alpha t)^2 = \lim_{t\to\infty} t^2\left(\frac{W_t}{t} + \alpha\right)^2 = +\infty \quad P\text{-a.s.}$$

by the law of large numbers for Brownian motion, and therefore $\lim_{t\to\infty} Z_t = 0$ P-a.s.

Second, assume that $\alpha = 0$. Then $\limsup_{t\to\infty} W_t = +\infty$ P-a.s. and $\liminf_{t\to\infty} W_t = -\infty$ P-a.s. by the (global) law of the iterated logarithm. Hence, $\limsup_{t\to\infty} W_t^2 = +\infty$ P-a.s. and $\liminf_{t\to\infty} W_t^2 = 0$ P-a.s. as W crosses 0 infinitely often. Therefore, $\limsup_{t\to\infty} Z_t = 1$ P-a.s. and $\liminf_{t\to\infty} Z_t = 0$ P-a.s.

Exercise 5

(a) Define the process $Z = (Z_t)_{t\in[0,T]}$ by $Z_t = S^1_t/S^1_0$. Then Z is a $(Q^*, \mathbb{F})$-martingale because Q^* is an equivalent martingale measure for S^1, $Z_0 = 1$ by construction, and as $S^1 > 0$ P-a.s., also $Z > 0$ P-a.s. Hence, Z is the density process of $\widehat{Q}$ with respect to Q^* and $\widehat{Q}$ is a probability measure equivalent to Q^*.

Now, let $\widetilde{H} \in L^0_+(\mathcal{F}_T)$ and fix $t \in [0,T]$. By the Bayes formula (Lemma 6.2.1 in the lecture notes),

$$\begin{aligned}\widetilde{S}^1_t E_{\widehat{Q}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_t\right] &= \frac{\widetilde{S}^1_t}{Z_t} E_{Q^*}\left[Z_T\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_t\right] = \frac{\widetilde{S}^1_t}{S^1_t} E_{Q^*}\left[S^1_T\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_t\right] \\ &= \widetilde{S}^0_t E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\,\mathcal{F}_t\right] \qquad \widehat{Q}\text{-a.s.}\end{aligned}$$

The assertion follows because $\widehat{Q} \approx Q^* \approx P$ on $\mathcal{F}_T$.

(b) It is known from the lecture notes that

$$W^*_t := W_t + \frac{\mu - r}{\sigma}t, \quad t \in [0,T],$$

defines a Q^*-Brownian motion W^* and that S^1 satisfies the SDE

$$\mathrm{d}S^1_t = S^1_t\sigma\,\mathrm{d}W^*_t.$$

Using Itô's formula, we can compute the dynamics of $\widehat{S}^0 = 1/S^1$ under Q^*:

$$\begin{aligned}\mathrm{d}\widehat{S}^0_t &= -\frac{1}{(S^1_t)^2}\,\mathrm{d}S^1_t + \frac{1}{(S^1_t)^3}\,\mathrm{d}\langle S^1\rangle_t = -\widehat{S}^0_t\sigma\,\mathrm{d}W^*_t + \widehat{S}^0_t\sigma^2\,\mathrm{d}t \\ &= \widehat{S}^0_t\sigma(\sigma\,\mathrm{d}t - \mathrm{d}W^*_t).\end{aligned}$$

Note that $Z = S^1/S^1_0 = \mathcal{E}(\sigma W^*)$. Hence, by Girsanov's theorem (Theorem 6.2.3 in the lecture notes),

$$W^{**}_t := W^*_t - \langle\sigma W^*, W^*\rangle_t = W^*_t - \sigma t, \quad t \in [0,T],$$

defines a $\widehat{Q}$-Brownian motion W^{**}. Thus,

$$\mathrm{d}\widehat{S}^0_t = -\widehat{S}^0_t\sigma\,\mathrm{d}W^{**}_t.$$

As $\widehat{W} := -W^{**}$ is again a $\widehat{Q}$-Brownian motion (Exercise 8-3), the assertion follows.

(c) **Method 1:** By part (b), $\widehat{S}^0$ has the explicit representation as a stochastic exponential

$$\widehat{S}^0_t = \widehat{S}^0_0\mathcal{E}(\sigma\widehat{W})_t = \frac{1}{S^1_0}\exp\left(\sigma\widehat{W}_t - \frac{1}{2}\sigma^2 t\right), \quad t \in [0,T],$$

and $\widetilde{S}^0_t = \exp(rt)$, $t \in [0,T]$. Thus,

$$\widetilde{S}^1_t = \frac{\widetilde{S}^0_t}{\widehat{S}^0_t} = S^1_0\exp\left(-\sigma\widehat{W}_t + \left(r + \frac{\sigma^2}{2}\right)t\right), \quad t \in [0,T],$$

and a standard application of Itô's formula yields

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t\left((r+\sigma^2)\,\mathrm{d}t - \sigma\,\mathrm{d}\widehat{W}_t\right).$$

Method 2: By Itô's formula and the dynamics of $\widehat{S}^0$ from part (b),

$$\mathrm{d}\left(\frac{1}{\widehat{S}^0}\right)_t = -\frac{1}{(\widehat{S}^0_t)^2}\,\mathrm{d}\widehat{S}^0_t + \frac{1}{(\widehat{S}^0_t)^3}\,\mathrm{d}\langle \widehat{S}^0\rangle_t = \frac{1}{\widehat{S}^0_t}\left(-\sigma\,\mathrm{d}\widehat{W}_t + \sigma^2\,\mathrm{d}t\right).$$

Using the product rule, the given dynamics of $\widetilde{S}^0$, and the fact that $\widetilde{S}^0$ is continuous and of finite variation, we then obtain

$$\mathrm{d}\widetilde{S}^1_t = \mathrm{d}\left(\frac{1}{\widehat{S}^0}\widetilde{S}^0\right)_t = \frac{1}{\widehat{S}^0_t}\,\mathrm{d}\widetilde{S}^0_t + \widetilde{S}^0_t\,\mathrm{d}\left(\frac{1}{\widehat{S}^0}\right)_t = \widetilde{S}^1_t\left((r+\sigma^2)\,\mathrm{d}t - \sigma\,\mathrm{d}\widehat{W}_t\right).$$

(d) Let $\widetilde{H} = \widetilde{S}^1_T \mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}}$ denote the undiscounted payoff of the asset-or-nothing call with strike $\widetilde{K} > 0$. To replicate this claim, we first have to compute the discounted value process $V_t := E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\,\mathcal{F}_t\right]$, $t \in [0,T]$. To this end, we first conclude from part (c) that

$$\widetilde{S}^1_t = \widetilde{S}^1_0 \mathcal{E}\left(\int_0^\cdot (r+\sigma^2)\,\mathrm{d}u - \sigma\widehat{W}\right)_t = S^1_0 \exp\left(\left(r + \frac{1}{2}\sigma^2\right)t - \sigma\widehat{W}_t\right), \quad t \in [0,T],$$

so that for $t \in [0,T]$,

$$\widetilde{S}^1_T = \widetilde{S}^1_t \exp\left(-\sigma(\widehat{W}_T - \widehat{W}_t) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right). \tag{5}$$

Now, fix $t \in [0,T]$. Using the result from part (a),

$$V_t = \frac{1}{\widetilde{S}^0_t}\widetilde{S}^0_t E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\,\mathcal{F}_t\right] = \frac{\widetilde{S}^1_t}{\widetilde{S}^0_t} E_{\widehat{Q}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_t\right] = S^1_t E_{\widehat{Q}}\left[\mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}}\,\middle|\,\mathcal{F}_t\right] \quad P\text{-a.s.}$$

Using (**??**), we find that

$$\{\widetilde{S}^1_T \geq \widetilde{K}\} = \left\{\frac{\widehat{W}_T - \widehat{W}_t}{\sqrt{T-t}} \leq d_1(t, \widetilde{S}^1_t)\right\}$$

where

$$d_1(t,\widetilde{s}) = \frac{\log\frac{\widetilde{s}}{\widetilde{K}} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad \widetilde{s} > 0.$$

As $\widehat{W}$ is a $\widehat{Q}$-Brownian motion, $\frac{\widehat{W}_T - \widehat{W}_t}{\sqrt{T-t}}$ is independent of $\mathcal{F}_t$ and standard normally distributed under $\widehat{Q}$. Using also that $\widetilde{S}^1_t$ is $\mathcal{F}_t$-measurable, we obtain

$$V_t = S^1_t \Phi(d_1(t, \widetilde{S}^1_t)) = S^1_t \Phi(d_1(t, S^1_t e^{rt})) = v(t, S^1_t) \quad P\text{-a.s.},$$

where Φ denotes the cumulative distribution function of the standard normal distribution and

$$v(t,x) = x\Phi(d_1(t, xe^{rt})), \quad t \in [0,T],\ x > 0.$$

By definition of V and Itô's formula, $\frac{\widetilde{S}^1_T}{\widetilde{S}^0_T}\mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}} = V_T = v(T, S^1_T)$ and for all $t \in [0,T]$

$$V_t = v(t, S^1_t) = v(0, S^1_0) + \int_0^t \frac{\partial v}{\partial x}(u, S^1_u)\,\mathrm{d}S^1_u \quad P\text{-a.s.}; \tag{6}$$

note that the finite variation terms must vanish since V and S^1 are continuous $(Q^*, \mathbb{F})$-martingales by construction. In particular, the stochastic integral in (**??**) is a $(Q^*, \mathbb{F})$-martingale. We can thus set

$$\begin{aligned}
V_0 &:= v(0, S_0^1) = S_0^1 \Phi(d_1(0, S_0^1)), \\
\vartheta_t &:= \frac{\partial v}{\partial x}(t, S_t^1) = \Phi(d_1(t, S_t^1 e^{rt})) + S_t^1 \varphi(d_1(t, S_t^1 e^{rt})) \frac{1}{S_t^1 e^{rt} \sigma \sqrt{T-t}} e^{rt} \\
&= \Phi(d_1(t, S_t^1 e^{rt})) + \frac{\varphi(d_1(t, S_t^1 e^{rt}))}{\sigma \sqrt{T-t}},
\end{aligned}$$

where φ denotes the density of the standard normal distribution. As ϑ is continuous and adapted, it is predictable and locally bounded.